Trace Theorems for a Class of Ramified Domains with Self-Similar Fractal Boundaries

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November 29, 2009

Abstract

This work deals with trace theorems for a class of ramified bidimensional domains Ω with a self-similar fractal boundary Γ^∞. The fractal boundary Γ^∞ is supplied with a probability measure μ called the self-similar measure. Emphasis is put on the case when the domain is not a $\epsilon - \delta$ domain as defined by Jones and the fractal set is not totally disconnected. In this case, the classical trace results cannot be used. Here, the Lipschitz spaces with jumps recently introduced by Jonsson play a crucial role. Indeed, it is proved in particular that if the Hausdorff dimension d of Γ^∞ is not smaller than one, then the space of the traces of functions in $W^{m+1,q}(\Omega)$, $m \in \mathbb{N}$, $1 < q < \infty$ is $JLip(m+1-\frac{2-d}{q}, q, q; m; \Gamma^\infty)$. The proof is elementary; a main step is a strengthened trace inequality in the norm $L^q_\mu(\Gamma^\infty)$.

1 Introduction

This work deals with some properties of the Sobolev spaces $W^{m+1,q}(\Omega)$, $m \in \mathbb{N}$, for a class of ramified domains Ω of $\mathbb{R}^2$ with a self-similar fractal boundary called Γ^∞ below, see for example Figures 1 and 3. The domain Ω essentially depends on a parameter a, $0 < a \leq a^*$. As explained below, the restriction $a \leq a^*$ allows for the construction of Ω as a union of non-overlapping sub-domains, see (10) below.

Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. Indeed, the present work is a continuation of [1] and of [2], which were part of a wider project aimed at simulating the diffusion of medical sprays in lungs. Since the exchanges between the lungs and the circulatory system take place only in the last generations of the bronchial tree (the smallest structures), reasonable models for the diffusion of, e.g., oxygen may involve a non-homogeneous Neumann or Robin condition on the boundary Γ^∞. Similarly, the lungs are mechanically coupled to the diaphragm, which also implies non-homogeneous boundary conditions on Γ^∞, if one is interested in a coupled fluid-structure model. It is therefore necessary to study traces of functions on Γ^∞; here, we will focus on functions belonging to $W^{m+1,q}(\Omega)$, for $m \in \mathbb{N}$ and $1 < q < \infty$.

Function spaces defined in irregular domains have been widely studied in the literature:

- Jones [8] (and Vodopjanov et al [25] in the case $n = 2$, see also [17, 11]) have studied the open bounded subsets Ω of $\mathbb{R}^n$ such that there exists a continuous extension operator from $W^{\ell,p}(\Omega)$ in $W^{\ell,p}(\mathbb{R}^n)$, for all nonnegative integers ℓ and real numbers p, $1 \leq p \leq \infty$.

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Jones has proved that if Ω is a $\epsilon - \delta$ domain for some parameters $\epsilon, \delta > 0$, see [8, 11] for the definition, then the above extension property is true. Moreover, in dimension two, if the extension property stated above is true, then Ω is a $\epsilon - \delta$ domain for some parameters $\epsilon, \delta > 0$. In dimension two, the definition of such domains is equivalent to that of quasi-disks, see [17].

- Jonsson and Wallin [11] have considered closed subsets F of $\mathbb{R}^n$ supplied with a Borel measure μ such that there exists a positive real number d and two positive constants c_1 and c_2 with
$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d,$$
for all $x \in F$ and $r < 1$ (here $B(x,r)$ is the ball in F with center x and radius r, with respect to the Euclidean distance in $\mathbb{R}^n$); in [11], these sets are called d-sets.
Concerning the set Γ^∞ studied in the present paper, we will recall in §3.2 below the notion of self-similar measure μ defined in the triplet $(\Gamma^\infty, F_1, F_2)$, see [13]. With the Borel regular probability measure μ, Γ^∞ is a d-set where d is the Hausdorff dimension of Γ^∞.
In [11], Sobolev and Besov spaces are defined on the d-sets, and extension and trace results for Besov and Sobolev spaces are proved using as a main ingredient Whitney extension theory. In particular, see [11] page 182, there exists a continuous trace operator from $W^{1,p}(\mathbb{R}^n)$ onto $B^{p,p}_{1-\frac{n-d}{p}}(F)$, if $\max(1, n-d) < p < \infty$, where, for $0 < s < 1$,
$$B^{p,p}_s(F) = \left\{ f \in L^p_\mu(F); \int_{x,y \in F, |x-y|<1} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d\mu(x) d\mu(y) < \infty \right\},$$
see the definition in [11] page 103. A more general trace theorem is available, see Theorem 1, page 141 in [11].
The approach of Triebel [24] is somewhat different. In [24] chapter IV, paragraph 18, it is proved that the space of the traces of functions in $B^{p,q}_{\frac{n-d}{p}}(\mathbb{R}^n)$ is $L^p_\mu(F)$ for $0 < d < n$, $d/n < p < \infty$ and $0 < q \leq \min(1, p)$; Besov spaces on F are then defined as the space of the traces of Besov spaces on $\mathbb{R}^n$ and embeddings properties are studied.

- There is also a growing interest in analysis on self-similar fractal sets, see for instance Kigami [13], Strichartz [22, 23], Mosco[20, 19] and references therein. These works aim at intrinsically defining function spaces using Dirichlet forms and a different metric from the Euclidean one. The results in this direction are often subject to the important assumption that the set is post-critically finite (or p.c.f.), see [13], page 23 for the definition. In a different direction, Jonsson has studied Lipschitz functions spaces on self-similar fractal sets under a technical condition which yields a Markov inequality at any order, see the pioneering works [9, 10]. This theory does not require the fractal set to be post-critically finite. In [10], Lipschitz functions spaces allowing jumps at some special points in the self-similar fractal set have been introduced, along with Haar wavelets of arbitrary order. The previously mentioned function spaces can be characterized using the coefficients of the expansion in some high order Haar wavelet bases. The theory in [10] is the cornerstone of the present paper. It will be briefly reviewed in §4.

- The question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Results in this direction have been given in [21, 15, 14] for the Koch flake. Here also, the assumption that the fractal set is p.c.f. is generally made.

Our goal here is to study the traces of functions of $W^{m+1,q}(\Omega)$ on the fractal boundary Γ^∞. Note that this is different from considering the traces of functions of $W^{m+1,q}(\mathbb{R}^2)$ on Γ^∞.
If $a < a^*$, then Ω is a $\epsilon-\delta$ domain. In this case, the results of Jones [8] and those of Jonsson and Wallin [11] can be combined to obtain trace results. This has been done in [2]: if $\max(1, 2-d) < q < \infty$, the space of the traces of functions in $W^{1,q}(\Omega)$ is $B^{q,q}_{1-(2-d)/q}(\Gamma^\infty)$, where d is the Hausdorff dimension of Γ^∞.
When $a = a^*$, the situation is more complicated because

- Ω is not a $\epsilon - \delta$ domain,
- Γ^∞ may not be totally disconnected. It may even be non post-critically finite.

In [2] we mainly focused on the case $a = a^*$, and especially on the geometry presented in § 2.2.2 below, see Figure 3. We studied some properties of the traces of functions in $H^1(\Omega)$, without completely characterizing the trace space though. The results proved in [2] are :

- if $a \leq a^*$ then the trace of a function in $H^1(\Omega)$ belongs to $L^p_\mu(\Gamma^\infty)$, for all real numbers p such that $1 \leq p < \infty$.
- in the case $a = a^*$, we gave an example of a function in $H^1(\Omega)$ whose trace on Γ^∞ has not a bounded mean oscillation with respect to μ, (thus does not belong to $L^\infty_\mu(\Gamma^\infty)$).
- for the geometry displayed in Figure 3 in the critical case $a = a^*$, the trace of a function in $H^1(\Omega)$ belongs to the Besov space $B^{2,2}_s(\Gamma^\infty, \mu)$ for all real numbers s such that $0 \leq s < d/4$, where d is the Hausdorff dimension of Γ^∞, and there exists a function in $H^1(\Omega)$ whose trace does not belong to $B^{2,2}_s(\Gamma^\infty, \mu)$ for all $s > d/4$. Note the important contrast with the case $a < a^*$ for which the trace of a function in $H^1(\Omega)$ belongs to $B^{2,2}_s(\Gamma^\infty, \mu)$ for $0 \leq s \leq d/2$.

In the present paper, we aim at charaterizing the traces of functions in $W^{m+1,q}(\Omega)$ by using the results contained in [9] and mostly [10].
The article is organized as follows: the geometry is presented in Section 2. In Section 3, we recall some of the results of [1] on the space $W^{1,q}(\Omega)$, concerning Poincaré inequality and the construction of the trace operator. The theory proposed in [10] is reviewed in § 4 where we mainly focus on the spaces $JLip(s, q, q; m; \Gamma^\infty)$, $m < s < m + 1$, $1 \leq q < \infty$, that we will use later on. For simplicity, we first investigate the traces of functions in $W^{1,q}(\Omega)$: the main result of the paper is Theorem 9 stated and proved in Section 5. The proof uses elementary ingredients, mainly the strengthened trace inequality stated in Theorem 11. The traces of functions in $W^{m+1,q}(\Omega)$ for a positive integer m are characterized in Section 6.
The results presented here can be generalized to functions in $W^{s,q}(\Omega)$ for s non integer or in Besov spaces: this topic is currently under investigation.

2 The Geometry

We first describe the general geometrical setting, then we give two relevant examples.

2.1 General Setting

2.1.1 The similitudes F_1 and F_2 and the self-similar set Γ^∞

Consider four real numbers a, α, β, θ such that $0 < a < 1/\sqrt{2}$, $\alpha > 0$, $\beta > 0$ and $0 \leq \theta < \pi/2$. Let F_i, $i = 1, 2$ be the two similitudes in $\mathbb{R}^2$ given by

$$\begin{aligned} F_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}, \\ F_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}. \end{aligned} \tag{1}$$

The two similitudes have the same dilation ratio a and opposite angles $\pm\theta$. One can obtain F_2 by composing F_1 with the symmetry with respect to the axis $\{x_1 = 0\}$.
We call Γ^∞ the self-similar set associated to the similitudes F_1 and F_2, *i.e.* the unique compact subset of $\mathbb{R}^2$ such that

$$\Gamma^\infty = F_1(\Gamma^\infty) \cup F_2(\Gamma^\infty).$$

For $n \geq 1$, we call $\mathcal{A}_n$ the set containing all the 2^n mappings from $\{1, \ldots, n\}$ to $\{1, 2\}$. We define

$$\mathcal{M}_\sigma = F_{\sigma(1)} \circ \cdots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n. \tag{2}$$

The definition of Γ^∞ implies that for all $n > 0$,

$$\Gamma^\infty = \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\Gamma^\infty).$$

We state without proof the following proposition, which says that, up to an affine map, the shape of Γ^∞ does not depend on α and β.

Proposition 1 *Let F_1 and F_2 be defined by (1) where the parameters a, θ, α, β satisfy the assumptions above. Let the similitudes G_1 and G_2 be defined by*

$$G_1 = \begin{pmatrix} -\gamma \\ \delta \end{pmatrix} + F_1, \qquad G_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + F_2,$$

where $\alpha + \gamma > 0$ and $\beta + \delta > 0$. The self-similar set associated to G_1 and G_2 is

$$\begin{pmatrix} 0 \\ \dfrac{\alpha\delta - \beta\gamma}{\beta a \sin\theta + \alpha(1 - a\cos\theta)} \end{pmatrix} + \left(1 + \frac{\delta a \sin\theta + \gamma(1 - a\cos\theta)}{\beta a \sin\theta + \alpha(1 - a\cos\theta)}\right) \Gamma^\infty,$$

where Γ^∞ is the self-similar set associated to the similitudes F_1 and F_2.

Note that the assumptions in Proposition 1 imply that the parameter $\left(1 + \dfrac{\delta a \sin\theta + \gamma(1 - a\cos\theta)}{\beta a \sin\theta + \alpha(1 - a\cos\theta)}\right)$ is positive.
The following theorem was stated by Mandelbrodt et al, [16]:

Theorem 1 *For any θ, $0 \leq \theta < \pi/2$, there exists a unique positive number $a^* < 1/\sqrt{2}$, (which depends on θ but not on (α, β) from Proposition 1) such that*

$$\begin{aligned} 0 < a < a^* \quad &\Rightarrow \quad F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) = \emptyset \quad \Rightarrow \quad \Gamma^\infty \text{ is totally disconnected,} \\ a = a^* \quad &\Rightarrow \quad F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \neq \emptyset \quad \Rightarrow \quad \Gamma^\infty \text{ is connected, (from Th. 1.6.2 in [13]).} \end{aligned} \tag{3}$$

If $\theta > 0$, then the critical parameter a^* is the unique positive root of the polynomial equation:

$$\sum_{i=0}^{\kappa-1} X^{i+2} \cos i\theta = \frac{1}{2}, \tag{4}$$

where κ is the smallest integer such that $\kappa\theta \geq \pi/2$. If $\theta = 0$ then $a^* = 1/2$.

Remark 1 *From (4), it can be seen that $\theta \to a^*(\theta)$ is a continuous and increasing function from $[0, \pi/2)$ onto $[1/2, 1/\sqrt{2})$.*

Hereafter, for a given θ, $0 \leq \theta < \pi/2$, we will only consider a such that $0 < a \leq a^*$.

2.1.2 The simplest possible construction with F_1 and F_2

Call $P_1 = (-1, 0)$ and $P_2 = (1, 0)$ and Γ^0 the line segment $\Gamma^0 = [P_1, P_2]$. We impose that $F_2(P_1)$, and $F_2(P_2)$ have positive coordinates, i.e. that

$$a\cos\theta < \alpha \quad \text{and} \quad a\sin\theta < \beta. \tag{5}$$

We also impose that the open domain $\widetilde{Y}^0$ inside the closed polygonal line joining the points P_1, P_2, $F_2(P_2)$, $F_2(P_1)$, $F_1(P_2)$, $F_1(P_1)$, P_1 in this order is convex. With (5), this is true if and only if

$$(\alpha - 1)\sin\theta + \beta\cos\theta \geq 0. \tag{6}$$

Under assumptions (5) and (6), the domain $\widetilde{Y}^0$ is either hexagonal or trapezoidal in the degenerate cases (in particular if $\theta = 0$), contained in the half-plane $x_2 > 0$ and symmetric w.r.t. the vertical axis $x_1 = 0$.
We first describe the simplest possible ramified domain containing the images of P_1 and P_2 by all the mappings $F_{i_1} \circ \cdots \circ F_{i_n}$, $i_j = 1, 2$. This domain will be noted $\widetilde{\Omega}$. The construction is as follows: it it possible to glue together $\overline{\widetilde{Y}^0}$, $F_1(\overline{\widetilde{Y}^0})$ and $F_2(\overline{\widetilde{Y}^0})$ and obtain a new polygonal domain, also symmetric with respect to the axis $\{x_1 = 0\}$. The assumptions (5) and (6) imply that $\widetilde{Y}^0 \cap F_1(\widetilde{Y}^0) = \emptyset$ and $\widetilde{Y}^0 \cap F_2(\widetilde{Y}^0) = \emptyset$. We also define the ramified open domain $\widetilde{\Omega}$, see Figures 1 and 3:

$$\widetilde{\Omega} = \text{Interior}\left(\overline{\widetilde{Y}^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(\overline{\widetilde{Y}^0})\right)\right). \tag{7}$$

Note that $\widetilde{\Omega}$ is symmetric with respect to the axis $x_1 = 0$, and that for $a < 1/\sqrt{2}$, the measure of $\widetilde{\Omega}$ is finite.

With a^* defined as above, we shall make the following assumption on α and β:
Assumption 1 For $0 < \theta < \pi/2$, the parameters α and β satisfy (6) and (5) for $a = a^*$, and are such that

1. for all a, $0 < a \leq a^*$, the sets $\widetilde{Y}^0$, $\mathcal{M}_\sigma(\widetilde{Y}^0)$, $\sigma \in \mathcal{A}_n$, $n > 0$ are disjoint,
2. for all a, $0 < a < a^*$, $F_1(\overline{\widetilde{\Omega}}) \cap F_2(\overline{\widetilde{\Omega}}) = \emptyset$,
3. for $a = a^*$, $F_1(\overline{\widetilde{\Omega}}) \cap F_2(\overline{\widetilde{\Omega}}) \neq \emptyset$.

In the case when $\theta = 0$, Assumption 1 is satisfied by any $\alpha > a^* = 1/2$ and $\beta > 0$. The following theorem asserts that for any θ, $0 < \theta < \pi/2$, there exist (α, β) satisfying Assumption 1.

Theorem 2 *If $\theta \in (0, \pi/2)$, then for all $\alpha > a^* \cos\theta$, there exists $\bar{\beta} > 0$ such that $\bar{\beta} > a^* \sin\theta$ and $(\alpha - 1)\sin\theta + \bar{\beta}\cos\theta \geq 0$ and for all $\beta \geq \bar{\beta}$, (α, β) satisfies Assumption 1.*

Proof. The proof of this result is based on elementary but tricky geometric considerations. We sketch it in Appendix A. ■

The following theorem gives different and explicit sufficient conditions for Assumption 1 to hold in the case when $\pi/6 \leq \theta \leq \pi/3$.

Theorem 3 *1. If $\theta \in [\pi/6, \pi/4]$, then the pairs (α, β) satisfying (5) for $a = a^*$ and*

$$\beta \cos\theta > \max(1 - \alpha, \alpha)\sin\theta \tag{8}$$

satisfy Assumption 1.

2. If $\theta \in [\pi/4, \pi/3]$, then the pairs (α, β) satisfying (5) for $a = a^*$, (6) and

$$\begin{cases} 2\cos\theta\,(\beta\sin\theta - \alpha\cos\theta) & > & -\cos 2\theta, \\ (1-\alpha)\cos 3\theta + \beta\sin 3\theta & > & 0 \end{cases} \tag{9}$$

satisfy Assumption 1.

Proof. See Appendix A. ■

Remark 2 *We have observed numerically that for any θ, $0 < \theta < \pi/6$, the pairs (α, β) satisfying (5) for $a = a^*$ and (8) do satisfy Assumption 1, but we did not obtain a reasonably short proof.*

Remark 3 *Note that for $\theta = \pi/4$, conditions (9) and (8) are equivalent.*

For a given θ, let (α, β) satisfy Assumption 1; for $0 \leq a < a^*$, the Moran condition (open set condition) see [18, 13] is satisfied (the Moran condition is that there exists a nonempty bounded open subset O of $\mathbb{R}^2$ such that $F_1(O) \cap F_2(O) = \emptyset$ and $F_1(O) \cup F_2(O) \subset O$). Indeed, one can take $O = \widetilde{\Omega}$. From this, the Hausdorff dimension of Γ^∞ is

$$\dim_H(\Gamma^\infty) = d \equiv -\log 2/\log a,$$

see [18, 13]. The assumption $a < 1/\sqrt{2}$ implies that $d < 2$.

2.1.3 More general constructions

Let (α, β) satisfy Assumption 1. We assume that for all a, $0 < a \leq a^*$, (where a^* has been introduced above), the polygonal cell Y^0 satisfies the following assumptions:

1. Y^0 is contained in the half-plane $x_2 > 0$.
2. $\partial Y^0 \cap \{x_2 = 0\} = \Gamma^0 = [P_1, P_2]$.
3. $F_1(\Gamma^0)$ and $F_2(\Gamma^0)$ are contained in some sides of ∂Y^0.
4. the sets Y^0, $\mathcal{M}_\sigma(Y^0)$, $\sigma \in \mathcal{A}_n$, $n > 0$ are disjoint.

With these assumptions, we can construct the ramified open domain

$$\Omega = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right), \tag{10}$$

see Figure 2 for an example. The self-similar fractal set Γ^∞ is a subset of $\partial\Omega$. We split the boundary of Ω into Γ^∞, $\Gamma^0 = [-1, 1] \times \{0\}$ and $\Sigma = \partial\Omega \backslash (\Gamma^0 \cup \Gamma^\infty)$.

Remark 4 *A further generalization is possible: all the assumptions above are maintained except that $\partial Y^0 \backslash \left(\Gamma^0 \cup F_1(\Gamma^0) \cup F_2(\Gamma^0)\right)$ may be made of curved lines. All what follows is valid in this case. For simplicity, we will still assume that Y^0 is polygonal.*

2.1.4 Additional notations

For what follows, it is important to define the polygonal open domain Y^N obtained by stopping the above construction at step $N+1$,

$$Y^N = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right). \tag{11}$$

We introduce the open domains $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$ and $\Omega^N = \cup_{\sigma\in\mathcal{A}_N}\Omega^\sigma = \Omega\backslash\overline{Y^{N-1}}$, for $N > 0$. When needed, we will agree to say that $\Omega^0 = \Omega$. We define the sets $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$ and $\Gamma^N = \cup_{\sigma\in\mathcal{A}_N}\Gamma^\sigma$. The one-dimensional Lebesgue measure of Γ^σ for $\sigma\in\mathcal{A}_N$ and of Γ^N are

$$|\Gamma^\sigma| = a^N|\Gamma^0| \quad \text{and} \quad |\Gamma^N| = (2a)^N|\Gamma^0|.$$

2.2 Two Examples

2.2.1 Example 1

Let F_1 and F_2 be the affine maps in $\mathbb{R}^2$

$$F_1(x) = (-\alpha + ax_1, \beta + ax_2), \qquad F_2(x) = (\alpha + ax_1, \beta + ax_2), \tag{12}$$

so $\theta = 0$, and (5) becomes $a < \alpha$ and $\beta > 0$. The domain $\widetilde{Y}^0$ is trapezoidal:

$$\widetilde{Y}^0 = \text{Interior}\Big(\text{ convex hull}(P_1, P_2, F_2(P_2), F_1(P_1))\Big).$$

The sets $\overline{\widetilde{Y}^0}$, $\mathcal{M}_\sigma(\overline{\widetilde{Y}^0})$, $\sigma\in\mathcal{A}_n$, $n > 0$ do not overlap if $a \le a^* = 1/2$. The domain $\widetilde{\Omega}$ described in § 2.1.2 is shown in Figure 1 for $\alpha = 3/2$, $\beta = 3$ and $a = a^* = 1/2$.
If $a < 1/2$, the Hausdorff dimension of Γ^∞ is smaller than 1; the set Γ^∞ is a Cantor set, totally disconnected, (i.e. $F_1(\Gamma^\infty)\cap F_2(\Gamma^\infty)$ is empty), and contained in a straight line; it can be proved that $\widetilde{\Omega}$ is a $\epsilon-\delta$ domain as defined by Jones [8], see also [11] and [17].
In the critical case when $a = a^* = 1/2$, Γ^∞ is the straight line segment $[-2\alpha, 2\alpha]\times\{2\beta\}$, so it is connected and its Hausdorff dimension is one. It is post-critically finite because $F_1(\Gamma^\infty)\cap F_2(\Gamma^\infty)$ is a singleton. The open set $\widetilde{\Omega}$ is not a $\epsilon-\delta$ domain. Indeed, take the point $X = (0, \beta/2)$ and call $A_n = (F_1\circ F_2^n)(X)$, $B_n = (F_2\circ F_1^n)(X)$, we have that

- $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = (0, 2\beta)$, therefore $\lim_{n\to\infty}\text{dist}(A_n, B_n) = 0$,
- $A_n\in\widetilde{\Omega}$ and $B_n\in\widetilde{\Omega}$,
- the length of any curve joining A_n and B_n that is contained in $\widetilde{\Omega}$, is greater than 2β.

As explained in § 2.1.3, one can construct a different domain Ω whose boundary contains Γ^∞ by choosing for example Y^0 as a T-shaped domain, see Figure 2.

2.2.2 Example 2

We make the choice $\theta = \pi/4$, $\alpha = 1 - a/\sqrt{2}$, $\beta = 1 + a/\sqrt{2}$, so the similitudes F_i, $i = 1, 2$ read

$$F_i(x) = \begin{pmatrix} (-1)^i\Big(1 - \frac{a}{\sqrt{2}}\Big) + \frac{a}{\sqrt{2}}\Big(x_1 + (-1)^i x_2\Big) \\ 1 + \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}\Big(x_2 + (-1)^{i+1}x_1\Big) \end{pmatrix}.$$

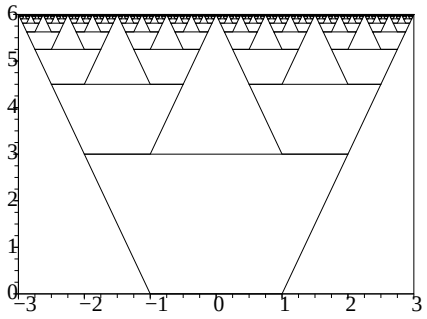

Figure 1: The ramified domain $\widetilde{\Omega}$ for the critical value $a = a^* = 1/2$, and for $\alpha = 3/2$, $\beta = 3$.

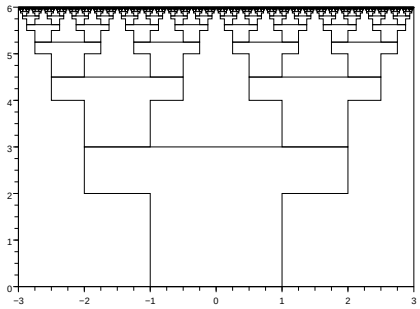

Figure 2: An example of a possible ramified domain Ω for the critical value $a = a^* = 1/2$.

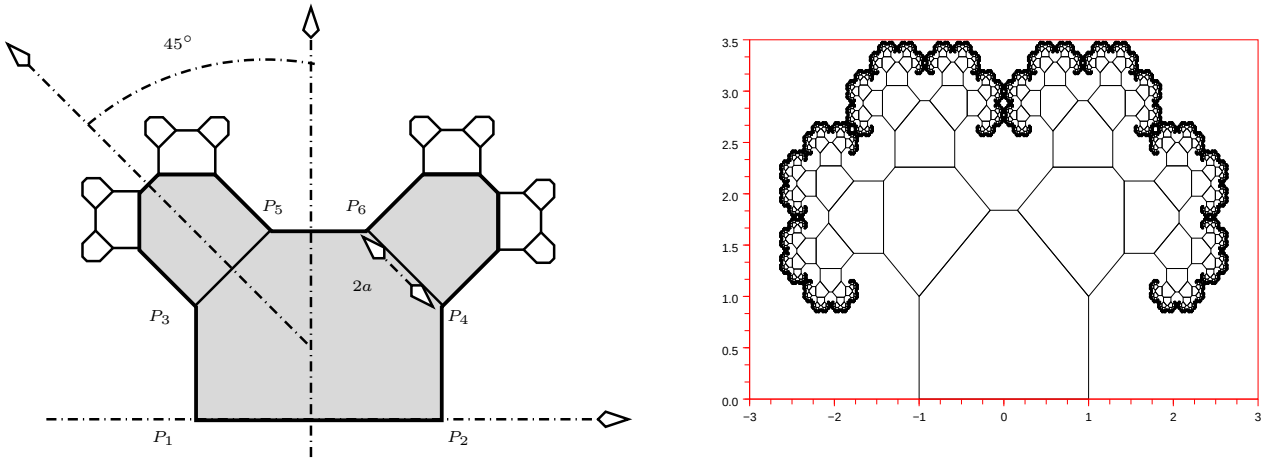

Figure 3: $\theta = \pi/4$, $a = a^*$, $\alpha = 1 - a^*/\sqrt{2}$, $\beta = 1 + a^*/\sqrt{2}$. Left, the construction (more exactly $\widetilde{Y}^3$). Right, the ramified domain $\widetilde{\Omega}$.

The critical parameter a^* is the unique positive solution of $X^3 + \sqrt{2}X^2 - \sqrt{2}/2 = 0$, *i.e.* $a \leq a^* \simeq 0.593465$. The construction described in § 2.1.2 with the critical value $a = a^*$ leads to the domain $\widetilde{\Omega}$ shown in Figure 3.
If $a > 1/2$, the Hausdorff dimension of Γ^∞ is larger than one. For instance, if $a = a^*$, then $\dim_H(\Gamma^\infty) \simeq 1.3284371$.
In the case when $a < a^*$, it can be proved that $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$ is empty and that Ω is a $\epsilon - \delta$ domain, see [2].
In the critical case when $a = a^*$, it can be proved, see [2], that $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \subset \Gamma^\infty$ is a non countable set, whose Hausdorff dimension is half the Hausdorff dimension of Γ^∞. The set Γ^∞ is not post-critically finite. It was also proved in [2] that $\widetilde{\Omega}$ is not a $\epsilon - \delta$ domain.

2.3 The self-similar measure μ

For defining traces on Γ^∞, we recall the classical result on self-similar measures, see [5, 7] and [13] page 26:

Theorem 4 *There exists a unique Borel regular probability measure μ on Γ^∞ such that for any Borel set $A \subset \Gamma^\infty$,*

$$\mu(A) = \frac{1}{2}\mu\left(F_1^{-1}(A)\right) + \frac{1}{2}\mu\left(F_2^{-1}(A)\right). \tag{13}$$

The measure μ is called the *self-similar measure defined in the self-similar triplet* $(\Gamma^\infty, F_1, F_2)$.

Proposition 2 *The measure μ is a d-measure on Γ^∞, with $d = -\log 2/\log a$, according to the definition in [11], page 28: there exists two positive constants c_1 and c_2 such that*

$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d,$$

for any r $0 < r < 1$ and $x \in \Gamma^\infty$, where $B(x,r)$ is the Euclidean ball in Γ^∞ centered at x and with radius r. In other words the closed set Γ^∞ is a d-set, see [11], page 28.

Proof. The proof stems from the Moran condition in § 2.1.2. It is due to Moran [18] and has been extended by Kigami, see [13], §1.5, especially Proposition 1.5.8 and Theorem 1.5.7. ∎

Remark 5 *In the first example, § 2.2.1, with $a = 1/2$, the measure μ is such that 6μ is the one-dimensional Lebesgue measure on the straight line Γ^∞.*

We define L^p_μ, $p \in [1, +\infty)$ as the space of the measurable functions v on Γ^∞ such that $\int_{\Gamma^\infty} |v|^p d\mu < \infty$, endowed with the norm $\|v\|_{L^p_\mu} = \left(\int_{\Gamma^\infty} |v|^p d\mu\right)^{1/p}$. We also introduce L^∞_μ, the space of essentially bounded functions with respect to the measure μ. A Hilbertian basis of L^2_μ can be constructed with e.g. Haar wavelets.

3 The space $W^{1,q}(\Omega)$

Hereafter, the parameters (α, β) satisfy Assumption 1. Except when explicitly mentioned, we will not distinguish between the simplest geometries presented in § 2.1.2 and the more general ones introduced in § 2.1.3. The former can be seen as a special case of the latter. Therefore, we will use the notation Ω for all the domains constructed in § 2.
For a real number $q \geq 1$, let $W^{1,q}(\Omega)$ be the space of functions in $L^q(\Omega)$ with first order partial derivatives in $L^q(\Omega)$. We also define $\mathcal{V}_q(\Omega) = \left\{v \in W^{1,q}(\Omega); v|_{\Gamma^0} = 0\right\}$ and $\mathcal{V}_q(Y^n) = \left\{v \in W^{1,q}(Y^n); v|_{\Gamma^0} = 0\right\}$.

If $a < 1/2$ then $d < 1$. In this case, Γ^∞ is totally disconnected, see [6], Lemma 4.1 page 54. This implies that $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) = \emptyset$, see [13], theorem 1.6.2 page 33. Therefore, $a < a^*$ and we can apply the result stated in the introduction: if $q > \max(1, 2-d)$, then the space of the traces on Γ^∞ of the functions $v \in W^{1,q}(\Omega)$ (see § 3 below for the definition) is $B^{q,q}_{1-\frac{2-d}{q}}(\Gamma^\infty)$ (see the introduction for the definition) and we will see in Theorem 8 below that in this case, $B^{q,q}_{1-\frac{2-d}{q}}(\Gamma^\infty) = JLip(1-\frac{2-d}{q}, q, q; 0; \Gamma^\infty)$.
The case $a < 1/2$ being understood, hereafter, we focus on a such that $1/2 \leq a \leq a^*$, so the Hausdorff dimension d of Γ^∞ is not smaller than 1.
The results stated below are important for the study of elliptic boundary value problems in Ω. Section 3.1 contains some Poincaré inequalities and §3.2 deals with the construction of a trace operator on Γ^∞. We refer to [1] for the proofs.
We will sometimes use the notation $\lesssim$ to indicate that there may arise constants in the estimates, which are independent of the index n in Ω^n (recall that Ω^n is the union of all $\mathcal{M}_\sigma(\Omega)$, $\sigma \in \mathcal{A}_n$) or Γ^n (Γ^n is the union of all $\mathcal{M}_\sigma(\Gamma^0)$, $\sigma \in \mathcal{A}_n$), $Y^n = Y^0 \cup \bigcup_{1\leq p\leq n} \bigcup_{\sigma\in\mathcal{A}_p} \mathcal{M}_\sigma(Y^0)$, or the index σ in $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$ or $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$.

3.1 Poincaré inequality and consequences

Theorem 5 *There exists a constant $C > 0$, such that*

$$\forall v \in \mathcal{V}_q(\Omega), \qquad \|v\|^q_{L^q(\Omega)} \leq C\|\nabla v\|^q_{L^q(\Omega)}.$$

Corollary 1 *There exists a positive constant C such that for all $v \in W^{1,q}(\Omega)$,*

$$\|v\|^q_{L^q(\Omega)} \leq C\left(\|\nabla v\|^q_{L^q(\Omega)} + \|v|_{\Gamma^0}\|^q_{L^q(\Gamma^0)}\right).$$

Corollary 2 *There exists a positive constant C such that for all integer $n \geq 0$ and for all $\sigma \in \mathcal{A}_n$, for all $v \in W^{1,q}(\Omega^\sigma)$,*

$$\|v\|^q_{L^q(\Omega^\sigma)} \leq C\left(a^{qn}\|\nabla v\|^q_{L^q(\Omega^\sigma)} + a^n\|v|_{\Gamma^\sigma}\|^q_{L^q(\Gamma^\sigma)}\right),$$

and for all $v \in W^{1,q}(\Omega^n)$

$$\|v\|^q_{L^q(\Omega^n)} \leq C\left(a^{qn}\|\nabla v\|^q_{L^q(\Omega^n)} + a^n\|v|_{\Gamma^n}\|^q_{L^q(\Gamma^n)}\right).$$

We need to estimate $\|v\|^q_{L^q(\Omega^n)}$ when $v \in W^{1,q}(\Omega)$:

Lemma 1 *There exists a positive constant C such that for all $v \in W^{1,q}(\Omega)$, for all $n \geq 0$,*

$$\|v\|^q_{L^q(\Omega^n)} \leq C\,(2a^2)^n\left(\|\nabla v\|^q_{L^q(\Omega)} + \|v|_{\Gamma^0}\|^q_{L^q(\Gamma^0)}\right). \tag{14}$$

Since $2a^2 < 1$, (14) implies the Rellich type theorem:

Theorem 6 (Compactness) *The imbedding of $W^{1,q}(\Omega)$ in $L^q(\Omega)$ is compact.*

The following lemma will be useful for defining a trace operator on Γ^∞:

Lemma 2 *There exists a positive constant C such that $\forall v \in W^{1,q}(\Omega)$, for all integers $p \geq 0$,*

$$\sum_{\sigma\in\mathcal{A}_p}\int_{\Gamma^\sigma}(v|_{\Gamma^\sigma})^q \leq C(2a)^p\left(\|\nabla v\|^q_{L^q(\Omega)} + \|v\|^q_{L^q(\Omega)}\right). \tag{15}$$

Remark 6 *Note that* $|\Gamma^p| = (2a)^p|\Gamma^0|$*, so (15) is equivalent to*

$$\frac{1}{|\Gamma^p|}\sum_{\sigma\in\mathcal{A}_p}\int_{\Gamma^\sigma}(v|_{\Gamma^\sigma})^q \lesssim \|\nabla v\|^q_{L^q(\Omega)} + \|v\|^q_{L^q(\Omega)}.$$

Corollary 3 *There exists a constant* $C > 0$ *such that for all* $v \in W^{1,q}(\Omega)$ *and all integers* $p \geq 0$*,*

$$\sum_{\sigma\in\mathcal{A}_p}\int_{\Gamma^\sigma}(v|_{\Gamma^\sigma} - \langle v|_{\Gamma^0}\rangle)^q \leq C(2a)^p\|\nabla v\|^q_{L^q(\Omega)},$$

where $\langle v|_{\Gamma^0}\rangle$ *is the mean value of* $v|_{\Gamma^0}$ *on* Γ^0*.*

Remark 7 *Similar results in a different geometry were proved by Berger [3] with different methods.*

3.2 A trace operator on Γ^∞

We construct a sequence $(\ell^n)_n$ of approximations of the trace operator: consider the sequence of linear operators $\ell^n : W^{1,q}(\Omega) \to L^q_\mu$,

$$\ell^n(v) = \sum_{\sigma\in\mathcal{A}_n}\left(\frac{1}{|\Gamma^\sigma|}\int_{\Gamma^\sigma} v\,dx\right)\mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}, \tag{16}$$

where $|\Gamma^\sigma|$ is the one-dimensional Lebesgue measure of Γ^σ, see [1].

Proposition 3 *The sequence* $(\ell^n)_n$ *converges in* $\mathcal{L}(W^{1,q}(\Omega), L^q_\mu)$ *to an operator that we call* ℓ^∞*.*

Proof. See [1]. ∎

4 Lipschitz Functions with Jumps on Γ^∞

In [10], A. Jonsson has introduced Haar wavelets of arbitrary order on self-similar fractal sets and has used these wavelets for constructiong a family of Lipschitz spaces. These function spaces are named $JLip(\alpha, p, q; m; K)$, where K is the fractal set, α is a nonnegative real number, p, q are two real numbers not smaller than 1 and m is an integer (m is the order of the Haar wavelets used for constructing the space). Here J stands for jumps, since the considered functions may jump at some points of K. If the fractal set K is totally disconnected, then these spaces coincide with the Lipschitz spaces $Lip(\alpha, p, q; m; K)$ also introduced in [10]. The latter are a generalization of the more classical spaces $Lip(\alpha, p, q; K)$ introduced in [11] since $Lip(\alpha, p, q; [\alpha]; K) = Lip(\alpha, p, q; K)$. Note that $Lip(\alpha, p, q; [\alpha]; K) = B^{p,q}_\alpha(K)$, see[12].
Since this is sufficient for what follows, we will focus on the case when $K = \Gamma^\infty$ and $p = q$.

4.1 Definition of $JLip(\alpha, q, q; 0; \Gamma^\infty)$ for $0 < \alpha < 1$.

We review the theory proposed in [10] in the special case when $0 < \alpha < 1$ and $m = 0$. Here, since we focus on the case $m = 0$, we do not need to suppose that Γ^∞ is not contained in a straight line, as it was done in [10] in order to obtain that the fractal set has a Markov property. For $f \in L^q_\mu$, a positive integer n and $\sigma \in \mathcal{A}_n$, we define $P_\sigma f$ as the projection of f on constants:

$$P_\sigma f = 2^n\int_{\mathcal{M}_\sigma(\Gamma^\infty)} f d\mu.$$

We also define $P_0 f$ as the projection of f on constants:

$$P_0 f = \int_{\Gamma^\infty} f d\mu.$$

For an integer ν, we define the set J_ν:

$$J_\nu = \{p \in \mathbb{N},\ \text{s.t. } \forall \sigma \in \mathcal{A}_p,\ 2^{-\nu} \leq \text{diam}(\mathcal{M}_\sigma(\Gamma^\infty)) < 2^{-\nu+1}\}, \tag{17}$$

agreeing that $\mathcal{M}_\sigma = Id$ if $p = 0$, $\sigma \in \mathcal{A}_p$.
Let ν_0 be the integer (possibly negative) such that $0 \in J_{\nu_0}$.
Consider a real number α, $0 < \alpha < 1$. For $f \in L^2_\mu$ and an integer $\nu \geq \nu_0$, we define $A_\nu(f)$:

$$\begin{aligned} A_{\nu_0}(f) &= 2^{\alpha\nu_0} \left(\int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{p \in J_{\nu_0}, p>0} \sum_{\sigma \in \mathcal{A}_p} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{1/q}, \\ A_\nu(f) &= 2^{\alpha\nu} \left(\sum_{p \in J_\nu} \sum_{\sigma \in \mathcal{A}_p} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{1/q}, \quad \text{if } \nu > \nu_0. \end{aligned} \tag{18}$$

Definition 1 *The function f belongs to $JLip(\alpha, q, q; 0; \Gamma^\infty)$ if the norm*

$$\|f\|_{JLip(\alpha,q,q;0;\Gamma^\infty)} = \left(\|f\|^q_{L^q_\mu} + \sum_{\nu=\nu_0}^{\infty} A^q_\nu(f) \right)^{\frac{1}{q}} \tag{19}$$

is finite.

Following [10], it is possible to characterize $JLip(\alpha, q, q; 0; \Gamma^\infty)$ by using expansions in the standard Haar wavelet basis on Γ^∞. Consider the Haar mother wavelet g^0 on Γ^∞,

$$g^0 = 1_{F_1(\Gamma^\infty)} - 1_{F_2(\Gamma^\infty)}, \tag{20}$$

and for $n \in \mathbb{N}$, $n > 0$, $\sigma \in \mathcal{A}_n$, let g^σ be given by

$$g^\sigma|_{\mathcal{M}_\sigma(\Gamma^\infty)} = 2^{n/2} g^0 \circ \mathcal{M}_\sigma^{-1}, \quad \text{and} \quad g^\sigma|_{\Gamma^\infty \setminus \mathcal{M}_\sigma(\Gamma^\infty)} = 0. \tag{21}$$

It is proved in [9] §5 that a function $f \in L^q_\mu$ can be expanded on the Haar basis as follows:

$$f = P_0 f + \beta_0 g^0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma} g^\sigma.$$

Let b_0 be a real number and $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$ be a sequence of real numbers, we define $\|(b_0, b_{n,\sigma})\|_{b^{q,q}_\alpha}$:

$$\|(b_0, b_{n,\sigma})\|_{b^{q,q}_\alpha} = \left(\begin{array}{l} 2^{\alpha\nu_0 q} 2^{d\nu_0(\frac{1}{2}-\frac{1}{q})q} \left(|b_0|^q + \displaystyle\sum_{p \in J_{\nu_0}, p>0} \sum_{\sigma \in \mathcal{A}_p} |b_{p,\sigma}|^q \right) \\ + \displaystyle\sum_{\nu=\nu_0+1}^{\infty} 2^{\alpha\nu q} 2^{d\nu(\frac{1}{2}-\frac{1}{q})q} \sum_{p \in J_\nu} \sum_{\sigma \in \mathcal{A}_p} |b_{p,\sigma}|^q \end{array} \right)^{\frac{1}{q}}. \tag{22}$$

Theorem 7 *(Jonsson) A function $f \in L^q_\mu$ belongs to $JLip(\alpha, q, q; 0; \Gamma^\infty)$ if and only if the coefficients of its expansion in the Haar wavelets basis satisfy $|P_0 f| + \|(\beta_0, \beta_{n,\sigma})\|_{b^{q,q}_\alpha}$ is finite; this defines a norm in $JLip(\alpha, q, q; 0; \Gamma^\infty)$ equivalent to the one in (19).*

Proof. This is a particular case of Theorem 1 in [10]. ∎

If the fractal set Γ^∞ is totally disconnected, then the jumps which are allowed in the space $JLip(\alpha, q, q; 0; \Gamma^\infty)$ disappear:

Theorem 8 *(Jonsson) If $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$ is empty, then*

$$JLip(\alpha, q, q; 0; \Gamma^\infty) = Lip(\alpha, q, q; 0; \Gamma^\infty) = B_\alpha^{q,q}(\Gamma^\infty),$$

where the Lipschitz space $Lip(\alpha, q, q; 0; \Gamma^\infty)$ and the Besov space $B_\alpha^{q,q}(\Gamma^\infty)$ are defined in [11].

Proof. This is a particular case of Theorem 2 in [10], see also [9] for a partial proof. ∎

Let us assume that $\mathrm{diam}(\Gamma^\infty) > 0$. Since in our case the ratii of the similitudes F_1 and F_2 both take the value a, we see that $\mathrm{diam}(\mathcal{M}_\sigma(\Gamma^\infty)) = a^n \mathrm{diam}(\Gamma^\infty)$ if $\sigma \in \mathcal{A}_n$ and $n > 0$. Therefore, for $n > 0$, $n \in J_\nu$ if and only if

$$2^{-\nu} \leq a^n \mathrm{diam}(\Gamma^\infty) < 2^{-\nu+1},$$

which can be written

$$-\nu \leq n \frac{\log a}{\log 2} + \frac{\log(\mathrm{diam}(\Gamma^\infty))}{\log 2} < -\nu + 1, \tag{23}$$

or

$$-\nu = \left[n \frac{\log a}{\log 2} + \frac{\log\left(\mathrm{diam}(\Gamma^\infty)\right)}{\log 2} \right] = \left[-n/d + \frac{\log\left(\mathrm{diam}(\Gamma^\infty)\right)}{\log 2} \right]. \tag{24}$$

where $d = -\log 2 / \log a$ is the Hausdorff dimension of Γ^∞ and $[\cdot]$ stands for the integer part. Let b_0 be a real number and $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$ be a sequence of real numbers; we define $\|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}}$:

$$\begin{aligned} \|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} &= \left(|b_0|^q + \sum_{n=1}^\infty 2^{q\alpha n/d} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}}. \\ &= \left(|b_0|^q + \sum_{n=1}^\infty a^{-q\alpha n} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}}. \end{aligned} \tag{25}$$

Assuming that $\mathrm{diam}(\Gamma^\infty) \neq 0$, there exist two positive constants c_1 and c_2 such that

$$c_1 \|(b_0, b_{n,\sigma})\|_{b_\alpha^{q,q}} \leq \|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} \leq c_2 \|(b_0, b_{n,\sigma})\|_{b_\alpha^{q,q}},$$

for any sequence $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$ such that $|b_0|^q + \sum_{n=1}^\infty 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q < \infty$. We have the corollary of Theorem 7:

Corollary 4 *A function $f \in L_\mu^q$ belongs to $JLip(\alpha, q, q; 0; \Gamma^\infty)$ if and only if the coefficients of its expansion in the Haar wavelets basis satisfy $|P_0 f| + \|(\beta_0, \beta_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}}$ is finite; the norm of f in $JLip(\alpha, q, q; 0; \Gamma^\infty)$ is equivalent to this sum. Another equivalent norm is*

$$\left(\|f\|_{L_\mu^q}^q + \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{n=1}^\infty a^{-\alpha q n} \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{\frac{1}{q}}. \tag{26}$$

Remark 8 *In the special case when* $\alpha = 1 - (2-d)/q$,

$$\|(b_0, b_{n,\sigma})\|_{\tilde{b}^{q,q}_\alpha} = \left(|b_0|^q + \sum_{n=1}^{\infty} 2^{\frac{nq}{2}} a^{-n(q-2)} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}},$$

and the norm in (26) reads

$$\left(\|f\|^q_{L^q_\mu} + \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{n=1}^{\infty} \left(2a^{2-q}\right)^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{\frac{1}{q}}.$$

These observations will be useful in the proof of Theorem 9 below.

4.2 Definition of $JLip(\alpha, q, q; m; \Gamma^\infty)$ for a positive integer m and $m \leq \alpha < m+1$.

Following [10], we suppose that Γ^∞ is not contained in a straight line (this rules out the geometry presented in § 2.2.1 above). This assumption implies that Γ^∞ preserves Markov's inequality, see [10].
Call $\mathcal{S}_0$ the subspace of L^2_μ made of polynomials of order not larger than m; thanks to Markov's inequality, its dimension is $M_0 = (m+2)(m+1)/2$; let $(\phi^i)_{1 \leq i \leq M_0}$ be a basis of $\mathcal{S}_0$ orthonormal with respect to $(\cdot,\cdot)_{L^2_\mu}$. For $f \in L^1_\mu$, we define by $P_0^{(m)} f$ the projection of f onto $\mathcal{P}_m$ defined as follows: let $(\phi^j)_{1 \leq j \leq M_0}$ be an orthonormal basis of $\mathcal{S}_0$ and $P_0^{(m)} f = \sum_i \phi^i \int_{\Gamma^\infty} f \phi^i d\mu$.
Similarly, for $f \in L^1_\mu(\mathcal{M}_\sigma(\Gamma^\infty))$, we define by $P_\sigma^{(m)} f$ the projection of f onto $\mathcal{P}_m$:

$$P_\sigma^{(m)} f = \left(P_0^{(m)} (f \circ \mathcal{M}_\sigma) \right) \circ \mathcal{M}_\sigma^{-1}.$$

The following definition of the space $JLip(\alpha, q, q; m; \Gamma^\infty)$ has to be compared to (26):

Definition 2 *Let m be a positive integer and α be a real number such that $m \leq \alpha < m+1$. The function $f \in L^q_\mu$ belongs to $JLip(\alpha, q, q; m; \Gamma^\infty)$ if the norm*

$$\|f\|_{JLip(\alpha,q,q;m;\Gamma^\infty)} = \left(\|f\|^q_{L^q_\mu} + \int_{\Gamma^\infty} |f - P_0^{(m)} f|^q d\mu + \sum_{n=1}^{\infty} a^{-\alpha q n} \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma^{(m)} f|^q d\mu \right)^{\frac{1}{q}}. \tag{27}$$

is finite.

Haar wavelets of order m can be used for an equivalent definition of $JLip(\alpha, q, q; m; \Gamma^\infty)$: call $\mathcal{S}_1$ the space of the functions f defined on Γ^∞ such that f coincides with a polynomial of degree m on $F_i(\Gamma^\infty) \setminus (F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty))$, $i = 1, 2$. The dimension of $\mathcal{S}_1$ is $2M_0$. The orthogonal complement $\mathcal{S}_1 \ominus \mathcal{S}_0$ of $\mathcal{S}_0$ in $\mathcal{S}_1$ has dimension M_0. Choose an orthonormal basis $(g^{0,i})_{1 \leq i \leq M_0}$ of $\mathcal{S}_1 \ominus \mathcal{S}_0$. Finally, for $n \geq 1$ and $\sigma \in \mathcal{A}_n$, define

$$g^{\sigma,i} = 2^{n/2} g^{0,i} \circ \mathcal{M}_\sigma^{-1}.$$

The family $(\phi^i, g^{0,i}, g^{\sigma,i})$, $1 \leq i \leq M_0$, $\sigma \in \mathcal{A}_n$, $n \geq 1$ is an orthonormal and complete family of L^2_μ. It is proved in [10] that a function $f \in L^q_\mu$ can be expanded as follows

$$f = \sum_{i=1}^{M_0} \lambda_i \phi^i + \sum_{i=1}^{M_0} \beta_{0,i} g^{0,i} + \sum_{i=1}^{M_0} \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma,i} g^{\sigma,i}. \tag{28}$$

Proposition 4 *(Jonsson, [10], Th. 1) A function $f \in L^q_\mu$ belongs to $JLip(\alpha, q, q; m; \Gamma^\infty)$ if and only if*

$$\left(\sum_{i=1}^{M_0} |\lambda_i|^q + \sum_{i=1}^{M_0} |\beta_{0,i}|^q + \sum_{i=1}^{M_0} \sum_{n=1}^{\infty} a^{-q\alpha n} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |\beta_{n,\sigma,i}|^q \right)^{\frac{1}{q}} < \infty, \tag{29}$$

where λ_i, $\beta_{0,i}$, $\beta_{n,\sigma,i}$ are the coefficients in (28). The norm in (29) is equivalent to that in (27).

5 Traces of functions belonging to $W^{1,q}(\Omega)$

We consider the case when $1/2 \leq a \leq a^*$; we are going to give a trace theorem with a direct proof (which differs from the previous argument when $a < a^*$).
The main theorem is as follows:

Theorem 9 *If $1/2 \leq a \leq a^*$, then for all q, $1 < q < \infty$,*

$$\ell^\infty \left(W^{1,q}(\Omega) \right) = JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty). \tag{30}$$

Proof. Theorem 9 is a straightforward consequence of Proposition 5 and Theorem 10 below. ∎

Remark 9 *Note the contrast with the fact that functions in $W^{1,q}(\mathbb{R}^2)$ have their traces in $Lip(1-(2-d)/q, q, q; 0; \Gamma^\infty)$, see [11] page 182 and [9, 10].*

Proposition 5 *For all real number $q \geq 1$, there exists a constant C such that for all $v \in JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$, there exists $\tilde{v} \in \mathcal{V}_q(\Omega)$ with the following properties: $\ell^\infty(\tilde{v}) = v$, and*

$$\|\tilde{v}\|^q_{\mathcal{V}_q(\Omega)} \approx \int_\Omega |\nabla \tilde{v}|^q \leq C \|v\|^q_{JLip(1-\frac{2-d}{q}, q, q; 0; \Gamma^\infty)}. \tag{31}$$

Proof. First, let us consider the case when v is a Haar wavelet on Γ^∞. We start with the mother wavelet g^0. One can construct a function $\tilde{g}^0$ such that $\tilde{g}^0 \in \mathcal{V}_q(\Omega)$ for all $1 \leq q < \infty$ and

$$\tilde{g}^0|_{F_1(\Omega)} = -1 \qquad \text{and} \qquad \tilde{g}^0(-x_1, x_2) = -\tilde{g}^0(x_1, x_2), \quad \forall x = (x_1, x_2) \in \Omega. \tag{32}$$

It is an easy matter to check that $\ell^\infty(\tilde{g}^0) = g^0$ and that $\int_\Omega |\nabla \tilde{g}^0|^q = \int_{Y^0} |\nabla \tilde{g}^0|^q$.
For a positive integer n and $\sigma \in \mathcal{A}_n$, let us define the function $\tilde{g}^\sigma$, $\tilde{g}^\sigma \in \mathcal{V}_q(\Omega)$ for all $1 \leq q < \infty$, by:

$$\tilde{g}^\sigma|_{\Omega^\sigma} = 2^{n/2}\, \tilde{g}^0 \circ \mathcal{M}_\sigma^{-1}, \tag{33}$$

$$\tilde{g}^\sigma|_{\Omega \setminus \Omega^\sigma} = 0. \tag{34}$$

From the definition of g^σ, we see that $\ell^\infty(\tilde{g}^\sigma) = g^\sigma$. The change of variable $x = \mathcal{M}_\sigma y$ yields

$$\int_\Omega |\nabla \tilde{g}^\sigma|^q = \int_{\Omega^\sigma} |\nabla \tilde{g}^\sigma|^q = 2^{n\frac{q}{2}} a^{(2-q)n} \int_\Omega |\nabla \tilde{g}^0|^q = 2^{n\frac{q}{2}} a^{(2-q)n} \int_{Y^0} |\nabla \tilde{g}^0|^q. \tag{35}$$

Moreover, the support of $\nabla \tilde{g}^\sigma$ is contained in Y^σ, which implies that for all positive integer n and n', for all $\sigma \in \mathcal{A}_n$, $\eta \in \mathcal{A}_{n'}$,

- the supports of $\nabla \tilde{g}^\sigma$ and $\nabla \tilde{g}^0$ are disjoint.
- if $(n, \sigma) \neq (n', \eta)$ then the supports of $\nabla \tilde{g}^\sigma$ and $\nabla \tilde{g}^\eta$ are disjoint.

We also introduce a function χ, $\chi \in \mathcal{V}_q(\Omega)$ for all $1 \le q < \infty$, such that $\chi|_{\Omega^1} = 1$ and that $\chi(x_1, x_2) = \chi(-x_1, x_2)$. It is clear that $\ell^\infty(\chi) = 1$. Moreover, it can be seen that

$$\left\{ \begin{array}{lll} & \displaystyle\int_\Omega \nabla\chi \cdot \nabla\tilde{g}^0 \quad = 0 \quad , \\ \text{and} & \displaystyle\int_\Omega \nabla\chi \cdot \nabla\tilde{g}^\sigma = \quad 0 \quad \text{for all integer } n > 0 \text{ and all } \sigma \in \mathcal{A}_n. \end{array} \right. \tag{36}$$

Consider now $v \in JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$. We can expand v on the Haar basis:

$$v = P_0 v + \beta_0 g^0 + \sum_{n\ge 1} \sum_{\sigma\in\mathcal{A}_n} \beta_{n,\sigma} g^\sigma,$$

where $\beta_0 = \int_{\Gamma^\infty} v g^0 d\mu$ and $\beta_{n,\sigma} = \int_{\Gamma^\infty} v g^\sigma d\mu$. We know from Theorem 1 and Corollary 4 (see Remark 8) that

$$|\beta_0|^q + \sum_{n=1}^{\infty} 2^{n\frac{q}{2}} a^{(2-q)n} \sum_{\sigma\in\mathcal{A}_n} |\beta_{n,\sigma}|^q < \infty. \tag{37}$$

Let us define the function $\tilde{v} : \Omega \to \mathbb{R}$, by

$$\tilde{v} = (P_0 v)\chi + \beta_0 \tilde{g}^0 + \sum_{n\ge 1} \sum_{\sigma\in\mathcal{A}_n} \beta_{n,\sigma} \tilde{g}^\sigma, \tag{38}$$

where we agree to identify the function $P_0 v$ with its real value. From a classical inequality, we obtain that

$$\int_\Omega |\nabla\tilde{v}|^q \lesssim |P_0 v|^q \int_\Omega |\nabla\chi|^q + \int_\Omega \left| \beta_0 \nabla\tilde{g}^0 + \sum_{n\ge 1} \sum_{\sigma\in\mathcal{A}_n} \beta_{n,\sigma} \nabla\tilde{g}^\sigma \right|^q .$$

From (35) and the observation on the supports of the functions $\nabla\tilde{g}^\sigma$, this yields that

$$\int_\Omega |\nabla\tilde{v}|^q \lesssim |P_0 v|^q \int_\Omega |\nabla\chi|^q + \left(|\beta_0|^q + \sum_{n\ge 1} \sum_{\sigma\in\mathcal{A}_n} 2^{n\frac{q}{2}} a^{(2-q)n} |\beta_{n,\sigma}|^q \right) \int_{Y^0} |\nabla\tilde{g}^0|^q,$$

which is finite from (37), and the right hand sides of the two above estimates coincide. Therefore, $\tilde{v} \in \mathcal{V}_q(\Omega)$, and it is an easy matter to see that $\ell^\infty(\tilde{v}) = v$ using the continuity of ℓ^∞ in $W^{1,q}(\Omega)$, see Proposition 3. Finally, (31) is a consequence of Theorem 1 and Corollary 4, (see Remark 8). ■

Theorem 10 *Assume that $1/2 \le a \le a^*$. For all q, $1 < q < \infty$ and $v \in W^{1,q}(\Omega)$, $\ell^\infty(v)$ belongs to $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ and there exists a constant c such that*

$$\|\ell^\infty(v)\|_{JLip(1-\frac{2-d}{q},q,q;0;\Gamma^\infty)} \le c\|v\|_{W^{1,q}(\Omega)}, \quad \forall v \in W^{1,q}(\Omega). \tag{39}$$

The proof of Theorem 10 is postponed to the end of § 5. It makes use of the strengthened trace inequality stated in Theorem 11 below. The following lemma (see, e.g., [4], p. 126) will play an important role in the proof of Theorem 11.

Lemma 3 (Peetre–Tartar) *Let V, V_1, V_2, W be Banach spaces, and let $A_i \in \mathcal{L}(V, V_i)$ be continuous linear maps for $i = 1, 2$, the map A_1 being compact. Suppose there exists $c_0 > 0$ such that*

$$\|v\|_V \le c_0\big(\|A_1 v\|_{V_1} + \|A_2 v\|_{V_2}\big), \qquad \text{for any } v \in V.$$

In addition, let $L \in \mathcal{L}(V, W)$ be a continuous linear map such that $L\big|_{\ker A_2} \equiv 0$. Then there exists $C > 0$ such that

$$\|Lv\|_W \le C\|A_2 v\|_{V_2}, \qquad \text{for any } v \in V.$$

Theorem 11 *Assume that $a \geq 1/2$.*

1. *For all real number ρ such that*
$$(2a^2)^{q-1} < \rho < 1, \tag{40}$$
there exists a constant C such that for all $v \in \mathcal{V}_q(\Omega)$,
$$\|\ell^\infty(v)\|_{L^q_\mu}^q \leq C\left(\|\nabla v\|_{L^q(Y^0)}^q + \sum_{n=1}^{\infty} \rho^{-n}(2a^2)^{n(q-1)} \sum_{\sigma\in\mathcal{A}_n} \|\nabla v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q\right). \tag{41}$$

2. *For ρ satisfying (40), there exists a constant C such that, for all $v \in W^{1,q}(\Omega)$,*
$$\|\ell^\infty(v) - P_0\left(\ell^\infty(v)\right)\|_{L^q_\mu}^q \leq C\left(\|\nabla v\|_{L^q(Y^0)}^q + \sum_{n=1}^{\infty} \rho^{-n}(2a^2)^{n(q-1)} \sum_{\sigma\in\mathcal{A}_n} \|\nabla v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q\right). \tag{42}$$

Proof.

The proofs in the cases when $a = 1/2$ and $a > 1/2$ slightly differ. Hereafter, we thoroughly discuss the case $a > 1/2$. For the ease of the reader, the case $a = 1/2$ will be discussed in Appendix B. In both cases, the proof of Theorem 11 is done in three steps.

The case when $a > 1/2$

Step 1 The first step consists of proving (41) for the constructions described in § 2.1.2: we take $P_1 = (-1, 0)$, $P_2 = (1, 0)$, and define $P_3 = (0, \beta + a\sin\theta)$. We partition $\widetilde{Y}^0$ into five nonoverlapping triangles: $\widetilde{Y}^0 = \cup_{i=1}^5 \widetilde{T}_i$, where $\widetilde{T}_1 = (P_1, P_2, P_3)$, $\widetilde{T}_2 = (P_1, P_3, F_1(P_1))$, $\widetilde{T}_3 = (F_1(P_1), F_1(P_2), P_3)$, $\widetilde{T}_4 = (P_2, F_2(P_2), P_3)$ and $\widetilde{T}_5 = (F_2(P_2), F_2(P_1), P_3)$, see Figure 4.
Let us introduce the two real numbers
$$\begin{array}{rcl} r_1 &=& (\alpha - 1 + a\cos\theta)(\beta + a\sin\theta) + (\beta - a\sin\theta), \\ r_2 &=& 2a\sin\theta(\alpha - a\cos\theta) + \beta + a\sin\theta. \end{array} \tag{43}$$
The assumptions (5) and (6) imply that $r_1 \geq 0$ and that $r_2 \geq \beta + a\sin\theta$. We introduce the points $\widehat{P}_4 = (-1, r_1)$, $\widehat{P}_5 = (0, r_2)$ and $\widehat{P}_6 = (1, r_1)$, and the triangles $\widehat{T}_1 = \widetilde{T}_1$, $\widehat{T}_2 = (P_1, P_3, \widehat{P}_4)$, $\widehat{T}_3 = (\widehat{P}_4, \widehat{P}_5, P_3)$, $\widehat{T}_4 = (P_2, \widehat{P}_6, P_3)$ and $\widehat{T}_5 = (\widehat{P}_6, \widehat{P}_5, P_3)$, see Figure 4.
The points $\widehat{P}_i$, $i = 4, 5, 6$ have been chosen in such a way that

- The triangles $\widetilde{T}_i$ and $\widehat{T}_i$ have the same measure, $i = 1, \ldots 5$.
- $\widehat{P}_4$ and P_1 have the same abcissa -1, $\widehat{P}_5$ and P_3 have the same abcissa 0, and $\widehat{P}_6$ and P_2 have the same abcissa 1.

The triangles $\widehat{T}_i$ $i = 1, \ldots, 5$ form a nonoverlapping partition of a new polygonal domain $\widehat{Y}_0$ with a vertical fracture namely $[P_3, \widehat{P}_5]$, see Figure 4.
Therefore it is possible to construct a continuous and piecewise affine map H from $\widetilde{Y}_0$ onto $\widehat{Y}_0$, in such a way that $H_{|\widetilde{T}_i}$ is an affine map from $\widetilde{T}_i$ onto $\widehat{T}_i$. It is clear that H is measure preserving.
Let G_1 and G_2 be the maps in $\mathbb{R}^2$ defined by
$$\begin{array}{rcl} G_1(x_1, x_2) &=& \left(\frac{x_1-1}{2}, \frac{r_1+r_2}{2} + \frac{r_2-r_1}{2}x_1 + 2a^2x_2\right), \\ G_2(x_1, x_2) &=& \left(\frac{x_1+1}{2}, \frac{r_1+r_2}{2} - \frac{r_2-r_1}{2}x_1 + 2a^2x_2\right). \end{array}$$

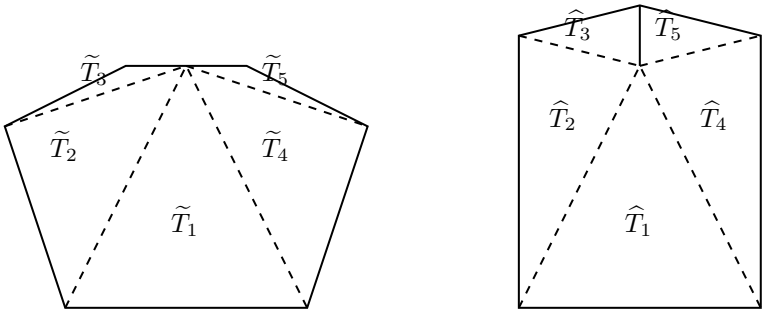

Figure 4: Left: the cell $\widetilde{Y}^0$. Right: the fractured open set $\widehat{Y}^0$. The Jacobian matrix of H on the left (resp. of H^{-1} on the right) jumps across the broken lines. In this figure, we have not respected the proportions (the areas of $\widetilde{T}_i$ and $\widehat{T}_i$ should coincide).

Note that for $i = 1, 2$, the abcissa of $G_i(x_1, x_2)$ only depends on x_1. This implies that for $i = 1, 2$, G_i maps part of $\partial\widehat{Y}^0$ to vertical lines. Moreover, $G_1(P_2) = G_2(P_1)$ is a point on the axis $x_1 = 0$. We define

$$\widehat{\mathcal{M}}_\sigma = G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n,$$

the sets

$$\widehat{Y}^N = \text{Interior}\left(\overline{\widehat{Y}^0} \cup \bigcup_{n=1}^{N} \bigcup_{\sigma\in\mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\overline{\widehat{Y}^0})\right), \qquad \widehat{\Omega} = \text{Interior}\left(\overline{\widehat{Y}^0} \cup \bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\overline{\widehat{Y}^0})\right),$$

see Figure 5, and the one to one mapping

$$\chi^N : \left| \begin{array}{rcl} \widehat{Y}^N & \to & \widetilde{Y}^N, \\ x & \mapsto & \mathcal{M}_\sigma \circ H^{-1} \circ \widehat{\mathcal{M}}_\sigma^{-1}(x) \quad \text{if } x \in \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0). \end{array} \right.$$

Note that χ^N is a piecewise affine map and that the Jacobian of χ^N is almost everywhere 1. Moreover, take $\sigma \in \mathcal{A}_n$ with $n \le N$, $(x_1, x_2) \in \widehat{\mathcal{M}}_\sigma(Y^0)$ and $h \in \mathbb{R}$ such that $(x_1, x_2 + h) \in \widehat{\mathcal{M}}_\sigma(Y^0)$. We aim at bounding $|\chi^N(x_1, x_2+h) - \chi^N(x_1, x_2)|$: call $(z_1, z_2) = \widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2)$. It can be easily seen that $\widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2 + h) = (z_1, z_2 + (2a^2)^{-n}h)$. Therefore,

$$\begin{aligned} |\chi^N(x_1, x_2 + h) - \chi^N(x_1, x_2)| &= |\mathcal{M}_\sigma \circ H^{-1}(z_1, z_2 + (2a^2)^{-n}h) - \mathcal{M}_\sigma \circ H^{-1}(z_1, z_2)| \\ &\le C_H a^n (2a^2)^{-n}|h| = C_H(2a)^{-n}|h|, \end{aligned}$$

where the constant C_H is the norm of H^{-1}, and where we have used the fact that $\mathcal{M}_\sigma$ is a similitude with dilation ratio a^n. But $2a > 1$. Passing to the limit as h tends to 0, we see that

$$\|\frac{\partial\chi^N}{\partial x_2}\|_\infty \le C_H. \tag{44}$$

Note that $\widehat{Y^N}$ is contained in the rectangle $[-1, 1] \times [0, \xi]$, where $\xi = r_2 \sum_{n=0}^{\infty}(2a^2)^n = \frac{r_2}{1-2a^2}$. Moreover $\widehat{Y^N}$ has $I^N = 2 + \sum_{n=0}^{N} 2^n$ vertical boundaries, (among which $\sum_{n=0}^{N} 2^n$ vertical fractures) see Figure 5. We order increasingly the abscissa $(\alpha_i)_{i=1,\dots,I^N}$ of these vertical segments, and we have $\alpha_{i+1} - \alpha_i = 2^{-N}$, $i = 1, \dots, I^N - 1$. Notice also that $\widehat{Y}^N$ can be seen as the epigraph of a function $\Phi^N : (-1, 1) \mapsto \mathbb{R}_+$, and that Φ^N is discontinuous at α_i, $i = 2, \dots, I^N - 1$, and linear in the intervals (α_i, α_{i+1}), $i = 1, \dots, I^N - 1$. Another important and natural property is

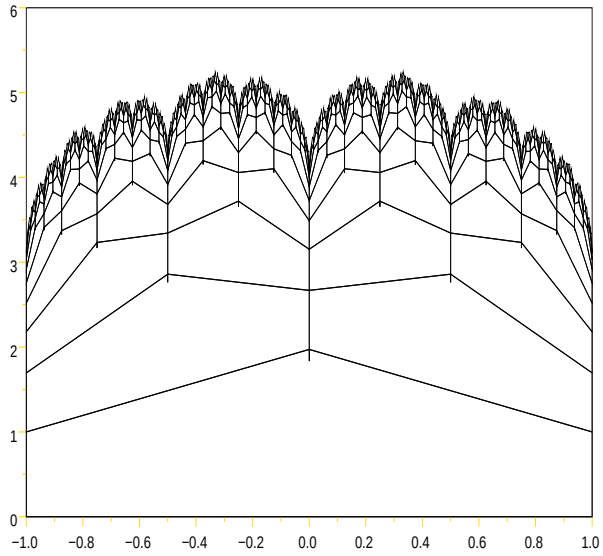

Figure 5: The open set $\widehat{\Omega}$ in the case when $\theta = \pi/4$, $a = a^* \approx 0.593465$, $\alpha = 1 - a\cos\theta$ and $\beta = 1 + a\sin\theta$.

that the sequence $(\Phi^N)_N$ is nondecreasing with respect to N.
Call $\widehat{\Gamma}^\infty$ the self-similar set associated to G_1 and G_2, ($\widehat{\Gamma}^\infty$ is part of the boundary of $\widehat{\Omega}$), and $\widehat{\mu}$ the self-similar measure associated to $(\widehat{\Gamma}^\infty, G_1, G_2)$. As above, it is possible to define a trace operator $\widehat{\ell}^\infty$ from $W^{1,q}(\widehat{\Omega})$ to $L^q(\widehat{\Gamma}^\infty)$.
Consider a function $v \in C^\infty(\Omega)$ such that $v|_{\Gamma^0} = 0$. Since χ^N is piecewise affine and continuous, and since $|\widetilde{\Gamma}^{N+1}| = 2^{N+1}|\widetilde{\Gamma}^\sigma|$, for all $\sigma \in \mathcal{A}_{N+1}$,

$$\frac{1}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q = \sum_{\sigma\in\mathcal{A}_{N+1}}\frac{1}{2^{N+1}\left|\widetilde{\Gamma}^\sigma\right|}\int_{\widetilde{\Gamma}^\sigma}|v|^q = \sum_{\sigma\in\mathcal{A}_{N+1}}\frac{1}{2^{N+1}\left|\widehat{\Gamma}^\sigma\right|}\int_{\widehat{\Gamma}^\sigma}|v\circ\chi^N|^q.$$

On the other hand,

$$\sum_{\sigma\in\mathcal{A}_{N+1}}\frac{1}{2^{N+1}\left|\widehat{\Gamma}^\sigma\right|}\int_{\widehat{\Gamma}^\sigma}|v\circ\chi^N|^q = \frac{1}{2}\sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left|v\circ\chi^N(x,\Phi^N(x))\right|^q\,dx.$$

Therefore,

$$\begin{aligned}\frac{2}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q &= \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left|v(\chi^N(x,\Phi^N(x)))\right|^q dx = \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left|\int_0^{\Phi^N(x)}\frac{d}{dt}(v(\chi^N(x,t)))\,dt\right|^q dx\\ &= \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left|\sum_{j=0}^{N}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\frac{d}{dt}(v(\chi^N(x,t)))\,dt\right|^q dx.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\frac{2}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q\\
&\leq \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{\frac{j}{q-1}}\right)^{q-1}\left(\sum_{j=0}^{N}\rho^{-j}\left|\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\frac{d}{dt}(v(\chi^N(x,t)))\,dt\right|^q\right)dx\\
&\lesssim \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\sum_{j=0}^{N}\rho^{-j}\left|\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\frac{d}{dt}(v(\chi^N(x,t)))\,dt\right|^q dx\\
&\leq \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{-j}(\Phi^j(x)-\Phi^{j-1}(x))^{q-1}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\left|\frac{d}{dt}(v(\chi^N(x,t)))\right|^q dt\right)dx\\
&\lesssim \sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{-j}(2a^2)^{j(q-1)}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\left|\frac{d}{dt}(v(\chi^N(x,t)))\right|^q dt\right)dx
\end{aligned}
$$

where we have used several Hölder's inequalities. This yields that

$$
\begin{aligned}
&\frac{2}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q\\
\lesssim &\left(\begin{array}{l}\displaystyle\sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{-j}(2a^2)^{j(q-1)}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\left|\frac{\partial v}{\partial x_1}(\chi^N(x,t))\frac{\partial\chi_1^N}{\partial t}(x,t)\right|^q dt\right)dx\\ \displaystyle+\sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{-j}(2a^2)^{j(q-1)}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\left|\frac{\partial v}{\partial x_2}(\chi^N(x,t))\frac{\partial\chi_2^N}{\partial t}(x,t)\right|^q dt\right)dx\end{array}\right).
\end{aligned}
$$

Using (44),

$$
\frac{1}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q\lesssim\sum_{i=0}^{I_N-1}\int_{\alpha_i}^{\alpha_{i+1}}\left(\sum_{j=0}^{N}\rho^{-j}(2a^2)^{j(q-1)}\int_{\Phi^{j-1}(x)}^{\Phi^j(x)}\left|\nabla v(\chi^N(x,t))\right|^q dt\right)dx.
$$

Performing the inverse change of variables, one obtains that

$$
\frac{1}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q\leq C\left(\int_{\widetilde{Y}^0}|\nabla v|^q+\sum_{n=1}^{N}\rho^{-n}(2a^2)^{n(q-1)}\sum_{\sigma\in\mathcal{A}_n}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla v|^q\right),
$$

where the constant C can be chosen independently of N.
Finally, since $\frac{1}{\left|\widetilde{\Gamma}^{N+1}\right|}\int_{\widetilde{\Gamma}^{N+1}}|v|^q$ tends to $\int_{\widetilde{\Gamma}^\infty}|\widetilde{\ell^\infty}(v)|^q d\widetilde{\mu}$ as $N\to\infty$, we find that

$$
\int_{\widetilde{\Gamma}^\infty}|\widetilde{\ell^\infty}(v)|^q d\widetilde{\mu}\leq C\left(\int_{\widetilde{Y}^0}|\nabla v|^q+\sum_{n=1}^{\infty}\rho^{-n}(2a^2)^{n(q-1)}\sum_{\sigma\in\mathcal{A}_n}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla v|^q\right),\tag{45}
$$

which is the desired result. We conclude by using the density of $\{v\in\mathcal{C}^\infty(\Omega);\ v|_{\Gamma^0}=0\}$ in $\mathcal{V}_q(\Omega)$.

Step 2 We aim at proving (42) for $\widetilde{\Omega}$ by using Lemma 3.
Let V_2 be the Banach space

$$V_2 = \left\{ v : \widetilde{\Omega} \to \mathbb{R}^2,\ v \text{ measurable } \ \|v\|^q_{L^q(\widetilde{Y}^0)} + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|v\|^q_{L^q(\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0))} < \infty \right\},$$

and let V be the Banach space

$$V = \left\{ v \in L^q_{loc}(\widetilde{\Omega}),\ v|_{\widetilde{Y}^0} \in L^q(\widetilde{Y}^0),\ \nabla v \in V_2 \right\},$$

endowed with the norm

$$\|v\|_V = \left(\left| \int_{\widetilde{\Gamma}_0} v|_{\widetilde{\Gamma}^0} \right|^q + \|\nabla v\|^q_{L^q(\widetilde{Y}^0)} + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|^q_{L^q(\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0))} \right)^{1/q}.$$

Remark 10 *It can be proved that*

$$V = \left\{ v \in L^q(\widetilde{\Omega}),\ \nabla v \in V_2 \right\}.$$

It is clear from (40) that $W^{1,q}(\widetilde{\Omega})$ is a subspace of V.
Furthermore, we will prove at the end of the paragraph that $W^{1,q}(\widetilde{\Omega})$ is dense in V.
As a consequence of (41), one can extend the operator $\widetilde{\ell}^\infty$ to a continuous linear operator from V to $W = L^q_{\widetilde{\mu}} = L^q(\widetilde{\Gamma}^\infty, d\widetilde{\mu})$, that we still denote $\widetilde{\ell}^\infty$.
Let $\widetilde{P}_0$ be the projection on constants: $\widetilde{P}_0 v = \int_{\widetilde{\Gamma}^\infty} v d\widetilde{\mu}$. It is straigthforward to see that the operator $L :\ v \mapsto \widetilde{\ell}^\infty(v) - \widetilde{P}_0 \left(\widetilde{\ell}^\infty(v) \right)$ is also a continuous linear operator from V to $W = L^q_{\widetilde{\mu}}$.
Define V_1 as the space of the constant functions on $\widetilde{\Omega}$. We introduce the linear operators

$$A_1 : V \mapsto \mathbb{R}, \quad A_1 v = \int_{\widetilde{\Gamma}_0} v, \qquad \text{and} \qquad A_2 : V \mapsto V_2, \quad A_2 v = \nabla v.$$

We can apply Lemma 3 since $L\big|_{\ker A_2} \equiv 0$. Therefore, there exists a constant $C > 0$ such that

$$\|\widetilde{\ell}^\infty(v) - \widetilde{P}_0 \left(\widetilde{\ell}^\infty(v) \right)\|^q_{L^q_{\widetilde{\mu}}} \le C \left(\|\nabla v\|^q_{L^q(\widetilde{Y}^0)} + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|^q_{L^q(\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0))} \right), \qquad \forall v \in V, \tag{46}$$

which implies (42) in the case when $\Omega = \widetilde{\Omega}$, since $W^{1,q}(\widetilde{\Omega}) \subset V$.

Proof of the density of $W^{1,q}(\widetilde{\Omega})$ in V. We introduce a smooth real valued function ϕ defined on $\widetilde{Y}^0$ such that

$$\begin{cases} 0 \le \phi \le 1, & \text{in } \widetilde{Y}^0, \\ \phi = 1, & \text{on } \widetilde{\Gamma}^0, \\ \phi = 0, & \text{on } \widetilde{\Sigma}^0, \end{cases}$$

where

$$\widetilde{\Sigma}^0 = \{ x \in \partial \widetilde{Y}^0 \text{ such that } x_2 > b \}.$$

For $n > 0$ and $\sigma \in \mathcal{A}_n$, we also introduce the function ϕ_σ defined on $\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)$ by $\phi_\sigma = \phi \circ (\widetilde{\mathcal{M}}_\sigma)^{-1}$.
For a function v in V, we define v_n by:

$$\begin{cases} v_n|_{\widetilde{Y}^{n-1}} = v|_{\widetilde{Y}^{n-1}}, \\ v_n|_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} = \phi_\sigma v|_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} + (1 - \phi_\sigma) \langle v \rangle_\sigma, \quad \forall \sigma \in \mathcal{A}_n, \\ v_n|_{\widetilde{\Omega}^\eta} = \langle v \rangle_\sigma, \quad \text{for} \quad \eta = \sigma \circ \widetilde{F}_i,\ \sigma \in \mathcal{A}_n,\ i = 1, 2, \end{cases}$$

where

$$\langle v\rangle_\sigma = \frac{1}{|\widetilde{\Sigma}\sigma|}\int_{\widetilde{\Sigma}\sigma} v \quad \text{and} \quad \widetilde{\Sigma}^\sigma = \widetilde{\mathcal{M}}_\sigma(\widetilde{\Sigma}^0).$$

It is easy to see that $u_n \in W^{1,q}(\widetilde{\Omega})$. Morover, $\lim_{n\to\infty}\|u-u_n\|_V = 0$; indeed,

$$\nabla u - \nabla u_n = \begin{cases} 0 & \text{in } \widetilde{Y}^{n-1}, \\ \phi_\sigma \nabla u + (u - \langle u\rangle_\sigma)\nabla\phi_\sigma & \text{in } \widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0),\ \forall\sigma\in\mathcal{A}_n, \\ \nabla u & \text{in } \widetilde{\Omega}\backslash\widetilde{Y}^n. \end{cases}$$

But $\|\nabla\phi_\sigma\|_\infty = a^{-n}\|\nabla\phi\|_\infty$. Therefore

$$\begin{aligned}\|u-u_n\|_V^q &\le c_q \left(\begin{array}{l} \displaystyle\sum_{p\ge n}\rho^{-p}(2a^2)^{p(q-1)}\sum_{\sigma\in\mathcal{A}_p}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla u|^q \\ \displaystyle +\rho^{-n}(2a^2)^{n(q-1)}\|\nabla\phi\|_\infty^q\sum_{\sigma\in\mathcal{A}_n}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} a^{-nq}\,|u-\langle u\rangle_\sigma|^q \end{array}\right) \\ &\le c_q\left(\sum_{p\ge n}\rho^{-p}(2a^2)^{p(q-1)}\sum_{\sigma\in\mathcal{A}_p}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla u|^q + C_q\rho^{-p}(2a^2)^{n(q-1)}\|\nabla\phi\|_\infty^q\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla u|^q\right),\end{aligned}$$

using the Poincaré-Wiertinger inequality in $\widetilde{\mathcal{M}}(\widetilde{Y}^0)$:

$$\int_{\widetilde{\mathcal{M}}(\widetilde{Y}^0)}|u-\langle u\rangle|^q \le C_q a^{nq}\int_{\widetilde{\mathcal{M}}(\widetilde{Y}^0)}|\nabla u|^q,$$

and where the constants c_q and C_q do not depend on n. From this and Lebesgue theorem, we see that $\lim_{n\to\infty}\|u-u_n\|_V = 0$.

Step 3 We now aim at proving (41) and (42) in the general case, i.e. for Y^0 satisfying the assumptions made in § 2.1.3. For that, we are going to construct a self-similar piecewise linear map from the domain Ω to $\widetilde{\Omega}$. It is easy to see that there exist

1. a partition $\mathcal{T}$ of Y^0 into p non-overlapping triangles, such that Γ^0, $F_1(\Gamma^0)$, $F_2(\Gamma^0)$ are the whole side of a triangle in the partition.
2. a partition $\widetilde{\mathcal{T}}$ of $\widetilde{Y}^0$ into p non-overlapping triangles, such that $\widetilde{\Gamma}^0$, $\widetilde{F}_1(\widetilde{\Gamma}^0)$, $\widetilde{F}_2(\widetilde{\Gamma}^0)$ are the whole side of a triangle in the partition.
3. a continuous, one to one and piecewise linear function ψ from Y^0 onto $\widetilde{Y}^0$, such that
   - its restriction to the triangles in $\mathcal{T}$ is linear,
   - ψ maps each triangle in $\mathcal{T}$ to a triangle in $\widetilde{\mathcal{T}}$,
   - the restriction of ψ to Γ^0 is the identity,
   - for $i=1,2$,
   $$F_i(\psi^{-1}(x)) = \psi^{-1}(\widetilde{F}_i(x)), \quad \forall x\in\widetilde{\Gamma}^0.$$

An example of such a construction is shown in Figure 6.
This construction allows for the definition of the continuous linear operator

$$\Psi : W^{1,q}(\Omega)\to W^{1,q}(\widetilde{\Omega}),$$

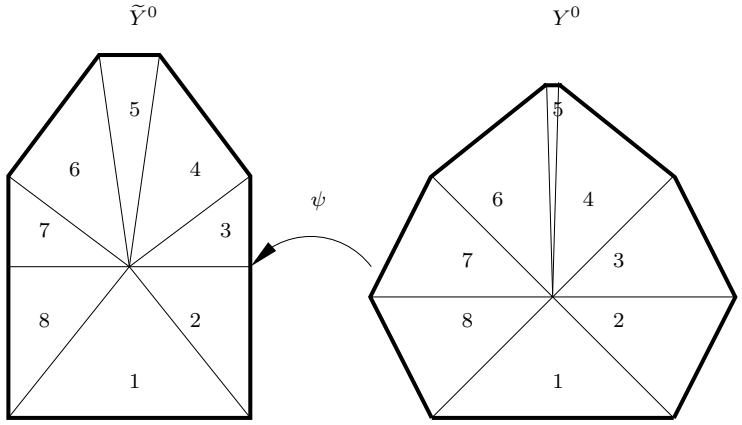

Figure 6: The domains Y^0 and $\widetilde{Y}^0$ are partitioned into eight triangles

$$\Psi(u)=\widetilde{u},\quad \widetilde{u}\text{ defined by }\left\{\begin{array}{rcll}\widetilde{u}|_{\widetilde{Y}^0} &=& u|_{Y^0}\circ\psi^{-1},&\\ \widetilde{u}|_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} &=& u|_{\mathcal{M}_\sigma(Y^0)}\circ\mathcal{M}_\sigma\circ\psi^{-1}\circ(\widetilde{\mathcal{M}}_\sigma)^{-1}, & \sigma\in\mathcal{A}_n,\ n\geq 1.\end{array}\right. \tag{47}$$

Using the same argument as in the proof of Theorem 4 in [2], we have that

$$\|\ell^\infty(v)\|_{L^q_\mu}\lesssim\|\widetilde{\ell}^\infty(\Psi(v))\|_{L^q_\mu},\quad\forall v\in\mathcal{V}_q(\Omega). \tag{48}$$

From (48) and (45), we see that

$$\|\ell^\infty(v)\|^q_{L^q_\mu}\lesssim\int_{\widetilde{Y}^0}|\nabla(\Psi(v))|^q+\sum_{n=1}^\infty\rho^{-n}(2a^2)^{n(q-1)}\sum_{\sigma\in\mathcal{A}_n}\int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)}|\nabla(\Psi(v))|^q,\quad\forall v\in\mathcal{V}_q(\Omega). \tag{49}$$

The inverse change of variables yields (41).
Similarly, the proof of (42) follows from the observation that $P_0(\ell^\infty(v))=\widetilde{P}_0(\widetilde{\ell}^\infty(\Psi(v)))$ and from (46).

The case $a=1/2$. See Appendix B. ■

Proof of Theorem 10 Call $f=\ell^\infty(v)$. Recall that

$$\|f\|^q_{JLip(1-\frac{2-d}{q},q,q;0;\Gamma^\infty)}=\|f\|^q_{L^q_\mu}+\int_{\Gamma^\infty}|f-P_0f|^qd\mu+\sum_{n=1}^\infty\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}\int_{\mathcal{M}_\sigma(\Gamma^\infty)}|f-P_\sigma f|^qd\mu. \tag{50}$$

We have to prove that the right side of (50) is finite. A change of variables leads to

$$\begin{aligned}&\sum_{n=1}^\infty\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}\int_{\mathcal{M}_\sigma(\Gamma^\infty)}|f-P_\sigma f|^qd\mu\\ =&\sum_{n=1}^\infty\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}2^{-n}\int_{\Gamma^\infty}|f\circ\mathcal{M}_\sigma-P_0(f\circ\mathcal{M}_\sigma)|^qd\mu.\end{aligned}$$

This and Theorem 11 imply that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}\int_{\mathcal{M}_\sigma(\Gamma^\infty)}|f-P_\sigma f|^q d\mu\\
\lesssim&\sum_{n=1}^{\infty}a^{n(2-q)}\sum_{\sigma\in\mathcal{A}_n}\left(\|\nabla(v\circ\mathcal{M}_\sigma)\|^q_{L^q(Y^0)}+\sum_{p=1}^{\infty}\rho^{-p}(2a^2)^{p(q-1)}\sum_{\eta\in\mathcal{A}_p}\|\nabla(v\circ\mathcal{M}_\sigma)\|^q_{L^q(\mathcal{M}_\eta(Y^0))}\right)\\
=&\sum_{n=1}^{\infty}a^{n(2-q)}\sum_{\sigma\in\mathcal{A}_n}\left(a^{n(q-2)}\left(\|\nabla v\|^q_{L^q(\mathcal{M}_\sigma(Y^0))}+\sum_{p=1}^{\infty}\rho^{-p}(2a^2)^{p(q-1)}\sum_{\eta\in\mathcal{A}_p}\|\nabla v\|^q_{L^q(\mathcal{M}_{\sigma\circ\eta}(Y^0))}\right)\right)
\end{aligned}$$

where ρ is such that $(2a^2)^{q-1}<\rho<1$. Thus,

$$\sum_{n=1}^{\infty}\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}\int_{\mathcal{M}_\sigma(\Gamma^\infty)}|f-P_\sigma f|^q d\mu\lesssim\sum_{i=1}^{\infty}\rho^{-i}(2a^2)^{i(q-1)}\left(\sum_{n=1}^{i}\rho^n(2a^2)^{n(1-q)}\right)\sum_{\mu\in\mathcal{A}_i}\|\nabla v\|^q_{L^q(\mathcal{M}_\mu(Y^0))}.$$

From $\rho(2a^2)^{1-q}>1$, we deduce that $\sum_{n=1}^{i}\rho^n(2a^2)^{n(1-q)}\lesssim\rho^i(2a^2)^{i(1-q)}$. Thus,

$$\sum_{n=1}^{\infty}\left(2a^{2-q}\right)^n\sum_{\sigma\in\mathcal{A}_n}\int_{\mathcal{M}_\sigma(\Gamma^\infty)}|f-P_\sigma f|^q d\mu\lesssim\sum_{i=1}^{\infty}\sum_{\mu\in\mathcal{A}_i}\|\nabla v\|^q_{L^q(\mathcal{M}_\mu(Y^0))},$$

and the desired result follows easily. ■

6 Traces of functions belonging to $W^{m+1,q}(\Omega)$ for a positive integer m.

We now assume that $a\geq 1/2$ and that Γ^∞ is not contained in a straight line. We use the notations defined in § 4.2. To keep the length of the paper reasonable, some proofs will be omitted or just sketched. For any real number $q\geq 1$ and any nonnegative integer m, the space $W^{m+1,q}(\Omega)$ is the space of the functions in $L^q(\Omega)$ such that all their partial derivatives of order less than or equal to $m+1$ (in the sense of distribution) belong to $L^q(\Omega)$.

Proposition 6 *For all real number $q\geq 1$, there exists a constant C such that for all $f\in JLip(m+1-\frac{2-d}{q},q,q;m;\Gamma^\infty)$, there exists $\tilde{f}\in W^{m+1,q}(\Omega)$ with the properties $\tilde{f}|_{Y^0}=0$, $\ell^\infty(\tilde{f})=f$ and*

$$\|\tilde{f}\|^q_{W^{m+1,q}(\Omega)}\approx\int_\Omega|D^{m+1}\tilde{f}|^q\leq C\|f\|^q_{JLip(m+1-\frac{2-d}{q},q,q;m;\Gamma^\infty)}.\tag{51}$$

Proof. The proof resembles very much that of Proposition 5. It mainly consists of constructing functions $\tilde{g}^{0,i}\in W^{m+1,q}(\Omega)$, $i=1,\dots M_0$ such that

- $\tilde{g}^{0,i}|_{Y^0}=0$,
- $\tilde{g}^{0,i}|_{\mathcal{M}_\sigma(\Omega)}$ belongs to $\mathcal{P}_m$, for all $\sigma\in\mathcal{A}_2$,
- $\ell^\infty(\tilde{g}^{0,i})=g^{0,i}$.

For a positive integer n and $\sigma \in \mathcal{A}_n$, let us define the function $\tilde{g}^{\sigma,i}$ by:

$$\tilde{g}^{\sigma,i}|_{\Omega^\sigma} = 2^{n/2}\ \tilde{g}^{0,i} \circ \mathcal{M}_\sigma^{-1}, \tag{52}$$

$$\tilde{g}^{\sigma,i}|_{\Omega\setminus\Omega^\sigma} = 0. \tag{53}$$

From the definition of $g^{\sigma,i}$, we see that $\ell^\infty(\tilde{g}^{\sigma,i}) = g^{\sigma,i}$. For a fixed $i \in \{1,\dots,M_0\}$,

- the supports of $D^{m+1}\tilde{g}^{\sigma,i}$ and $D^{m+1}\tilde{g}^{0,i}$ are disjoint,
- if $(n,\sigma) \neq (n',\eta)$ then the supports of $D^{m+1}\tilde{g}^{\sigma,i}$ and $D^{m+1}\tilde{g}^{\eta,i}$ are disjoint.

Consider also M_0 functions $\tilde{\phi}_i \in W^{m+1,q}(\Omega)$ such that

- $\tilde{\phi}^i|_{Y^0} = 0$,
- $\ell^\infty(\tilde{\phi}^i) = \phi^i$.

For a funtion $f \in JLip(m+1-(2-d)/q, q, q; m; \Gamma^\infty)$ given by the expansion (28), it can be shown as in the proof of Proposition 5 that the function $\tilde{f}$ given by

$$\tilde{f} = \sum_{i=1}^{M_0} \lambda_i \tilde{\phi}^i + \sum_{i=1}^{M_0} \beta_{0,i}\tilde{g}^{0,i} + \sum_{i=1}^{M_0}\sum_{n\geq 1}\sum_{\sigma\in\mathcal{A}_n} \beta_{n,\sigma,i}\tilde{g}^{\sigma,i}$$

has the desired properties. ■
We have the analogue of Theorem 11:

Theorem 12 *For all q, $1 < q < \infty$ and all real number ρ satisfying (40), there exists a constant C such that, for all $v \in W^{1,q}(\Omega)$,*

$$\|\ell^\infty(v) - P_0^{(m)}\left(\ell^\infty(v)\right)\|_{L^q_\mu}^q \leq C \left(\begin{array}{l} \|D^{m+1}v\|^q_{L^q(Y^0)} \\ +\displaystyle\sum_{n=1}^{\infty} \rho^{-n(m+1)}(2a^2)^{n((m+1)q-1)} \sum_{\sigma\in\mathcal{A}_n} \|D^{m+1}v\|^q_{L^q(\mathcal{M}_\sigma(Y^0))} \end{array} \right). \tag{54}$$

Proof. We skip the proof, because it is long and resembles that of Theorem 11. ■
Theorem 12 makes it possible to prove the analogue of Theorem 10:

Theorem 13 *For all q, $1 < q < \infty$ and $v \in W^{m+1,q}(\Omega)$, $\ell^\infty(v)$ belongs to $JLip(m+1-\frac{2-d}{q}, q, q; m; \Gamma^\infty)$ and there exists a constant c such that*

$$\|\ell^\infty(v)\|_{JLip(m+1-\frac{2-d}{q},q,q;m;\Gamma^\infty)} \leq c\|v\|_{W^{m+1,q}(\Omega)}, \quad \forall v \in W^{m+1,q}(\Omega). \tag{55}$$

Proof. The proof is very similar to that of Theorem 10. ■
From Proposition 6 and Theorem 13, we obtain that for all q, $1 < q < \infty$,

$$\ell^\infty\left(W^{m+1,q}(\Omega)\right) = JLip(m+1-\frac{2-d}{q}, q, q; m; \Gamma^\infty). \tag{56}$$

Appendix

A Sketched proofs of Theorems 2 and 3

A.1 Preliminary lemmas

Lemma 4 *For $0 < \theta < \pi/2$, assume that (α, β) satisfy (5) for $a = a^*$ and (6). Then, for all a, $0 \leq a < a^*$, the convex hull of $\left(\overline{\widetilde{Y}^0} \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{\widetilde{Y}^0})\right)\right)$ coincides with the convex hull of $\{P_1, P_2\} \cup \left(\bigcup_{\sigma \in \mathcal{A}_{N+1}} \{\mathcal{M}_\sigma(P_1), \mathcal{M}_\sigma(P_2)\}\right)$.*

Proof. First step Call χ^N the convex hull of $\overline{\widetilde{Y}^0} \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{\widetilde{Y}^0})\right)$.
We first observe that the points $F_1(P_1)$, $F_1(P_2)$, $F_2(P_1)$, and $F_2(P_2)$ are strictly contained in χ^1: this can be checked by simple geometric arguments: for example, one sees that the angle between the line segments $[P_2, F_2(P_2)]$ and $F_2([P_2, F_2(P_2)])$ is $\pi - \theta < \pi$, from which we deduce that $F_2(P_2)$ is strictly contained in χ^1. From this, we deduce that the points $F_1(P_1)$, $F_1(P_2)$, $F_2(P_1)$, and $F_2(P_2)$ are strictly contained in χ^N for all $N \geq 1$, because $\chi^1 \subset \chi^N$.
Second step We prove Lemma 4 by induction. We have just seen that the claim is true for $N = 1$. Assume that the claim is true up to N. We have

$$\chi^{N+1} = \text{convex hull}\left(\overline{\widetilde{Y}^0} \cup F_1(\chi^N) \cup F_2(\chi^N)\right).$$

From the induction hypothesis, we deduce

$$\chi^{N+1} = \text{convex hull}\left(\{P_0, P_1, F_1(P_1), F_1(P_2), F_2(P_1), F_2(P_2)\} \cup \left(\bigcup_{\sigma \in \mathcal{A}_{N+2}} \{\mathcal{M}_\sigma(P_1), \mathcal{M}_\sigma(P_2)\}\right)\right).$$

But we know that $F_1(P_1)$, $F_1(P_2)$, $F_2(P_1)$, and $F_2(P_2)$ are strictly contained in χ^{N+1}: therefore,

$$\chi^{N+1} = \text{convex hull}\left(\{P_0, P_1\} \cup \bigcup_{\sigma \in \mathcal{A}_{N+2}} \{\mathcal{M}_\sigma(P_1), \mathcal{M}_\sigma(P_2)\}\right).$$

■

By passing to the limit as $N \to \infty$, we obtain that:

Lemma 5 *Assume that (α, β) satisfy (5) for $a = a^*$ and (6). Then, for all a, $0 \leq a < a^*$, the convex hull of $\overline{\widetilde{\Omega}}$ is the convex hull of $\{P_1, P_2\} \cup \Gamma^\infty$.*

A.2 Proof of Theorem 2

With an argument similar to those used in [16] and elementary calculus, one can prove:

Lemma 6 *For θ, $0 < \theta < \pi/2$, call $(F_2 \circ F_1)^\infty$ the fixed point of $F_2 \circ F_1$ and ζ the smallest integer such that $\zeta\theta \geq \pi$. The point $F_1^\zeta(F_2 \circ F_1)^\infty$ is a point in Γ^∞ with minimal ordinate. Its ordinate Z takes the value*

$$Z = \beta - \alpha\frac{\cos\theta}{\sin\theta} + \frac{1}{1-a^2}\left(\beta a - \frac{\alpha(1 - a\cos\theta)}{\sin\theta}\right)\left(\sum_{i=1}^{\zeta} a^{i-1}\cos i\theta - a^2\sum_{i=1}^{\zeta-2} a^{i-1}\cos i\theta\right). \quad (57)$$

Lemma 7 *For θ, $0 < \theta < \pi/2$, for any $\alpha > a^* \cos\theta$, there exists $\bar\beta > 0$ such that $\bar\beta > a^* \sin\theta$ and $(\alpha - 1)\sin\theta + \bar\beta\cos\theta \geq 0$ and for any $\beta \geq \bar\beta$, Γ^∞ is contained in the half-plane $\{x : x_2 > 0\}$ for all a, $0 < a \leq a^*$.*

Proof. From (57), we see that Z is affine with respect to β. Thus, Lemma 7 will be proved if show that for a positive constant c depending on θ but not on a, $0 \leq a < a^*$,

$$1 + \frac{a}{1-a^2}\left(\sum_{i=1}^{\zeta} a^{i-1}\cos i\theta - a^2\sum_{i=1}^{\zeta-2} a^{i-1}\cos i\theta\right) \geq c,$$

or in an equivalent manner

$$\frac{1}{1-a^2}\left(\sum_{i=0}^{\zeta} a^{i}\cos i\theta - \sum_{i=0}^{\zeta-2} a^{i+2}\cos i\theta\right) \geq c. \tag{58}$$

Since a^* is the positive solution of (4), $0 \leq a < a^*$ implies that $-\sum_{i=0}^{\kappa-1} a^{i+2}\cos i\theta \geq -1/2$. Since $\zeta \geq \kappa + 1$, and since $a^* \leq 1/\sqrt{2}$, we see that a sufficient condition for (58) is

$$\frac{1}{2} + \sum_{i=1}^{\zeta} a^{i}\cos i\theta - a^2\sum_{i=\kappa}^{\zeta-2} a^{i}\cos i\theta \geq c, \tag{59}$$

for a different constant c. We make out two cases:
a) the case when $(2\kappa-1) < \pi$ which implies $\zeta = 2\kappa$ and $\theta < \pi/3$; b) the case when $(2\kappa-1)\theta \geq \pi$ which implies $\zeta = 2\kappa - 1$; for technical reasons, we split the case b) into two subcases: b1) $\kappa = 2$ and $\zeta = 3$, which occurs for $\pi/3 \leq \theta < \pi/2$; b2) $\kappa \geq 3$ so $\theta < \pi/4$.

a) $(2\kappa - 1)\theta < \pi$**.** In this case, $0 < \theta < \pi/3$ and $\zeta = 2\kappa$. We have

$$\frac{1}{2} + \sum_{i=1}^{\zeta} a^{i}\cos i\theta - a^2\sum_{i=\kappa}^{\zeta-2} a^{i}\cos i\theta = T_0 + \sum_{i=1}^{\kappa-1} T_i + S,$$

where $S = -a^2\sum_{i=\kappa}^{\zeta-2} a^i\cos i\theta \geq 0$, $T_i = a^i\cos i\theta + a^{2\kappa-i-1}\cos((2\kappa-i-1)\theta)$, $i = 1, \ldots, \kappa-1$, and $T_0 = 1/2 + a^{2\kappa-1}\cos((2\kappa-1)\theta) + a^{2\kappa}\cos(2\kappa\theta)$. One sees that for $i = 1, \ldots, \kappa-1$, $-\cos((2\kappa - i - 1)\theta) \leq \cos i\theta$, so $T_i \geq 0$. On the other hand $T_0 \geq 1/2 - a^{2\kappa-1} - a^{2\kappa} \geq 1/2 - (a^*)^3 - (a^*)^4$ and since $a^*(\theta) < a^*(\pi/3) \sim 0.62$, $1/2 - (a^*)^3 - (a^*)^4 \geq c > 0$.

b1) $\kappa = 2$ **and** $\zeta = 3$**.** In this case, $\pi/3 \leq \theta < \pi/2$ and the left hand side of (59) takes the simple form $1/2(1 + 2a\cos\theta + 2a^2\cos 2\theta + 2a^3\cos 3\theta)$. We have

$$\begin{aligned}1 + 2a\cos\theta + 2a^2\cos 2\theta + 2a^3\cos 3\theta &= 1 + 2a\cos\theta + 2a^2\cos 2\theta + 2a^3(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)\\ &\geq 1 + 2a\cos\theta + 2a^2\cos 2\theta + (1 - 2a^2)\cos 2\theta - 2a^3\sin\theta\sin 2\theta,\end{aligned}$$

where the latter estimate comes from the fact that $a \leq a^*$, $2(a^*)^3\cos\theta = 1 - 2(a^*)^2$ and $\cos 2\theta < 0$. Thus,

$$\begin{aligned}1 + 2a\cos\theta + 2a^2\cos 2\theta + 2a^3\cos 3\theta &\geq 1 + 2a\cos\theta + \cos 2\theta - 2a^3\sin\theta\sin 2\theta\\ &= \cos\theta\left(2\cos\theta + 2a - 4a^3\sin^2\theta\right)\\ &= \cos\theta\left(2\cos\theta + 2a(1 - 2a^2) + 4a^3\cos^2\theta\right) \geq c > 0,\end{aligned}$$

because $a \leq a^* < 1/\sqrt{2}$.

b2) $\zeta = 2\kappa - 1$ **and** $\kappa \geq 3$**.** In this case, $0 < \theta < \pi/4$

$$\frac{1}{2} + \sum_{i=1}^{\zeta} a^i \cos i\theta - a^2 \sum_{i=\kappa}^{\zeta-2} a^i \cos i\theta = T_0 + \sum_{i=1}^{\kappa-1} T_i + S,$$

where $S = a^{\kappa-1}\cos((\kappa-1)\theta) - a^2 \sum_{i=\kappa}^{\zeta-2} a^i \cos i\theta \geq 0$, $T_i = a^i \cos i\theta + a^{2\kappa-i-2}\cos((2\kappa - i - 2)\theta)$, $i = 1, \ldots, \kappa - 2$, and $T_0 = 1/2 + a^{2\kappa-2}\cos((2\kappa-2)\theta) + a^{2\kappa-1}\cos((2\kappa-1)\theta)$. One sees that for $i = 1, \ldots, \kappa - 2$, $-\cos((2\kappa - i - 2)\theta) \leq \cos i\theta$, so $T_i \geq 0$. On the other hand $T_0 \geq 1/2 - a^{2\kappa-2} - a^{2\kappa-1} \geq 1/2 - (a^*)^4 - (a^*)^5$ and the latter is greater than $c > 0$. ■

Lemma 8 *For θ, $0 < \theta < \pi/2$, for any $\alpha > a^* \cos\theta$, there exists $\bar{\beta} > 0$ such that $\bar{\beta} > a \sin\theta$ and $(\alpha - 1)\sin\theta + \bar{\beta}\cos\theta \geq 0$ and for any $\beta \geq \bar{\beta}$, $\widetilde{\Omega}$ is contained in the half-plane $\{x : x_2 > 0\}$ for all a, $0 < a \leq a^*$.*

Proof. This is a consequence of Lemmas 7 and 5. ■

Proof of Theorem 2 Take $a \leq a^*$: from Lemma 8, $\widetilde{\Omega}$ lies in the half-plane $\{x : x_2 > 0\}$; by similarity, $F_1(\widetilde{\Omega})$ is contained in the open half-plane above the straight line joining $F_1(P_1)$ and $F_1(P_2)$. Therefore $F_1(\widetilde{\Omega}) \cap \widetilde{Y}^0 = \emptyset$. Similarly, $F_2(\widetilde{\Omega}) \cap \widetilde{Y}^0 = \emptyset$.
Moreover for $a < a^*$, $F_1(\Gamma^\infty)$ is contained in the half-plane $\{x : x_1 < 0\}$ otherwise, by symmetry, $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$ would not be empty. By continuity, $F_1(\Gamma^\infty)$ is contained in the half-plane $\{x : x_1 \leq 0\}$ for $a = a^*$. This implies by Lemma 5 and self-similarity that $F_1(\widetilde{\Omega})$ is contained in the open half-space $\{x : x_1 < 0\}$ for $a \leq a^*$. Similarly, $F_2(\widetilde{\Omega})$ is contained in the open half-space $\{x : x_1 > 0\}$ for $a \leq a^*$. Therefore $F_1(\widetilde{\Omega}) \cap F_2(\widetilde{\Omega}) = \emptyset$ for $a \leq a^*$.
Collecting the above results and using self-similarity, it is easy to check that (α, β) satisfies Assumption 1. ■

A.3 Proof of Theorem 3

We define some useful notations: we note L the axis $\{x : x_1 = 0\}$. By symmetry, the intersections of the lines L, $F_1(L)$ and $F_2(L)$ is a point called A, see Figure 7. We call L_1' (resp. L_2') the symmetric of L with respect to $F_1(L)$ (resp. $F_2(L)$). We call S_1 (resp. S_2) the sector limited by the lines L and L_1' (resp L and L_2'): the vertex of S_1 is A, S_1 is symmetric with respect to $F_1(L)$ and the aperture of S_1 is 2θ. We introduce $S = S_1 \cup S_2$ the sector of vertex A, of aperture 4θ which is symmetric w.r.t. L, see Figure 7.

A.3.1 The case $\pi/6 \leq \theta \leq \pi/4$

If $\pi/6 \leq \theta \leq \pi/4$, a^* is the unique positive solution of the equation

$$-1 + 2X^2 + 2\cos\theta X^3 + 2\cos 2\theta X^4 = 0. \tag{60}$$

Lemma 9 *If F_1 and F_2 are given by (1) with $\alpha = a\cos\theta$ and $\beta = a\sin\theta$, $\pi/6 \leq \theta \leq \pi/4$, then for all a, $0 < a < a^*$, Γ^∞ is contained in the half-plane $\{x : x_2 > 0\}$.*

Proof. (sketched) Here, $A = (0, -a\frac{\cos 2\theta}{\sin\theta})$. Since Γ^∞ is symmetric with respect to L, $F_1(\Gamma^\infty)$ is symmetric with respect to $F_1(L)$. Similarly $F_2(\Gamma^\infty)$ is symmetric with respect to $F_2(L)$. For $a < a^*$, $F_1(\Gamma^\infty)$ lies strictly on side of the axis L (because $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) = \emptyset$). By symmetry with respect to $F_1(L)$, $F_1(\Gamma^\infty)$ is contained in the sector S_1. Similarly, $F_2(\Gamma^\infty)$ is contained in the sector S_2. Therefore $\Gamma^\infty \subset S = S_1 \cup S_2$.
If $\theta = \pi/4$, S is the half-plane $\{x : x_2 > 0\}$ and the proof is finished.

If $\pi/6 \le \theta < \pi/4$, by similarity, $F_1(\Gamma^\infty)$ is contained in $S_1 \cap F_1(S)$. Easy, but long calculations using (60) show that $S_1 \cap F_1(S)$ is contained in the half-plane $\{x : x_2 > 0\}$, see Figure 7. We do not write them explicitly. Similarly $F_2(\Gamma^\infty)$ is contained in $S_1 \cap F_2(S)$, which is also contained in $\{x : x_2 > 0\}$. This concludes the proof.

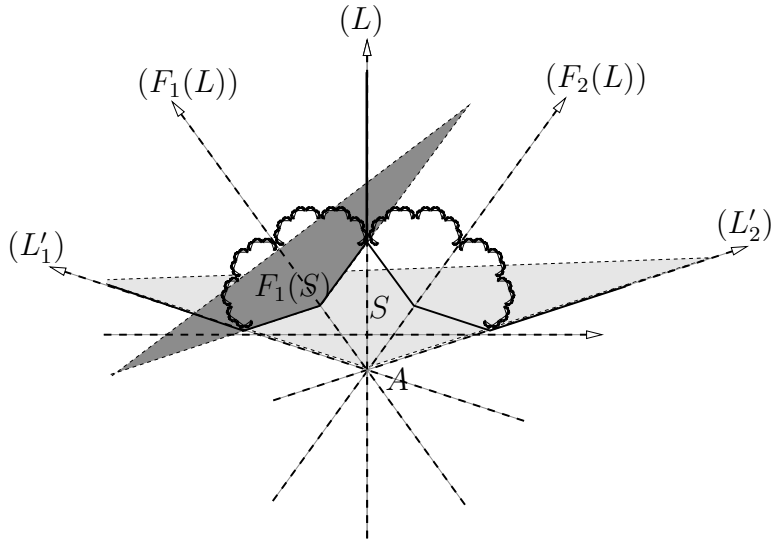

Figure 7: The case $\theta = \pi/5$, $(\alpha, \beta) = (a\cos\theta, a\sin\theta)$ with $a = a^* \approx 0.56658$.

■

Remark 11 *We have checked numerically that Lemma 9 holds for $0 \le \theta \le \pi/6$, but we have not found a short proof of this. It seems that one needs to iterate the argument above $\kappa - 1$ times, where κ was introduced in (4), which leads to intricate algebra.*

Lemma 10 *If F_1 and F_2 are given by (1) with (α, β) satisfying (5) for $a = a^*$ and $\beta\cos\theta - \alpha\sin\theta \ge 0$, then for all a, $0 < a < a^*$, Γ^∞ is contained in the half-plane $\{x : x_2 > 0\}$.*

Proof. This is an immediate consequence of Proposition 1 and of Lemma 9. ■
The proof of Theorem 3 in the case $\pi/6 \le \theta < \pi/4$ follows exactly as that of Theorem 2. ■

A.3.2 The case $\pi/4 \le \theta \le \pi/3$

Lemma 11 *Assume that (α, β) satisfy (5) for $a = a^*$, (6) and (9). Then $\Gamma^\infty \cap \widetilde{Y}^0 = \emptyset$ for $a \le a^*$.*

Proof. (sketched) Since $\Gamma^\infty \subset \overline{F_1(S) \cup F_2(S)}$ if $a \le a^*$, the desired result will be proved if we show that $\widetilde{Y}^0 \cap \overline{F_1(S)} = \emptyset$ because by symmetry this implies that $\widetilde{Y}^0 \cap \overline{F_1(S) \cup F_2(S)} = \emptyset$.
The sector $F_1(S)$ is delimited by the two lines $F_1(L'_2)$ and $F_1(L'_1)$.
The line $F_1(L'_2)$ is parallel to the vector $(\sin\theta, \cos\theta)$ and contains $F_1(A)$. Since $A = (0, \beta - \alpha\cos\theta/\sin\theta) \in \widetilde{Y}^0$ from (9), we also have $F_1(A) \in F_1(\widetilde{Y}^0)$ thus $F_1(A) \notin \overline{\widetilde{Y}^0}$. It can thus be seen that $\widetilde{Y}_0$ is strictly on one side of $F_1(L'_2)$.
Thus to see that $\widetilde{Y}^0 \cap \overline{F_1(S)} = \emptyset$, we just need to prove that all the vertices of $\widetilde{Y}_0$ are strictly contained on one side of $F_1(L'_1)$. The equation of $F_1(L'_1)$ is

$$\cos 3\theta(x_1 + \alpha - a(\alpha\cos\theta - \beta\sin\theta)) + \sin 3\theta\left(x_2 - \beta - a\left(-\frac{\alpha}{\sin\theta} + \alpha\sin\theta + \beta\cos\theta\right)\right) = 0.$$

The desired result will be obtained if injecting $P_1 = (-1, 0)$ and $F_1(P_1) = (-\alpha - a\cos\theta, \beta - a\sin\theta)$ in the equation of $F_1(L_1')$ yields two negative numbers for all a, $0 \le a \le a^*$: this reads

$$\begin{aligned}(\alpha - 1)\cos 3\theta - \beta \sin 3\theta + 2a\cos\theta(\alpha\cos\theta - \beta\sin\theta) &< 0,\\ 2(-\beta\sin\theta + \alpha\cos\theta)\cos\theta - \cos 2\theta &< 0.\end{aligned}$$

and these conditions are implied by (9). This concludes the proof. ■
The proof of Theorem 3 for $\pi/4 \le \theta \le \pi/3$ then follows as that of Theorem 2, by proving that $\widetilde{Y}^0 \cap F_1(\widetilde{\Omega}) = \emptyset$. This comes from the fact that the convex hull of $F_1(\widetilde{\Omega})$ is the convex hull of $F_1(P_1)$, $F_1(P_2)$ and $F_1(\Gamma^\infty)$ by Lemma 5, which, from Lemma 11, is separated from $\widetilde{Y}^0$ by the straight line joining $F_1(P_1)$ and $F_1(P_2)$. ■

B Some details in the proof of Theorem 11 for $a = 1/2$

The proof of Theorem 11 is similar to that given in the case $a > 1/2$, except that the set $\widetilde{\Omega}$ and $\widehat{\Omega}$ are defined in a slightly different manner. For the set $\widetilde{\Omega}$, we choose it as in § 2.2.1 with $\theta = 0$ (see Figure 1), so the similitudes are the affine maps given by (12) with $a = 1/2$. The polygonal cell $\widetilde{Y}^0$ is the convex hull of the points $P_1 = (-1, 0)$, $P_2 = (1, 0)$, $F_2(P_2)$, and $F_1(P_1)$.

Step 1 We limit ourselves to describing the set $\widehat{\Omega}$. It is possible to define a piecewise affine one to one and measure preserving map H from $\widetilde{Y}^0$ onto the fractured set

$$\widehat{Y}^0 = \text{Interior}\left(((-1, 1) \times (0, h)) \setminus \left(\{0\} \times (h', h)\right)\right),$$

where $h = \frac{\beta}{2}\left(\alpha + \frac{3}{2}\right)$, and h' is a real number smaller than h. The map H is such that

- the restriction of H to Γ^0 is the identity,
- for $i = 1, 2$, H induces a one to one map from $F_1(\Gamma^0)$ onto $[-1, 0] \times \{h\}$ and from $F_2(\Gamma^0)$ onto $[0, 1] \times \{h\}$.

The domains $\widetilde{Y}^0$ and $\widehat{Y}^0$ are shown in Figure 8.
In Figure 8, we also show a partition of $\widetilde{Y}^0$ into triangles corresponding to a possible map H with H linear on the triangles. The images of the triangles by H are triangles which form a partition of $\widehat{Y}^0$. The points on the figure are $I = \left(1 + \frac{1}{8}(\alpha - \frac{1}{2}), \frac{\beta}{8}\right)$, $J = \left(0, \frac{2\beta}{3}\right)$, $K = \left(1 + \frac{7}{8}(\alpha - \frac{1}{2}), \frac{7\beta}{8}\right)$, $L = \left(\alpha - \frac{1}{2}, \beta\right)$, $I' = \left(1, \frac{\beta}{8}\right)$, $J' = \left(0, \frac{2\beta}{3}(1 + \frac{1}{8}(\alpha - \frac{1}{2}))\right)$, $K' = \left(1, h - \frac{\beta}{8}\right)$, $M' = \left(0, h - \frac{\beta}{8}(\alpha - \frac{1}{2})\right)$. In this construction, h' is $h' = h - \frac{\beta}{8}(\alpha - \frac{1}{2}) = \beta\left(\frac{3\alpha}{8} + \frac{13}{16}\right)$.

Let us introduce $\widehat{\mathcal{Q}}^0 = ((-1, 1) \times (0, h]) \setminus (\{0\} \times [h', h])$. Let G_1 and G_2 be the maps in $\mathbb{R}^2$ defined by

$$G_1(x_1, x_2) = \left(\frac{1}{2}(x_1 - 1), h + \frac{x_2}{2}\right), \quad G_2(x_1, x_2) = \left(\frac{1}{2}(x_1 + 1), h + \frac{x_2}{2}\right).$$

We define

$$\widehat{\mathcal{M}}_\sigma = G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n,$$

the sets

$$\widehat{Y}^N = \text{Interior}\left(\widehat{\mathcal{Q}}^0 \cup \bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\widehat{\mathcal{Q}}^0)\right), \quad \widehat{\Omega} = \text{Interior}\left(\overline{\widehat{\mathcal{Q}}^0} \cup \bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\widehat{\mathcal{Q}}^0)\right),$$

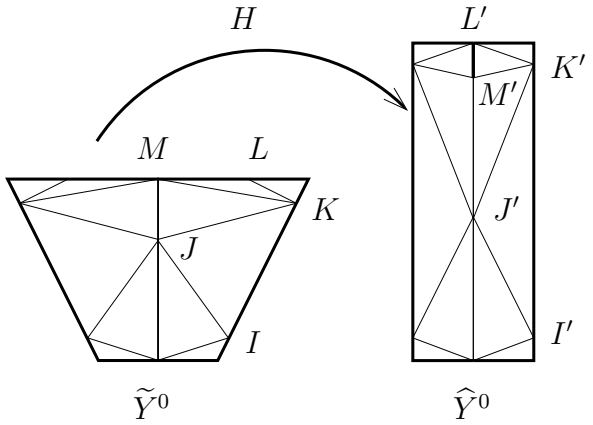

Figure 8: The domains $\widetilde{Y}^0$ (left) and $\widehat{Y}^0$ (right). The restriction of the map H to the triangles covering $\widetilde{Y}^0$ is linear.

see Figure 9, and the one to one mapping

$$\chi^N : \left| \begin{array}{rcl} \widehat{Y}^N & \to & \widetilde{Y}^N, \\ x & \mapsto & \mathcal{M}_\sigma \circ H^{-1} \circ \widehat{\mathcal{M}}_\sigma^{-1}(x) \quad \text{if } x \in \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0). \end{array} \right.$$

The set $\widehat{\Omega}$ is obtained by removing an infinite number of vertical segments from the rectangle $(-1,1)\times(0,2h)$. It is constructed by assembling translated/dilated copies of the set $\widehat{\mathcal{Q}}^0$ displayed on the left of Figure 9. Note that χ^N is a piecewise affine function and that the Jacobian of χ^N

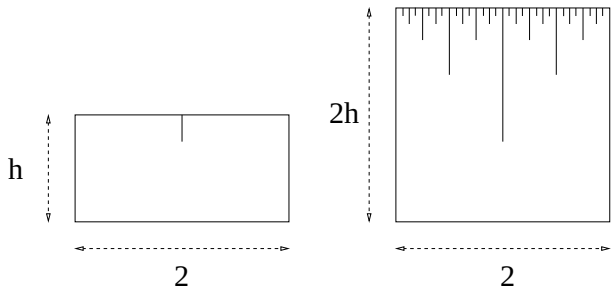

Figure 9: Left: the set $\widehat{\mathcal{Q}}^0$. Right: the open set $\widehat{\Omega}$ (only the longest fractures are displayed)

is almost everywhere 1. Similarly, we can construct a one to one map χ^∞ from $\widehat{\Omega}$ to $\widetilde{\Omega}$, which preserves the measure.

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