

The infinitesimal Hopf algebra and the operads of planar forests

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ABSTRACT. We introduce two operads which own the set of planar forests as a basis. With its usual product and two other products defined by different types of graftings, the algebra of planar rooted trees \mathcal{H} becomes an algebra over these operads. The compatibility with the infinitesimal coproduct of \mathcal{H} and these structures is studied. As an application, an inductive way of computing the dual basis of \mathcal{H} for its infinitesimal pairing is given. Moreover, three Cartier-Quillen-Milnor-Moore theorems are given for the operads of planar forests and a rigidity theorem for one of them.

KEYWORDS. Infinitesimal Hopf algebra, Planar rooted trees, Operads.

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Introduction

The Connes-Kreimer Hopf algebra of rooted trees, introduced in [1, 6, 7, 8], is a commutative, non cocommutative Hopf algebra, its coproduct being given by admissible cuts of trees. A non commutative version, the Hopf algebra of planar rooted trees, is introduced in [3, 5]. We furthermore introduce in [4] an infinitesimal version of this object, replacing admissible cuts by left admissible cuts: this last object is here denoted by \mathcal{H} . Similarly with the Hopf case, \mathcal{H} is a self-dual object and it owns a non-degenerate, symmetric Hopf pairing, denoted by $\langle -, - \rangle$. This pairing is related to a partial order on the set of planar rooted forests, making it isomorphic to the Tamari poset. As a consequence, \mathcal{H} is given a dual basis denoted by $(f_F)_{F \in \mathbf{F}}$, indexed by the set \mathbf{F} of planar forest. In particular, the sub-family $(f_t)_{t \in \mathbf{T}}$ indexed by the set of planar rooted trees \mathbf{T} is a basis of the space of primitive elements of \mathcal{H} .

The aim of this text is to introduce two structures of operad on the space of planar forests. We introduce two (non-symmetric) operads \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow} defined in the following way:

1. \mathbb{P}_{\searrow} is generated by m and $\searrow \in \mathbb{P}_{\searrow}(2)$, with relations:

$$\begin{cases} m \circ (\searrow, I) &= \searrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \searrow \circ (m, I) &= \searrow \circ (I, \searrow). \end{cases}$$

2. \mathbb{P}_{\nearrow} is generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\begin{cases} m \circ (\nearrow, I) &= \nearrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \nearrow \circ (\nearrow, I) &= \nearrow \circ (I, \nearrow). \end{cases}$$

We then introduce two products on \mathcal{H} or on its augmentation ideal \mathcal{M} , denoted by \nearrow and \searrow . The product $F \nearrow G$ consists of grafting F on the left leaf of G and the product $F \searrow G$ consists of grafting F on the left root of G . Together with its usual product m , \mathcal{M} becomes both a \mathbb{P}_{\searrow} - and a \mathbb{P}_{\nearrow} -algebra. More precisely, \mathcal{M} is the free \mathbb{P}_{\searrow} - and \mathbb{P}_{\nearrow} -algebra generated by a single element \bullet . As a consequence, \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow} inherits a combinatorial representation using planar forests, with composition iteratively described using the products \searrow and \nearrow .

We then give several applications of these operadic structures. For example, the antipode of \mathcal{H} is described in term of the operad \mathbb{P}_{\searrow} . We show how to compute elements f_t 's, with $t \in \mathbf{T}$, using the action of \mathbb{P}_{\searrow} , and the elements f_F 's, $F \in \mathbf{F}$ from the preceding ones using the action of \mathbb{P}_{\nearrow} . Combining all these results, it is possible to compute by induction the basis $(f_F)_{F \in \mathbf{F}}$.

We finally study the compatibilities of products m , \nearrow , \searrow , the coproduct $\tilde{\Delta}$ and the coproduct $\tilde{\Delta}_{\nearrow}$ dual of \nearrow . This leads to the definition of two types of \mathbb{P}_{\nearrow} -bialgebras, and one type of \mathbb{P}_{\searrow} -bialgebras. Each type then define a suboperad of \mathbb{P}_{\nearrow} or \mathbb{P}_{\searrow} corresponding to primitive elements of \mathcal{M} , which are explictively described:

1. The first one is a free operad, generated by the element $\mathbf{1} - \bullet \bullet \in \mathbb{P}_{\nearrow}(2)$. As a consequence, the space of primitive elements of \mathcal{H} admits a basis $(p_t)_{t \in \mathbf{T}_b}$ indexed by the set of planar binary trees. The link with the basis $(f_t)_{t \in \mathbf{T}}$ is given with the help of the Tamari order.
2. The second one admits a combinatorial representation in terms of planar rooted trees. It is generated by the corollas $c_n \in \mathbb{P}_{\nearrow}(n)$, $n \geq 2$, with the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \dots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \dots, I}_{l-1 \text{ times}}, c_k).$$

- The third one admits a combinatorial representation in terms of planar rooted trees, and is freely generated by $\mathbf{1} \in \mathbb{P}_{\searrow}(2)$.

We also give the definition of a double \mathbb{P}_{\nearrow} -bialgebra, combining the two types of \mathbb{P}_{\nearrow} -bialgebras already introduced. We then prove a rigidity theorem: any double \mathbb{P}_{\nearrow} -bialgebra connected as a coalgebra is isomorphic to a decorated version of \mathcal{M} .

This text is organised as follows: the first section gives several recalls on the infinitesimal Hopf algebra of planar rooted trees and its pairing. The two products \searrow and \nearrow are introduced in section 2, as well as the combinatorial representation of the two associated operads. The applications to the computation of $(f_F)_{F \in \mathbf{F}}$ is given in section 3. Section 4 is devoted to the study of the suboperads of primitive elements and the last section deals with the rigidity theorem for double \mathbb{P}_{\nearrow} -bialgebras.

Notations.

- We shall denote by K a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc, will be taken over K .
- Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, non zero, such that $\Delta(1) = 1 \otimes 1$. We then define the non counitary coproduct:

$$\tilde{\Delta} : \begin{cases} Ker(\varepsilon) & \longrightarrow Ker(\varepsilon) \otimes Ker(\varepsilon) \\ a & \longrightarrow \tilde{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a. \end{cases}$$

We shall use the Sweedler notations $\Delta(a) = a^{(1)} \otimes a^{(2)}$ and $\tilde{\Delta}(a) = a' \otimes a''$.

1 Planar rooted forests and their infinitesimal Hopf algebra

We here recall some results and notations of [4].

1.1 Planar trees and forests

- The set of planar trees is denoted by \mathbf{T} , and the set of planar forests is denoted by \mathbf{F} . The weight of a planar forest is the number of its vertices. For all $n \in \mathbb{N}$, we denote by $\mathbf{F}(n)$ the set of planar forests of weight n .

Examples. Planar rooted trees of weight ≤ 5 :

Planar rooted forests of weight ≤ 4 :

- The algebra \mathcal{H} is the free associative, unitary algebra generated by \mathbf{T} . As a consequence, a linear basis of \mathcal{H} is given by \mathbf{F} , and its product is given by the concatenation of planar forests.
- We shall also need two partial orders and a total order on the set $Vert(F)$ of vertices of $F \in \mathbf{F}$, defined in [3, 4]. We put $F = t_1 \dots t_n$, and let s, s' be two vertices of F .

- We shall say that $s \geq_{high} s'$ if there exists a path from s' to s in F , the edges of F being oriented from the roots to the leaves. Note that \geq_{high} is a partial order, whose Hasse graph is the forest F .

(b) If s and s' are not comparable for \geq_{high} , we shall say that $s \geq_{left} s'$ if one of these assertions is satisfied:

- i. s is a vertex of t_i and s' is a vertex of t_j , with $i < j$.
- ii. s and s' are vertices of the same t_i , and $s \geq_{left} s'$ in the forest obtained from t_i by deleting its root.

This defines the partial order \geq_{left} for all forests F , by induction on the the weight.

(c) We shall say that $s \geq_{h,l} s'$ if $s \geq_{high} s'$ or $s \geq_{left} s'$. This defines a total order on the vertices of F .

1.2 Infinitesimal Hopf algebra of planar forests

1. Let $F \in \mathbf{F}$. An *admissible cut* is a non empty cut of certain edges and trees of F , such that each path in a non-cut tree of F meets at most one cut edge. The set of admissible cuts of F will be denoted by $Adm(F)$. If c is an admissible cut of F , the forest of the vertices which are over the cuts of c will be denoted by $P^c(t)$ (branch of the cut c), and the remaining forest will be denoted by $R^c(t)$ (trunk of the cut). An admissible cut of F will be said to be *left-admissible* if, for all vertices x and y of F , $x \in P^c(F)$ and $x \leq_{left} y$ imply that $y \in P^c(F)$. The set of left-admissible cuts of F will be denoted by $Adm^l(F)$.

2. \mathcal{H} is given a coproduct by the following formula: for all $F \in \mathbf{F}$,

$$\Delta(F) = \sum_{c \in Adm^l(F)} P^c(F) \otimes R^c(F) + F \otimes 1 + 1 \otimes F.$$

Then (\mathcal{H}, Δ) is an infinitesimal bialgebra, that is to say: for all $x, y \in \mathcal{H}$,

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y.$$

Examples.

$$\begin{aligned} \Delta(\cdot) &= \cdot \otimes 1 + 1 \otimes \cdot, \\ \Delta(\cdot\cdot) &= \cdot\cdot \otimes 1 + 1 \otimes \cdot\cdot + \cdot \otimes \cdot, \\ \Delta(\mathbf{!}) &= \mathbf{!} \otimes 1 + 1 \otimes \mathbf{!} + \cdot \otimes \cdot, \\ \Delta(\mathbf{!}\cdot) &= \mathbf{!}\cdot \otimes 1 + 1 \otimes \mathbf{!}\cdot + \cdot \otimes \cdot\cdot + \mathbf{!} \otimes \cdot, \\ \Delta(\mathbf{V}) &= \mathbf{V} \otimes 1 + 1 \otimes \mathbf{V} + \cdot\cdot \otimes \cdot + \cdot \otimes \mathbf{!}, \\ \Delta(\mathbf{!}\mathbf{!}) &= \mathbf{!}\mathbf{!} \otimes 1 + 1 \otimes \mathbf{!}\mathbf{!} + \mathbf{!} \otimes \cdot + \cdot \otimes \mathbf{!}, \\ \Delta(\cdot\cdot\cdot\cdot) &= \cdot\cdot\cdot\cdot \otimes 1 + 1 \otimes \cdot\cdot\cdot\cdot + \cdot\cdot \otimes \cdot\cdot + \cdot\cdot \otimes \cdot\cdot + \cdot\cdot \otimes \cdot, \\ \Delta(\mathbf{!}\cdot\cdot) &= \mathbf{!}\cdot\cdot \otimes 1 + 1 \otimes \mathbf{!}\cdot\cdot + \cdot \otimes \cdot\cdot\cdot + \mathbf{!} \otimes \cdot\cdot + \mathbf{!}\cdot \otimes \cdot, \\ \Delta(\cdot\mathbf{!}\cdot) &= \cdot\mathbf{!}\cdot \otimes 1 + 1 \otimes \cdot\mathbf{!}\cdot + \cdot \otimes \mathbf{!}\cdot + \cdot\cdot \otimes \cdot\cdot + \mathbf{!} \otimes \cdot, \\ \Delta(\cdot\cdot\mathbf{!}) &= \cdot\cdot\mathbf{!} \otimes 1 + 1 \otimes \cdot\cdot\mathbf{!} + \cdot \otimes \cdot\mathbf{!} + \cdot\cdot \otimes \mathbf{!} + \cdot\cdot \otimes \cdot, \\ \Delta(\cdot\mathbf{V}) &= \cdot\mathbf{V} \otimes 1 + 1 \otimes \cdot\mathbf{V} + \cdot \otimes \mathbf{V} + \cdot\cdot \otimes \mathbf{!} + \cdot\cdot \otimes \cdot, \\ \Delta(\cdot\mathbf{!}\mathbf{!}) &= \cdot\mathbf{!}\mathbf{!} \otimes 1 + 1 \otimes \cdot\mathbf{!}\mathbf{!} + \cdot \otimes \mathbf{!}\mathbf{!} + \cdot\cdot \otimes \mathbf{!} + \mathbf{!}\mathbf{!} \otimes \cdot, \\ \Delta(\mathbf{V}\cdot) &= \mathbf{V}\cdot \otimes 1 + 1 \otimes \mathbf{V}\cdot + \cdot \otimes \mathbf{!}\cdot + \cdot\cdot \otimes \cdot\cdot + \mathbf{V} \otimes \cdot, \\ \Delta(\mathbf{!}\cdot\cdot) &= \mathbf{!}\cdot\cdot \otimes 1 + 1 \otimes \mathbf{!}\cdot\cdot + \cdot \otimes \mathbf{!}\cdot + \mathbf{!} \otimes \cdot\cdot + \mathbf{!}\cdot \otimes \cdot, \\ \Delta(\mathbf{!}\mathbf{!}\mathbf{!}) &= \mathbf{!}\mathbf{!}\mathbf{!} \otimes 1 + 1 \otimes \mathbf{!}\mathbf{!}\mathbf{!} + \cdot \otimes \cdot\mathbf{!}\mathbf{!} + \mathbf{!} \otimes \mathbf{!}\mathbf{!} + \mathbf{!}\cdot \otimes \cdot, \end{aligned}$$

$$\begin{aligned}
\Delta(\Psi) &= \Psi \otimes 1 + 1 \otimes \Psi + \cdot \otimes \Psi + \dots \otimes \mathbf{1} + \dots \otimes \cdot, \\
\Delta(\downarrow \mathbf{V}) &= \downarrow \mathbf{V} \otimes 1 + 1 \otimes \downarrow \mathbf{V} + \cdot \otimes \mathbf{V} + \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \cdot, \\
\Delta(\mathbf{V} \downarrow) &= \mathbf{V} \downarrow \otimes 1 + 1 \otimes \mathbf{V} \downarrow + \cdot \otimes \mathbf{1} + \dots \otimes \mathbf{1} + \cdot \otimes \mathbf{1}, \\
\Delta(\Upsilon) &= \Upsilon \otimes 1 + 1 \otimes \Upsilon + \cdot \otimes \mathbf{1} + \dots \otimes \mathbf{1} + \mathbf{V} \otimes \cdot, \\
\Delta(\mathbf{1}) &= \mathbf{1} \otimes 1 + 1 \otimes \mathbf{1} + \cdot \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \cdot.
\end{aligned}$$

We proved in [4] that \mathcal{H} is an infinitesimal Hopf algebra, that is to say has an antipode S . This antipode satisfies $S(1) = 1$, $S(t) \in \text{Prim}(\mathcal{H})$ for all $t \in \mathbf{T}$, and $S(F) = 0$ for all $F \in \mathbf{F} - (\mathbf{T} \cup \{1\})$.

1.3 Pairing on \mathcal{H}

1. We define the operator $B^+ : \mathcal{H} \rightarrow \mathcal{H}$, which associates, to a forest $F \in \mathbf{F}$, the tree obtained by grafting the roots of the trees of F on a common root. For example, $B^+(\mathbf{V} \cdot) = \mathbf{V} \downarrow$, and $B^+(\cdot \mathbf{V}) = \mathbf{V} \downarrow$.

2. The application γ is defined by:

$$\gamma : \begin{cases} \mathcal{H} & \longrightarrow \mathcal{H} \\ t_1 \dots t_n \in \mathbf{F} & \longrightarrow \delta_{t_1, \cdot} t_2 \dots t_n. \end{cases}$$

3. There exists a unique pairing $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \rightarrow K$, satisfying:

- i. $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{H}$.
- ii. $\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle$ for all $x, y, z \in \mathcal{H}$.
- iii. $\langle B^+(x), y \rangle = \langle x, \gamma(y) \rangle$ for all $x, y \in \mathcal{H}$.

Moreover:

- iv. $\langle -, - \rangle$ is symmetric and non-degenerate.
- v. If x and y are homogeneous of different weights, $\langle x, y \rangle = 0$.
- vi. $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{H}$.

This pairing admits a combinatorial interpretation using the partial orders \geq_{left} and \geq_{high} and is related to the Tamari order on planar binary trees, see [4].

4. We denote by $(f_F)_{F \in \mathbf{F}}$ the dual basis of the basis of forests for the pairing $\langle -, - \rangle$. In other terms, for all $F \in \mathbf{F}$, f_F is defined by $\langle f_F, G \rangle = \delta_{F,G}$, for all forest $G \in \mathbf{F}$. The family $(f_t)_{t \in \mathbf{T}}$ is a basis of the space $\text{Prim}(\mathcal{H})$ of primitive elements of \mathcal{H} .

2 The operads of forests and graftings

2.1 A few recalls on non- Σ -operads

1. We shall work here with non- Σ -operads [11]. Recall that such an object is a family $\mathbb{P} = (\mathbb{P}(n))_{n \in \mathbb{N}}$ of vector spaces, together with a composition for all $n, k_1, \dots, k_n \in \mathbb{N}$:

$$\begin{cases} \mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \dots \otimes \mathbb{P}(k_n) & \longrightarrow \mathbb{P}(k_1 + \dots + k_n) \\ p \otimes p_1 \otimes \dots \otimes p_n & \longrightarrow p \circ (p_1, \dots, p_n). \end{cases}$$

The following associativity condition is satisfied: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, \dots , $p_n \in \mathbb{P}(k_n)$, $p_{1,1}, \dots, p_{n,k_n} \in \mathbb{P}$,

$$\begin{aligned} & (p \circ (p_1, \dots, p_n)) \circ (p_{1,1}, \dots, p_{1,k_1}, \dots, p_{n,1}, \dots, p_{n,k_n}) \\ &= p \circ (p_1 \circ (p_{1,1}, \dots, p_{1,k_1}), \dots, p_n \circ (p_{n,1}, \dots, p_{n,k_n})). \end{aligned}$$

Moreover, there exists a unit element $I \in \mathbb{P}(1)$, satisfying: for all $p \in \mathbb{P}(n)$,

$$\begin{cases} p \circ (I, \dots, I) = p, \\ I \circ p = p. \end{cases}$$

An operad is a non- Σ -operad \mathbb{P} with a right action of the symmetric group S_n on $\mathbb{P}(n)$ for all n , satisfying a certain compatibility with the composition.

2. Let \mathbb{P} be a non- Σ -operad. A \mathbb{P} -algebra is a vector space A , together with an action of \mathbb{P} :

$$\begin{cases} \mathbb{P}(n) \otimes A^{\otimes n} \longrightarrow A \\ p \otimes a_1 \otimes \dots \otimes a_n \longrightarrow p.(a_1, \dots, a_n), \end{cases}$$

satisfying the following compatibility: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, \dots , $p_n \in \mathbb{P}(k_n)$, for all $a_{1,1}, \dots, a_{n,k_n} \in A$,

$$\begin{aligned} & (p \circ (p_1, \dots, p_n)).(a_{1,1}, \dots, a_{1,k_1}, \dots, a_{n,1}, \dots, a_{n,k_n}) \\ &= p.(p_1.(a_{1,1}, \dots, a_{1,k_1}), \dots, p_n.(a_{n,1}, \dots, a_{n,k_n})). \end{aligned}$$

Moreover, $I.a = a$ for all $a \in A$.

In particular, if V is a vector space, the free \mathbb{P} -algebra generated by V is:

$$F_{\mathbb{P}}(V) = \bigoplus_{n \in \mathbb{N}} \mathbb{P}(n) \otimes V^{\otimes n},$$

with the action of \mathbb{P} given by:

$$\begin{aligned} & p.((p_1 \otimes a_{1,1} \otimes \dots \otimes a_{1,k_1}), \dots, (p_n \otimes a_{n,1} \otimes \dots \otimes a_{n,k_n})) \\ &= (p \circ (p_1, \dots, p_n)) \otimes a_{1,1} \otimes \dots \otimes a_{1,k_1} \otimes \dots \otimes a_{n,1} \otimes \dots \otimes a_{n,k_n}. \end{aligned}$$

3. Let \mathbf{T}_b be the set of planar binary trees:

$$\mathbf{T}_b = \left\{ \mathbf{1}, \Upsilon, \begin{array}{c} \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \dots \right\}.$$

For all $n \in \mathbb{N}$, $\mathbb{T}_b(n)$ is the vector space generated by the elements of \mathbf{T}_b with n leaves:

$$\begin{aligned} \mathbb{T}_b(0) &= (0), \\ \mathbb{T}_b(1) &= \text{Vect}(\mathbf{1}), \\ \mathbb{T}_b(2) &= \text{Vect}\left(\begin{array}{c} \Upsilon \\ \Upsilon \end{array}\right), \\ \mathbb{T}_b(3) &= \text{Vect}\left(\begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \end{array}\right), \\ \mathbb{T}_b(4) &= \text{Vect}\left(\begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}, \begin{array}{c} \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \\ \Upsilon \end{array}\right). \end{aligned}$$

The family of vector spaces \mathbb{T}_b is given a structure of non- Σ -operad by graftings on the leaves. More precisely, if $t, t_1, \dots, t_n \in \mathbf{T}_b$, t with n leaves, then $t \circ (t_1, \dots, t_n)$ is the binary tree obtained by grafting t_1 on the first leaf of t , t_2 on the second leaf of t , and so on (note that the leaves of t are ordered from left to right). The unit is $\mathbf{1}$.

It is known that \mathbb{T}_b is the free non- Σ -operad generated by $\mathbf{Y} \in \mathbb{T}_b(2)$. Similarly, given elements m_1, \dots, m_k in $\mathbb{P}(2)$, it is possible to describe the free non- Σ -operad \mathbb{P} generated by these elements in terms of planar binary trees whose internal vertices are decorated by m_1, \dots, m_k .

2.2 Presentations of the operads of forests

Definition 1

1. \mathbb{P}_{\searrow} is the non- Σ -operad generated by m and $\searrow \in \mathbb{P}_{\searrow}(2)$, with relations:

$$\begin{cases} m \circ (\searrow, I) = \searrow \circ (I, m), \\ m \circ (m, I) = m \circ (I, m), \\ \searrow \circ (m, I) = \searrow \circ (I, \searrow). \end{cases}$$

2. \mathbb{P}_{\nearrow} is the non- Σ -operad generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\begin{cases} m \circ (\nearrow, I) = \nearrow \circ (I, m), \\ m \circ (m, I) = m \circ (I, m), \\ \nearrow \circ (\nearrow, I) = \nearrow \circ (I, \nearrow). \end{cases}$$

Remark. We shall prove in [2] that these quadratic operads are Koszul.

2.3 Grafting on the root

Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$ and $t_1 = B^+(G_1)$. We define:

$$F \searrow G = B^+(FG_1)t_2 \dots t_n.$$

In other terms, F is grafted on the root of the first tree of G , on the left. In particular, $F \searrow \cdot = B^+(F)$.

Examples.

$$\begin{array}{l} \dots \searrow \mathbf{1} = \begin{array}{c} \diagdown \quad \diagup \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \dots = \mathbf{1} \cdot \dots \quad \left| \quad \dots \searrow \dots = \begin{array}{c} \mathbf{V} \dots \end{array} \quad \left| \quad \dots \searrow \dots = \begin{array}{c} \mathbf{V} \cdot \end{array} \\ \cdot \mathbf{1} \searrow \mathbf{1} = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \cdot \mathbf{1} = \mathbf{1} \mathbf{1} \quad \left| \quad \dots \searrow \cdot \mathbf{1} = \begin{array}{c} \mathbf{V} \mathbf{1} \end{array} \quad \left| \quad \cdot \mathbf{1} \searrow \dots = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \\ \mathbf{1} \cdot \searrow \mathbf{1} = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \mathbf{1} \cdot = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \dots \searrow \mathbf{1} \cdot = \begin{array}{c} \mathbf{V} \cdot \end{array} \quad \left| \quad \mathbf{1} \cdot \searrow \dots = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \\ \mathbf{V} \searrow \mathbf{1} = \begin{array}{c} \mathbf{V} \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \mathbf{V} = \begin{array}{c} \mathbf{1} \\ \mathbf{V} \end{array} \quad \left| \quad \dots \searrow \mathbf{V} = \begin{array}{c} \mathbf{V} \end{array} \quad \left| \quad \mathbf{V} \searrow \dots = \begin{array}{c} \mathbf{Y} \cdot \end{array} \\ \mathbf{1} \searrow \mathbf{1} = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \mathbf{1} = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \dots \searrow \mathbf{1} = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \quad \left| \quad \mathbf{1} \searrow \dots = \begin{array}{c} \mathbf{1} \\ \mathbf{1} \end{array} \end{array}$$

Obviously, \searrow can be inductively defined in the following way: for $F, G, H \in \mathbf{F} - \{1\}$,

$$\begin{cases} F \searrow \cdot = B^+(F), \\ F \searrow (GH) = (F \searrow G)H \\ F \searrow B^+(G) = B^+(FG). \end{cases}$$

We denote by \mathcal{M} the augmentation ideal of \mathcal{H} , that is to say the vector space generated by the elements of $\mathbf{F} - \{1\}$. We extend $\searrow: \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}$ by linearity.

Proposition 2 For all $x, y, z \in \mathcal{M}$:

$$x \searrow (yz) = (x \searrow y)z, \quad (1)$$

$$x \searrow (y \searrow z) = (xy) \searrow z. \quad (2)$$

Proof. We can restrict ourselves to $x, y, z \in \mathbf{F} - \{1\}$. Then (1) is immediate. In order to prove (2), we put $z = B^+(z_1)z_2$, $z_1, z_2 \in \mathbf{F}$. Then:

$$x \searrow (y \searrow z) = x \searrow (B^+(yz_1)z_2) = B^+(xyz_1)z_2 = (xy) \searrow (B^+(z_1)z_2) = (xy) \searrow z,$$

which proves (2). \square

Corollary 3 \mathcal{M} is given a graded \mathbb{P}_{\searrow} -algebra structure by its products m and by \searrow .

Proof. Immediate, by proposition 2. \square

2.4 Grafting on the left leave

Let $F, G \in \mathbf{F}$. Suppose that $G \neq 1$. Then $F \nearrow G$ is the planar forest obtained by grafting F on the leave of G which is at most on the left. For $G = 1$, we put $F \nearrow 1 = F$. In particular, $F \nearrow \cdot = B^+(F)$.

Examples.

$$\begin{array}{c} \dots \nearrow \mathbf{1} = \mathbb{Y} \\ \cdot \mathbf{1} \nearrow \mathbf{1} = \mathbb{Y}^{\downarrow} \\ \mathbf{1} \cdot \nearrow \mathbf{1} = \mathbb{Y}^{\downarrow} \\ \mathbb{V} \nearrow \mathbf{1} = \mathbb{Y} \\ \mathbf{1} \nearrow \mathbf{1} = \mathbf{1} \end{array} \left| \begin{array}{c} \mathbf{1} \nearrow \dots = \mathbf{1} \cdot \\ \mathbf{1} \nearrow \cdot \mathbf{1} = \mathbf{1} \mathbf{1} \\ \mathbf{1} \nearrow \mathbf{1} \cdot = \mathbf{1} \cdot \\ \mathbf{1} \nearrow \mathbb{V} = \mathbb{V} \\ \mathbf{1} \nearrow \mathbf{1} = \mathbf{1} \end{array} \right| \begin{array}{c} \dots \nearrow \dots = \mathbb{V} \cdot \\ \dots \nearrow \cdot \mathbf{1} = \mathbb{V} \mathbf{1} \\ \dots \nearrow \mathbf{1} \cdot = \mathbb{Y} \cdot \\ \dots \nearrow \mathbb{V} = \mathbb{V} \\ \dots \nearrow \mathbf{1} = \mathbb{Y} \end{array} \left| \begin{array}{c} \dots \nearrow \cdot \cdot = \mathbb{V} \cdot \\ \cdot \mathbf{1} \nearrow \cdot \cdot = \mathbb{V}^{\downarrow} \\ \mathbf{1} \cdot \nearrow \cdot \cdot = \mathbb{V}^{\downarrow} \\ \mathbb{V} \nearrow \cdot \cdot = \mathbb{Y} \cdot \\ \mathbf{1} \nearrow \cdot \cdot = \mathbf{1} \cdot \end{array} \right.$$

In an obvious way, \nearrow can be inductively defined in the following way: for $F, G, H \in \mathbf{F}$,

$$\begin{cases} F \nearrow 1 = F, \\ F \nearrow (GH) = (F \nearrow G)H \text{ if } G \neq 1, \\ F \nearrow B^+(G) = B^+(F \nearrow G). \end{cases}$$

We extend $\nearrow: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ by linearity.

Proposition 4 1. For all $x, z \in \mathcal{H}$, $y \in \mathcal{M}$:

$$x \nearrow (yz) = (x \nearrow y)z. \quad (3)$$

2. For all $x, y, z \in \mathcal{H}$:

$$x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z.$$

So (\mathcal{H}, \nearrow) is an associative algebra, with unitary element 1.

Proof. Note that (3) is immediate for $x, y, z \in \mathbf{F}$, with $y \neq 1$. This implies the first point. In order to prove the second point, we consider:

$$Z = \{z \in \mathcal{H} / \forall x, y \in \mathcal{H}, x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z\}.$$

Let us first prove that $1 \in Z$: for all $x, y \in \mathcal{H}$,

$$x \nearrow (y \nearrow 1) = x \nearrow y = (x \nearrow y) \nearrow 1.$$

Let $z_1, z_2 \in Z$. Let us show that $z_1 z_2 \in Z$. By linearity, we can separate the proof into two cases:

1. $z_1 = 1$. Then it is obvious.
2. $\varepsilon(z_1) = 0$. Let $x, y \in \mathcal{H}$. By the first point:

$$\begin{aligned} x \nearrow (y \nearrow (z_1 z_2)) &= x \nearrow ((y \nearrow z_1) z_2) \\ &= (x \nearrow (y \nearrow z_1)) z_2 \\ &= ((x \nearrow y) \nearrow z_1) z_2 \\ &= (x \nearrow y) \nearrow (z_1 z_2). \end{aligned}$$

So Z is a subalgebra of \mathcal{H} . Let us show that it is stable by B^+ . Let $z \in Z$, $x, y \in \mathcal{H}$. Then:

$$\begin{aligned} x \nearrow (y \nearrow B^+(z)) &= x \nearrow B^+(y \nearrow z) \\ &= B^+(x \nearrow (y \nearrow z)) \\ &= B^+((x \nearrow y) \nearrow z) \\ &= (x \nearrow y) \nearrow B^+(z). \end{aligned}$$

So Z is a subalgebra of \mathcal{H} , stable by B^+ . Hence, $Z = \mathcal{H}$. □

Remarks.

1. (3) is equivalent to: for any $x, y, z \in \mathcal{H}$,

$$x \nearrow (yz) - \varepsilon(y)x \nearrow z = (x \nearrow y)z - \varepsilon(y)xz.$$

2. Let $F \in \mathbf{F} - \{1\}$. There exists a unique family $(\cdot F_1, \dots, \cdot F_n)$ of elements of \mathbf{F} such that:

$$F = (\cdot F_1) \nearrow \dots \nearrow (\cdot F_n).$$

For example, $\mathbb{V} \mathbf{1} \cdot = (\cdot \cdot) \nearrow (\cdot \cdot) \nearrow (\cdot \mathbf{1} \cdot)$. As a consequence, (\mathcal{H}, \nearrow) is freely generated by $\cdot \mathbf{F}$ as an associative algebra.

Corollary 5 \mathcal{M} is given a graded \mathbb{P}_{\nearrow} -algebra structure by its product m and by \nearrow .

Proof. Immediate, by proposition 4. □

2.5 Dimensions of \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow}

We now compute the dimensions of $\mathbb{P}_{\searrow}(n)$ and $\mathbb{P}_{\nearrow}(n)$ for all n and deduce that \mathcal{M} is the free \mathbb{P}_{\searrow} - and \mathbb{P}_{\nearrow} -algebra generated by $\cdot \cdot$.

Notation. We denote by r_n the number of planar rooted forests and we put $R(X) = \sum_{n=1}^{+\infty} r_n X^n$. It is well-known (see [3, 13]) that $R(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X}$.

Proposition 6 For $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$ and all $n \in \mathbb{N}^*$, in the $\mathbb{P}_{\overset{?}{\rightarrow}}$ -algebra \mathcal{M} :

$$\mathbb{P}_{\overset{?}{\rightarrow}}(n).(\cdot, \dots, \cdot) = \text{Vect}(\text{planar forests of weight } n).$$

As a consequence, \mathcal{M} is generated as a $\mathbb{P}_{\overset{?}{\rightarrow}}$ -algebra by \cdot .

Proof. \subseteq . Immediate, as \mathcal{M} is a graded $\mathbb{P}_{\overset{?}{\rightarrow}}$ -algebra.

\supseteq . Induction on n . For $n = 1$, $I.(\cdot) = \cdot$. For $n \geq 2$, two cases are possible.

1. $F = F_1 F_2$, $\text{weight}(F_i) = n_i < n$. By the induction hypothesis, there exists $p_1, p_2 \in \mathbb{P}_{\overset{?}{\rightarrow}}$, such that $F_1 = p_1.(\cdot, \dots, \cdot)$ and $F_2 = p_2.(\cdot, \dots, \cdot)$. Then $(m \circ (p_1, p_2)).(\cdot, \dots, \cdot) = m.(F_1, F_2) = F_1 F_2$.
2. $F \in \mathbf{T}$. Let us put $F = B^+(G)$. Then there exists $p \in \mathbb{P}_{\overset{?}{\rightarrow}}$, such that $p.(\cdot, \dots, \cdot) = G$. Then:

$$\begin{cases} (\searrow \circ (p, I)).(\cdot, \dots, \cdot) = G \searrow \cdot = F, \\ (\nearrow \circ (p, I)).(\cdot, \dots, \cdot) = G \nearrow \cdot = F. \end{cases}$$

Hence, in both cases, $F \in \mathbb{P}_{\overset{?}{\rightarrow}}(n).(\cdot, \dots, \cdot)$. □

Corollary 7 For all $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\overset{?}{\rightarrow}}(n)) \geq r_n$.

Proof. Because we proved the surjectivity of the following application:

$$\text{ev}_{\overset{?}{\rightarrow}} : \begin{cases} \mathbb{P}_{\overset{?}{\rightarrow}}(n) & \longrightarrow & \text{Vect}(\text{planar forests of weight } n) \\ p & \longrightarrow & p.(\cdot, \dots, \cdot). \end{cases}$$

□

Lemma 8 For all $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\overset{?}{\rightarrow}}(n)) \leq r_n$.

Proof. We prove it for $\overset{?}{\rightarrow} = \nearrow$. Let us fix $n \in \mathbb{N}^*$. Then $\mathbb{P}_{\nearrow}(n)$ is linearly generated by planar binary trees whose internal vertices are decorated by m and \nearrow . The following relations hold:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array}, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ m \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ m \end{array}, \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ m \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ m \end{array}.$$

In the sequel of the proof, we shall say that such a tree is *admissible* if it satisfies the following conditions:

1. For each internal vertex s decorated by m , the left child of s is a leaf.
2. For each internal vertex s decorated by \nearrow , the left child of s is a leaf or is decorated by m .

For example, here are the admissible trees with 1, 2 or 3 leaves:

$$1, \begin{array}{c} \diagdown \diagup \\ m \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ m \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ m \end{array}, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \nearrow \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ m \end{array}, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \nearrow \end{array}.$$

The preceding relations imply that $\mathbb{P}_{\nearrow}(n)$ is linearly generated by admissible trees with n leaves. So $\dim(\mathbb{P}_{\nearrow}(n))$ is smaller than a_n , the number of admissible trees with n leaves. For $n \geq 2$, we denote by b_n the number of admissible trees with n leaves whose root is decorated by

m , and by c_n the number of admissible trees with n leaves whose root is decorated by \nearrow . We also put $b_1 = 1$ and $c_1 = 0$. Finally, we define:

$$A(X) = \sum_{n \geq 1} a_n X^n, \quad B(X) = \sum_{n \geq 1} b_n X^n, \quad C(X) = \sum_{n \geq 1} c_n X^n.$$

Immediately, $A(X) = B(X) + C(X)$. Every admissible tree with $n \geq 2$ leaves whose root is decorated by m is of the form $m \circ (I, t)$, where t is an admissible tree with $n - 1$ leaves. Hence, $B(X) = XA(X) + X$. Moreover, every admissible tree with $n \geq 2$ leaves whose root is decorated by \nearrow is of the form $\nearrow \circ (t_1, t_2)$, where t_1 is an admissible tree with k leaves whose eventual root is decorated by m and t_2 an admissible tree with $n - k$ leaves ($1 \leq k \leq n - 1$). Hence, for all $n \geq 2$, $c_n = \sum_{k=1}^{n-1} b_k a_{n-k}$, so $C(X) = B(X)A(X)$. As a conclusion:

$$\begin{cases} A(X) &= B(X) + C(X), \\ B(X) &= XA(X) + X, \\ C(X) &= B(X)A(X). \end{cases}$$

So $A(X) = XA(X) + X + B(X)A(X) = XA(X) + X + XA(X)^2 + XA(X)$, and:

$$XA(X)^2 + (2X - 1)A(X) + X = 0.$$

As $a_1 = 1$:

$$A(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X} = R(X).$$

So, for all $n \geq 1$, $\dim(\mathbb{P}_{\nearrow}(n)) \leq a_n = r_n$. The proof is similar for \mathbb{P}_{\searrow} . \square

As immediate consequences:

Theorem 9 For $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\overset{?}{\rightarrow}}(n)) = r_n$. Moreover, the following application is bijective:

$$ev_{\overset{?}{\rightarrow}} : \begin{cases} \mathbb{P}_{\overset{?}{\rightarrow}}(n) &\longrightarrow Vect(\text{planar forests of weight } n) \subseteq \mathcal{M} \\ p &\longrightarrow p.(\cdot, \dots, \cdot). \end{cases}$$

Corollary 10 1. $(\mathcal{M}, m, \searrow)$ is the free \mathbb{P}_{\searrow} -algebra generated by \cdot .

2. $(\mathcal{M}, m, \nearrow)$ is the free \mathbb{P}_{\nearrow} -algebra generated by \cdot .

2.6 A combinatorial description of the composition

Let $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$. We identify $\mathbb{P}_{\overset{?}{\rightarrow}}$ and the vector space of non-empty planar forests via theorem 9. In other terms, we identify $F \in \mathbf{F}(n)$ and $ev_{\overset{?}{\rightarrow}}^{-1}(F) \in \mathbb{P}_{\overset{?}{\rightarrow}}(n)$.

Notations.

1. In order to distinguish the compositions in \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow} , we now denote:

- (a) $F^{\searrow} (F_1, \dots, F_n)$ the composition of \mathbb{P}_{\searrow} ,
- (b) $F^{\nearrow} (F_1, \dots, F_n)$ the composition of \mathbb{P}_{\nearrow} .

2. In order to distinguish the action of the operads \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow} on \mathcal{M} , we now denote:

- (a) $F^{\searrow} (x_1, \dots, x_n)$ the action of \mathbb{P}_{\searrow} on \mathcal{M} ,
- (b) $F^{\nearrow} (x_1, \dots, x_n)$ the action of \mathbb{P}_{\nearrow} on \mathcal{M} .

Our aim in this paragraph is to describe the compositions of \mathbb{P}_{\searrow} and \mathbb{P}_{\nearrow} in term of forests. We shall prove the following result:

Theorem 11 1. The composition of \mathbb{P}_{\searrow} in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \bullet \searrow (H) = H, \\ B^+(F) \searrow (H_1, \dots, H_{n+1}) = (F \searrow (H_1, \dots, H_n)) \searrow H_{n+1}, \\ FG \searrow (H_1, \dots, H_{n_1+n_2}) = F \searrow (H_1, \dots, H_{n_1}) G \searrow (H_{n_1+1}, \dots, H_{n_1+n_2}). \end{cases}$$

2. The composition of \mathbb{P}_{\nearrow} in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \bullet \nearrow (H) = H, \\ B^+(F) \nearrow (H_1, \dots, H_{n+1}) = (F \nearrow (H_1, \dots, H_n)) \nearrow H_{n+1}, \\ FG \nearrow (H_1, \dots, H_{n_1+n_2}) = F \nearrow (H_1, \dots, H_{n_1}) G \nearrow (H_{n_1+1}, \dots, H_{n_1+n_2}). \end{cases}$$

Examples. Let $F_1, F_2, F_3 \in \mathbf{F} - \{1\}$.

$$\begin{array}{l|l} \bullet \nearrow (F_1, F_2) = F_1 F_2, & \bullet \searrow (F_1, F_2) = F_1 F_2, \\ \mathbf{1} \nearrow (F_1, F_2) = F_1 \nearrow F_2, & \mathbf{1} \searrow (F_1, F_2) = F_1 \searrow F_2, \\ \dots \nearrow (F_1, F_2, F_3) = F_1 F_2 F_3, & \dots \searrow (F_1, F_2, F_3) = F_1 F_2 F_3, \\ \bullet \mathbf{1} \nearrow (F_1, F_2, F_3) = F_1 (F_2 \nearrow F_3), & \bullet \mathbf{1} \searrow (F_1, F_2, F_3) = F_1 (F_2 \searrow F_3), \\ \mathbf{1} \bullet \nearrow (F_1, F_2, F_3) = (F_1 \nearrow F_2) F_3, & \mathbf{1} \bullet \searrow (F_1, F_2, F_3) = (F_1 \searrow F_2) F_3, \\ \mathbf{V} \nearrow (F_1, F_2, F_3) = (F_1 F_2) \nearrow F_3, & \mathbf{V} \searrow (F_1, F_2, F_3) = (F_1 F_2) \searrow F_3, \\ \mathbf{1} \nearrow (F_1, F_2, F_3) = (F_1 \nearrow F_2) \nearrow F_3, & \mathbf{1} \searrow (F_1, F_2, F_3) = (F_1 \searrow F_2) \searrow F_3. \end{array}$$

Proposition 12 Let $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$.

1. \bullet is the unit element of $\mathbb{P}_{\overset{?}{\rightarrow}}$.
2. $\bullet \bullet = m$ in $\mathbb{P}_{\overset{?}{\rightarrow}}(2)$. Consequently, in $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\bullet \bullet \circ (F, G) = FG$ for all $F, G \in \mathbf{F} - \{1\}$.
3. Let $F, G \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\mathbf{1} = \overset{?}{\rightarrow}$. Consequently, $\mathbf{1} \overset{?}{\rightarrow} (F, G) = F \overset{?}{\rightarrow} G$ for all $F, G \in \mathbf{F} - \{1\}$.

Proof.

1. Indeed, $ev_{\overset{?}{\rightarrow}}(\bullet) = \bullet = ev_{\overset{?}{\rightarrow}}(I)$. Hence, $\bullet = I$.
2. By definition, $ev_{\overset{?}{\rightarrow}}(\bullet \bullet) = \bullet \bullet = ev_{\overset{?}{\rightarrow}}(m)$. So $\bullet \bullet = m$ in $\mathbb{P}_{\overset{?}{\rightarrow}}(2)$. Moreover, for all $F, G \in \mathbf{F} - \{1\}$:

$$\begin{aligned} ev_{\overset{?}{\rightarrow}}(FG) &= FG \\ &= m_{\bullet \bullet}^{\overset{?}{\rightarrow}}(F, G) \\ &= m_{\bullet \bullet}^{\overset{?}{\rightarrow}}(F \overset{?}{\rightarrow} (\bullet, \dots, \bullet), G \overset{?}{\rightarrow} (\bullet, \dots, \bullet)) \\ &= \left(m_{\bullet \bullet}^{\overset{?}{\rightarrow}}(F, G) \right) \overset{?}{\rightarrow} (\bullet, \dots, \bullet) \\ &= ev_{\overset{?}{\rightarrow}}(m_{\bullet \bullet}^{\overset{?}{\rightarrow}}(F, G)). \end{aligned}$$

So $FG = m_{\bullet \bullet}^{\overset{?}{\rightarrow}}(F, G) = \bullet \bullet \overset{?}{\rightarrow} (F, G)$.

3. Indeed, $ev_{\underline{\rightarrow}}(\mathbf{1}) = \bullet \xrightarrow{\cdot} \bullet = ev_{\underline{\rightarrow}}(\xrightarrow{\cdot})$. So $\mathbf{1} = \xrightarrow{\cdot}$ in $\mathbb{P}_{\underline{\rightarrow}}(2)$. Moreover:

$$\begin{aligned} ev_{\underline{\rightarrow}}(F \xrightarrow{\cdot} G) &= F \xrightarrow{\cdot} G \\ &= \xrightarrow{\cdot} \bullet (F, G) \\ &= \xrightarrow{\cdot} \bullet (F \xrightarrow{\cdot} (\cdot, \dots, \cdot), G \xrightarrow{\cdot} (\cdot, \dots, \cdot)) \\ &= (\xrightarrow{\cdot} \bullet (F, G)) \cdot (\cdot, \dots, \cdot) \\ &= ev_{\underline{\rightarrow}}(\xrightarrow{\cdot} \bullet (F, G)). \end{aligned}$$

$$\text{So, } F \xrightarrow{\cdot} G = \xrightarrow{\cdot} \bullet (F, G) = \mathbf{1} \bullet (F, G).$$

□

Proposition 13 1. Let $F, G \in \mathbf{F}$, different from $\mathbf{1}$, of respective weights n_1 and n_2 . Let $H_{1,1}, \dots, H_{1,n_1}$ and $H_{2,1}, \dots, H_{2,n_2} \in \mathbf{F} - \{\mathbf{1}\}$. Let $\xrightarrow{\cdot} \in \{\searrow, \nearrow\}$. Then, in $\mathbb{P}_{\underline{\rightarrow}}$:

$$(FG) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) = F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}) G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2}).$$

2. Let $F \in \mathbf{F}$, of weight $n \geq 1$. Let $H_1, \dots, H_{n+1} \in \mathbf{F}$. In $\mathbb{P}_{\underline{\rightarrow}}$:

$$B^+(F) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) = (F \xrightarrow{\cdot} (H_1, \dots, H_n)) \xrightarrow{\cdot} H_{n+1}.$$

Proof.

1. Indeed, in $\mathbb{P}_{\underline{\rightarrow}}$:

$$\begin{aligned} &(FG) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) \\ &= (m \xrightarrow{\cdot} (F, G)) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) \\ &= m \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}), G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2})) \\ &= F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}) G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2}). \end{aligned}$$

2. In $\mathbb{P}_{\underline{\rightarrow}}$:

$$\begin{aligned} B^+(F) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) &= (F \xrightarrow{\cdot} \bullet) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) \\ &= (\mathbf{1} \xrightarrow{\cdot} (F, \bullet)) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) \\ &= \mathbf{1} \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_1, \dots, H_n), \bullet \xrightarrow{\cdot} (H_{n+1})) \\ &= \mathbf{1} \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_1, \dots, H_n), H_{n+1}) \\ &= (F \xrightarrow{\cdot} (H_1, \dots, H_n)) \xrightarrow{\cdot} H_{n+1}. \end{aligned}$$

□

Combining propositions 12 and 13, we obtain theorem 11.

3 Applications to the infinitesimal Hopf algebra \mathcal{H}

3.1 Antipode of \mathcal{H}

We here give a description of the antipode of \mathcal{H} in terms of the action \searrow_{\bullet} of the operad $\mathbb{P}_{\searrow_{\bullet}}$.

Notations. For all $n \in \mathbb{N}^*$, we denote $l_n = (B^+)^n(1) \in \mathbf{F}(n)$. For example:

$$l_1 = \bullet, l_2 = \mathbf{1}, l_3 = \mathbf{1} \cdot, l_4 = \mathbf{1} \cdot \mathbf{1}, l_5 = \mathbf{1} \cdot \mathbf{1} \cdot \dots$$

Lemma 14 *Let $t \in \mathbf{T}$. There exists a unique $k \in \mathbb{N}^*$, and a unique family $(t_2, \dots, t_k) \in \mathbf{T}^{k-1}$ such that:*

$$t = l_k \searrow_{\mathbf{q}} (\bullet, t_2, \dots, t_k).$$

Proof. Induction on the weight n of t . If $n = 1$, then $t = \bullet$, so $k = 1$ and the family is empty. We suppose the result at all rank $< n$. We put $t = B^+(s_1 \dots s_m)$. Necessarily, $t_k = B^+(s_2 \dots s_m)$ and $l_{n-1} \searrow_{\mathbf{q}} (\bullet, t_2, \dots, t_{k-1}) = s_1$. We conclude with the induction hypothesis on s_1 . \square

Example.

$$\begin{array}{c} \mathbf{1} \\ \swarrow \searrow \\ \mathbf{1} \cdot \mathbf{1} \\ \swarrow \searrow \\ \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \end{array} = l_4 \searrow_{\mathbf{q}} (\bullet, \mathbf{1}, \mathbf{1}, \mathbf{1} \cdot \mathbf{1}).$$

Definition 15 For all $n \in \mathbb{N}^*$, we put $p_n = \sum_{k=1}^n \sum_{\substack{a_1 + \dots + a_k = n \\ \forall i, a_i > 0}} (-1)^k l_{a_1} \dots l_{a_k}$.

Examples.

$$\begin{aligned} p_1 &= \bullet, \\ p_2 &= -\mathbf{1} + \bullet, \\ p_3 &= -\mathbf{1} \cdot + \mathbf{1} \cdot \mathbf{1} + \mathbf{1} \cdot \mathbf{1} \cdot - \dots, \\ p_4 &= -\mathbf{1} \cdot \mathbf{1} \cdot + \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot + \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot - \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot - \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1} \cdot + \dots \end{aligned}$$

Remark that p_n is in fact the antipode of l_n in \mathcal{H} . It is also the antipode of l_n in the non commutative Connes-Kreimer Hopf algebra of planar trees [3].

Corollary 16 *Let $t \in \mathbf{T}$, written under the form $t = l_k \searrow_{\mathbf{q}} (t_1, \dots, t_k)$, with $t_1 = \bullet$. Then:*

$$S(t) = p_k \searrow_{\mathbf{q}} (t_1, \dots, t_k).$$

Proof. Corollary of proposition 15 of [4], observing that left cuts are cuts on edges from the root of t_i to the root of t_{i+1} in t , for $i = 1, \dots, n-1$. \square

3.2 Inverse of the application γ

Proposition 17 *The restriction $\gamma : \text{Prim}(\mathcal{H}) \longrightarrow \mathcal{H}$ is bijective.*

Proof. By proposition 21 of [4]:

$$\gamma|_{\text{Prim}(\mathcal{H})} : \begin{cases} \text{Prim}(\mathcal{H}) & \longrightarrow \mathcal{H} \\ f_{B^+(F)} (F \in \mathbf{F}) & \longrightarrow f_F. \end{cases}$$

So this restriction is clearly bijective. \square

We shall denote $\gamma|_{\text{Prim}(\mathcal{H})}^{-1} : \mathcal{H} \longrightarrow \text{Prim}(\mathcal{H})$ the inverse of this restriction. Then, for all $F \in \mathbf{F}$, $\gamma|_{\text{Prim}(\mathcal{H})}^{-1}(f_F) = f_{B^+(F)}$. Our aim is to express $\gamma|_{\text{Prim}(\mathcal{H})}^{-1}$ in the basis of forests.

We define inductively a sequence $(q_n)_{n \in \mathbb{N}^*}$ of elements of \mathbb{P}_{\searrow} :

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_{\searrow}(1), \\ q_2 &= \bullet\bullet - \mathbf{1} \in \mathbb{P}_{\searrow}(2), \\ q_{n+1} &= (\bullet\bullet - \mathbf{1}) \searrow_{\mathfrak{Q}} (q_n, \bullet) \in \mathbb{P}_{\searrow}(n+1) \text{ for } n \geq 1. \end{cases}$$

For all $F \in \mathbf{F}$, $\bullet\bullet \searrow_{\mathfrak{Q}} (F, \bullet) = F$. and $\mathbf{1} \searrow_{\mathfrak{Q}} (F, \bullet) = B^+(F)$. So, q_n can also be defined in the following way:

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_{\searrow}(1), \\ q_{n+1} &= q_n \bullet - B^+(q_n) \in \mathbb{P}_{\searrow}(n+1) \text{ for } n \geq 1. \end{cases}$$

Examples.

$$\begin{aligned} q_3 &= \dots - \mathbf{1} \bullet - \mathbb{V} + \mathbf{1}, \\ q_4 &= \dots - \mathbf{1} \bullet\bullet - \mathbb{V} \bullet + \mathbf{1} \bullet - \mathbb{W} + \mathbb{V} + \mathbb{Y} - \mathbf{1}, \\ q_5 &= \dots - \mathbf{1} \bullet\bullet\bullet - \mathbb{V} \bullet\bullet + \mathbf{1} \bullet\bullet - \mathbb{W} \bullet + \mathbb{V} \bullet + \mathbb{Y} \bullet - \mathbf{1} \bullet \\ &\quad - \mathbb{V} \bullet + \mathbb{W} + \mathbb{V} - \mathbb{V} + \mathbb{W} - \mathbb{Y} - \mathbf{1} + \mathbf{1}. \end{aligned}$$

Lemma 18 *Let $F \in \mathbf{F} - \{1\}$, and $t \in \mathbf{T}$. Then, in \mathcal{H} :*

$$\Delta(F \searrow t) = (F \searrow t) \otimes 1 + 1 \otimes (F \searrow t) + F' \otimes F'' \searrow t + Ft' \otimes t'' + F \otimes t.$$

Proof. The non-empty and non-total left-admissible cuts of the tree $F \searrow t$ are:

- The cut on the edges relating F to t . For this cut c , $P^c(F \searrow t) = F$ and $R^c(F \searrow t) = t$.
- Cuts acting only on edges of F or on edges relating F to t , at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of F , such that $P^c(F \searrow t) = P^{c'}(F)$ and $R^c(F \searrow t) = R^{c'}(F) \searrow t$.
- Cuts acting on edges of t . Then necessarily $F \subseteq P^c(F \searrow t)$. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of t , such that $P^c(F \searrow t) = FP^{c'}(t)$ and $R^c(F \searrow t) = R^{c'}(t)$.

Summing these cuts, we obtain the announced compatibility. \square

Proposition 19 *Let $F = t_1 \dots t_n \in \mathbf{F}$. Then:*

$$\gamma_{|Prim(\mathcal{H})}^{-1}(F) = q_{n+1} \searrow_{\mathfrak{Q}} (\bullet, t_1, \dots, t_n).$$

Proof. *First step.* Let us show the following property: for all $x \in Prim(\mathcal{H})$, $t \in \mathbf{T}$, $q_2 \searrow_{\mathfrak{Q}} (x, t)$ is primitive. By lemma 18, using the linearity in F :

$$\begin{aligned} \Delta(x \searrow t) &= (x \searrow t) \otimes 1 + 1 \otimes (x \searrow t) + x \otimes t + xt' \otimes t'', \\ \Delta(xt) &= xt \otimes 1 + 1 \otimes xt + x \otimes t + xt' \otimes t'', \\ \Delta(q_2 \searrow_{\mathfrak{Q}} (x, t)) &= \Delta(xt - x \searrow t) \\ &= (xt - x \searrow t) \otimes 1 + 1 \otimes (xt - x \searrow t). \end{aligned}$$

Second step. Let us show that for all $x \in Prim(\mathcal{H})$, $t_1, \dots, t_n \in \mathbf{T}$, $q_{n+1} \searrow_{\mathfrak{Q}} (x, t_1, \dots, t_n) \in Prim(\mathcal{H})$ by induction on n . This is obvious for $n = 0$, as $q_1 \searrow_{\mathfrak{Q}} (x) = x$. Suppose the result at rank $n - 1$. Then:

$$\begin{aligned} q_{n+1} \searrow_{\mathfrak{Q}} (x, t_1, \dots, t_n) &= (q_2 \searrow_{\mathfrak{Q}} (q_n, I)) \searrow_{\mathfrak{Q}} (x, t_1, \dots, t_n) \\ &= q_2 \searrow_{\mathfrak{Q}} (\underbrace{q_n \searrow_{\mathfrak{Q}} (x, t_1, \dots, t_{n-1})}_{\in Prim(\mathcal{H})}, t_n) \in Prim(\mathcal{H}), \end{aligned}$$

by the first step. As the tree \cdot is primitive, we deduce that, for all forest $F = t_1 \dots t_n \in \mathbf{F}$, $q_{n+1} \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_n) \in \text{Prim}(\mathcal{H})$.

Third step. Let us show that for all $x, y \in \mathcal{M}$, $\gamma(q_2 \searrow_{\mathfrak{A}} (x, y)) = \gamma(x)y$. We can limit ourselves to $x, y \in \mathbf{F} - \{1\}$. Then $q_2 \searrow_{\mathfrak{A}} (x, y) = xy - x \searrow y$. Moreover, by definition of \searrow , $x \searrow y$ is a forest whose first tree is not equal to \cdot . Hence, $\gamma(q_2 \searrow_{\mathfrak{A}} (x, y)) = \gamma(xy) - 0 = \gamma(x)y$.

Last step. Let us show by induction on n that $\gamma(q_{n+1} \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_n)) = t_1 \dots t_n$. As $q_1 \searrow_{\mathfrak{A}} (\cdot) = \cdot$, this is obvious if $n = 0$. Let us suppose the result at rank $n - 1$. By the third step:

$$\begin{aligned} \gamma(q_{n+1} \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_n)) &= \gamma(q_2 \searrow_{\mathfrak{A}} (q_n \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_{n-1}), t_n)) \\ &= \gamma(q_n \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_{n-1}))t_n \\ &= t_1 \dots t_n. \end{aligned}$$

Consequently, $x = q_{n+1} \searrow_{\mathfrak{A}} (\cdot, t_1, \dots, t_n) \in \text{Prim}(\mathcal{H})$, and satisfies $\gamma(x) = t_1 \dots t_n$, which proves proposition 19. \square

Examples. Let $t_1, t_2, t_3 \in \mathbf{T}$.

$$\begin{aligned} \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1) &= \cdot t_1 - \cdot \searrow t_1, \\ \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1 t_2) &= \cdot t_1 t_2 - (\cdot \searrow t_1)t_2 - (\cdot t_1) \searrow t_2 + (\cdot \searrow t_1) \searrow t_2, \\ \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1 t_2 t_3) &= \cdot t_1 t_2 t_3 - (\cdot \searrow t_1)t_2 t_3 - (\cdot t_1) \searrow t_2 t_3 + (\cdot \searrow t_1) \searrow t_2 t_3 - (\cdot t_1 t_2) \searrow t_3 \\ &\quad + (\cdot \searrow t_1 t_2) \searrow t_3 + ((\cdot t_1) \searrow t_2) \searrow t_3 - ((\cdot \searrow t_1) \searrow t_2) \searrow t_3. \end{aligned}$$

3.3 Elements of the dual basis

Lemma 20 For all $x, y \in \mathcal{H}$, $\Delta(x \nearrow y) = x \nearrow y^{(1)} \otimes y^{(2)} + x^{(1)} \otimes x^{(2)} \nearrow y - x \otimes y$. In other terms, $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal Hopf algebra.

Proof. We restrict to $x = F \in \mathbf{F} - \{1\}$, $y = G \in \mathbf{F} - \{1\}$. The non-empty and non-total left-admissible cuts of the tree $F \nearrow G$ are:

- The cut on the edges relating F to G . For this cut c , $P^c(F \nearrow G) = F$ and $R^c(F \nearrow G) = G$.
- Cuts acting only on edges of F or on edges relating F to G , at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of F , such that $P^c(F \nearrow G) = P^{c'}(F)$ and $R^c(F \nearrow G) = R^{c'}(F) \nearrow G$.
- Cuts acting on edges of G . Then necessarily $F \subseteq P^c(F \nearrow G)$. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of t , such that $P^c(F \nearrow G) = F \nearrow P^{c'}(G)$ and $R^c(F \nearrow G) = R^{c'}(G)$.

Summing these cuts, we obtain, denoting $\Delta(F) = F \otimes 1 + 1 \otimes F + F' \otimes F''$ and $\Delta(G) = G \otimes 1 + 1 \otimes G + G' \otimes G''$:

$$\begin{aligned} \tilde{\Delta}(F \nearrow G) &= (F \nearrow G) \otimes 1 + 1 \otimes (F \nearrow G) + F \otimes G + F' \otimes F'' \nearrow G + F \nearrow G' \otimes G'' \\ &= (F \otimes 1) \nearrow \Delta(G) + \Delta(F) \nearrow (1 \otimes G) - F \otimes G. \end{aligned}$$

So $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal bialgebra. As it is graded and connected, it has an antipode. \square

Proposition 21 Let $F = t_1 \dots t_n \in \mathbf{F}$. Then $f_F = f_{t_n} \nearrow \dots \nearrow f_{t_1}$.

Proof. *First step.* We show the following result: for all $F \in \mathbf{F}$, $t \in \mathbf{T}$, $f_F \nearrow f_t = f_{tF}$. We proceed by induction on the weight n of F . If $n = 0$, then $F = 1$ and the result is obvious. We now suppose that the result is true at all rank $< n$. Let be $G \in \mathbf{F}$, and let us prove that $\langle f_F \nearrow f_t, G \rangle = \delta_{tF, G}$. Three cases are possible.

1. $G = 1$. Then $\langle f_F \nearrow f_t, G \rangle = \langle f_F \nearrow f_t, 1 \rangle = \varepsilon(f_F \nearrow f_t) = 0 = \delta_{tF, G}$.

2. $G = G_1 G_2$, $G_i \neq 1$. Then, by lemma 20:

$$\begin{aligned}
\langle f_F \nearrow f_t, G \rangle &= \langle \Delta(f_F \nearrow f_t), G_2 \otimes G_1 \rangle \\
&= \sum_{F_1 F_2 = F} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1 \rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1 + f_F \nearrow 1 \otimes f_t, G_2 \otimes G_1 \rangle - \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1 \rangle + \langle 1 \otimes f_F \nearrow f_t, G_2 \otimes G_1 \rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1, G_2 \otimes G_1 \rangle + \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle - \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{t_{F_1}}, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \delta_{F_2, G_2} \delta_{t_{F_1}, G_1} \\
&= \delta_{tF, G}.
\end{aligned}$$

3. $G = B^+(G_1)$. Note that $f_F \nearrow f_t$ is a linear span of forests $H_1 \nearrow H_2$, with $H_1, H_2 \neq 1$. By definition of \nearrow , the first tree of such a forest is not \bullet . Hence, $\gamma(f_F \nearrow f_t) = 0$ and:

$$\langle f_F \otimes f_t, G \rangle = \langle \gamma(f_F \otimes f_t), G_1 \rangle = 0 = \delta_{tF, G},$$

as $tF \notin \mathbf{T}$ because $F \neq 1$.

Second step. We now prove proposition 21 by induction on n . It is obvious for $n = 1$. Suppose the result at rank $n - 1$. By the first step:

$$f_{t_1 \dots t_n} = f_{t_2 \dots t_n} \nearrow f_{t_1} = (f_{t_n} \nearrow \dots \nearrow f_{t_2}) \nearrow f_{t_1} = f_{t_n} \nearrow \dots \nearrow f_{t_2} \nearrow f_{t_1},$$

using the induction hypothesis for the second equality. \square

Remarks.

1. As an immediate corollary, because \nearrow is associative, for all forests $F_1, \dots, F_k \in \mathbf{F}$, $f_{F_1 \dots F_k} = f_{F_k} \nearrow \dots \nearrow f_{F_1}$.
2. In term of operads, proposition 21 can be rewritten in the following way:

Corollary 22 *Let $F_1, \dots, F_n \in \mathbf{F}$. Then $f_{F_1 \dots F_n} = l_n \not\circ (f_{F_n}, \dots, f_{F_1})$.*

Remark. Hence, the dual basis $(f_F)_{F \in \mathbf{F}}$ can be inductively computed, using proposition 21 of [4], together with propositions 19 and 21 of the present text:

$$\left\{ \begin{array}{l} f_1 = 1, \\ f_{t_1 \dots t_n} = f_{t_n} \nearrow \dots \nearrow f_{t_1}, \\ f_{B^+(t_1 \dots t_n)} = \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(f_{t_1 \dots t_n}). \end{array} \right.$$

For example:

$f_1 = 1$	$f. = .$
$f.. = \downarrow$	$f\downarrow = -\downarrow + ..$
$f... = \downarrow\downarrow$	$f.\downarrow = -\downarrow\downarrow + \downarrow\downarrow$
$f\downarrow. = -\downarrow + \downarrow.$	$f\downarrow\downarrow = -\downarrow\downarrow + \downarrow.$
$f\downarrow\downarrow = \downarrow - \downarrow. - \downarrow.\downarrow + \dots$	$f.\downarrow\downarrow = \downarrow\downarrow$
$f.\downarrow\downarrow = -\downarrow\downarrow + \downarrow\downarrow$	$f.\downarrow.\downarrow = -\downarrow\downarrow + \downarrow\downarrow$
$f.\downarrow\downarrow\downarrow = -\downarrow\downarrow\downarrow + \downarrow\downarrow\downarrow$	$f.\downarrow.\downarrow\downarrow = \downarrow\downarrow - \downarrow\downarrow - \downarrow\downarrow + \downarrow\downarrow$
$f\downarrow.\downarrow = -\downarrow\downarrow + \downarrow.$	$f\downarrow\downarrow\downarrow = \downarrow\downarrow - \downarrow\downarrow - \downarrow.\downarrow + \downarrow.$
$f\downarrow\downarrow. = -\downarrow\downarrow + \downarrow\downarrow$	$f\downarrow.\downarrow. = \downarrow\downarrow - \downarrow. - \downarrow\downarrow + \downarrow.\downarrow$
$f\downarrow\downarrow\downarrow = -\downarrow\downarrow\downarrow + \downarrow\downarrow\downarrow$	$f\downarrow.\downarrow\downarrow = \downarrow\downarrow - \downarrow\downarrow - \downarrow.\downarrow + \downarrow.\downarrow$
$f\downarrow\downarrow\downarrow\downarrow = \downarrow\downarrow\downarrow - \downarrow\downarrow\downarrow - \downarrow.\downarrow\downarrow + \downarrow.\downarrow.$	$f\downarrow.\downarrow\downarrow\downarrow = \downarrow\downarrow - \downarrow\downarrow - \downarrow.\downarrow + \downarrow.\downarrow$
$f_{\downarrow\downarrow} = -\downarrow\downarrow + \downarrow.\downarrow + \downarrow\downarrow\downarrow - \downarrow.\downarrow\downarrow + \downarrow.\downarrow - \downarrow.\downarrow - \dots + \dots$	

4 Primitive suboperads

4.1 Compatibilities between products and coproducts

We define another coproduct Δ_{\nearrow} on \mathcal{H} in the following way: for all $x, y, z \in \mathcal{H}$,

$$\langle \Delta_{\nearrow}(x), y \otimes z \rangle = \langle x, z \nearrow y \rangle.$$

Lemma 23 For all forest $F \in \mathbf{F}$, $\Delta_{\nearrow}(F) = \sum_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} F_1 \otimes F_2$.

Proof. Let $F, G, H \in \mathbf{F}$. Then:

$$\begin{aligned} \langle \Delta_{\nearrow}(F), f_G \otimes f_H \rangle &= \langle F, f_H \nearrow f_G \rangle \\ &= \langle F, f_{GH} \rangle \\ &= \delta_{F, GH} \\ &= \sum_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} \langle F_1 \otimes F_2, f_G \otimes f_H \rangle. \end{aligned}$$

As $(f_F)_{F \in \mathbf{F}}$ is a basis of \mathcal{H} and $\langle -, - \rangle$ is non degenerate, this proves the result. \square

Remark. As a consequence, the elements of \mathbf{T} are primitive for this coproduct.

We now have defined three products, namely m , \nearrow , and \searrow , and two coproducts, namely $\tilde{\Delta}$ and $\tilde{\Delta}_{\nearrow}$, on \mathcal{M} , obtained from Δ and Δ_{\nearrow} by subtracting their primitive parts. The following properties sum up the different compatibilities.

Proposition 24 For all $x, y \in \mathcal{M}$:

$$\tilde{\Delta}(xy) = (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y, \quad (4)$$

$$\tilde{\Delta}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}(y) + \tilde{\Delta}(x) \nearrow (1 \otimes y) + x \otimes y, \quad (5)$$

$$\tilde{\Delta}_{\nearrow}(xy) = (x \otimes 1)\tilde{\Delta}_{\nearrow}(y) + \tilde{\Delta}_{\nearrow}(x)(1 \otimes y) + x \otimes y, \quad (6)$$

$$\tilde{\Delta}_{\nearrow}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(y), \quad (7)$$

$$\tilde{\Delta}_{\nearrow}(x \searrow y) = (x \otimes 1) \searrow \tilde{\Delta}_{\nearrow}(y). \quad (8)$$

Proof. It remains to consider the compatibility between \nearrow or \searrow and $\tilde{\Delta}_{\nearrow}$. Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$, where the t_i 's are trees. Then $F \nearrow G = (F \nearrow t_1)t_2 \dots t_n$, and $F \nearrow t_1$ is a tree. Hence:

$$\begin{aligned} \tilde{\Delta}_{\nearrow}(F \nearrow G) &= \sum_{i=1}^{n-1} (F \nearrow t_1)t_2 \dots t_i \otimes t_{i+1} \dots t_n \\ &= \sum_{i=1}^{n-1} F \nearrow (t_1 t_2 \dots t_i) \otimes t_{i+1} \dots t_n \\ &= (F \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(G). \end{aligned}$$

The proof is similar for $F \searrow G$. So all these compatibilities are satisfied. \square

Remark. There is no similar compatibility between $\tilde{\Delta}$ and \searrow . In particular, lemma 19 is not available for $t \notin \mathbf{T}$.

This justifies the following definitions:

Definition 25

1. A \mathbb{P}_{\nearrow} -bialgebra of type 1 is a family $(A, m, \nearrow, \tilde{\Delta})$, such that:

- (a) (A, m, \nearrow) is a \mathbb{P}_{\nearrow} -algebra.
- (b) $(A, \tilde{\Delta})$ is a coassociative, non counitary coalgebra.
- (c) Compatibilities (4) and (5) are satisfied.

2. A \mathbb{P}_{\nearrow} -bialgebra of type 2 is a family $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$, such that:

- (a) (A, m, \nearrow) is a \mathbb{P}_{\nearrow} -algebra.
- (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
- (c) Compatibilities (6) and (7) are satisfied.

3. A \mathbb{P}_{\searrow} -bialgebra is a family $(A, m, \searrow, \tilde{\Delta}_{\nearrow})$, such that:

- (a) (A, m, \searrow) is a \mathbb{P}_{\searrow} -algebra.
- (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
- (c) Compatibilities (6) and (8) are satisfied.

Example. The augmentation ideal \mathcal{M} of the infinitesimal Hopf algebra of trees \mathcal{H} is both a \mathbb{P}_{\nearrow} -infinitesimal bialgebra of type 1 and 2, and also a \mathbb{P}_{\searrow} -infinitesimal bialgebra.

If A is a bialgebra of such a type, we denote by $Prim(A)$ the kernel of the coproduct. We deduce the definition of the following suboperads:

Definition 26 Let $n \in \mathbb{N}$. We put:

$$\left\{ \begin{array}{l} \text{PRIM}_{\nearrow}^{(1)}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 1,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}(A). \end{array} \right\}, \\ \text{PRIM}_{\nearrow}^{(2)}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 2,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}_{\nearrow}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}_{\nearrow}(A). \end{array} \right\}, \\ \text{PRIM}_{\searrow}(n) = \left\{ p \in \mathbb{P}_{\searrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\searrow}\text{-infinitesimal bialgebra,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}_{\nearrow}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}_{\nearrow}(A). \end{array} \right\}. \end{array} \right.$$

We identify $\mathbb{P}_{\nearrow}(n)$ and $\mathbb{P}_{\searrow}(n)$ with the homogeneous component of weight n of \mathcal{M} . We put $\text{Prim}(\mathcal{M}) = \text{Ker}(\tilde{\Delta})$ and $\text{Prim}_{\nearrow}(\mathcal{M}) = \text{Ker}(\tilde{\Delta}_{\nearrow})$. We obtain:

Proposition 27 1. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\nearrow}^{(1)}(n) = \{p \in \mathbb{P}_{\nearrow}(n) / p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}(\mathcal{M}).$$

2. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\nearrow}^{(2)}(n) = \{p \in \mathbb{P}_{\nearrow}(n) / p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}_{\nearrow}(\mathcal{M}).$$

3. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\searrow}(n) = \{p \in \mathbb{P}_{\searrow}(n) / p_{\searrow}(\cdot, \dots, \cdot) \in \text{Prim}_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\searrow}(n) \cap \text{Prim}_{\nearrow}(\mathcal{M}).$$

Proof. As \mathcal{M} is a \mathbb{P}_{\nearrow} -infinitesimal bialgebra, by definition:

$$\text{PRIM}_{\nearrow}^{(1)}(n) \subseteq \{p \in \mathbb{P}_{\nearrow}(n) / p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\}.$$

Moreover, $\{p \in \mathbb{P}_{\nearrow}(n) / p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}(\mathcal{M})$, as, for all $p \in \mathbb{P}_{\nearrow}(n)$, $p_{\nearrow}(\cdot, \dots, \cdot) = p \in \mathcal{M}$.

We now show that $\{p \in \mathbb{P}_{\nearrow}(n) / p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} \subseteq \text{PRIM}_{\nearrow}^{(1)}(n)$. We take $p \in \mathbb{P}_{\nearrow}(n)$, such that $p_{\nearrow}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})$. Let $\mathcal{D} = \{1, \dots, n\}$ and let A be the free \mathbb{P}_{\nearrow} -algebra generated by \mathcal{D} (with a unit). It can be described as the associative algebra $\mathcal{H}^{\mathcal{D}}$ generated by the set of planar rooted trees decorated by \mathcal{D} , and can be given a structure of \mathbb{P}_{\nearrow} -infinitesimal bialgebra. As \mathcal{M} is freely generated by \cdot as a \mathbb{P}_{\nearrow} -algebra, there exists a unique morphism of \mathbb{P}_{\nearrow} -algebras from \mathcal{M} to $\mathcal{M}^{\mathcal{D}}$, augmentation ideal of $\mathcal{H}^{\mathcal{D}}$:

$$\xi : \begin{cases} \mathcal{M} & \longrightarrow & \mathcal{M}^{\mathcal{D}} \\ \cdot & \longrightarrow & \cdot_1 + \dots + \cdot_n. \end{cases}$$

As $\cdot \in \text{Prim}(\mathcal{M})$ and $\cdot_1 + \dots + \cdot_n \in \text{Prim}(A)$, ξ is a \mathbb{P}_{\nearrow} -infinitesimal bialgebra morphism from \mathcal{M} to $\mathcal{M}^{\mathcal{D}}$. So, $\xi(p_{\nearrow}(\cdot, \dots, \cdot)) \in \text{Prim}(A)$.

Let $F \in A$ be a forest, and $s_1 \geq_{h,l} \dots \geq_{h,l} s_k$ its vertices. For all $i \in \{1, \dots, k\}$, we put d_i the decoration of s_i . The *decoration word* associated to F is the word $d_1 \dots d_n$. It belongs to $M(\mathcal{D})$, the free monoid generated by the elements of \mathcal{D} . For all $w \in M(\mathcal{D})$, Let A_w be the subspace of A generated by forests whose decoration word is w . This defines a $M(\mathcal{D})$ -gradation of A , as a \mathbb{P}_{\nearrow} -infinitesimal bialgebra of type 1.

Consider the projection $\pi_{1,\dots,n}$ onto $A_{1,\dots,n}$. We get:

$$\begin{aligned}\pi_{1,\dots,n} \circ \xi(p_{\bullet}^{\nearrow}(\cdot, \dots, \cdot)) &\in Prim(A), \\ &= \pi_{1,\dots,n}(p_{\bullet}^{\nearrow}(\xi(\cdot), \dots, \xi(\cdot))) \\ &= \pi_{1,\dots,n}(p_{\bullet}^{\nearrow}(\cdot_1 + \dots + \cdot_n, \dots, \cdot_1 + \dots + \cdot_n)) \\ &= p_{\bullet}^{\nearrow}(\cdot_1, \dots, \cdot_n).\end{aligned}$$

So $p_{\bullet}^{\nearrow}(\cdot_1, \dots, \cdot_n) \in Prim(A)$.

Let B be a \mathbb{P}_{\nearrow} -infinitesimal bialgebra and let $a_1, \dots, a_n \in Prim(B)$. As $\mathcal{M}^{\mathcal{D}}$ is freely generated by the \cdot_i 's, there exists a unique morphism of \mathbb{P}_{\nearrow} -algebras:

$$\chi: \begin{cases} A &\longrightarrow B \\ \cdot_i &\longrightarrow a_i. \end{cases}$$

As the \cdot_i and the a_i 's are primitive, χ is a \mathbb{P} -infinitesimal bialgebra morphism. So:

$$\xi(p_{\bullet}^{\nearrow}(\cdot_1, \dots, \cdot_n)) = p.(\xi(\cdot_1), \dots, \xi(\cdot_n)) = p.(a_1, \dots, a_n) \in \chi(prim(\mathcal{M}^{\mathcal{D}})) \subseteq Prim(A).$$

Hence, $p \in \mathbb{P}RIM_{\nearrow}^{(1)}(n)$. The proof is similar for $\mathbb{P}RIM_{\nearrow}^{(2)}$ and $\mathbb{P}RIM_{\searrow}$. \square

4.2 Suboperad $\mathbb{P}RIM_{\nearrow}^{(1)}$

Lemma 28 *We define inductively the following elements of \mathbb{P}_{\nearrow} :*

$$\begin{cases} q_1 &= \cdot, \\ q_{n+1} &= (\cdot\cdot - \mathbf{1})_{\bullet}^{\nearrow}(q_n, \cdot) = q_n \cdot - B^+(q_n), \text{ for } n \geq 1. \end{cases}$$

Then, for all $n \geq 1$, q_n belongs to $\mathbb{P}RIM_{\nearrow}^{(1)}$. Moreover, for all $x_1, \dots, x_n \in Prim(\mathcal{M})$:

$$\gamma(q_n_{\bullet}^{\nearrow}(x_1, \dots, x_n)) = \gamma(x_1)x_2 \dots x_n.$$

Remark. These q_n 's are the same as the q_n 's defined in section 3.2.

Proof. Let us remark that $f_{\mathbf{1}} = \cdot\cdot - \mathbf{1} \in Prim(\mathcal{M})$. By proposition 27, $\cdot\cdot - \mathbf{1} \in \mathbb{P}RIM_{\nearrow}^{(1)}(2)$. As $\mathbb{P}RIM_{\nearrow}^{(1)}$ is a suboperad of \mathbb{P}_{\nearrow} , it follows that all the q_n 's belongs to $\mathbb{P}RIM_{\nearrow}^{(1)}(n)$.

Let $x_1, \dots, x_n \in Prim(\mathcal{M})$. Let us show that $\gamma(q_n_{\bullet}^{\nearrow}(x_1, \dots, x_n)) = \gamma(x_1)x_2 \dots x_n$ by induction on n . If $n = 1$, this is immediate. For $n = 2$, $q_2_{\bullet}^{\nearrow}(x_1x_2) = x_1x_2 - x_1 \nearrow x_2$. Moreover, $x_1 \nearrow x_2$ is a linear span of forests whose first tree is not $\cdot\cdot$. So $\gamma(q_2_{\bullet}^{\nearrow}(x_1, x_2)) = \gamma(x_1x_2) - 0 = \gamma(x_1)x_2$.

Suppose now the result true at rank $n - 1$. Then:

$$\begin{aligned}q_n_{\bullet}^{\nearrow}(x_1, \dots, x_n) &= q_2_{\bullet}^{\nearrow}(\underbrace{q_{n-1}_{\bullet}^{\nearrow}(x_1, \dots, x_{n-1})}_{\in Prim(\mathcal{M})}, x_n), \\ \gamma(q_n_{\bullet}^{\nearrow}(x_1, \dots, x_n)) &= \gamma(q_2_{\bullet}^{\nearrow}(q_{n-1}_{\bullet}^{\nearrow}(x_1, \dots, x_{n-1}), x_n)) \\ &= \gamma(q_{n-1}_{\bullet}^{\nearrow}(x_1, \dots, x_{n-1}))x_n \\ &= \gamma(x_1)x_2 \dots x_n.\end{aligned}$$

\square

Theorem 29 *The non- Σ -operad $\mathbb{P}RIM_{\nearrow}^{(1)}$ is freely generated by $\mathbf{1} - \dots$*

Proof. Let us first show that the family $(q_n)_{n \geq 1}$ generates $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}$. Let \mathbb{P} be the suboperad of $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}$ generated by the q_n 's. Let us prove by induction on k that $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}(k) = \mathbb{P}(k)$. If $k = 1$, $\mathbb{P}(1) = \mathbb{P}\text{RIM}_{\nearrow}^{(1)}(1) = K \cdot$. Suppose the result at all ranks $\leq k-1$. By the rigidity theorem for infinitesimal bialgebra of [10], a basis of \mathcal{H} is $(f_{t_1} \cdots f_{t_n})_{t_1 \dots t_n \in \mathbf{F}}$, so a basis of $\text{Prim}(\mathcal{M})$ is:

$$\left(\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) \right)_{t_1 \dots t_n \in \mathbf{F}}.$$

So, a basis of $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}(k)$ is $\left(\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) \right)_{\substack{t_1 \dots t_n \in \mathbf{F} \\ \text{weight}(t_1 \dots t_n) = k-1}}$. By lemma 28:

$$\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) = q_{n+1} \nearrow (\cdot, f_{t_1}, \dots, f_{t_n}).$$

By the induction hypothesis, the f_{t_i} 's belongs to \mathbb{P} . So:

$$\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) = q_{n+1} \nearrow (\cdot, f_{t_1}, \dots, f_{t_n}) \in \mathbb{P}(n).$$

So $\mathbb{P}\text{RIM}_{\nearrow}^{(1)} = \mathbb{P}$.

Moreover, if we denote by \mathbb{P}' the suboperad of $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}$ generated by q_2 , then, immediately, $\mathbb{P}' \subseteq \mathbb{P}$. Finally, by induction on n , $q_n \in \mathbb{P}'(n)$ for all $n \geq 1$ and $\mathbb{P} \subseteq \mathbb{P}'$. So $\mathbb{P}' = \mathbb{P} = \mathbb{P}\text{RIM}_{\nearrow}^{(1)}$ is generated by q_2 .

Let \mathbb{P}_{q_2} be the non- Σ -operad freely generated by q_2 . There is a non- Σ -operad epimorphism:

$$\Psi : \begin{cases} \mathbb{P}_{q_2} & \longrightarrow \mathbb{P}\text{RIM}_{\nearrow}^{(1)} \\ q_2 & \longrightarrow q_2. \end{cases}$$

The dimension of $\mathbb{P}_{q_2}(n)$ is the number of planar binary rooted trees with n leaves, that is to say the Catalan number $c_n = \frac{(2n-2)!}{(n-1)!n!}$. On the other side, the dimension of $\mathbb{P}\text{RIM}_{\nearrow}^{(1)}(n)$ is the number of planar rooted trees with n vertices, that is to say c_n . So Ψ is an isomorphism. \square

In other terms, in the language of [9]:

Theorem 30 *The triple of operads $(\text{Ass}, \mathbb{P}_{\nearrow}^{\Sigma}, \text{FREE}_2)$, where $\mathbb{P}_{\nearrow}^{\Sigma}$ is the symmetrisation of \mathbb{P}_{\nearrow} and FREE_2 is the free operad generated by an element in $\text{FREE}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a \mathbb{P}_{\nearrow} -bialgebra of type 1, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S , then $-S$ is an eulerian idempotent for A .

4.3 Another basis of $\text{Prim}(\mathcal{H})$

Recall that \mathbf{T}_b is freely generated (as a non- Σ -operad) by Υ . In particular, if $t_1, t_2 \in \mathbf{T}_b$, we denote:

$$t_1 \vee t_2 = \Upsilon \circ (t_1, t_2).$$

Every element $t \in \mathbf{T}_b - \{1\}$ can be uniquely written as $t = t^l \vee t^r$.

There exists a morphism of operads:

$$\Theta : \begin{cases} \mathbf{T}_b & \longrightarrow \mathbb{P}_{\nearrow} \\ \Upsilon & \longrightarrow \cdot - \cdot. \end{cases}$$

By theorem 29, Θ is injective and its image is $\text{PRIM}^{\nearrow(1)}$. So, we obtain a basis of $\text{PRIM}^{\nearrow(1)}$ indexed by \mathbf{T}_b , given by $p_t = \Theta(t)$. It is also a basis of $\text{Prim}(\mathcal{M})$, which can be inductively computed by:

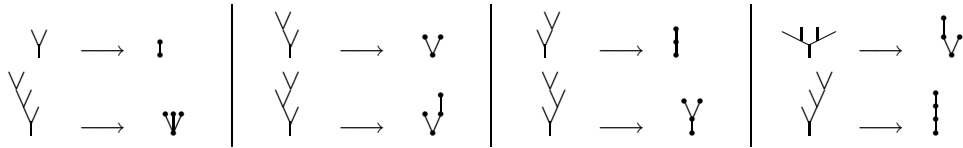
$$\begin{cases} p_1 &= \bullet, \\ p_{t_1 \vee t_2} &= (\bullet - \mathbf{!}) \nearrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}. \end{cases}$$

4.4 From the basis $(f_t)_{t \in \mathbf{T}}$ to the basis $(p_t)_{t \in \mathbf{T}_b}$

We define inductively the application $\kappa : \mathbf{T}_b \rightarrow \mathbf{T}$ in the following way:

$$\kappa : \begin{cases} \mathbf{T}_b &\rightarrow \mathbf{T} \\ \mathbf{!} &\rightarrow \bullet, \\ t_1 \vee t_2 &\rightarrow \kappa(t_2) \searrow \kappa(t_1). \end{cases}$$

Examples.



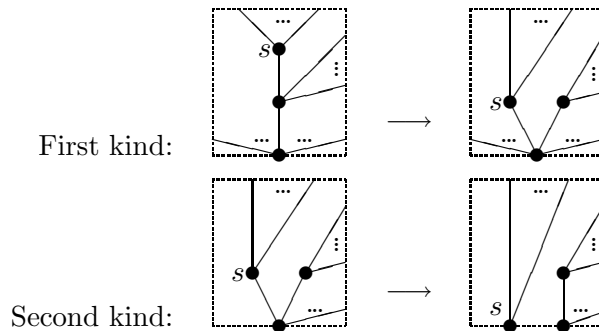
It is easy to show that κ is bijective, with inverse given by:

$$\kappa^{-1} : \begin{cases} \mathbf{T} &\rightarrow \mathbf{T}_b \\ \bullet &\rightarrow \mathbf{!}, \\ B^+(s_1 \dots s_m) &\rightarrow \kappa^{-1}(B^+(s_2 \dots s_m)) \vee \kappa^{-1}(s_1). \end{cases}$$

Let us recall the partial order \leq , defined in [4], on the set \mathbf{F} of planar forests, making it isomorphic to the Tamari poset.

Definition 31 Let $F \in \mathbf{F}$.

1. An *admissible transformation* on F is a local transformation of F of one of the following types (the part of F which is not in the frame remains unchanged):



2. Let F and $G \in \mathbf{F}$. We shall say that $F \leq G$ if there exists a finite sequence F_0, \dots, F_k of elements of \mathbf{F} such that:

- (a) For all $i \in \{0, \dots, k-1\}$, F_{i+1} is obtained from F_i by an admissible transformation.
- (b) $F_0 = F$.
- (c) $F_k = G$.

The aim of this section is to prove the following result:

Theorem 32 Let $t \in \mathbf{T}_b$. Then $p_t = \sum_{\substack{s \in \mathbf{T} \\ s \leq \kappa(t)}} f_s$.

Proof. By induction on the number n of leaves of t . If $n = 1$, then $t = \mathbf{1}$ and $p_{\mathbf{1}} = \bullet = f_{\bullet}$. Suppose the result at all ranks $\leq n - 1$. As p_t is primitive, we can put:

$$p_t = \sum_{s \in \mathbf{T}} a_s f_s.$$

Write $t = t_1 \vee t_2$. By the induction hypothesis:

$$p_{t_1} = \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} f_{s_1} \text{ and } p_{t_2} = \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} f_{s_2}.$$

As $t = t_1 \vee t_2$, $p_t = (\bullet - \mathbf{1}) \not\rightarrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \not\rightarrow p_{t_2}$. So, for all $s \in \mathbf{T}$, as s is primitive for $\Delta_{\not\rightarrow}$:

$$\begin{aligned} a_s &= \langle p_t, s \rangle \\ &= \langle p_{t_1} p_{t_2} - p_{t_1} \not\rightarrow p_{t_2}, s \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) - \Delta_{\not\rightarrow}(s) \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) \rangle \\ &= \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} \langle f_{s_2} \otimes f_{s_1}, \Delta(s) \rangle. \end{aligned}$$

So a_s is the number of left-admissible cuts c of s , such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$.

Suppose that $a_s \neq 0$. Then, there exists a left-admissible cut c of s , such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. As s is a tree, $s \leq \kappa(t_2) \searrow \kappa(t_1) = \kappa(t)$. Moreover, by considering the degree of $P^c(s)$, this cut c is unique, so $a_s = 1$. Reciproquely, if $s \leq \kappa(t)$, if c is the unique left admissible cut such that $weight(P^c(s)) = weight(t_2)$, then $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. So $a_s \neq 0$. Hence, $(s \leq \kappa(t)) \implies (a_s \neq 0) \implies (a_s = 1) \implies (s \leq \kappa(t))$. This proves theorem 32. \square

Let μ be the Möbius function of the poset \mathbf{F} ([12, 13]). By the Möbius inversion formula:

Corollary 33 Let $s \in \mathbf{T}$. Then $f_s = \sum_{t \in T_b, \kappa(t) \leq s} \mu(\kappa(t), s) p_t$.

4.5 Suboperad $\text{PRIM}_{\not\rightarrow}^{(2)}$

For all $n \in \mathbb{N}$, we put $c_{n+1} = B^+(\bullet^n)$. In other terms, c_{n+1} is the corolla tree with $n + 1$ vertices, or equivalently with n leaves.

Examples. $c_1 = \bullet$, $c_2 = \mathbf{1}$, $c_3 = \mathbf{V}$, $c_4 = \mathbf{\Psi}$, $c_5 = \mathbf{\heartsuit} \dots$

Lemma 34 The set \mathbf{T} is a basis of the operad $\text{PRIM}_{\not\rightarrow}^{(2)}$. As an operad, $\text{PRIM}_{\not\rightarrow}^{(2)}$ is generated by the c_n 's, $n \geq 2$. Moreover, for all $k, l \geq 2$,

$$c_k \not\rightarrow (c_l, \underbrace{\bullet, \dots, \bullet}_{k-1 \text{ times}}) = c_l \not\rightarrow (\underbrace{\bullet, \dots, \bullet}_{l-1 \text{ times}}, c_k).$$

Proof. The operad $\text{PRIM}_{\nearrow}^{(2)}$ is identified with $\text{Prim}_{\nearrow}(\mathcal{M})$ by proposition 27. So $\text{Prim}_{\nearrow}(\mathcal{M})$ is equal to $\text{Vect}(\mathbf{T})$. Let \mathbb{P} be the suboperad of $\text{PRIM}_{\nearrow}^{(2)}$ generated by the corollas. Let $t \in \mathbf{T}$, of weight n . Let us prove that $t \in \mathbb{P}$ by induction on n . If $n = 1$, then $t = \cdot \in \mathbb{P}$. If $n \geq 2$, we can suppose that $t = B^+(t_1 \dots t_k)$, with $t_1, \dots, t_k \in \mathbb{P}$. Then, by theorem 11:

$$c_{k+1} \nearrow (t_1, \dots, t_k, \cdot) = (\cdot^k \nearrow (t_1, \dots, t_k)) \nearrow \cdot = (t_1 \dots t_k) \nearrow \cdot = B^+(t_1 \dots t_k) = t.$$

So $t \in \mathbb{P}$. hence, $\mathbb{P} = \text{PRIM}_{\nearrow}^{(2)}$.

Let $k, l \geq 2$. Then, by theorem 11:

$$\begin{aligned} c_k \nearrow (c_l, \cdot, \dots, \cdot) &= (\cdot^{k-1} \nearrow (c_l, \cdot, \dots, \cdot)) \nearrow \cdot \\ &= (c_l \cdot^{k-2}) \nearrow \cdot \\ &= B^+(c_l \cdot^{k-2}) \\ &= B^+(B^+(\cdot^{l-1}) \cdot^{k-2}). \end{aligned}$$

On the other hand:

$$\begin{aligned} c_l \nearrow (\cdot, \dots, \cdot, c_k) &= (\cdot^{l-1} \nearrow (\cdot, \dots, \cdot)) \nearrow c_k \\ &= (\cdot^{l-1}) \nearrow c_k \\ &= (\cdot^{l-1}) \nearrow B^+(c^{k-1}) \\ &= B^+(((\cdot^{l-1}) \nearrow \cdot) \cdot^{k-2}) \\ &= B^+(B^+(\cdot^{l-1}) \cdot^{k-2}). \end{aligned}$$

So $c_k \nearrow (c_l, \cdot, \dots, \cdot) = c_l \nearrow (\cdot, \dots, \cdot, c_k)$. □

Definition 35 The operad \mathbb{T} is the non- Σ -operad generated by elements $c_n \in \mathbb{T}(n)$, for $n \geq 2$, and the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \dots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \dots, I}_{l-1 \text{ times}}, c_k).$$

In other terms, a \mathbb{T} -algebra A has a family of n -multilinear products $[\cdot, \dots, \cdot] : A^{\otimes n} \longrightarrow A$ for all $n \geq 2$, with the associativity condition:

$$[[a_1, \dots, a_l], a_{l+1}, \dots, a_{l+k}] = [a_1, \dots, a_{l-1}, [a_l, \dots, a_{l+k}]].$$

In particular, $[\cdot, \cdot]$ is associative.

Theorem 36 The operads \mathbb{T} and $\text{PRIM}_{\nearrow}^{(2)}$ are isomorphic.

Proof. By lemma 34, there is an epimorphism of operads:

$$\begin{cases} \mathbb{T} & \longrightarrow & \text{PRIM}_{\nearrow}^{(2)} \\ c_n & \longrightarrow & c_n. \end{cases}$$

In order to prove this is an isomorphism, it is enough to prove that $\dim(\mathbb{T}(n)) \leq \dim(\text{PRIM}_{\nearrow}^{(2)}(n))$ for all $n \geq 2$. By lemma 34, $\dim(\text{PRIM}_{\nearrow}^{(2)}(n))$ is the n -th Catalan number. Because of the defining relations, $\mathbb{T}(n)$ is generated as a vector space by elements of the form $c_l \circ (I, b_2, \dots, b_l)$, with $b_i \in \mathbb{T}(n_i)$, such that $n_1 + \dots + n_l = n - 1$. Hence, we define inductively the following subsets the free non- Σ -operad generated by the c_n 's, $n \geq 2$:

$$X(n) = \begin{cases} \{I\} & \text{if } n = 1, \\ \bigcup_{l=2}^n \bigcup_{i_2+\dots+i_l=n-1} c_l \circ (I, X(i_2), \dots, X(i_l)) & \text{if } n \geq 2. \end{cases}$$

Then the images of the elements of $X(n)$ linearly generate $\mathbb{T}(n)$, so $\dim(\mathbb{T}(n)) \leq \text{card}(X(n))$ for all n . We now put $a_n = \text{card}(X(n))$ and prove that a_n is the n -th Catalan number. We denote by $A(h)$ their generating formal series. Then:

$$\begin{cases} a_1 = 1, \\ a_n = \sum_{l=2}^n \sum_{i_2+\dots+i_l=n-1} a_{i_1} \dots a_{i_l} \text{ if } n \geq 2. \end{cases}$$

In terms of generating series:

$$A(h) - a_1 h = h \frac{A(x)}{1 - A(x)}.$$

So $A(h)^2 - A(h) + h = 0$. As $A(h) = 1$:

$$A(h) = \frac{1 - \sqrt{1 - 4h}}{2}.$$

So a_n is the n -th Catalan number for all $n \geq 1$. □

In other terms:

Theorem 37 *The triple of operads $(\text{Ass}, \mathbb{P}_{\nearrow}^{\Sigma}, \mathbb{T}^{\Sigma})$ is a good triple of operads.*

Remark. Note that if A is a \mathbb{P}_{\nearrow} -bialgebra of type 2, then $(A, m, \tilde{\Delta}_{\nearrow})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta_{\nearrow})$ has an antipode S_{\nearrow} , then $-S_{\nearrow}$ is an eulerian idempotent for A .

4.6 Suboperad PRIM_{\searrow}

Lemma 38 *The set \mathbf{T} is a basis of the operad PRIM_{\searrow} . As an operad, PRIM_{\searrow} is generated by $\mathbf{1}$.*

Proof. Let \mathbb{P} be the suboperad of PRIM_{\searrow} generated by $\mathbf{1}$. Let $t \in \mathbf{T}$, of weight n . Let us prove that $t \in \mathbb{P}$ by induction on n . If $n = 1$ or 2 , this is obvious. If $n \geq 2$, suppose that $t = B^+(t_1 \dots t_k)$. By the induction hypothesis, t_1 and $B^+(t_2 \dots t_k)$ belong to \mathbb{P} . Then:

$$t = t_1 \searrow B^+(t_2 \dots t_k) = \mathbf{1} \searrow_{\searrow} (t_1, B^+(t_2 \dots t_k)).$$

So $t \in \mathbb{P}$. □

Theorem 39 *The non- Σ -operad PRIM_{\searrow} is freely generated by $\mathbf{1}$.*

Proof. Similar as the proof of theorem 29. □

In other terms:

Theorem 40 *The triple of operads $(\text{Ass}, \mathbb{P}_{\searrow}^{\Sigma}, \mathbb{F}_2)$, where \mathbb{F}_2 is the free operad generated by an element in $\mathbb{F}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a \mathbb{P}_{\searrow} -bialgebra, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S , then $-S$ is an eulerian idempotent for A .

5 A rigidity theorem for \mathbb{P}_{\nearrow} -algebras

5.1 Double \mathbb{P}_{\nearrow} -infinitesimal bialgebras

Definition 41 A double \mathbb{P}_{\nearrow} -infinitesimal bialgebra is a family $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ where $m, \nearrow: A \otimes A \rightarrow A$, $\tilde{\Delta}, \tilde{\Delta}_{\nearrow}: A \rightarrow A \otimes A$, with the following compatibilities:

1. (A, m, \nearrow) is a (non unitary) \mathbb{P}_{\nearrow} -algebra.
2. For all $x \in A$:

$$\begin{cases} (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(x) = (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(x), \\ (\tilde{\Delta}_{\nearrow} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) = (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}_{\nearrow}(x), \\ (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) = (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(x). \end{cases}$$

In other terms, $(A, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is a \mathbb{P}_{\nearrow} -coalgebra.

3. $(A, m, \nearrow, \tilde{\Delta})$ is a \mathbb{P}_{\nearrow} -bialgebra of type 1.
4. $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$ is a \mathbb{P}_{\nearrow} -bialgebra of type 2.

Remark. If $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a graded double \mathbb{P}_{\nearrow} -infinitesimal bialgebra, with finite-dimensional homogeneous components, then its graded dual $(A^*, \tilde{\Delta}^{*,op}, \tilde{\Delta}_{\nearrow}^{*,op}, m^{*,cop}, \nearrow^{*,cop})$ also is.

Theorem 42 $(\mathcal{M}, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a double \mathbb{P} -infinitesimal bialgebra.

Proof. We already now that $(\mathcal{M}, m, \nearrow)$ is a \mathbb{P}_{\nearrow} -algebra. Moreover, $(\mathcal{M}, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is isomorphic to $(\mathcal{M}^*, m^*, \nearrow^*)$ via the pairing $\langle -, - \rangle$, so it is a \mathbb{P}_{\nearrow} -coalgebra. It is already known that $(\mathcal{M}, m, \tilde{\Delta})$ and $(\mathcal{M}, \nearrow, \tilde{\Delta})$ are infinitesimal bialgebras. As $(\mathcal{M}, \nearrow, \tilde{\Delta})$ is isomorphic to $(\mathcal{M}^{op}, m^{op}, \tilde{\Delta}_{\nearrow}^{cop})$ via the pairing $\langle -, - \rangle$, it is also an infinitesimal bialgebra. So all the needed compatibilities are satisfied. \square

Remarks.

1. Via the pairing $\langle -, - \rangle$, \mathcal{M} is isomorphic to its graded dual as an double \mathbb{P}_{\nearrow} -infinitesimal bialgebra. As a consequence, as \mathcal{M} is the free \mathbb{P}_{\nearrow} -algebra generated by \bullet , then \mathcal{M}^{cop} is also the cofree \mathbb{P}_{\nearrow} -coalgebra cogenerated by \bullet .
2. All these results can be easily extended to infinitesimal Hopf algebras of decorated planar rooted trees, in other terms to every free \mathbb{P}_{\nearrow} -algebras.

Lemma 43 In the double infinitesimal \mathbb{P}_{\nearrow} -algebra \mathcal{M} , $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = Vect(\bullet)$.

Proof. \supseteq . Obvious.

\subseteq . Let $x \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(x) = 0$, so x is a linear span of trees. We can write:

$$x = \sum_{t \in \mathbf{T}} a_t t.$$

Consider the terms in $\mathcal{M} \otimes \bullet$ of $\tilde{\Delta}(x)$. We get $\sum_{t \in \mathbf{T} - \{\bullet\}} a_t B^-(t) \otimes \bullet = 0$, where $B^-(t)$ is the forest obtained by deleting the root of t . So, if $t \neq \bullet$, then $a_t = 0$. So $x \in vect(\bullet)$. \square

Remark. This lemma can be extended to any free \mathbb{P}_{\nearrow} -algebra: if V is a vector space, then the free \mathbb{P}_{\nearrow} -algebra $F_{\mathbb{P}_{\nearrow}}(V)$ generated by V is given a structure of double \mathbb{P}_{\nearrow} -infinitesimal bialgebra by $\tilde{\Delta}(v) = \tilde{\Delta}_{\nearrow}(v) = 0$ for all $v \in V$. In this case, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = V$ for $F_{\mathbb{P}_{\nearrow}}(V)$.

5.2 Connected double \mathbb{P}_{\nearrow} -infinitesimal bialgebras

Notations. Let A be a double \mathbb{P}_{\nearrow} -infinitesimal bialgebra. The iterated coproducts will be denoted in the following way: for all $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\Delta}^n &: \begin{cases} A & \longrightarrow A^{\otimes(n+1)} \\ a & \longrightarrow a^{(1)} \otimes \dots \otimes a^{(n+1)}, \end{cases} \\ \tilde{\Delta}_{\nearrow}^n &: \begin{cases} A & \longrightarrow A^{\otimes(n+1)} \\ a & \longrightarrow a_{\nearrow}^{(1)} \otimes \dots \otimes a_{\nearrow}^{(n+1)}. \end{cases}\end{aligned}$$

Definition 44 Let A be a double \mathbb{P}_{\nearrow} -infinitesimal bialgebra. It will be said *connected* if, for any $a \in A$, every iterated coproduct $A \longrightarrow A^{\otimes(n+1)}$ vanishes on a for a great enough n .

Theorem 45 *Let A be a connected double \mathbb{P}_{\nearrow} -infinitesimal bialgebra. Then A is isomorphic to the free \mathbb{P}_{\nearrow} -algebra generated by $\text{Prim}(A) = \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_{\nearrow})$ as a double \mathbb{P}_{\nearrow} -infinitesimal bialgebra.*

Proof. *First step.* We shall use the results on infinitesimal Hopf algebras of [4]. We show that $A = \text{Prim}(A) + A.A + A \nearrow A$. As $(A, \nearrow, \tilde{\Delta})$ is a connected non unitary infinitesimal bialgebra, it (or more precisely its unitarisation) has an antipode S_{\nearrow} , defined by:

$$S_{\nearrow} : \begin{cases} A & \longrightarrow A \\ a & \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)} \nearrow \dots \nearrow a^{(i+1)}. \end{cases}$$

As $(A, \tilde{\Delta})$ is connected, this makes sense. Moreover, $-S_{\nearrow}$ is the projector on $\text{Ker}(\tilde{\Delta})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}) \oplus A \nearrow A$.

In the same order of idea, as $(A, m, \tilde{\Delta}_{\nearrow})$ is a connected infinitesimal bialgebra, we can define its antipode S^{\nearrow} by:

$$S^{\nearrow} : \begin{cases} A & \longrightarrow A \\ a & \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a_{\nearrow}^{(1)} \dots a_{\nearrow}^{(i+1)}, \end{cases}$$

and $-S^{\nearrow}$ is the projector on $\text{Ker}(\tilde{\Delta}_{\nearrow})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}_{\nearrow}) \oplus A.A$.

Let $a \in A$, $b \in \text{Ker}(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(a \nearrow b) = (a \otimes 1)\tilde{\Delta}_{\nearrow}(b) = 0$. So $A \nearrow \text{Ker}(\tilde{\Delta}_{\nearrow})$ is a subset of $\text{Ker}(\tilde{\Delta}_{\nearrow})$. Moreover, if $\tilde{\Delta}_{\nearrow}(a) = 0$, then $(\text{Id} \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(a) = (\tilde{\Delta} \otimes \text{Id}) \circ \tilde{\Delta}_{\nearrow}(a) = 0$. So $\tilde{\Delta}(a) \in A \otimes \text{Ker}(\tilde{\Delta}_{\nearrow})$. As a consequence, if $n \geq 1$:

$$\tilde{\Delta}^n(a) = (\tilde{\Delta}^{n-1} \otimes \text{Id}) \circ \tilde{\Delta}(a) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_{\nearrow}).$$

Hence, for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(\text{Ker}(\tilde{\Delta}_{\nearrow})) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_{\nearrow})$. Finally, we deduce that $S_{\nearrow}(\text{Ker}(\tilde{\Delta}_{\nearrow})) \subseteq \text{Ker}(\tilde{\Delta}_{\nearrow})$.

Let $a \in A$. Then $S^{\nearrow}(a) \in \text{Ker}(\tilde{\Delta}_{\nearrow})$ and $S_{\nearrow} \circ S^{\nearrow}(a) \in \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_{\nearrow}) = \text{Prim}(A)$ by the preceding point. Moreover:

$$\begin{aligned}S^{\nearrow}(a) &= -a + A.A, \\ S_{\nearrow} \circ S^{\nearrow}(a) &= -S^{\nearrow}(a) + A \nearrow A, \\ S_{\nearrow} \circ S^{\nearrow}(a) &= a + A.A + A \nearrow A.\end{aligned}$$

This proves the first step.

Second step. As A is connected, it classically inherits a filtration of \mathbb{P}_{\nearrow} -algebra given by the kernels of the iterated coproducts. We denote by deg_p the associated degree. In particular, for

all $x \in A$, $\deg_p(x) \leq 1$ if, and only if, $x \in \text{Prim}(A)$. Let B be the \mathbb{P}_{\nearrow} -subalgebra of A generated by $\text{Prim}(A)$. Let $a \in A$, let us show that $a \in B$ by induction on $n = \deg_p(a)$. If $n \leq 1$, then $a \in \text{Prim}(A) \subseteq B$. Suppose the result true at all ranks $\leq n - 1$. Then, by the first step, we can write:

$$a = b + \sum_i a_i b_i + \sum_j c_j d_j,$$

with $b \in \text{Prim}(A)$, $a_i, b_i, c_j, d_j \in A$. Because of the filtration, we can suppose that $\deg_p(a_i), \deg_p(b_i), \deg_p(c_j), \deg_p(d_j) < n$. By the induction hypothesis, they belong to B , so $a \in B$.

Last step. So, there is an epimorphism of \mathbb{P}_{\nearrow} -algebras:

$$\phi : \begin{cases} F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A)) & \longrightarrow A \\ a \in \text{Prim}(A) & \longrightarrow a, \end{cases}$$

where $F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A))$ is the free \mathbb{P}_{\nearrow} -algebra generated by $\text{Prim}(A)$. As the elements of $\text{Prim}(A)$ are primitive both in A and $F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A))$, this is a morphism of double \mathbb{P}_{\nearrow} -infinitesimal bialgebras. Suppose that it is not monic. Take then $x \in \text{Ker}(\phi)$, non-zero, of minimal degree. Then it is primitive, so belongs to $\text{Prim}(A)$ (lemma 43). Hence, $\phi(x) = x = 0$: this is a contradiction. So ϕ is a bijection. \square

In other terms:

Corollary 46 *The triple of operads $\left((\mathbb{P}_{\nearrow}^{\Sigma})^{op}, \mathbb{P}_{\nearrow}^{\Sigma}, \text{VECT} \right)$ is a good triple. Here, VECT denotes the operad of vector spaces:*

$$\text{VECT}(k) = \begin{cases} KI & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}$$

where I is the unit of VECT .

We also showed that $S_{\nearrow} \circ S_{\nearrow}$ is the projection on $\text{Prim}(A)$ in the direct sum $A = \text{Prim}(A) \oplus (A.A + A \nearrow A)$. So $S_{\nearrow} \circ S_{\nearrow}$ is the eulerian idempotent.

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