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A process very similar to multifractional Brownian motion

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Abstract : In [AT05], the multifractional Brownian (mBm) motion is obtained by replacing the constant parameter H of the fractional Brownian motion (fBm) by a smooth enough functional parameter $H(\cdot)$ depending on the time t . Here, we consider the process Z obtained by replacing in the wavelet expansion of the fBm the index H by a function $H(\cdot)$ depending on the dyadic point $k/2^j$. This process was introduced in [BBCI00] to model fBm with piece-wise constant Hurst index and continuous paths. In this work, we investigate the case where the functional parameter satisfies a uniform Hölder condition of order $\beta > \sup_{t \in \mathbb{R}} H(t)$ and one shows that, in this case, the process Z is very similar to the mBm in the following senses: i) the difference between Z and a mBm satisfies a uniform Hölder condition of order $d > \sup_{t \in \mathbb{R}} H(t)$; ii) as a by product, one deduces that at each point $t \in \mathbb{R}$ the pointwise Hölder exponent of Z is $H(t)$ and that Z is tangent to a fBm with Hurst parameter $H(t)$.

1 Introduction and statement of the main results

Throughout this article we denote by $H(\cdot)$ a function defined on the real line and with values in an arbitrary fixed compact interval $[a, b] \subset (0, 1)$. We will always assume that $H(\cdot)$ satisfies a uniform Hölder condition of order $\beta > b$ on each compact $\mathcal{K} \subset \mathbb{R}$ i.e. there is a constant $c_1 > 0$ (which a priori depends on \mathcal{K}) such that for every $t_1, t_2 \in \mathcal{K}$ one has,

$$|H(t_1) - H(t_2)| \leq c_1 |t_1 - t_2|^\beta. \quad (1)$$

We will also assume that $a = \inf\{H(t) : t \in \mathbb{R}\}$ and $b = \sup\{H(t) : t \in \mathbb{R}\}$. Recall that the multifractional Brownian motion (mBm) of functional parameter $H(\cdot)$, which we denote by $X = \{X(t) : t \in \mathbb{R}\}$, is the continuous and nowhere differentiable Gaussian process obtained by replacing the Hurst parameter in the harmonizable representation of fractional Brownian motion

(fBm) by the function $H(\cdot)$; namely, the process X can be represented for each $t \in \mathbb{R}$ as the following stochastic integral

$$X(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} d\widehat{W}(\xi), \quad (2)$$

where $d\widehat{W}$ is “the Fourier transform” of the real-valued white-noise dW in the sense that for any function $f \in L^2(\mathbb{R})$ one has a.s.

$$\int_{\mathbb{R}} f(x) dW(x) = \int_{\mathbb{R}} \widehat{f}(\xi) d\widehat{W}(\xi). \quad (3)$$

Observe that (3) implies that (see [C99, ST06]) the following equality holds a.s., for every t , up to a deterministic smooth, bounded and non-vanishing deterministic function

$$\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} d\widehat{W}(\xi) = \int_{\mathbb{R}} \left\{ |t-s|^{H(t)-1/2} - |s|^{H(t)-1/2} \right\} dW(s).$$

Therefore X is a real-valued process. MbM was introduced independently in [PLV95] and [BJR97]; its main three features of mBm are the following:

- (a) X reduces to a fBm when the function $H(\cdot)$ is constant.
- (b) Contrarily to fBm, $\alpha_X = \{\alpha_X(t) : t \in \mathbb{R}\}$ the pointwise Hölder exponent of X may depend on the location and can be prescribed via the functional parameter $H(\cdot)$; actually one has (see [PLV95, BJR97, AT05, AJT07]) a.s. for each t ,

$$\alpha_X(t) = H(t). \quad (4)$$

Recall that α_X the pointwise Hölder exponent of an arbitrary continuous and nowhere differentiable process X is defined for each $t \in \mathbb{R}$, as

$$\alpha_X(t) = \sup \left\{ \alpha \in \mathbb{R}_+ : \limsup_{h \rightarrow 0} \frac{|X(t+h) - X(t)|}{|h|^\alpha} = 0 \right\}. \quad (5)$$

- (c) At any point $t \in \mathbb{R}$, there is an fBm of Hurst parameter $H(t)$, which is tangent to mBm [BJR97, F02, F03] i.e. for each sequence (ρ_n) of positive real numbers converging to 0, one has,

$$\lim_{n \rightarrow \infty} \text{law} \left\{ \frac{X(t + \rho_n u) - X(t)}{\rho_n^{H(t)}} : u \in \mathbb{R} \right\} = \text{law} \{ B_{H(t)}(u) : u \in \mathbb{R} \}, \quad (6)$$

where the convergence holds in distribution for the topology of uniform convergence on compact sets.

The main goal of our article is to give a natural wavelet construction of a continuous and nowhere differentiable Gaussian process $Z = \{Z(t)\}_{t \in \mathbb{R}}$ which has the same features (a), (b) and (c) as mBm X and which differs from it by a smoother stochastic process $R = \{R(t) : t \in \mathbb{R}\}$ (see Theorem 1).

In order to be able to construct Z , first we need to introduce some notations. In all the sequel we denote by $\{2^{j/2}\psi(2^jx - k) : (j, k) \in \mathbb{Z}^2\}$ a Lemarié-Meyer wavelet basis of $L^2(\mathbb{R})$ [LM86] and we denote by Ψ , the function defined for each $(x, \theta) \in \mathbb{R} \times (0, 1)$ as

$$\Psi(x, \theta) = \int_{\mathbb{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{|\xi|^{\theta+1/2}} d\xi. \quad (7)$$

By using the fact that $\widehat{\psi}$ is a compactly supported C^∞ -function vanishing on a neighborhood of the origin, it follows that Ψ is a well-defined C^∞ -function satisfying the following localization property (see [AT05]): For any $(l, m, n) \in \mathbb{N}^3$ with $l \geq 2$, there is a constant $c_2 > 0$ only depending on (l, m, n) , such that one has

$$\sup_{\theta \in [a, b], x \in \mathbb{R}} |(\partial_x^m \partial_\theta^n \Psi)(x, \theta)| \leq c_2(2 + |x|)^{-l}, \quad (8)$$

where $\partial_x^m \partial_\theta^n \Psi$ denotes the function obtained by differentiating the function Ψ , n times with respect to the variable θ and m times with respect to the variable x . For the sake of the convenience, let us introduce the Gaussian field $B = \{B(t, \theta) : (t, \theta) \in \mathbb{R} \times (0, 1)\}$ defined for each $(t, \theta) \in \mathbb{R} \times (0, 1)$ as

$$B(t, \theta) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{\theta+1/2}} d\widehat{W}(\xi). \quad (9)$$

Observe that for every fixed θ , the Gaussian process $B(\cdot, \theta)$ is a fBm of Hurst parameter θ on the real line. Also observe that the mBm X satisfies for each $t \in \mathbb{R}$,

$$X(t) = B(t, H(t)). \quad (10)$$

By expanding for every fixed (t, θ) , the kernel function $\xi \mapsto \frac{e^{it\xi} - 1}{|\xi|^{\theta+1/2}}$ in the orthonormal basis of $L^2(\mathbb{R})$, $\{2^{-j/2}(2\pi)^{1/2}e^{i2^{-j}k\xi}\widehat{\psi}(-2^{-j}\xi) : (j, k) \in \mathbb{Z}^2\}$ and by using the isometry property of the stochastic integral in (9), it follows that

$$B(t, \theta) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j\theta} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, \theta) - \Psi(-k, \theta) \right\}, \quad (11)$$

where $\{\varepsilon_{j,k} : (j, k) \in \mathbb{Z}^2\}$ is a sequence of independent $\mathcal{N}(0, 1)$ Gaussian random variables and where the series is, for every fixed (t, θ) , convergent in $L^2(\Omega)$, throughout this article Ω denotes the underlying probability space (in fact this series is also convergent in a much stronger sense, see part (i) of the following remark).

Remark 1. The field B has already been introduced and studied in [AT05]; let us here recall some its useful properties:

- (i) The series in (11) is a.s. uniformly convergent in (t, θ) on each compact subset of $\mathbb{R} \times (0, 1)$, B is therefore a continuous Gaussian field.
- (ii) The low frequency component of B , namely the field $\dot{B} = \{\dot{B}(t, \theta) : (t, \theta) \in \mathbb{R} \times (0, 1)\}$ defined for all $(t, \theta) \in \mathbb{R} \times (0, 1)$ as

$$\dot{B}(t, \theta) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-j\theta} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, \theta) - \Psi(-k, \theta) \right\}, \quad (12)$$

is a C^∞ Gaussian field. Therefore (1) and (10) imply that the low frequency component of the mBm X , namely the Gaussian process $\dot{X} = \{\dot{X}(t)\}_{t \in \mathbb{R}}$ defined for each $t \in \mathbb{R}$ as

$$\dot{X}(t) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-jH(t)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(t)) - \Psi(-k, H(t)) \right\}, \quad (13)$$

satisfies a uniform Hölder condition of order β on each compact subset of \mathbb{R} . Thus, in view of (b) and the assumption $\sup\{H(t) : t \in \mathbb{R}\} < \beta$ the pointwise Hölder exponent of X is only determined by its high frequency component, namely the continuous Gaussian process $\ddot{X} = \{\ddot{X}(t)\}_{t \in \mathbb{R}}$ defined for each $t \in \mathbb{R}$ as

$$\ddot{X}(t) = \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{\infty} 2^{-jH(t)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(t)) - \Psi(-k, H(t)) \right\}. \quad (14)$$

Definition 1. *The process $Z = \{Z(t) : t \in \mathbb{R}\}$ is defined for each $t \in \mathbb{R}$ as*

$$Z(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(k/2^j)) - \Psi(-k, H(k/2^j)) \right\}. \quad (15)$$

In view of (11) it is clear that the process Z reduces to a fBm when the function $H(\cdot)$ is constant; this means that the process Z has the same feature (a) as mBm.

Remark 2. Using the same technics as in [AT05] one can show that:

- (i) The series in (15) is a.s. uniformly convergent in t on each compact interval of \mathbb{R} ; therefore Z is a well-defined continuous Gaussian process.
- (ii) The low frequency component of the process Z , namely the process $\dot{Z} = \{\dot{Z}(t) : t \in \mathbb{R}\}$ defined for all $t \in \mathbb{R}$ as

$$\dot{Z}(t) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(k/2^j)) - \Psi(-k, H(k/2^j)) \right\}, \quad (16)$$

is a C^∞ Gaussian process. The pointwise Hölder exponent of Z is therefore only determined by its high frequency component, namely the continuous Gaussian process $\check{Z} = \{\check{Z}(t) : t \in \mathbb{R}\}$ defined for all $t \in \mathbb{R}$ as

$$\check{Z}(t) = \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(k/2^j)) - \Psi(-k, H(k/2^j)) \right\}. \quad (17)$$

It is worth noticing that if one replaces in (17) the Hölder function $H(\cdot)$ by a step function then one recovers the step fractional Brownian motion which has been studied in [BBCI00, ABLV07].

Let us now state our main results.

Theorem 1. *Let $R = \{R(t) : t \in \mathbb{R}\}$ be the process defined for any $t \in \mathbb{R}$ as*

$$R(t) = Z(t) - X(t) \quad (18)$$

and let us assume that $a = \inf\{H(t) : \mathbb{R}\}$ and $b = \sup\{H(t) : \mathbb{R}\}$ verify the following condition:

$$1 - b > (1 - a)(1 - ab^{-1}). \quad (19)$$

Then there exists a constant $d \in (b, 1]$, such that the process R satisfies a uniform Hölder condition of order d on each compact subset of \mathbb{R} . More precisely, there is Ω^ an event of probability 1 such that all $\omega \in \Omega$, for any positive real number K and for each $t_0, t_1 \in [-K, K]$, one has*

$$|R(t_1, \omega) - R(t_0, \omega)| \leq C_1(\omega) |t_1 - t_0|^d, \quad (20)$$

where C_1 is a nonnegative random variable of finite moment of any order only depending on Ω^ and K .*

Corollary 1. *The process Z has the same features (a), (b) and (c) as mBm.*

Let us precise some notations: one will denote c_1, \dots, c_{12} some deterministic constants, $C_1(\omega), C_2(\omega)$ some random constants and by $[x]$ the entire part of the real number x .

2 Proofs of the main results

The following lemma is the key tool for the obtention of results on an event Ω^* of probability 1. It is a classical result, we refer for example to [MST99] or [AT03] for its proof.

Lemma 1. *[MST99, AT03] There are an event Ω^* of probability 1 and a non-negative random variable C_2 of finite moment of any order such the inequality*

$$|\varepsilon_{j,k}(\omega)| \leq C_2(\omega) \sqrt{\log(3 + |j| + |k|)}, \quad (21)$$

holds for all $\omega \in \Omega^$ and $j, k \in \mathbb{Z}$.*

Proof. (of Theorem of 1) In view of Remark 1 (ii) and of Remark 2 it is sufficient to prove that the theorem holds when the process R is replaced by its high frequency component, namely the process $\ddot{R} = \{\ddot{R}(t) : t \in \mathbb{R}\}$ defined for each $t \in \mathbb{R}$ as

$$\ddot{R}(t) = \ddot{Z}(t) - \ddot{X}(t). \quad (22)$$

To lighten the notations, for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $g_{j,k}$ be the function defined on $\mathbb{R} \times (0, 1)$ as

$$g_{j,k}(t, \theta) = 2^{-j\theta} \left\{ \Psi(2^j t - k, \theta) - \Psi(-k, \theta) \right\}. \quad (23)$$

One first gives a useful upper bound of the quantity $|\ddot{R}(t_1, \omega) - \ddot{R}(t_0, \omega)|$. It follows from (22), (14), (17), (23) and (21) that for any $\omega \in \Omega^*$,

$$\begin{aligned} |\ddot{R}(t_1, \omega) - \ddot{R}(t_0, \omega)| &\leq C_2(\omega) \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \sqrt{\log(3 + j + |k|)} \\ &\times \left| g_{j,k}(t_1, H(k/2^j)) - g_{j,k}(t_0, H(k/2^j)) - g_{j,k}(t_1, H(t_1)) + g_{j,k}(t_0, H(t_0)) \right|. \end{aligned} \quad (24)$$

Next, one expands the term $g_{j,k}(t_i, H(\tau))$ with $i = 0$ or 1 and $\tau = t_1$ or $k/2^j$ with respect to the second variable in the neighborhood of $H(t_0)$ by using the Taylor-Lagrange formula of order 1 with an integral reminder. Indeed, since the function Ψ is \mathcal{C}^∞ , the functions $g_{j,k}$ are also \mathcal{C}^∞ . Thus, one gets,

$$\begin{aligned} g_{j,k}(t_1, H(t_1)) &= g_{j,k}(t_1, H(t_0)) + (H(t_1) - H(t_0))(\partial_\theta g_{j,k})(t_1, H(t_0)) \\ &+ (H(t_1) - H(t_0))^2 \int_0^1 (1 - \tau)(\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(t_1) - H(t_0))) d\tau, \end{aligned} \quad (25)$$

$$\begin{aligned} g_{j,k}(t_0, H(k/2^j)) &= g_{j,k}(t_0, H(t_0)) + (H(k/2^j) - H(t_0))(\partial_\theta g_{j,k})(t_0, H(t_0)) \\ &+ (H(k/2^j) - H(t_0))^2 \int_0^1 (1 - \tau)(\partial_\theta^2 g_{j,k})(t_0, H(t_0) + \tau(H(k/2^j) - H(t_0))) d\tau. \end{aligned} \quad (26)$$

and

$$\begin{aligned} g_{j,k}(t_1, H(k/2^j)) &= g_{j,k}(t_1, H(t_0)) + (H(k/2^j) - H(t_0))(\partial_\theta g_{j,k})(t_1, H(t_0)) \\ &+ (H(k/2^j) - H(t_0))^2 \int_0^1 (1 - \tau)(\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(k/2^j) - H(t_0))) d\tau. \end{aligned} \quad (27)$$

By adding or subtracting Relations (25), (26) and (27) the constant terms disappear and one gets the following upper bound

$$\begin{aligned} &\left| g_{j,k}(t_1, H(k/2^j)) - g_{j,k}(t_0, H(k/2^j)) - g_{j,k}(t_1, H(t_1)) + g_{j,k}(t_0, H(t_0)) \right| \\ &\leq |H(t_1) - H(t_0)| \left| (\partial_\theta g_{j,k})(t_1, H(t_0)) \right| \end{aligned} \quad (28)$$

$$\begin{aligned}
 & + |H(t_1) - H(t_0)|^2 \int_0^1 (1 - \tau) \left| (\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(t_1) - H(t_0))) \right| d\tau, \\
 & + |H(k/2^j) - H(t_0)| \left| (\partial_\theta g_{j,k})(t_1, H(t_0)) - (\partial_\theta g_{j,k})(t_0, H(t_0)) \right| \\
 & + |H(k/2^j) - H(t_0)|^2 \int_0^1 (1 - \tau) \left| (\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(k/2^j) - H(t_0))) \right. \\
 & \quad \left. - (\partial_\theta^2 g_{j,k})(t_0, H(t_0) + \tau(H(k/2^j) - H(t_0))) \right| d\tau.
 \end{aligned}$$

Then, one plugs the previous bound into the inequality (24). One stresses that the quantities $|H(t_1) - H(t_0)|$ and $|H(t_1) - H(t_0)|^2$ can be factorized outside the sum whereas the quantities $|H(k/2^j) - H(t_0)|$ and $|H(k/2^j) - H(t_0)|^2$ remain inside the sum. Moreover, since the function $H(\cdot)$ satisfies an uniform Hölder condition of order $\beta > b$, the two first terms also satisfy the same Hölder condition provided one can prove an uniform bound of the sums which multiply these terms. This is the aim of lemma 3 in which one shows that for $n = 1$ or 2

$$A_n(K) := \sup_{(t,\theta) \in [-K,K] \times [a,b]} \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} |(\partial_\theta^n g_{j,k})(t, \theta)| \sqrt{\log(3 + j + |k|)} < \infty.$$

From the other hand, the Hölder condition mainly follows from the localization property and the regularity of the function $g_{j,k}(t, \theta)$ with respect to the first variable t . Indeed, from (23) and (8), the functions $g_{j,k}$ inherit the localization and regularity properties from the function Ψ . This is the aim of Lemma 4 which shows that when condition (19) is fulfilled there exists a real number $d \in (b, 1]$ and a deterministic constant $c_4 > 0$ such that for $n = 1$ or 2 , one has

$$\sup_{(t_0, \theta) \in [-K, K] \times [a, b], |h| < 1/4} B_n(t_0, h, \theta) \leq c_4 |h|^d$$

where

$$\begin{aligned}
 B_n(t_0, h, \theta) := & \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} |H(k/2^j) - H(t_0)|^n \times \sqrt{\log(3 + j + |k|)} \\
 & \times \left| (\partial_\theta^n g_{j,k})(2^j(t_0 + h) - k, \theta) - (\partial_\theta^n g_{j,k})(2^j t_0 - k, \theta) \right|
 \end{aligned}$$

With these notations, by putting together the inequalities (28), (24), (32), (33) and (37) one obtains that

$$\begin{aligned}
 |\check{R}(t_1, \omega) - \check{R}(t_0, \omega)| \leq & C_2(\omega) \left\{ |H(t_1) - H(t_0)| A_1(K) + |H(t_1) - H(t_0)|^2 A_2(K) \right. \\
 & \left. + \sup_{\theta \in [a, b]} B_1(t_0, t_1 - t_0, \theta) + \sup_{\theta \in [a, b]} B_2(t_0, t_1 - t_0, \theta) \right\}. \quad (29)
 \end{aligned}$$

Finally, combining (29) with (1), Lemma 3 and Lemma 4 one gets Theorem 1. Note in passing that there is no restriction to assume that $|t_1 - t_0| \leq 1/4$ since the function $\check{R}(\cdot, \omega)$ is continuous.

To complete the proof of Theorem 1, it remains to prove Lemma 3 and Lemma 4. To begin with, the following lemma conveniently express the $\partial_\theta^n g_{j,k}$'s.

Lemma 2. *For any integer $n \geq 0$ and any $(t, \theta) \in \mathbb{R} \times (0, 1)$ one has*

$$\begin{aligned} & (\partial_\theta^n g_{j,k})(t, \theta) \\ &= \sum_{p=0}^n C_n^p (-j \log 2)^p 2^{-j\theta} \left\{ (\partial_\theta^{n-p}) \Psi(2^j t - k, \theta) - (\partial_\theta^{n-p}) \Psi(-k, \theta) \right\}. \end{aligned} \quad (30)$$

Proof. (of Lemma 2) The lemma can easily be obtained by applying the Leibniz formula for the n 's derivative of a product of two functions.

In the calculations, one will use many times that

$$\log(3 + x + y) \leq \log(3 + x) \times \log(3 + y) \quad \text{for all } (x, y) \in \mathbb{R}_+^2 \quad (31)$$

By using these two ingredients, one can prove the following Lemma:

Lemma 3. *For any integer $n \geq 0$ and $(t, \theta) \in \mathbb{R} \times (0, 1)$ one sets*

$$A_n(t, \theta) := \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} |(\partial_\theta^n g_{j,k})(t, \theta)| \sqrt{\log(3 + j + |k|)}. \quad (32)$$

Then, for any real $K > 0$ one has

$$A_n(K) := \sup \left\{ A_n(t, \theta) : (t, \theta) \in [-K, K] \times [a, b] \right\} < \infty. \quad (33)$$

Proof. (of Lemma 3) From Lemma 2, one can deduce

$$\begin{aligned} A_n(t, \theta) &\leq \sum_{p=0}^n C_n^p |\log 2|^p \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} j^p 2^{-j\theta} \sqrt{\log(3 + j + |k|)} \\ &\quad \times \left\{ |(\partial_\theta^{n-p}) \Psi(2^j t - k, \theta)| + |(\partial_\theta^{n-p}) \Psi(-k, \theta)| \right\} \end{aligned} \quad (34)$$

Remark that the deepest bracket in (34) contains two terms: the first one is depending on $t \in [-K, K]$ when the second one no more depends on t . Therefore, it suffices to obtain a bound of the supremum for $t \in [-K, K]$ of the sum corresponding to the first term, next one can use it in the special case $K = 0$ to bound the sum corresponding to the second term. By using (8) and the convention that $0^0 = 1$, then by making the change of variable $k = k' + [2^j t]$ and by using $|t| \leq K$, eventually by using (31) and by remarking that $z = 2^j t - [2^j t]$ belongs to $[0, 1]$, one has the following calculations for each $p \in \{0, \dots, n\}$ and $(t, \theta) \in [-K, K] \times [a, b]$,

$$\begin{aligned}
 & \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} j^p 2^{-j\theta} \sqrt{\log(3+j+|k|)} |(\partial_\theta^{n-p}\Psi)(2^j t - k, \theta)| \\
 & \leq c_2 \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j+|k|)} \cdot (2 + |2^j t - k|)^{-l} \\
 & \leq c_2 \sum_{j=0}^{+\infty} \sum_{k'=-\infty}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j+|k'|+2^j K)} \cdot (2 + |2^j t - [2^j t] - k'|)^{-l} \\
 & \leq c_2 c_3 \sum_{j=0}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j+2^j K)} < \infty, \tag{35}
 \end{aligned}$$

where

$$c_3 = \sup \left\{ \sum_{k=-\infty}^{+\infty} (2 + |z - k|)^{-l} \sqrt{\log(3 + |k|)} : z \in [0, 1] \right\} < \infty. \tag{36}$$

Clearly, (35) combined with (34) implies that (33) holds. This finishes the proof of Lemma 3.

Lemma 4. *For any integer $n \geq 1$ and $(t, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, 1)$ one sets*

$$\begin{aligned}
 B_n(t_0, h, \theta) := & \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} |H(k/2^j) - H(t_0)|^n \times \sqrt{\log(3+j+|k|)} \\
 & \times \left| (\partial_\theta^n g_{j,k})(2^j(t_0+h) - k, \theta) - (\partial_\theta^n g_{j,k})(2^j t_0 - k, \theta) \right|
 \end{aligned}$$

Assume that the condition (19) holds. Then, for any real $K > 0$, for any integer $n \geq 1$, there are two constants $d \in (b, 1]$ and $c_4 > 0$ such that

$$\sup_{(t_0, \theta) \in [-K, K] \times [a, b], |h| < 1/4} B_n(t_0, h, \theta) \leq c_4 |h|^d. \tag{37}$$

To prove Lemma 4, one splits the set of indices $(j, k) \in \mathbb{N} \times \mathbb{Z}$ in three different subsets, namely $\mathcal{V}(t_0, h, \eta)$, $\mathcal{W}(t_0, h, \eta, \gamma)$, $\mathcal{W}^c(t_0, h, \eta, \gamma)$, and one obtains three different upper bounds for the corresponding sums. This is the aim of Lemma 5, Lemma 6 and Lemma 7.

The subset $\mathcal{V}(t_0, h, \eta)$ corresponds to the indices (j, k) for which the dyadic number $k/2^j$ stays in a neighborhood of size $|h|^\eta$ of t_0 . The second and the third subsets correspond to the indices (j, k) for which the dyadic number $k/2^j$ stay outside this neighborhood with a splitting following whether the index j is greater or smaller than a critical index j_1 .

This critical index j_1 depends on the parameter γ with $0 < \eta < \gamma < 1$. Eventually, when Condition (19) is fulfilled, one shows that a "good choice" of the parameters (η, γ) is possible in order to obtain Lemma 4.

To begin with, let fix $\eta \in (0, 1)$ (η will be defined more precisely later) and let $\mathcal{V}(t_0, h, \eta)$ and $\mathcal{V}^c(t_0, h, \eta)$ be the sets of indices defined as

$$\mathcal{V}(t_0, h, \eta) = \{(j, k) \in \mathbb{N} \times \mathbb{Z} : |k/2^j - t_0| \leq |h|^\eta\} \quad (38)$$

and

$$\mathcal{V}^c(t_0, h, \eta) = \{(j, k) \in \mathbb{N} \times \mathbb{Z} : |k/2^j - t_0| > |h|^\eta\}. \quad (39)$$

Lemma 5. *For any integer $n \geq 1$ and $(t, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, 1)$ one sets*

$$B_{1,n,p}(t_0, h, \theta) := \sum_{(j,k) \in \mathcal{V}(t_0, h, \eta)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3 + j + |k|)} \\ \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0 + h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right|$$

Then, for any real $K > 0$, for any integers $n \geq 1$ and $0 \leq p \leq n$, there exists a constant $c_5 > 0$ such that

$$\sup_{(t_0, h, \theta) \in [-K, K] \times \mathbb{R} \times [a, b]} B_{1,n,p}(t_0, h, \theta) \leq c_5 |h|^{\alpha+n\eta\beta} \log^{p+1/2}(1/|h|). \quad (40)$$

Proof. (of Lemma 5) It follows from (1) and (38) that

$$\sum_{(j,k) \in \mathcal{V}(t_0, h, \eta)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3 + j + |k|)} \quad (41) \\ \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0 + h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right| \\ \leq c_1 |h|^{n\beta\eta} \sum_{(j,k) \in \mathcal{V}(t_0, h, \eta)} j^p 2^{-ja} \times \sqrt{\log(3 + j + |k|)} \\ \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0 + h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right|.$$

Now let $j_0 \geq 2$ be the unique integer such that

$$2^{-j_0-1} < |h| \leq 2^{-j_0}. \quad (42)$$

By using (8), (42), the change of variable $k = k' + [2^j y]$, (31) and $|t_0| \leq K$, one has that for any $y \in [t_0 - 1, t_0 + 1]$,

$$\sum_{j=j_0+1}^{+\infty} \sum_{k=-\infty}^{+\infty} j^p 2^{-ja} |(\partial_\theta^{n-p} \Psi)(2^j y - k, \theta)| \sqrt{\log(3 + j + |k|)} \\ \leq c_2 \sum_{j=j_0+1}^{+\infty} \sum_{k'=-\infty}^{+\infty} j^p 2^{-ja} (2 + |2^j y - [2^j y] - k'|)^{-l} \sqrt{\log(3 + j + 2^j(|t_0| + 1) + |k'|)} \\ \leq c_2 c_3 \sum_{j=j_0+1}^{+\infty} j^p 2^{-ja} \sqrt{\log(3 + j + 2^j(K + 1))} \\ \leq c_6 |h|^\alpha \log^{p+1/2}(1/|h|), \quad (43)$$

where c_3 is the constant defined by (36) and the last inequality follows from (42) and some straightforward but tedious calculations. Remark that the constant c_6 depends on K .

From the other hand, by using the Mean-value Theorem applied to the function $\partial_\theta^{n-p} g_{j,k}$ with respect to the first variable combined with (8), the fact that for all $2^j|h| \leq 1$ for all $j \in \{0, \dots, j_0\}$, (42), (31), (36) and $|t_0| \leq K$, one gets that

$$\begin{aligned}
 & \sum_{j=0}^{j_0} \sum_{k=-\infty}^{+\infty} j^p 2^{-ja} \times \sqrt{\log(3+j+|k|)} \\
 & \quad \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0+h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right| \\
 & \leq c_2 |h| \sum_{j=0}^{j_0} \sum_{k=-\infty}^{+\infty} j^p 2^{j(1-a)} (1 + |2^j t_0 - [2^j t_0] - k|)^{-l} \sqrt{\log(4+j+2^j|t_0|+|k|)} \\
 & \leq c_2 c_3 |h| \sum_{j=0}^{j_0} j^p 2^{j(1-a)} \sqrt{\log(4+j+2^j K)} \\
 & \leq c_7 |h|^\alpha \log^{p+1/2}(1/|h|), \tag{44}
 \end{aligned}$$

Where the constant c_7 depends on K . Finally, by combining (41) with (43) and (44), one can deduce (40). This finishes the proof of Lemma 5.

Let us now fix $\gamma \in (\eta, 1)$ (γ will be defined more precisely later) and denote by j_1 the unique nonnegative integer satisfying

$$2^{-j_1-1} < |h|^\gamma \leq 2^{-j_1}. \tag{45}$$

Let $\mathcal{W}(t_0, h, \eta, \gamma)$ and $\mathcal{W}^c(t_0, h, \eta, \gamma)$ be the sets of indices defined as

$$\mathcal{W}(t_0, h, \eta, \gamma) = \{(j, k) \in \mathcal{V}(t_0, h, \eta) : 0 \leq j \leq j_1\} \tag{46}$$

and

$$\mathcal{W}^c(t_0, h, \eta, \gamma) = \{(j, k) \in \mathcal{V}^c(t_0, h, \eta) : j \geq j_1 + 1\}. \tag{47}$$

Observe that

$$\mathcal{V}^c(t_0, h, \eta) = \mathcal{W}(t_0, h, \eta, \gamma) \cup \mathcal{W}^c(t_0, h, \eta, \gamma). \tag{48}$$

Lemma 6. *For any integer $n \geq 1$ and $(t, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, 1)$ one sets*

$$\begin{aligned}
 B_{2,n,p}(t_0, h, \theta) := & \sum_{(j,k) \in \mathcal{W}(t_0, h, \eta, \gamma)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3+j+|k|)} \\
 & \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0+h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right|
 \end{aligned}$$

Then, for any real $K > 0$, for any integers $n \geq 1$ and $0 \leq p \leq n$, there exists a constant $c_8 > 0$ such that

$$\sup_{(t_0, \theta) \in (-K, K] \times [a, b], |h| < 1/4} B_{2,n,p}(t_0, h, \theta) \leq c_8 |h|^{(1-\gamma)+\gamma a} \log^{p+1/2}(1/|h|). \tag{49}$$

Proof. (of Lemma 6) To begin with, remark that for any pair of real numbers $(\theta_0, \theta_1) \in]0, 1[^2$, one has $|\theta_1 - \theta_0| < 1$. Therefore,

$$\text{for all } (j, k) \in \mathbb{N} \times \mathbb{Z}, \quad |H(t_0) - H(k/2^j)|^n \leq 1 \quad (50)$$

Next, by using the Mean-value Theorem applied to the function $\partial_\theta^{n-p} g_{j,k}$ with respect to the first variable combined with (8), the fact that for all $2^j |h| \leq 1$ for all $j \in \{0, \dots, j_1\}$, the change of variable $k = k' + [2^j t_0]$ and (45), one gets that

$$\begin{aligned} & \sum_{(j,k) \in \mathcal{W}(t_0, h, \eta, \gamma)} j^p 2^{-j\theta} \times \sqrt{\log(3 + j + |k|)} \\ & \quad \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0 + h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right| \\ & \leq c_2 |h| \sum_{j=0}^{j_1} \sum_{k=-\infty}^{+\infty} j^p 2^{j(1-a)} (1 + |2^j t_0 - k|)^{-l} \sqrt{\log(4 + j + 2^j |t_0| + |k|)} \\ & \leq c_2 |h| \sum_{j=0}^{j_1} \sum_{k'=-\infty}^{+\infty} j^p 2^{j(1-a)} (1 + |2^j t_0 - [2^j t_0] - k'|)^{-l} \\ & \quad \times \sqrt{\log(4 + j + 2^{j+1} |t_0| + |k'|)} \\ & \leq c_2 c_3 |h| \sum_{j=0}^{j_1} j^p 2^{j(1-a)} \sqrt{\log(4 + j + 2^{j+1} K)} \\ & \leq c_9 |h|^{(1-\gamma)+\gamma a} \log^{p+1/2}(1/|h|), \end{aligned} \quad (51)$$

where the constant c_9 depends on K . Finally, by combining (51) and (50), one can deduce (49). This finishes the proof of Lemma 6.

Lemma 7. *For any integer $n \geq 1$ and $(t, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, 1)$ one sets*

$$\begin{aligned} B_{3,n,p}(t_0, h, \theta) := & \sum_{(j,k) \in \mathcal{W}^c(t_0, h, \eta, \gamma)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3 + j + |k|)} \\ & \times \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0 + h) - k, \theta) - (\partial_\theta^{n-p} \Psi)(2^j t_0 - k, \theta) \right| \end{aligned}$$

Then, for any arbitrarily small real $\varepsilon > 0$ and for any integers $l \geq 2$, $n \geq 1$ and $0 \leq p \leq n$, there exists a constant $c_{10} > 0$ such that, one has

$$\sup_{(t_0, \theta) \in \mathbb{R} \times [a, b], |h| < 1/4} B_{2,n,p}(t_0, h, \theta) \leq c_{10} |h|^{(\gamma-n)(l-1-\varepsilon)+\gamma a} \log^{p+1/2}(|h|^{-1}). \quad (52)$$

Proof. (of Lemma 7) By using the triangular inequality combined with (47) and (39), one gets, for any real h verifying $|h| \leq 1/4$ and all $(j, k) \in \mathcal{W}^c(t_0, h, \eta, \gamma)$,

$$\begin{aligned} |2^j(t_0 + h) - k| & \geq |2^j t_0 - k| - 2^j |h| \geq 2^j |h|^\eta - 2^j |h| \\ & \geq c_{11} 2^j |h|^\eta \end{aligned} \quad (53)$$

where $c_{11} = 1 - 4^{\eta-1} > 0$. More precisely, (53) is a lower bound, for a fixed index $j \in \mathbb{N}$, when k varies with $(j, k) \in \mathcal{W}^c(t_0, h, \eta, \gamma)$, one has

$$k = c_{11}2^j|h|^\eta + z + q \quad \text{or} \quad k = -c_{11}2^j|h|^\eta - z - q$$

for a fixed real number $z \in]0, 1[$ and $q \in \mathbb{N}$. By combining this remark with (8), (53), the inequality $(2+x)^{-l}\sqrt{\log(4+x)} \leq c_{12}(2+x)^{-l+\varepsilon}$ valid for all nonnegative real number x (here ε is a fixed arbitrarily small positive real number and c_{12} is a constant only depending on ε), $l \geq 2$ and (45) that

$$\begin{aligned} & \sum_{(j,k) \in \mathcal{W}^c(t_0, h, \eta, \gamma)} j^p 2^{-j\theta} \left| (\partial_\theta^{n-p} \Psi)(2^j(t_0+h) - k, \theta) \right| \sqrt{\log(3+j+|k|)} \\ & \leq 2c_2 \sum_{j=j_1+1}^{+\infty} \sum_{q=0}^{+\infty} j^p 2^{-ja} \left(2+q+c_{11}2^j|h|^\eta\right)^{-l} \times \sqrt{\log(4+j+q+c_{11}2^j|h|^\eta)} \\ & \leq 2c_2 \sum_{j=j_1+1}^{+\infty} \sum_{q=0}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j)} \\ & \quad \times \left(2+q+c_{11}2^j|h|^\eta\right)^{-l} \times \sqrt{\log(4+q+c_{11}2^j|h|^\eta)} \\ & \leq 2c_2 c_{12} \sum_{j=j_1+1}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j)} \times \left(\int_0^{+\infty} (1+y+c_{11}2^j|h|^\eta)^{-l+\varepsilon} dy\right) \\ & \leq c_{13} \sum_{j=j_1+1}^{+\infty} j^p 2^{-ja} \sqrt{\log(3+j)} \times \left(1+c_{11}2^j|h|^\eta\right)^{-(l-1-\varepsilon)} \\ & \leq c_{14} |h|^{-\eta(l-1-\varepsilon)} \sum_{j=j_1+1}^{+\infty} j^p 2^{-j(a+l-1-\varepsilon)} \sqrt{\log(3+j)} \\ & \leq c_{15} |h|^{(\gamma-\eta)(l-1-\varepsilon)+\gamma a} \log^{p+1/2}(1/|h|). \end{aligned} \tag{54}$$

Stress that the constant c_{15} does not depends on t_0 . Then, remark that when one replace $2^j(t_0+h)$ by $2^j t_0$, one has $|2^j t_0 - k| \geq 2^j |h|^\eta$, which is the same lower bound than (53) with c_{11} replace by 1. Thus, one can repeat the same calculations and one obtains (54) with a different constant. Eventually, by using triangular inequality, (50) and (54), one can deduce (52). This finishes the proof of Lemma 7.

We are now in position to prove Lemma 4.

Proof. (of Lemma 4) It follows from Lemmas 4 to 6 that for any integer $n \geq 1$ and reals $0 < \eta < \gamma < 1$, one has

$$\begin{aligned} B_n(t_0, h, \theta) & \leq \sum_{p=0}^n \sum_{m=1}^3 C_n^p (\log 2)^p B_{m,n,p}(t_0, h, \theta) \\ & \leq c_{15} \left(|h|^{a+\eta\beta} + |h|^{(1-\gamma)+\gamma a} + |h|^{(\gamma-\eta)(l-1-\varepsilon)+\gamma a} \right) \log^{n+1/2}(1/|h|). \end{aligned} \tag{55}$$

Moreover, the condition (19) is equivalent to $(1-a)^{-1}(1-b) > b^{-1}(b-a)$. Thus, assuming that it is verified then there exist 3 reals η_0 , γ_0 and β_0 such that $\beta_0 \in (b, \beta]$ and

$$1 > (1-a)^{-1}(1-b) > \gamma_0 > \eta_0 > \beta_0^{-1}(b-a).$$

One has therefore,

$$a + \eta_0\beta_0 > b \text{ and } (1 - \gamma_0) + \gamma_0a > b.$$

From the other hand, assuming that l_0 is big enough, then one has that

$$(\gamma_0 - \eta_0)(l_0 - 1 - \varepsilon) + \gamma_0a > b.$$

Finally, taking in (55) $\eta = \eta_0$, $\gamma = \gamma_0$ and $l = l_0$ one obtains the lemma.

Let us now prove that Corollary 1 holds.

Proof. (of Corollary 1) Let us first show that Z has the same feature (b) as mBm. In view of Theorem 1, it is clear that α_R the pointwise Hölder exponent of R satisfies a.s. for all $t \in \mathbb{R}$,

$$\alpha_R(t) \geq d. \quad (56)$$

Next putting together (56), the fact that $d > b$, (18) and (4) it follows that a.s. for all $t \in \mathbb{R}$,

$$\alpha_Z(t) = H(t).$$

Let us now show that Z has the same feature (c) as mBm. Let (ρ_n) be an arbitrary sequence of positive real numbers converging to 0. In view of (18) and (6), for proving that for each $t \in \mathbb{R}$ one has

$$\lim_{n \rightarrow \infty} \text{law} \left\{ \frac{Z(t + \rho_n u) - Z(t)}{\rho_n^{H(t)}} : u \in \mathbb{R} \right\} = \text{law} \{ B_{H(t)}(u) : u \in \mathbb{R} \}, \quad (57)$$

in the sense of finite dimensional distribution, it is sufficient to prove that for any $u \in \mathbb{R}$ one has

$$\lim_{n \rightarrow +\infty} E \left\{ \left(\frac{R(t + \rho_n u) - R(t)}{\rho_n^{H(t)}} \right)^2 \right\} = 0. \quad (58)$$

Observe that for all n big enough one has $\rho_n |u| \leq 1$. Therefore, setting in Theorem 1, $K = |t| + 1$, it follows that for n big enough,

$$E \left\{ \left(\frac{R(t + \rho_n u) - R(t)}{\rho_n^{H(t)}} \right)^2 \right\} \leq \rho_n^{2(d-H(t))} E(C_1^2) \quad (59)$$

and the latter inequality clearly implies that (58) holds. To have in (57) the convergence in distribution for the topology of the uniform convergence on

compact sets it is sufficient to show that for any positive real L , the sequence of continuous Gaussian processes,

$$\left\{ \frac{Z(t + \rho_n u) - Z(t)}{\rho_n^{H(t)}} : u \in [-L, L] \right\}, n \in \mathbb{N},$$

is tight. This tightness result can be obtained (see [B68]) by proving that there exists a constant $c_{16} > 0$ only depending on L and t such that for all $n \in \mathbb{N}$ and each $u_1, u_2 \in [-L, L]$ one has

$$E \left\{ \left(\frac{Z(t + \rho_n u_1) - Z(t)}{\rho_n^{H(t)}} - \frac{Z(t + \rho_n u_2) - Z(t)}{\rho_n^{H(t)}} \right)^2 \right\} \leq c_{16} |u_1 - u_2|^{2H(t)}. \tag{60}$$

There is no restriction to assume that for every $n \in \mathbb{N}$, $\rho_n \in (0, 1]$. Then by using the fact that (60) is verified when Z is replaced by X (see [BCI98] Proposition 2) as well as the fact it is also verified when Z is replaced by R (this can be done similarly to (59)), one can establish that this inequality holds.

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