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# Trace Theorems for a Class of Ramified Domains with Self-Similar Fractal Boundaries

Yves Achdou \*, Nicoletta Tchou †

January 14, 2009

## Abstract

This work deals with trace theorems for a class of ramified bidimensional domains  $\Omega$  with a self-similar fractal boundary  $\Gamma^\infty$ . The fractal boundary  $\Gamma^\infty$  is supplied with a probability measure  $\mu$  called the self-similar measure. Emphasis is put on the case when the domain is not a  $\epsilon - \delta$  domain as defined by Jones and the fractal set is not totally disconnected. In this case, the classical trace results cannot be used. Here, the Lipschitz spaces with jumps recently introduced by Jonsson play a crucial role. Indeed, it is proved in particular that if the Hausdorff dimension  $d$  of  $\Gamma^\infty$  is not smaller than one, then the space of the traces of functions in  $W^{m+1,q}(\Omega)$ ,  $m \in \mathbb{N}$ ,  $1 < q < \infty$  is  $JLip(m + 1 - \frac{2-d}{q}, q, q; m; \Gamma^\infty)$ . The proof is elementary and uses as a main ingredient a strengthened trace inequality in the norm  $L_\mu^q(\Gamma^\infty)$ .

## 1 Introduction

This work deals with some properties of the Sobolev spaces  $W^{m+1,q}(\Omega)$ ,  $m \in \mathbb{N}$ , for a class of ramified domains  $\Omega$  of  $\mathbb{R}^2$  with a self-similar fractal boundary called  $\Gamma^\infty$  below, see for example Figures 1 and 3. The domain  $\Omega$  essentially depends on a parameter  $a$ ,  $0 < a \leq a^*$ . As explained below, the restriction  $a \leq a^*$  allows for the construction of  $\Omega$  as a union of non-overlapping sub-domains, see (5) below.

Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. Indeed, the present work is a continuation of [1] and of [2], which were part of a wider project aimed at simulating the diffusion of medical sprays in lungs. Since the exchanges between the lungs and the circulatory system take place only in the last generations of the bronchial tree (the smallest structures), reasonable models for the diffusion of, e.g., oxygen may involve a non-homogeneous Neumann or Robin condition on the boundary  $\Gamma^\infty$ . Similarly, the lungs are mechanically coupled to the diaphragm, which also implies non-homogeneous boundary conditions on  $\Gamma^\infty$ , if one is interested in a coupled fluid-structure model. It is therefore necessary to study traces of functions on  $\Gamma^\infty$ ; here, we will focus on functions belonging to  $W^{m+1,q}(\Omega)$ , for  $m \in \mathbb{N}$  and  $1 < q < \infty$ .

Function spaces defined in irregular domains have been widely studied in the literature:

- Jones [8] (and Vodopjanov et al [24] in the case  $n = 2$ , see also [16, 11]) have studied the open bounded subsets  $\Omega$  of  $\mathbb{R}^n$  such that there exists a continuous extension operator

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from  $W^{\ell,p}(\Omega)$  in  $W^{\ell,p}(\mathbb{R}^n)$ , for all nonnegative integers  $\ell$  and real numbers  $p$ ,  $1 \leq p \leq \infty$ . Jones has proved that if  $\Omega$  is a  $\epsilon - \delta$  domain for some parameters  $\epsilon, \delta > 0$ , see [8, 11] for the definition, then the above extension property is true. Moreover, in dimension two, if the extension property stated above is true, then  $\Omega$  is a  $\epsilon - \delta$  domain for some parameters  $\epsilon, \delta > 0$ . In dimension two, the definition of such domains is equivalent to that of quasi-disks, see [16].

- Jonsson and Wallin [11] have considered closed subsets  $F$  of  $\mathbb{R}^n$  supplied with a Borel measure  $\mu$  such that there exists a positive real number  $d$  and two positive constants  $c_1$  and  $c_2$  with

$$c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d,$$

for all  $x \in F$  and  $r < 1$  (here  $B(x, r)$  is the ball in  $F$  with center  $x$  and radius  $r$ , with respect to the Euclidean distance in  $\mathbb{R}^n$ ); in [11], these sets are called  $d$ -sets.

Concerning the set  $\Gamma^\infty$  under investigation in the present paper, we will recall in §3.2 below the notion of self-similar measure  $\mu$  defined in the triplet  $(\Gamma^\infty, F_1, F_2)$ , see [13]. With the Borel regular probability measure  $\mu$ ,  $\Gamma^\infty$  is a  $d$ -set where  $d$  is the Hausdorff dimension of  $\Gamma^\infty$ .

In [11], Sobolev and Besov spaces are defined on the  $d$ -sets, and extension and trace results for Besov and Sobolev spaces are proved using as a main ingredient Whitney extension theory. In particular, see [11] page 182, there exists a continuous trace operator from  $W^{1,p}(\mathbb{R}^n)$  onto  $B_{1-\frac{n-d}{p}}^{p,p}(F)$ , if  $\max(1, n-d) < p < \infty$ , where, for  $0 < s < 1$ ,

$$B_s^{p,p}(F) = \left\{ f \in L_\mu^p(F); \int_{x,y \in F, |x-y| < 1} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\mu(x) d\mu(y) < \infty \right\},$$

see the definition in [11] page 103. A more general trace theorem is available, see Theorem 1, page 141 in [11].

The approach of Triebel [23] is somewhat different. In [23] chapter IV, paragraph 18, it is proved that the space of the traces of functions in  $B_{\frac{n-d}{p}}^{p,q}(\mathbb{R}^n)$  is  $L_\mu^p(F)$  for  $0 < d < n$ ,  $d/n < p < \infty$  and  $0 < q \leq \min(1, p)$ ; Besov spaces on  $F$  are then defined as the space of the traces of Besov spaces on  $\mathbb{R}^n$  and embeddings properties are studied.

- There is also a growing interest in analysis on self-similar fractal sets, see for instance Kigami [13], Strichartz [21, 22], Mosco [19, 18] and references therein. These works aim at intrinsically defining function spaces using Dirichlet forms and a different metric from the Euclidean one. The results in this direction are often subject to the important assumption that the set is post-critically finite (or p.c.f.), see [13], page 23 for the definition. In a different direction, Jonsson has studied Lipschitz functions spaces on self-similar fractal sets under a technical condition which yields a Markov inequality at any order, see the pioneering works [9, 10]. This theory does not require the fractal set to be post-critically finite. In [10], Lipschitz functions spaces allowing jumps at some special points in the self-similar fractal set have been introduced, along with Haar wavelets of arbitrary order. The previously mentioned function spaces can be characterized using the coefficients of the expansion in some higher Haar wavelet bases. The theory in [10] is the cornerstone of the present paper. It will be briefly reviewed in §4.
- The question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Results in this direction have been given in

[20, 15, 14] for the Koch flake. Here also, the assumption that the fractal set is p.c.f. is generally made.

Our goal here is to study the traces of functions of  $W^{m+1,q}(\Omega)$  on the fractal boundary  $\Gamma^\infty$ . Note that this is different from considering the traces of functions of  $W^{m+1,q}(\mathbb{R}^2)$  on  $\Gamma^\infty$ .

If  $a < a^*$ , then  $\Omega$  is a  $\epsilon - \delta$  domain. In this case, the results of Jones [8] and those of Jonsson and Wallin [11] can be combined to obtain trace results. This has been done in [2]: if  $\max(1, 2-d) < q < \infty$ , the space of the traces of functions in  $W^{1,q}(\Omega)$  is  $B_{1-(2-d)/q}^{q,q}(\Gamma^\infty)$ , where  $d$  is the Hausdorff dimension of  $\Gamma^\infty$ .

When  $a = a^*$ , the situation is more complicated because

- $\Omega$  is not a  $\epsilon - \delta$  domain,
- $\Gamma^\infty$  may not be totally disconnected. It may even be non post-critically finite.

In [2] we mainly focused on the case  $a = a^*$ , and especially on the geometry presented in § 2.2.2 below, see Figure 3. We studied some properties of the traces of functions in  $H^1(\Omega)$ , without completely characterizing the trace space. The results proved in [2] are :

- if  $a \leq a^*$  then the trace of a function in  $H^1(\Omega)$  belongs to  $L_\mu^p(\Gamma^\infty)$ , for all real numbers  $p$  such that  $1 \leq p < \infty$ .
- in the case  $a = a^*$ , we gave an example of a function in  $H^1(\Omega)$  whose trace on  $\Gamma^\infty$  has not a bounded mean oscillation with respect to  $\mu$ , (and, as a consequence, does not belong to  $L_\mu^\infty(\Gamma^\infty)$ ).
- for the geometry displayed in Figure 3 in the critical case  $a = a^*$ , the trace of a function in  $H^1(\Omega)$  belongs to the Besov space  $B_s^{2,2}(\Gamma^\infty, \mu)$  for all real numbers  $s$  such that  $0 \leq s < d/4$ , where  $d$  is the Hausdorff dimension of  $\Gamma^\infty$ , and there exists a function in  $H^1(\Omega)$  whose trace does not belong to  $B_s^{2,2}(\Gamma^\infty, \mu)$  for all  $s > d/4$ . Note the important contrast with the case  $a < a^*$  for which the trace of a function in  $H^1(\Omega)$  belongs to  $B_s^{2,2}(\Gamma^\infty, \mu)$  for  $0 \leq s \leq d/2$ .
- Similar results concerning the spaces  $W^{1,p}(\Omega)$ ,  $1 \leq p < \infty$  were also given.

In the present paper, we aim at characterizing the traces of functions in  $W^{m+1,q}(\Omega)$  by using the results contained in [9] and mostly [10].

The article is organized as follows: the geometry is presented in section 2. In Section 3, we recall some of the results of [1] on the space  $W^{1,q}(\Omega)$ , concerning Poincaré inequality and the construction of the trace operator. The theory proposed in [10] is reviewed in § 4 where we mainly focus on the spaces  $JLip(s, q, q; m; \Gamma^\infty)$ ,  $m < s < m+1$ ,  $1 \leq q < \infty$ , that we will use later on. For simplicity, we first investigate the traces of functions in  $W^{1,q}(\Omega)$ : the main result of the paper is Theorem 6 stated and proved in Section 5. The proof uses elementary ingredients, mainly the strengthened trace inequality stated in Theorem 8. The traces of functions in  $W^{m+1,q}(\Omega)$  for a positive integer  $m$  are characterized in Section 6.

The results presented here can be generalized to functions in  $W^{s,q}(\Omega)$  for  $s$  non integer or in Besov spaces: this topic is currently under investigation.

## 2 The Geometry

We first describe the general geometrical setting, then we give two relevant examples.

## 2.1 General Setting

### 2.1.1 The similitudes $F_1$ and $F_2$ and the self-similar set $\Gamma^\infty$

Consider four real numbers  $a, \alpha, \beta, \theta$  such that  $0 < a < 1/\sqrt{2}$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $0 \leq \theta < \pi/2$ . Let  $F_i$ ,  $i = 1, 2$  be the two similitudes in  $\mathbb{R}^2$  given by

$$\begin{aligned} F_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}, \\ F_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}. \end{aligned} \tag{1}$$

The two similitudes have the same dilation ratio  $a$  and opposite angles  $\pm\theta$ . One can obtain  $F_2$  by composing  $F_1$  with the symmetry with respect to the axis  $\{x_1 = 0\}$ .

We call  $\Gamma^\infty$  the self-similar set associated to the similitudes  $F_1$  and  $F_2$ , i.e. the unique compact subset of  $\mathbb{R}^2$  such that

$$\Gamma^\infty = F_1(\Gamma^\infty) \cup F_2(\Gamma^\infty).$$

For  $n \geq 1$ , we call  $\mathcal{A}_n$  the set containing all the  $2^n$  mappings from  $\{1, \dots, n\}$  to  $\{1, 2\}$ . We define

$$\mathcal{M}_\sigma = F_{\sigma(1)} \circ \dots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n. \tag{2}$$

The definition of  $\Gamma^\infty$  implies that for all  $n > 0$ ,

$$\Gamma^\infty = \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\Gamma^\infty).$$

We state without proof the following proposition, which says that, up to an affine map, the shape of  $\Gamma^\infty$  does not depend on  $\alpha$  and  $\beta$ .

**Proposition 1** *Let  $F_1$  and  $F_2$  be defined by (1) where the parameters  $a, \theta, \alpha, \beta$  satisfy the assumptions above. Let the similitudes  $G_1$  and  $G_2$  be defined by*

$$G_1 = \begin{pmatrix} -\gamma \\ \delta \end{pmatrix} + F_1, \quad G_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + F_2,$$

where

$$\alpha + \gamma > 0 \quad \text{and} \quad \beta + \delta > 0.$$

The self-similar set associated to  $G_1$  and  $G_2$  is

$$\left( \begin{pmatrix} 0 \\ \frac{\alpha\delta - \beta\gamma}{\beta a \sin \theta + \alpha(1 - a \cos \theta)} \end{pmatrix} + \left( 1 + \frac{\delta a \sin \theta + \gamma(1 - a \cos \theta)}{\beta a \sin \theta + \alpha(1 - a \cos \theta)} \right) \Gamma^\infty \right)$$

where  $\Gamma^\infty$  is the self-similar set associated to the similitudes  $F_1$  and  $F_2$ .

Note that the assumptions in Proposition 1 imply that the parameter

$$\left( 1 + \frac{\delta a \sin \theta + \gamma(1 - a \cos \theta)}{\beta a \sin \theta + \alpha(1 - a \cos \theta)} \right)$$

is positive.

### 2.1.2 The simplest possible construction with $F_1$ and $F_2$

Call  $P_1 = (-1, 0)$  and  $P_2 = (1, 0)$  and  $\Gamma^0$  the line segment  $\Gamma^0 = [P_1, P_2]$ . We impose that  $F_2(P_1)$  and  $F_2(P_2)$  have positive coordinates, i.e. that

$$a \cos \theta < \alpha \quad \text{and} \quad a \sin \theta < \beta. \quad (3)$$

We first describe the simplest possible ramified domain containing the images of  $P_1$  and  $P_2$  by all the mappings  $F_{i_1} \circ \dots \circ F_{i_n}$ ,  $i_j = 1, 2$ . This domain will be noted  $\tilde{\Omega}$ . Let  $\tilde{Y}^0$  be the open polygonal set obtained as the convex hull of the six points

$$P_1, P_2, F_2(P_2), F_2(P_1), F_1(P_2), F_1(P_1).$$

The domain  $\tilde{Y}^0$  is hexagonal if  $\theta > 0$  and trapezoidal if  $\theta = 0$ , contained in the half plane  $x_2 > 0$  and symmetric w.r.t. the vertical axis  $x_1 = 0$ . It is possible to glue together  $\tilde{Y}^0$ ,  $F_1(\tilde{Y}^0)$  and  $F_2(\tilde{Y}^0)$  and obtain a new polygonal domain, also symmetric with respect to the axis  $\{x_1 = 0\}$ . The assumption (3) implies that  $\tilde{Y}^0$ ,  $F_1(\tilde{Y}^0)$  and  $F_2(\tilde{Y}^0)$  do not overlap.

We also define the ramified open domain  $\tilde{\Omega}$ , see Figures 1 and 3:

$$\tilde{\Omega} = \text{Interior} \left( \overline{\tilde{Y}^0} \cup \left( \bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_{\sigma}(\overline{\tilde{Y}^0}) \right) \right). \quad (4)$$

Note that  $\tilde{\Omega}$  is symmetric with respect to the axis  $x_1 = 0$ , and that for  $a < 1/\sqrt{2}$ , the measure of  $\tilde{\Omega}$  is finite.

It can be proved that there exists  $a^* < 1/\sqrt{2}$  (depending on  $\theta$  only) such that

1. for all  $a$ ,  $0 < a \leq a^*$  and  $\alpha, \beta$  satisfying (3), the sets  $\tilde{Y}^0$ ,  $\mathcal{M}_{\sigma}(\tilde{Y}^0)$ ,  $\sigma \in \mathcal{A}_n$ ,  $n > 0$  are disjoint,
2. for all  $a$ ,  $0 < a < a^*$ , for all  $\alpha, \beta$  satisfying (3),  $F_1(\tilde{\Omega}) \cap F_2(\tilde{\Omega}) = \emptyset$ ,
3. for  $a = a^*$  and  $\alpha, \beta$  satisfying (3),  $F_1(\tilde{\Omega}) \cap F_2(\tilde{\Omega}) \neq \emptyset$ .

The proof of this result is elementary but long. We skip it for brevity.

Hereafter, we will only consider  $a$  such that  $0 < a \leq a^*$ . It can be proved that  $\Gamma^{\infty} \subset \partial\Omega$  and that

$$\begin{aligned} 0 < a < a^* &\Rightarrow F_1(\Gamma^{\infty}) \cap F_2(\Gamma^{\infty}) = \emptyset &\Rightarrow \Gamma^{\infty} \text{ is totally disconnected,} \\ a = a^* &\Rightarrow F_1(\Gamma^{\infty}) \cap F_2(\Gamma^{\infty}) \neq \emptyset &\Rightarrow \Gamma^{\infty} \text{ is connected, see Th. 1.6.2 in [13].} \end{aligned}$$

We are able to compute the Hausdorff dimension of  $\Gamma^{\infty}$  because the Moran condition (open set condition) see [17, 13] is satisfied: the Moran condition is that there exists a nonempty bounded open subset  $O$  of  $\mathbb{R}^2$  such that  $F_1(O) \cap F_2(O) = \emptyset$  and  $F_1(O) \cup F_2(O) \subset O$ . Here, one can take  $O = \tilde{\Omega}$ . From this, the Hausdorff dimension of  $\Gamma^{\infty}$  is

$$d = \dim_H(\Gamma^{\infty}) = -\log 2 / \log a,$$

see [17, 13]. The assumption on  $a < 1/\sqrt{2}$  implies that  $d < 2$ .

### 2.1.3 More general constructions

We also consider more general polygonal domains  $Y^0$ : we assume that for all  $a$ ,  $0 < a \leq a^*$ , (where  $a^*$  has been introduced above),

1.  $Y^0$  is contained in the half-plane  $x_2 > 0$ .
2.  $\partial Y^0 \cap \{x_2 = 0\} = \Gamma^0 = [P_1, P_2]$ .
3.  $F_1(\Gamma^0)$  and  $F_2(\Gamma^0)$  are contained in some sides of  $\partial Y^0$ .
4. the sets  $Y^0$ ,  $\mathcal{M}_\sigma(Y^0)$ ,  $\sigma \in \mathcal{A}_n$ ,  $n > 0$  are disjoint.

With these assumptions, we can construct the ramified open domain

$$\Omega = \text{Interior} \left( \overline{Y^0} \cup \left( \bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0}) \right) \right), \quad (5)$$

see Figure 2 for an example. The self-similar fractal set  $\Gamma^\infty$  is contained in  $\partial\Omega$ . We split the boundary of  $\Omega$  into  $\Gamma^\infty$ ,  $\Gamma^0 = [-1, 1] \times \{0\}$  and  $\Sigma = \partial\Omega \setminus (\Gamma^0 \cup \Gamma^\infty)$ .

**Remark 1** *A further generalization is possible: all the assumptions above are maintained except that  $\partial Y^0 \setminus (\Gamma^0 \cup F_1(\Gamma^0) \cup F_2(\Gamma^0))$  may be made of curved lines. All what follows is valid in this case. For simplicity, we will still assume that  $Y^0$  is polygonal.*

### 2.1.4 Additional notations

For what follows, it is important to define the polygonal open domain  $Y^N$  obtained by stopping the above construction at step  $N + 1$ ,

$$Y^N = \text{Interior} \left( \overline{Y^0} \cup \left( \bigcup_{n=1}^N \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0}) \right) \right). \quad (6)$$

We introduce the open domains  $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$  and  $\Omega^N = \bigcup_{\sigma \in \mathcal{A}_N} \Omega^\sigma = \Omega \setminus \overline{Y^{N-1}}$ , for  $N > 0$ . When needed, we will agree to say that  $\Omega^0 = \Omega$ . We define the sets  $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$  and  $\Gamma^N = \bigcup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$ . The one-dimensional Lebesgue measure of  $\Gamma^\sigma$  for  $\sigma \in \mathcal{A}_N$  and of  $\Gamma^N$  are given by

$$|\Gamma^\sigma| = a^N |\Gamma^0| \quad \text{and} \quad |\Gamma^N| = (2a)^N |\Gamma^0|.$$

## 2.2 Two Examples

### 2.2.1 Example 1

Let  $F_1$  and  $F_2$  be the affine maps in  $\mathbb{R}^2$

$$F_1(x) = (-\alpha + ax_1, \beta + ax_2), \quad F_2(x) = (\alpha + ax_1, \beta + ax_2), \quad (7)$$

so  $\theta = 0$ , and (3) becomes  $a < \alpha$  and  $\beta > 0$ . The domain  $\tilde{Y}^0$  is trapezoidal:

$$\tilde{Y}^0 = \text{Interior} \left( \text{Conv}(P_1, P_2, F_2(P_2), F_1(P_1)) \right).$$

The sets  $\overline{\tilde{Y}^0}$ ,  $\mathcal{M}_\sigma(\overline{\tilde{Y}^0})$ ,  $\sigma \in \mathcal{A}_n$ ,  $n > 0$  do not overlap if

$$a \leq a^* = \frac{1}{2}.$$

The domain  $\tilde{\Omega}$  described in § 2.1.2 is shown in Figure 1 for  $\alpha = 3/2$ ,  $\beta = 3$  and  $a = a^* = 1/2$ . If  $a < \frac{1}{2}$ , the Hausdorff dimension of  $\Gamma^\infty$  is smaller than 1; the set  $\Gamma^\infty$  is totally disconnected, (i.e.  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$  is empty), and contained in a straight line; it can be proved that  $\tilde{\Omega}$  is a  $\epsilon - \delta$  domain as defined by Jones [8], see also [11] and [16].

In the critical case when  $a = a^* = \frac{1}{2}$ ,  $\Gamma^\infty$  is the straight line segment  $[-2\alpha, 2\alpha] \times \{2\beta\}$ , so it is connected and its Hausdorff dimension is one. It is post-critically finite because  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$  is a singleton. The open set  $\tilde{\Omega}$  is not a  $\epsilon - \delta$  domain. Indeed, take the point  $X = (0, \beta/2)$  and call  $A_n = (F_1 \circ F_2^n)(X)$ ,  $B_n = (F_2 \circ F_1^n)(X)$ , we have that

- $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n = (0, 2\beta)$ , therefore  $\lim_{n \rightarrow \infty} |A_n B_n| = 0$ ,
- $A_n \in \tilde{\Omega}$  and  $B_n \in \tilde{\Omega}$ ,
- the length of any curve joining  $A_n$  and  $B_n$  that is contained in  $\tilde{\Omega}$ , is greater than  $2\beta$ .

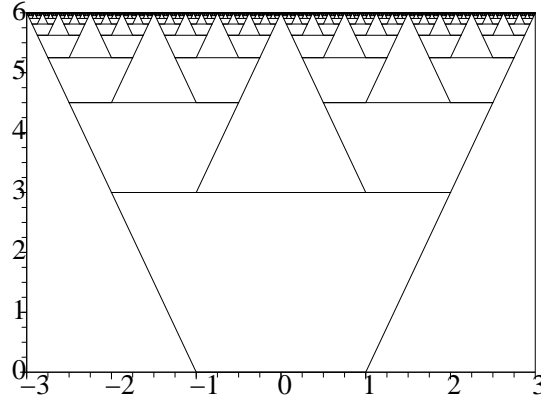


Figure 1: The ramified domain  $\tilde{\Omega}$  for the critical value  $a = a^* = 1/2$ , and for  $\alpha = 3/2$ ,  $\beta = 3$ .

As described in § 2.1.3, one can construct a different domain  $\Omega$  whose boundary contains  $\Gamma^\infty$  by choosing for example  $Y^0$  as a T-shaped domain, see Figure 2.

### 2.2.2 Example 2

We make the choice  $\theta = \pi/4$ ,  $\alpha = 1 - \frac{a}{\sqrt{2}}$ ,  $\beta = 1 + \frac{a}{\sqrt{2}}$ , so the similitudes  $F_i$ ,  $i = 1, 2$  are defined by

$$F_i(x) = \begin{pmatrix} (-1)^i \left(1 - \frac{a}{\sqrt{2}}\right) + \frac{a}{\sqrt{2}} \left(x_1 + (-1)^i x_2\right) \\ 1 + \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}} \left(x_2 + (-1)^{i+1} x_1\right) \end{pmatrix}.$$

It can be proved by elementary geometrical arguments that the condition to prevent the sets  $\mathcal{M}_\sigma(\overline{Y^0})$ ,  $\sigma \in \mathcal{A}_n$ ,  $n > 0$ , from overlapping is

$$2\sqrt{2}a^5 + 2a^4 + 2a^2 + \sqrt{2}a - 2 \leq 0, \quad (8)$$

i.e.

$$a \leq a^* \simeq 0.593465.$$



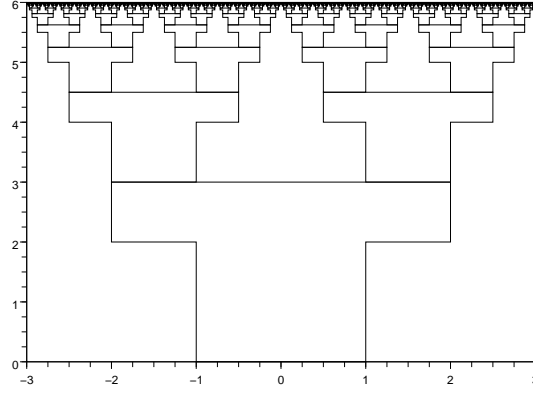


Figure 2: An example of a possible ramified domain  $\Omega$  for the critical value  $a = a^* = 1/2$ .

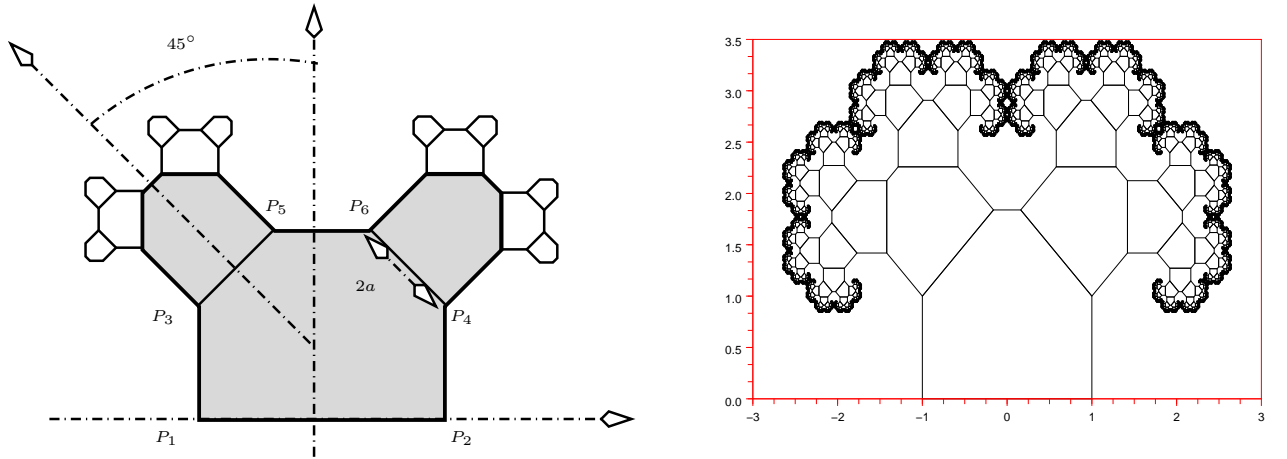


Figure 3: The angle  $\theta$  is  $\pi/4$ . Left, the construction (more exactly  $\tilde{Y}^3$ ). Right, the ramified domain  $\tilde{\Omega}$  for the critical value  $a = a^*$ .

The domain  $\tilde{\Omega}$  constructed in § 2.1.2 is shown in Figure 3.

If  $a > 1/2$ , the Hausdorff dimension of  $\Gamma^\infty$  is larger than one. For instance, if  $a = a^*$ , then  $\dim_H(\Gamma^\infty) \simeq 1.3284371$ .

In the case when  $a < a^*$ , it can be proved that  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$  is empty and that  $\Omega$  is a  $\epsilon - \delta$  domain, see [2].

In the critical case when  $a = a^*$ , it can be proved, see [2], that  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \subset \Gamma^\infty$  is a non countable set, whose Hausdorff dimension is half the Hausdorff dimension of  $\Gamma^\infty$ . The set  $\Gamma^\infty$  is not post-critically finite. It was also proved in [2] that  $\tilde{\Omega}$  is not a  $\epsilon - \delta$  domain.

## 2.3 The self-similar measure $\mu$

### 2.3.1 Basic facts

For defining traces on  $\Gamma^\infty$ , we recall the classical result on self-similar measures, see [5, 7] and [13] page 26:

**Theorem 1** *There exists a unique Borel regular probability measure  $\mu$  on  $\Gamma^\infty$  such that for any Borel set  $A \subset \Gamma^\infty$ ,*

$$\mu(A) = \frac{1}{2}\mu(F_1^{-1}(A)) + \frac{1}{2}\mu(F_2^{-1}(A)). \quad (9)$$

The measure  $\mu$  is called the *self-similar measure defined in the self-similar triplet*  $(\Gamma^\infty, F_1, F_2)$ .

**Proposition 2** *The measure  $\mu$  is a  $d$ -measure on  $\Gamma^\infty$ , with  $d = -\log 2 / \log a$ , according to the definition in [11], page 28: there exists two positive constants  $c_1$  and  $c_2$  such that*

$$c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d,$$

for any  $r$   $0 < r < 1$  and  $x \in \Gamma^\infty$ , where  $B(x, r)$  is the Euclidean ball in  $\Gamma^\infty$  centered at  $x$  and with radius  $r$ . In other words the closed set  $\Gamma^\infty$  is a  $d$ -set, see [11], page 28.

**Proof.** The proof stems from the Moran condition in § 2. It is due to Moran [17] and has been extended by Kigami, see [13], §1.5, especially Proposition 1.5.8 and Theorem 1.5.7. ■

**Remark 2** *In the first example with  $a = 1/2$ , the measure  $\mu$  is such that  $6\mu$  is the one-dimensional Lebesgue measure on the straight line  $\Gamma^\infty$ .*

We define  $L_\mu^p$ ,  $p \in [1, +\infty)$  as the space of the measurable functions  $v$  on  $\Gamma^\infty$  such that  $\int_{\Gamma^\infty} v^p d\mu < \infty$ , endowed with the norm  $\|v\|_{L_\mu^p} = (\int_{\Gamma^\infty} v^p d\mu)^{1/p}$ . We also introduce  $L_\mu^\infty$ , the space of essentially bounded functions with respect to the measure  $\mu$ . A Hilbertian basis of  $L_\mu^2$  can be constructed with e.g. Haar wavelets.

### 2.3.2 The case when $a < \frac{1}{2}$

The condition  $a < \frac{1}{2}$  implies  $d < 1$ . In this case,  $\Gamma^\infty$  is totally disconnected, see [6], Lemma 4.1 page 54. This implies that  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) = \emptyset$ , see [13], theorem 1.6.2 page 33. Therefore,  $a < a^*$  and we can apply the result stated in the introduction: if  $q > \max(1, 2 - d)$ , then the space of the traces on  $\Gamma^\infty$  of the functions  $v \in W^{1,q}(\Omega)$  is  $B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$  (see the introduction for the definition) and we will see in Theorem 5 below that in this case,  $B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty) = J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ .

### 3 The space $W^{1,q}(\Omega)$

Hereafter, except when explicitly mentioned, we will not distinguish between the simplest geometries presented in § 2.1.2 and the more general ones introduced in § 2.1.3. The former can be seen as a special case of the latter. Therefore, we will use the notation  $\Omega$  for all the domains constructed in § 2. We also take  $a$  such that  $1/2 \leq a \leq a^*$ , so the Hausdorff dimension  $d$  of  $\Gamma^\infty$  is not smaller than 1, because the case  $a < 1/2$  has already been discussed in § 2.3.2.

For a real number  $q \geq 1$ , let  $W^{1,q}(\Omega)$  be the space of functions in  $L^q(\Omega)$  with first order partial derivatives in  $L^q(\Omega)$ . We also define  $\mathcal{V}_q(\Omega) = \{v \in W^{1,q}(\Omega); v|_{\Gamma^0} = 0\}$  and  $\mathcal{V}_q(Y^n) = \{v \in W^{1,q}(Y^n); v|_{\Gamma^0} = 0\}$ .

The results stated below are important for the study of elliptic boundary value problems in  $\Omega$ . Section 3.1 contains some Poincaré inequalities and §3.2 deals with the construction of a trace operator on  $\Gamma^\infty$ . We refer to [1] for the proofs.

We will sometimes use the notation  $\lesssim$  to indicate that there may arise constants in the estimates, which are independent of the index  $n$  in  $\Omega^n$  (recall that  $\Omega^n$  is the union of all  $\mathcal{M}_\sigma(\Omega)$ ,  $\sigma \in \mathcal{A}_n$ ) or  $\Gamma^n$  ( $\Gamma^n$  is the union of all  $\mathcal{M}_\sigma(\Gamma^0)$ ,  $\sigma \in \mathcal{A}_n$ ),  $Y^n = Y^0 \cup \bigcup_{1 \leq p \leq n} \bigcup_{\sigma \in \mathcal{A}_p} \mathcal{M}_\sigma(Y^0)$ , or the index  $\sigma$  in  $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$  or  $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$ .

#### 3.1 Poincaré inequality and consequences

**Theorem 2** *There exists a constant  $C > 0$ , such that*

$$\forall v \in \mathcal{V}_q(\Omega), \quad \|v\|_{L^q(\Omega)}^q \leq C \|\nabla v\|_{L^q(\Omega)}^q.$$

**Corollary 1** *There exists a positive constant  $C$  such that for all  $v \in W^{1,q}(\Omega)$ ,*

$$\|v\|_{L^q(\Omega)}^q \leq C \left( \|\nabla v\|_{L^q(\Omega)}^q + \|v|_{\Gamma^0}\|_{L^q(\Gamma^0)}^q \right).$$

**Corollary 2** *There exists a positive constant  $C$  such that for all integer  $n \geq 0$  and for all  $\sigma \in \mathcal{A}_n$ , for all  $v \in W^{1,q}(\Omega^\sigma)$ ,*

$$\|v\|_{L^q(\Omega^\sigma)}^q \leq C \left( a^{qn} \|\nabla v\|_{L^q(\Omega^\sigma)}^q + a^n \|v|_{\Gamma^\sigma}\|_{L^q(\Gamma^\sigma)}^q \right),$$

and for all  $v \in W^{1,q}(\Omega^n)$

$$\|v\|_{L^q(\Omega^n)}^q \leq C \left( a^{qn} \|\nabla v\|_{L^q(\Omega^n)}^q + a^n \|v|_{\Gamma^n}\|_{L^q(\Gamma^n)}^q \right).$$

We need to estimate  $\|v\|_{L^q(\Omega^n)}^q$  when  $v \in W^{1,q}(\Omega)$ :

**Lemma 1** *There exists a positive constant  $C$  such that for all  $v \in W^{1,q}(\Omega)$ , for all  $n \geq 0$ ,*

$$\|v\|_{L^q(\Omega^n)}^q \leq C (2a^2)^n \left( \|\nabla v\|_{L^q(\Omega)}^q + \|v|_{\Gamma^0}\|_{L^q(\Gamma^0)}^q \right). \quad (10)$$

Since  $2a^2 < 1$ , (10) implies the Rellich type theorem:

**Theorem 3 (Compactness)** *The imbedding of  $W^{1,q}(\Omega)$  in  $L^q(\Omega)$  is compact.*

The following lemma will be useful for defining a trace operator on  $\Gamma^\infty$ :

**Lemma 2** *There exists a positive constant  $C$  such that for all  $v \in W^{1,q}(\Omega)$ , for all integers  $p \geq 0$ ,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^q \leq C(2a)^p \left( \|\nabla v\|_{L^q(\Omega)}^q + \|v\|_{L^q(\Omega)}^q \right). \quad (11)$$

**Remark 3** *Note that  $\frac{|\Gamma^p|}{|\Gamma^0|} = (2a)^p$ , so (11) is equivalent to*

$$\frac{1}{|\Gamma^p|} \sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^q \lesssim \|\nabla v\|_{L^q(\Omega)}^q + \|v\|_{L^q(\Omega)}^q.$$

**Corollary 3** *There exists a positive constant  $C$  such that for all  $v \in W^{1,q}(\Omega)$ , for all integers  $p \geq 0$ ,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma} - \langle v|_{\Gamma^0} \rangle)^q \leq C(2a)^p \|\nabla v\|_{L^q(\Omega)}^q,$$

where  $\langle v|_{\Gamma^0} \rangle$  is the mean value of  $v|_{\Gamma^0}$  on  $\Gamma^0$ .

**Remark 4** *Similar results in a different geometry were proved by Berger [3] with different methods.*

### 3.2 A trace operator on $\Gamma^\infty$

We construct a sequence  $(\ell^n)_n$  of approximations of the trace operator: consider the sequence of linear operators  $\ell^n : W^{1,q}(\Omega) \rightarrow L_\mu^q$ ,

$$\ell^n(v) = \sum_{\sigma \in \mathcal{A}_n} \left( \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}, \quad (12)$$

where  $|\Gamma^\sigma|$  is the one-dimensional Lebesgue measure of  $\Gamma^\sigma$ , see [1].

**Proposition 3** *The sequence  $(\ell^n)_n$  converges in  $\mathcal{L}(W^{1,q}(\Omega), L_\mu^q)$  to an operator that we call  $\ell^\infty$ .*

**Proof.** See [1]. ■

## 4 Lipschitz Functions with Jumps on $\Gamma^\infty$

In [10], A. Jonsson has introduced Haar wavelets of arbitrary order on self-similar fractal sets and has used these wavelets for constructing a family of Lipschitz spaces. These function spaces are named  $JLip(\alpha, p, q; m; K)$ , where  $K$  is the fractal set,  $\alpha$  is a nonnegative real number,  $p, q$  are two real numbers not smaller than 1 and  $m$  is an integer ( $m$  is the order of the Haar wavelets used for constructing the space). Here  $J$  stands for jumps, since the considered functions may jump at some points of  $K$ . If the fractal set  $K$  is totally disconnected, then these spaces coincide with the Lipschitz spaces  $Lip(\alpha, p, q; m; K)$  also introduced in [10]. The latter are a generalization of the more classical spaces  $Lip(\alpha, p, q; K)$  introduced in [11] since  $Lip(\alpha, p, q; [\alpha]; K) = Lip(\alpha, p, q; K)$ . Note that  $Lip(\alpha, p, q; [\alpha]; K) = B_\alpha^{p,q}(K)$ , see [12].

Since this is sufficient for what follows, we are going to focus on the case when  $K = \Gamma^\infty$  and  $p = q$ .

#### 4.1 Definition of $JLip(\alpha, q, q; 0; \Gamma^\infty)$ for $0 < \alpha < 1$ .

We review the theory proposed in [10] in the special case when  $0 < \alpha < 1$  and  $m = 0$ . Here, since we focus on the case  $m = 0$ , we do not need to suppose that  $\Gamma^\infty$  is not contained in a straight line, as it was done in [10] in order to obtain that the fractal set has a Markov property. For a function  $f \in L_\mu^q$ , a positive integer  $n$  and  $\sigma \in \mathcal{A}_n$  if  $n > 0$ , we define  $P_\sigma f$  as the projection of  $f$  on constants defined as follows:

$$P_\sigma f = 2^n \int_{\mathcal{M}_\sigma(\Gamma^\infty)} f d\mu.$$

We also define  $P_0 f$  as the projection of  $f$  on constants:

$$P_0 f = \int_{\Gamma^\infty} f d\mu.$$

For an integer  $\nu$ , we define the set  $J_\nu$ :

$$J_\nu = \{p \in \mathbb{N}, \text{ s.t. } \forall \sigma \in \mathcal{A}_p, 2^{-\nu} \leq \text{diam}(\mathcal{M}_\sigma(\Gamma^\infty)) < 2^{-\nu+1}\}, \quad (13)$$

agreeing that  $\mathcal{M}_\sigma = Id$  if  $p = 0$ ,  $\sigma \in \mathcal{A}_p$ .

Let  $\nu_0$  be the integer (possibly negative) such that  $0 \in J_{\nu_0}$ .

Consider a real number  $\alpha$ ,  $0 < \alpha < 1$ . For a function  $f \in L_\mu^2$  and an integer  $\nu \geq \nu_0$ , we define  $A_\nu(f)$  by

$$\begin{aligned} A_{\nu_0}(f) &= 2^{\alpha\nu_0} \left( \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{p \in J_{\nu_0}, p > 0} \sum_{\sigma \in \mathcal{A}_p} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{1/q}, \\ A_\nu(f) &= 2^{\alpha\nu} \left( \sum_{p \in J_\nu} \sum_{\sigma \in \mathcal{A}_p} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{1/q}, \quad \text{if } \nu > \nu_0. \end{aligned} \quad (14)$$

**Definition 1** *The function  $f$  belongs to  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  if the norm*

$$\|f\|_{JLip(\alpha, q, q; 0; \Gamma^\infty)} = \left( \|f\|_{L_\mu^q}^q + \sum_{\nu=\nu_0}^{\infty} A_\nu^q(f) \right)^{\frac{1}{q}} \quad (15)$$

*is finite.*

Following [10], it is possible to characterize  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  by using expansions in the standard Haar wavelet basis on  $\Gamma^\infty$ . Consider the Haar mother wavelet  $g^0$  on  $\Gamma^\infty$ ,

$$g^0 = 1_{F_1(\Gamma^\infty)} - 1_{F_2(\Gamma^\infty)}, \quad (16)$$

and for  $n \in \mathbb{N}$ ,  $n > 0$ ,  $\sigma \in \mathcal{A}_n$ , let  $g^\sigma$  be given by

$$g^\sigma|_{\mathcal{M}_\sigma(\Gamma^\infty)} = 2^{n/2} g^0 \circ \mathcal{M}_\sigma^{-1}, \quad \text{and} \quad g^\sigma|_{\Gamma^\infty \setminus \mathcal{M}_\sigma(\Gamma^\infty)} = 0. \quad (17)$$

It is proved in [9] §5 that a function  $f$  in  $L_\mu^q$  can be expanded on the Haar wavelet basis as follows:

$$f = P_0 f + \beta_0 g^0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma} g^\sigma.$$

Let  $b_0$  be a real number and  $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$  be a sequence of real numbers, we define  $\|(b_0, b_{n,\sigma})\|_{b_\alpha^{q,q}}$  by

$$\|(b_0, b_{n,\sigma})\|_{b_\alpha^{q,q}} = \left( 2^{\alpha\nu_0 q} 2^{d\nu_0(\frac{1}{2}-\frac{1}{q})q} \left( |b_0|^q + \sum_{p \in J_{\nu_0}, p > 0} \sum_{\sigma \in \mathcal{A}_p} |b_{p,\sigma}|^q \right) + \sum_{\nu=\nu_0+1}^{\infty} 2^{\alpha\nu q} 2^{d\nu(\frac{1}{2}-\frac{1}{q})q} \sum_{p \in J_\nu} \sum_{\sigma \in \mathcal{A}_p} |b_{p,\sigma}|^q \right)^{\frac{1}{q}}. \quad (18)$$

**Theorem 4** (Jonsson) *A fonction  $f \in L_\mu^q$  belongs to  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  if and only if the coefficients of its expansion in the Haar wavelets basis satisfy  $|P_0 f| + \|(\beta_0, \beta_{n,\sigma})\|_{b_\alpha^{q,q}}$  is finite; this defines a norm in  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  equivalent to the one in (15).*

**Proof.** This is a particular case of Theorem 1 in [10]). ■

If the fractal set  $\Gamma^\infty$  is totally disconnected, then the jumps which are allowed in the space  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  disappear:

**Theorem 5** (Jonsson) *If  $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$  is empty, then*

$$JLip(\alpha, q, q; 0; \Gamma^\infty) = Lip(\alpha, q, q; 0; \Gamma^\infty) = B_\alpha^{q,q}(\Gamma^\infty),$$

where the Lipschitz space  $Lip(\alpha, q, q; 0; \Gamma^\infty)$  and the Besov space  $B_\alpha^{q,q}(\Gamma^\infty)$  are defined in [11].

**Proof.** This is a particular case of Theorem 2 in [10], see also [9] for a partial proof. ■

Let us assume that  $\text{diam}(\Gamma^\infty) > 0$ . Since in our case the ratii of the similitudes  $F_1$  and  $F_2$  both take the value  $a$ , we see that  $\text{diam}(\mathcal{M}_\sigma(\Gamma^\infty)) = a^n \text{diam}(\Gamma^\infty)$  if  $\sigma \in \mathcal{A}_n$  and  $n > 0$ . Therefore, for  $n > 0$ ,  $n \in J_\nu$  if and only if

$$2^{-\nu} \leq a^n \text{diam}(\Gamma^\infty) < 2^{-\nu+1},$$

which can be written

$$-\nu \leq n \frac{\log a}{\log 2} + \frac{\log(\text{diam}(\Gamma^\infty))}{\log 2} < -\nu + 1, \quad (19)$$

or

$$\begin{aligned} -\nu &= \left\lfloor n \frac{\log a}{\log 2} + \frac{\log(\text{diam}(\Gamma^\infty))}{\log 2} \right\rfloor \\ &= \left\lfloor -n/d + \frac{\log(\text{diam}(\Gamma^\infty))}{\log 2} \right\rfloor. \end{aligned} \quad (20)$$

where  $d = -\log 2 / \log a$  is the Hausdorff dimension of  $\Gamma^\infty$  and where  $[\cdot]$  stands for the integer part.

Let  $b_0$  be a real number and  $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$  be a sequence of real numbers; we define  $\|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}}$  by

$$\begin{aligned} \|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} &= \left( |b_0|^q + \sum_{n=1}^{\infty} 2^{q\alpha n/d} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}} \\ &= \left( |b_0|^q + \sum_{n=1}^{\infty} a^{-q\alpha n} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}}. \end{aligned} \quad (21)$$

Assuming that  $\text{diam}(\Gamma^\infty) \neq 0$ , there exist two positive constants  $c_1$  and  $c_2$  such that

$$c_1 \|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} \leq \|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} \leq c_2 \|(b_0, b_{n,\sigma})\|_{b_\alpha^{q,q}},$$

for any sequence  $(b_{n,\sigma})_{n \geq 1, \sigma \in \mathcal{A}_n}$  such that  $|b_0|^q + \sum_{n=1}^\infty 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q < \infty$ . We have the corollary of Theorem 4:

**Corollary 4** *A fonction  $f \in L_\mu^q$  belongs to  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  if and only if the coefficients of its expansion in the Haar wavelets basis satisfy  $|P_0 f| + \|(\beta_0, \beta_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}}$  is finite; the norm of  $f$  in  $JLip(\alpha, q, q; 0; \Gamma^\infty)$  is equivalent to this sum. Another equivalent norm is*

$$\left( \|f\|_{L_\mu^q}^q + \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{n=1}^\infty a^{-\alpha q n} \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{\frac{1}{q}}. \quad (22)$$

**Remark 5** *In the special case when  $\alpha = 1 - \frac{2-d}{q}$ ,*

$$\|(b_0, b_{n,\sigma})\|_{\tilde{b}_\alpha^{q,q}} = \left( |b_0|^q + \sum_{n=1}^\infty 2^{\frac{nq}{2}} a^{-n(q-2)} \sum_{\sigma \in \mathcal{A}_n} |b_{n,\sigma}|^q \right)^{\frac{1}{q}},$$

*and the norm in (22) reads*

$$\left( \|f\|_{L_\mu^q}^q + \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{n=1}^\infty (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \right)^{\frac{1}{q}}.$$

*These observations will be useful in the proof of Theorem 6 below.*

## 4.2 Definition of $JLip(\alpha, q, q; m; \Gamma^\infty)$ for a positive integer $m$ and $m \leq \alpha < m+1$ .

Following [10], we suppose that  $\Gamma^\infty$  is not contained in a straight line (this rules out the geometry presented in § 2.2.1 above). This assumption implies that  $\Gamma^\infty$  preserves Markov's inequality, see [10].

Call  $\mathcal{S}_0$  the subspace of  $L_\mu^2$  made of polynomials of order not larger than  $m$ ; thanks to Markov's inequality, its dimension is  $M_0 = \frac{(m+2)(m+1)}{2}$ ; let  $(\phi^i)_{1 \leq i \leq M_0}$  be a basis of  $\mathcal{S}_0$  orthonormal with respect to  $(\cdot, \cdot)_{L_\mu^2}$ . For  $f \in L_\mu^1$ , we define by  $P_0^{(m)} f$  the projection of  $f$  onto  $\mathcal{P}_m$  defined as follows: let  $(\phi^j)_{1 \leq j \leq M_0}$  be an orthonormal basis of  $\mathcal{S}_0$  and  $P_0^{(m)} f = \sum_i \phi^i \int_{\Gamma^\infty} f \phi^i d\mu$ .

Similarly, for  $f \in L_\mu^1(\mathcal{M}_\sigma(\Gamma^\infty))$ , we define by  $P_\sigma^{(m)} f$  the projection of  $f$  onto  $\mathcal{P}_m$  defined as follows:

$$P_\sigma^{(m)} f = \left( P_0^{(m)} (f \circ \mathcal{M}_\sigma) \right) \circ \mathcal{M}_\sigma^{-1}.$$

The following definition of the space  $JLip(\alpha, q, q; m; \Gamma^\infty)$  has to be compared to (22):

**Definition 2** *Let  $m$  be a positive integer and  $\alpha$  be a real number such that  $m \leq \alpha < m+1$ . The function  $f \in L_\mu^q$  belongs to  $JLip(\alpha, q, q; m; \Gamma^\infty)$  if the norm*

$$\|f\|_{JLip(\alpha, q, q; m; \Gamma^\infty)} = \left( \|f\|_{L_\mu^q}^q + \int_{\Gamma^\infty} |f - P_0^{(m)} f|^q d\mu + \sum_{n=1}^\infty a^{-\alpha q n} \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma^{(m)} f|^q d\mu \right)^{\frac{1}{q}}. \quad (23)$$

*is finite.*

Haar wavelets of order  $m$  can be used for an equivalent definition of  $JLip(\alpha, q, q; m; \Gamma^\infty)$ : call  $\mathcal{S}_1$  the space of the functions  $f$  defined on  $\Gamma^\infty$  such that  $f$  coincides with a polynomial of degree  $m$  on  $F_i(\Gamma^\infty) \setminus (F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty))$ ,  $i = 1, 2$ . The dimension of  $\mathcal{S}_1$  is  $2M_0$ . The orthogonal complement  $\mathcal{S}_1 \ominus \mathcal{S}_0$  of  $\mathcal{S}_0$  in  $\mathcal{S}_1$  has dimension  $M_0$ . Choose an orthonormal basis  $(g^{0,i})_{1 \leq i \leq M_0}$  of  $\mathcal{S}_1 \ominus \mathcal{S}_0$ . Finally, for  $n \geq 1$  and  $\sigma \in \mathcal{A}_n$ , define

$$g^{\sigma,i} = 2^{n/2} g^{0,i} \circ \mathcal{M}_\sigma^{-1}.$$

The family  $(\phi^i, g^{0,i}, g^{\sigma,i})$ ,  $1 \leq i \leq M_0$ ,  $\sigma \in \mathcal{A}_n$ ,  $n \geq 1$  is an orthonormal and complete family of  $L_\mu^2$ . It is proved in [10] that a function  $f \in L_\mu^q$  can be expanded as follows

$$f = \sum_{i=1}^{M_0} \lambda_i \phi^i + \sum_{i=1}^{M_0} \beta_{0,i} g^{0,i} + \sum_{i=1}^{M_0} \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma,i} g^{\sigma,i}. \quad (24)$$

**Proposition 4** (Jonsson, [10], Th. 1) *A function  $f \in L_\mu^q$  belongs to  $JLip(\alpha, q, q; m; \Gamma^\infty)$  if and only if*

$$\left( \sum_{i=1}^{M_0} |\lambda_i|^q + \sum_{i=1}^{M_0} |\beta_{0,i}|^q + \sum_{i=1}^{M_0} \sum_{n=1}^{\infty} a^{-q\alpha n} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |\beta_{n,\sigma,i}|^q \right)^{\frac{1}{q}} < \infty, \quad (25)$$

where  $\lambda_i, \beta_{0,i}, \beta_{n,\sigma,i}$  are the coefficients in (24). The norm in (25) is equivalent to that in (23).

## 5 Traces of functions belonging to $W^{1,q}(\Omega)$

We consider the case when  $1/2 \leq a \leq a^*$ ; we are going to give a trace theorem with a direct proof (which differs from the previous argument when  $a < a^*$ ).

The main theorem is as follows:

**Theorem 6** *If  $1/2 \leq a \leq a^*$ , then for all  $q$ ,  $1 < q < \infty$ ,*

$$\ell^\infty(W^{1,q}(\Omega)) = JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty). \quad (26)$$

**Proof.** Theorem 6 is a straightforward consequence of Proposition 5 and Theorem 7 below. ■

**Remark 6** *Note the contrast with the fact that functions in  $W^{1,q}(\mathbb{R}^2)$  have their traces in  $Lip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ , see [11] page 182 and [9, 10].*

**Proposition 5** *For all real number  $q \geq 1$ , there exists a constant  $C$  such that for all  $v \in JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ , there exists  $\tilde{v} \in \mathcal{V}_q(\Omega)$  with the following properties:  $\ell^\infty(\tilde{v}) = v$ , and*

$$\|\tilde{v}\|_{\mathcal{V}_q(\Omega)}^q \approx \int_\Omega |\nabla \tilde{v}|^q \leq C \|v\|_{JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)}^q. \quad (27)$$

**Proof.** First, let us consider the case when  $v$  is a Haar wavelet on  $\Gamma^\infty$ . We start with the mother wavelet  $g^0$ . One can construct a function  $\tilde{g}^0$  such that  $\tilde{g}^0 \in \mathcal{V}_q(\Omega)$  for all  $1 \leq q < \infty$  and

$$\tilde{g}^0|_{F_1(\Omega)} = -1 \quad \text{and} \quad \tilde{g}^0(-x_1, x_2) = -\tilde{g}^0(x_1, x_2), \quad \forall x = (x_1, x_2) \in \Omega. \quad (28)$$

It is an easy matter to check that  $\ell^\infty(\tilde{g}^0) = g^0$  and that  $\int_\Omega |\nabla \tilde{g}^0|^q = \int_{Y^0} |\nabla \tilde{g}^0|^q$ .

For a positive integer  $n$  and  $\sigma \in \mathcal{A}_n$ , let us define the function  $\tilde{g}^\sigma, \tilde{g}^\sigma \in \mathcal{V}_q(\Omega)$  for all  $1 \leq q < \infty$ , by:

$$\tilde{g}^\sigma|_{\Omega^\sigma} = 2^{n/2} \tilde{g}^0 \circ \mathcal{M}_\sigma^{-1}, \quad (29)$$

$$\tilde{g}^\sigma|_{\Omega \setminus \Omega^\sigma} = 0. \quad (30)$$



From the definition of  $g^\sigma$ , we see that  $\ell^\infty(\tilde{g}^\sigma) = g^\sigma$ . The change of variable  $x = \mathcal{M}_\sigma y$  yields that

$$\int_{\Omega} |\nabla \tilde{g}^\sigma|^q = \int_{\Omega^\sigma} |\nabla \tilde{g}^\sigma|^q = 2^{n\frac{q}{2}} a^{(2-q)n} \int_{\Omega} |\nabla \tilde{g}^0|^q = 2^{n\frac{q}{2}} a^{(2-q)n} \int_{Y^0} |\nabla \tilde{g}^0|^q. \quad (31)$$

Moreover, the support of  $\nabla \tilde{g}^\sigma$  is contained in  $Y^\sigma$ , which implies that for all positive integer  $n$  and  $n'$ , for all  $\sigma \in \mathcal{A}_n$ ,  $\eta \in \mathcal{A}_{n'}$ ,

- the supports of  $\nabla \tilde{g}^\sigma$  and  $\nabla \tilde{g}^0$  are disjoint.
- if  $(n, \sigma) \neq (n', \eta)$  then the supports of  $\nabla \tilde{g}^\sigma$  and  $\nabla \tilde{g}^\eta$  are disjoint.

We also introduce a function  $\chi$ ,  $\chi \in \mathcal{V}_q(\Omega)$  for all  $1 \leq q < \infty$ , such that  $\chi|_{\Omega^1} = 1$  and that  $\chi(x_1, x_2) = \chi(-x_1, x_2)$ . It is clear that  $\ell^\infty(\chi) = 1$ . Moreover, it can be seen that

$$\left\{ \begin{array}{l} \int_{\Omega} \nabla \chi \cdot \nabla \tilde{g}^0 = 0, \\ \text{and} \quad \int_{\Omega} \nabla \chi \cdot \nabla \tilde{g}^\sigma = 0 \end{array} \right. \quad \text{for all integer } n > 0 \text{ and all } \sigma \in \mathcal{A}_n. \quad (32)$$

Consider now  $v \in J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ . We can expand  $v$  on the Haar basis:

$$v = P_0 v + \beta_0 g^0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma} g^\sigma,$$

where  $\beta_0 = \int_{\Gamma^\infty} v g^0 d\mu$  and  $\beta_{n,\sigma} = \int_{\Gamma^\infty} v g^\sigma d\mu$ . We know from Theorem 1 and Corollary 4 (see Remark 5) that

$$|\beta_0|^q + \sum_{n=1}^{\infty} 2^{n\frac{q}{2}} a^{(2-q)n} \sum_{\sigma \in \mathcal{A}_n} |\beta_{n,\sigma}|^q < \infty. \quad (33)$$

Let us define the function  $\tilde{v} : \Omega \rightarrow \mathbb{R}$ , by

$$\tilde{v} = (P_0 v) \chi + \beta_0 \tilde{g}^0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma} \tilde{g}^\sigma, \quad (34)$$

where we agree to identify the function  $P_0 v$  with its real value. From a classical inequality, we obtain that

$$\int_{\Omega} |\nabla \tilde{v}|^q \lesssim |P_0 v|^q \int_{\Omega} |\nabla \chi|^q + \int_{\Omega} \left| \beta_0 \nabla \tilde{g}^0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma} \nabla \tilde{g}^\sigma \right|^q.$$

From (31) and the observation on the supports of the functions  $\nabla \tilde{g}^\sigma$ , this yields that

$$\int_{\Omega} |\nabla \tilde{v}|^q \lesssim |P_0 v|^q \int_{\Omega} |\nabla \chi|^q + \left( |\beta_0|^q + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} 2^{n\frac{q}{2}} a^{(2-q)n} |\beta_{n,\sigma}|^q \right) \int_{Y^0} |\nabla \tilde{g}^0|^q,$$

which is finite from (33), and the right hand sides of the two above estimates coincide. Therefore,  $\tilde{v} \in \mathcal{V}_q(\Omega)$ , and it is an easy matter to see that  $\ell^\infty(\tilde{v}) = v$  using the continuity of  $\ell^\infty$  in  $W^{1,q}(\Omega)$ , see Proposition 3. Finally, (27) is a consequence of Theorem 1 and Corollary 4, (see Remark 5).  $\blacksquare$

**Theorem 7** Assume that  $1/2 \leq a \leq a^*$ . For all  $q$ ,  $1 < q < \infty$  and  $v \in W^{1,q}(\Omega)$ ,  $\ell^\infty(v)$  belongs to  $J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$  and there exists a constant  $c$  such that

$$\|\ell^\infty(v)\|_{J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)} \leq c \|v\|_{W^{1,q}(\Omega)}, \quad \forall v \in W^{1,q}(\Omega). \quad (35)$$

The proof of Theorem 7 is postponed to the end of the section. It makes use of the strengthened trace inequality stated in Theorem 8 below. The following lemma (see, e.g., [4], p. 126) will play an important role in the proof of Theorem 8.

**Lemma 3 (Peetre–Tartar)** *Let  $V, V_1, V_2, W$  be Banach spaces, and let  $A_i \in \mathcal{L}(V, V_i)$  be continuous linear maps for  $i = 1, 2$ , the map  $A_1$  being compact. Suppose there exists  $c_0 > 0$  such that*

$$\|v\|_V \leq c_0(\|A_1 v\|_{V_1} + \|A_2 v\|_{V_2}),$$

*for any  $v \in V$ . In addition, let  $L \in \mathcal{L}(V, W)$  be a continuous linear map such that*

$$L|_{\ker A_2} \equiv 0.$$

*Then there exists  $C > 0$  such that*

$$\|Lv\|_W \leq C\|A_2 v\|_{V_2},$$

*for any  $v \in V$ .*

**Theorem 8** *Assume that  $a \geq 1/2$ .*

1. *For all real number  $\rho$  such that*

$$(2a^2)^{q-1} < \rho < 1, \tag{36}$$

*there exists a constant  $C$  such that for all  $v \in \mathcal{V}_q(\Omega)$ ,*

$$\|\ell^\infty(v)\|_{L_\mu^q}^q \leq C \left( \|\nabla v\|_{L^q(Y^0)}^q + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q \right). \tag{37}$$

2. *For all real number  $\rho$  satisfying (36), there exists a constant  $C$  such that, for all  $v \in W^{1,q}(\Omega)$ ,*

$$\|\ell^\infty(v) - P_0(\ell^\infty(v))\|_{L_\mu^q}^q \leq C \left( \|\nabla v\|_{L^q(Y^0)}^q + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q \right). \tag{38}$$

**Proof.**

The proofs in the cases when  $a = 1/2$  and  $a > 1/2$  slightly differ. Hereafter, we thoroughly discuss the case  $a > 1/2$ . For the ease of the reader, the case  $a = 1/2$  will be discussed in the appendix A. In both cases, the proof of Theorem 8 is done in three steps.

**The case when  $a > 1/2$**

**Step 1** The first step consists of proving (37) in one of the simplest case introduced in § 2.1.2: the two similitudes are the maps  $\tilde{F}_1$  and  $\tilde{F}_2$

$$\begin{aligned} \tilde{F}_1(x) &= (a \cos \theta - 1 + a \cos \theta x_1 - a \sin \theta x_2, b + a \sin \theta + a \sin \theta x_1 + a \cos \theta x_2), \\ \tilde{F}_2(x) &= (1 - a \cos \theta + a \cos \theta x_1 + a \sin \theta x_2, b + a \sin \theta - a \sin \theta x_1 + a \cos \theta x_2), \end{aligned} \tag{39}$$

for  $\theta$ ,  $0 < \theta < \pi/2$ ,  $1/2 < a \leq a^*$ ,  $b > 0$ . This corresponds to (1) with  $\alpha = 1 - 2a \cos \theta$  and  $\beta = b + a \sin \theta$ .

We recall the construction of  $\tilde{\Omega}$  described in § 2.1.2. We take  $P_1 = (-1, 0)$ ,  $P_2 = (1, 0)$ , and define  $P_3 = \tilde{F}_1(P_1) = (-1, b)$ ,  $P_4 = \tilde{F}_2(P_2) = (1, b)$ ,  $P_5 = \tilde{F}_1(P_2) = (2a \cos \theta - 1, b + 2a \sin \theta)$  and  $P_6 = \tilde{F}_2(P_1) = (1 - 2a \cos \theta, b + 2a \sin \theta)$ . Let  $\tilde{Y}^0$  be the open hexagonal subset of  $\mathbb{R}^2$  defined as the convex hull of these six points.

$$\tilde{Y}^0 = \text{Interior} \left( \text{Conv}(P_1, P_2, P_3, P_4, P_5, P_6) \right).$$

For  $n \geq 1$  and  $\sigma \in \mathcal{A}_n$ , we define

$$\tilde{\mathcal{M}}_\sigma = \tilde{F}_{\sigma(1)} \circ \cdots \circ \tilde{F}_{\sigma(n)}$$

and

$$\tilde{\Omega} = \text{Interior} \left( \overline{\tilde{Y}^0 \cup \left( \bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \tilde{\mathcal{M}}_\sigma \tilde{Y}^0 \right)} \right)$$

The domain  $\tilde{\Omega}$  is displayed in Figure 3 for  $\theta = \pi/4$ ,  $b = 1$  and  $a \simeq 0.593465$ .

The proof of (37) in this particular case is done by explicitly constructing a measure preserving and one to one mapping from  $\tilde{\Omega}$  onto the fractured set  $\hat{\Omega}$  displayed in Figure 5. The important feature of  $\hat{\Omega}$  is that it lies under the graph of a function. Consider the piecewise linear map  $H$ :

$$H(x_1, x_2) = \begin{cases} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x_2 \leq b + 2a \sin \theta(1 - |x_1|), \\ \begin{pmatrix} 1 + \frac{1}{2a \sin \theta}(b - x_2) \\ b + 2a \sin \theta(x_1 - 1) + 2(x_2 - b) \end{pmatrix} & \text{if } x_1 > 0, x_2 > b + 2a \sin \theta(1 - x_1), \\ \begin{pmatrix} -1 - \frac{1}{2a \sin \theta}(b - x_2) \\ b - 2a \sin \theta(x_1 + 1) + 2(x_2 - b) \end{pmatrix} & \text{if } x_1 < 0, x_2 > b + 2a \sin \theta(1 + x_1). \end{cases}$$

It can be seen that  $H$  maps the set  $\tilde{Y}^0$  to the fractured domain  $\hat{Y}^0$  displayed on figure 4. By

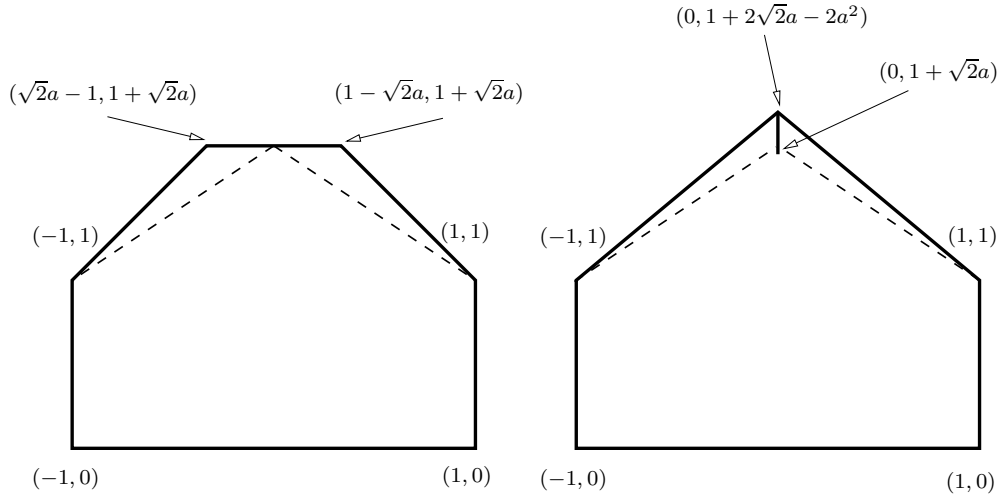


Figure 4: Left: the cell  $\tilde{Y}^0$  for  $\theta = \pi/4$  and  $b = 1$ . Right: the fractured open set  $\hat{Y}^0$ . The Jacobian matrix of  $H$  on the left (resp. of  $H^{-1}$  on the right) jumps across the broken lines.

computing the Jacobian of  $H$  ( $H$  is piecewise smooth), we see that that  $H$  is measure preserving.

Let  $G_1$  and  $G_2$  be the maps in  $\mathbb{R}^2$  defined by

$$\begin{aligned} G_1(x_1, x_2) &= \left( \frac{1}{2}(x_1 - 1), b + (2a \sin \theta - a^2 \sin(2\theta))(1 + x_1) + 2a^2 x_2 \right), \\ G_2(x_1, x_2) &= \left( \frac{1}{2}(x_1 + 1), b + (2a \sin \theta - a^2 \sin(2\theta))(1 - x_1) + 2a^2 x_2 \right). \end{aligned}$$

Note that for  $i = 1, 2$ , the abscissa of  $G_i(x_1, x_2)$  only depends on  $x_1$ . This implies that for  $i = 1, 2$ ,  $G_i$  maps the vertical boundaries of  $\widehat{Y}^0$  to vertical lines. Moreover,  $G_1(P_2) = G_2(P_1)$  is a point on the axis  $\{x_1 = 0\}$ .

We define

$$\widehat{\mathcal{M}}_\sigma = G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n,$$

the sets

$$\begin{aligned} \widehat{Y}^N &= \text{Interior} \left( \widehat{Y}^0 \cup \bigcup_{n=1}^N \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0) \right), \\ \widehat{\Omega} &= \text{Interior} \left( \widehat{Y}^0 \cup \bigcup_{n=1}^\infty \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0) \right), \end{aligned}$$

see Figure 5, and the one to one mapping

$$\chi^N : \begin{cases} \widehat{Y}^N & \rightarrow \widetilde{Y}^N, \\ x & \mapsto \mathcal{M}_\sigma \circ H^{-1} \circ \widehat{\mathcal{M}}_\sigma^{-1}(x) \quad \text{if } x \in \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0). \end{cases}$$

Note that  $\chi^N$  is a piecewise linear function and that the Jacobian of  $\chi^N$  is almost everywhere 1. Moreover, take  $\sigma \in \mathcal{A}_n$  with  $n \leq N$ ,  $(x_1, x_2) \in \widehat{\mathcal{M}}_\sigma(Y^0)$  and  $h \in \mathbb{R}$  such that  $(x_1, x_2 + h) \in \widehat{\mathcal{M}}_\sigma(Y^0)$ . We aim at bounding  $|\chi^N(x_1, x_2 + h) - \chi^N(x_1, x_2)|$ : call  $(z_1, z_2) = \widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2)$ . It can be easily seen that  $\widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2 + h) = (z_1, z_2 + (2a^2)^{-n}h)$ . Therefore,

$$\begin{aligned} |\chi^N(x_1, x_2 + h) - \chi^N(x_1, x_2)| &= |\mathcal{M}_\sigma \circ H^{-1}(z_1, z_2 + (2a^2)^{-n}h) - \mathcal{M}_\sigma \circ H^{-1}(z_1, z_2)| \\ &\leq C_H a^n (2a^2)^{-n} |h| = C_H (2a)^{-n} |h|, \end{aligned}$$

where the constant  $C_H$  is the norm of  $H^{-1}$ , and where we have used the fact that  $\mathcal{M}_\sigma$  is a similitude with dilation ratio  $a^n$ . But  $2a > 1$ . Passing to the limit as  $h$  tends to zero, we see that

$$\left\| \frac{\partial \chi^N}{\partial x_2} \right\|_\infty \leq C_H. \quad (40)$$

Note that  $\widehat{Y}^N$  is contained in the rectangle  $[-1, 1] \times [0, \zeta]$ , where

$$\zeta = (b + 4a \sin \theta - 2a^2 \sin(2\theta)) \sum_{n=0}^{\infty} (2a^2)^n = \frac{b + 4a \sin \theta - 2a^2 \sin(2\theta)}{1 - 2a^2}.$$

Moreover  $\widehat{Y}^N$  has  $I^N = 2 + \sum_{n=0}^N 2^n$  vertical boundaries, (among which  $\sum_{n=0}^N 2^n$  vertical fractures) see Figure 5. We order increasingly the abscissa  $(\alpha_i)_{i=1, \dots, I^N}$  of these vertical segments, and we have  $\alpha_{i+1} - \alpha_i = 2^{-N}$ ,  $i = 1, \dots, I^N - 1$ . Notice also that  $\widehat{Y}^N$  can be seen as the epigraph of a function  $\Phi^N : (-1, 1) \mapsto \mathbb{R}_+$ , and that  $\Phi^N$  is discontinuous at  $\alpha_i$ ,  $i = 2, \dots, I^N - 1$ , and linear in the intervals  $(\alpha_i, \alpha_{i+1})$ ,  $i = 1, \dots, I^N - 1$ . Another important and natural property is that the sequence  $(\Phi^N)_N$  is nondecreasing with respect to  $N$ .

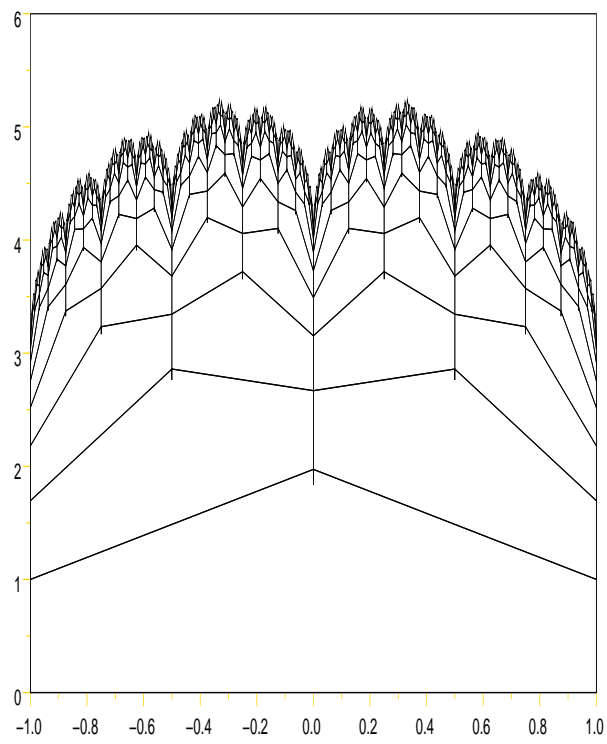


Figure 5: The open set  $\hat{\Omega}$

Call  $\widehat{\Gamma}^\infty$  the self-similar set associated to  $G_1$  and  $G_2$ , ( $\widehat{\Gamma}^\infty$  is part of the boundary of  $\widehat{\Omega}$ ), and  $\widehat{\mu}$  the self-similar measure associated to  $(\widehat{\Gamma}^\infty, G_1, G_2)$ . As above, it is possible to define a trace operator  $\widehat{\ell}^\infty$  from  $W^{1,q}(\widehat{\Omega})$  to  $L^q(\widehat{\Gamma}^\infty)$ .

Consider a function  $v \in C^\infty(\Omega)$  such that  $v|_{\Gamma^0} = 0$ . Since  $\chi^N$  is piecewise affine and continuous, and since  $|\widetilde{\Gamma}^{N+1}| = 2^{N+1}|\widetilde{\Gamma}^\sigma|$ , for all  $\sigma \in \mathcal{A}_{N+1}$ ,

$$\frac{1}{|\widetilde{\Gamma}^{N+1}|} \int_{\widetilde{\Gamma}^{N+1}} |v|^q = \sum_{\sigma \in \mathcal{A}_{N+1}} \frac{1}{2^{N+1}|\widetilde{\Gamma}^\sigma|} \int_{\widetilde{\Gamma}^\sigma} |v|^q = \sum_{\sigma \in \mathcal{A}_{N+1}} \frac{1}{2^{N+1}|\widehat{\Gamma}^\sigma|} \int_{\widehat{\Gamma}^\sigma} |v \circ \chi^N|^q.$$

On the other hand,

$$\sum_{\sigma \in \mathcal{A}_{N+1}} \frac{1}{2^{N+1}|\widehat{\Gamma}^\sigma|} \int_{\widehat{\Gamma}^\sigma} |v \circ \chi^N|^q = \frac{1}{2} \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} |v \circ \chi^N(x, \Phi^N(x))|^q dx.$$

Therefore,

$$\begin{aligned} \frac{2}{|\widetilde{\Gamma}^{N+1}|} \int_{\widetilde{\Gamma}^{N+1}} |v|^q &= \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} |v(\chi^N(x, \Phi^N(x)))|^q dx \\ &= \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left| \int_0^{\Phi^N(x)} \frac{d}{dt}(v(\chi^N(x, t))) dt \right|^q dx \\ &= \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left| \sum_{j=0}^N \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \frac{d}{dt}(v(\chi^N(x, t))) dt \right|^q dx. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{2}{|\widetilde{\Gamma}^{N+1}|} \int_{\widetilde{\Gamma}^{N+1}} |v|^q \\ &\leq \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{\frac{j}{q-1}} \right)^{q-1} \left( \sum_{j=0}^N \rho^{-j} \left| \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \frac{d}{dt}(v(\chi^N(x, t))) dt \right|^q \right) dx \\ &\lesssim \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \sum_{j=0}^N \rho^{-j} \left| \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \frac{d}{dt}(v(\chi^N(x, t))) dt \right|^q dx \\ &\leq \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{-j} (\Phi^j(x) - \Phi^{j-1}(x))^{q-1} \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \left| \frac{d}{dt}(v(\chi^N(x, t))) \right|^q dt \right) dx \\ &\lesssim \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{-j} (2a^2)^{j(q-1)} \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \left| \frac{d}{dt}(v(\chi^N(x, t))) \right|^q dt \right) dx \end{aligned}$$

where we have used several Hölder's inequalities. This yields that

$$\begin{aligned} & \frac{2}{|\tilde{\Gamma}^{N+1}|} \int_{\tilde{\Gamma}^{N+1}} |v|^q \\ & \lesssim \left( \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{-j} (2a^2)^{j(q-1)} \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \left| \frac{\partial v}{\partial x_1} (\chi^N(x, t)) \frac{\partial \chi_1^N}{\partial t} (x, t) \right|^q dt \right) dx \right. \\ & \quad \left. + \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{-j} (2a^2)^{j(q-1)} \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} \left| \frac{\partial v}{\partial x_2} (\chi^N(x, t)) \frac{\partial \chi_2^N}{\partial t} (x, t) \right|^q dt \right) dx \right). \end{aligned}$$

Using (40),

$$\frac{1}{|\tilde{\Gamma}^{N+1}|} \int_{\tilde{\Gamma}^{N+1}} |v|^q \lesssim \sum_{i=0}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \left( \sum_{j=0}^N \rho^{-j} (2a^2)^{j(q-1)} \int_{\Phi^{j-1}(x)}^{\Phi^j(x)} |\nabla v(\chi^N(x, t))|^q dt \right) dx.$$

Performing the inverse change of variables, one obtains that

$$\frac{1}{|\tilde{\Gamma}^{N+1}|} \int_{\tilde{\Gamma}^{N+1}} |v|^q \leq C \left( \int_{\tilde{Y}^0} |\nabla v|^q + \sum_{n=1}^N \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \int_{\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0)} |\nabla v|^q \right),$$

where the constant  $C$  can be chosen independently of  $N$ .

Finally, since  $\frac{1}{|\tilde{\Gamma}^{N+1}|} \int_{\tilde{\Gamma}^{N+1}} |v|^q$  tends to  $\int_{\tilde{\Gamma}^\infty} |\tilde{\ell}^\infty(v)|^q d\tilde{\mu}$  as  $N \rightarrow \infty$ , we find that

$$\int_{\tilde{\Gamma}^\infty} |\tilde{\ell}^\infty(v)|^q d\tilde{\mu} \leq C \left( \int_{\tilde{Y}^0} |\nabla v|^q + \sum_{n=1}^\infty \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \int_{\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0)} |\nabla v|^q \right), \quad (41)$$

which is the desired result. We conclude by using the density of  $\{v \in \mathcal{C}^\infty(\Omega); v|_{\Gamma^0} = 0\}$  in  $\mathcal{V}_q(\Omega)$ .

**Step 2** We aim at proving (38) for the special geometry described in Step 1 by using Lemma 3. Let  $V_2$  be the Banach space

$$V_2 = \left\{ v : \tilde{\Omega} \rightarrow \mathbb{R}^2, v \text{ measurable } \|v\|_{L^q(\tilde{Y}^0)}^q + \sum_{n=1}^\infty \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|v\|_{L^q(\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0))}^q < \infty \right\},$$

and let  $V$  be the Banach space

$$V = \left\{ v \in L_{loc}^q(\tilde{\Omega}), v|_{\tilde{Y}^0} \in L^q(\tilde{Y}^0), \nabla v \in V_2 \right\},$$

endowed with the norm

$$\|v\|_V = \left( \left| \int_{\tilde{\Gamma}^0} v|_{\tilde{\Gamma}^0} \right|^q + \|\nabla v\|_{L^q(\tilde{Y}^0)}^q + \sum_{n=1}^\infty \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|_{L^q(\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0))}^q \right)^{1/q}.$$

**Remark 7** It can be proved that

$$V = \left\{ v \in L^q(\tilde{\Omega}), \nabla v \in V_2 \right\}.$$

It is clear from (36) that  $W^{1,q}(\tilde{\Omega})$  is a subspace of  $V$ .

Furthermore, we will prove at the end of the paragraph that  $W^{1,q}(\tilde{\Omega})$  is dense in  $V$ .

As a consequence of (37), one can extend the operator  $\tilde{\ell}^\infty$  to a continuous linear operator from  $V$  to  $W = L_\mu^q = L^q(\tilde{\Gamma}^\infty, d\tilde{\mu})$ , that we still denote  $\tilde{\ell}^\infty$ .

Let  $\tilde{P}_0$  be the projection on constants:  $\tilde{P}_0 v = \int_{\tilde{\Gamma}^\infty} v d\tilde{\mu}$ . It is straightforward to see that the operator  $L : v \mapsto \tilde{\ell}^\infty(v) - \tilde{P}_0(\tilde{\ell}^\infty(v))$  is also a continuous linear operator from  $V$  to  $W = L_\mu^q$ .

Define  $V_1$  as the space of the constant functions on  $\tilde{\Omega}$ . We introduce the linear operators

$$A_1 : V \mapsto \mathbb{R}, \quad A_1 v = \int_{\tilde{\Gamma}_0} v,$$

and

$$A_2 : V \mapsto V_2, \quad A_2 v = \nabla v.$$

We can apply Lemma 3 since  $L|_{\ker A_2} \equiv 0$ . Therefore, there exists a constant  $C > 0$  such that

$$\|\tilde{\ell}^\infty(v) - \tilde{P}_0(\tilde{\ell}^\infty(v))\|_{L_\mu^q}^q \leq C \left( \|\nabla v\|_{L^q(\tilde{Y}^0)}^q + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \|\nabla v\|_{L^q(\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0))}^q \right), \quad \forall v \in V, \quad (42)$$

which implies (38) in the case when  $\Omega = \tilde{\Omega}$ , since  $W^{1,q}(\tilde{\Omega}) \subset V$ .

**Proof of the density of  $W^{1,q}(\tilde{\Omega})$  in  $V$ .** We introduce a smooth real valued function  $\phi$  defined on  $\tilde{Y}^0$  such that

$$\begin{cases} 0 \leq \phi \leq 1, & \text{in } \tilde{Y}^0, \\ \phi = 1, & \text{on } \tilde{\Gamma}^0, \\ \phi = 0, & \text{on } \tilde{\Sigma}^0, \end{cases}$$

where

$$\tilde{\Sigma}^0 = \{x \in \partial \tilde{Y}^0 \text{ such that } x_2 > b\}.$$

For  $n > 0$  and  $\sigma \in \mathcal{A}_n$ , we also introduce the function  $\phi_\sigma$  defined on  $\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0)$  by  $\phi_\sigma = \phi \circ (\tilde{\mathcal{M}}_\sigma)^{-1}$ .

For a function  $v$  in  $V$ , we define  $v_n$  by:

$$\begin{cases} v_n|_{\tilde{Y}^{n-1}} &= v|_{\tilde{Y}^{n-1}}, \\ v_n|_{\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0)} &= \phi_\sigma v|_{\tilde{\mathcal{M}}_\sigma(\tilde{Y}^0)} + (1 - \phi_\sigma) \langle v \rangle_\sigma, \quad \forall \sigma \in \mathcal{A}_n, \\ v_n|_{\tilde{\Omega}^\eta} &= \langle v \rangle_\sigma, \quad \text{for } \eta = \sigma \circ \tilde{F}_i, \sigma \in \mathcal{A}_n, i = 1, 2, \end{cases}$$

where

$$\langle v \rangle_\sigma = \frac{1}{|\tilde{\Sigma}^\sigma|} \int_{\tilde{\Sigma}^\sigma} v \quad \text{and} \quad \tilde{\Sigma}^\sigma = \tilde{\mathcal{M}}_\sigma(\tilde{\Sigma}^0).$$

It is easy to see that  $u_n \in W^{1,q}(\tilde{\Omega})$ . Moreover,  $\lim_{n \rightarrow \infty} \|u - u_n\|_V = 0$ ; indeed,

$$\nabla u - \nabla u_n = \begin{cases} 0 & \text{in } \tilde{Y}^{n-1}, \\ \phi_\sigma \nabla u + (u - \langle u \rangle_\sigma) \nabla \phi_\sigma & \text{in } \tilde{\mathcal{M}}_\sigma(\tilde{Y}^0), \forall \sigma \in \mathcal{A}_n, \\ \nabla u & \text{in } \tilde{\Omega} \setminus \tilde{Y}^n. \end{cases}$$



But  $\|\nabla\phi_\sigma\|_\infty = a^{-n}\|\nabla\phi\|_\infty$ . Therefore

$$\begin{aligned} \|u - u_n\|_V^q &\leq c_q \left( \sum_{p \geq n} \rho^{-p} (2a^2)^{p(q-1)} \sum_{\sigma \in \mathcal{A}_p} \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} |\nabla u|^q \right. \\ &\quad \left. + \rho^{-n} (2a^2)^{n(q-1)} \|\nabla\phi\|_\infty^q \sum_{\sigma \in \mathcal{A}_n} \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} a^{-nq} |u - \langle u \rangle_\sigma|^q \right) \\ &\leq c_q \left( \sum_{p \geq n} \rho^{-p} (2a^2)^{p(q-1)} \sum_{\sigma \in \mathcal{A}_p} \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} |\nabla u|^q + C_q \rho^{-p} (2a^2)^{n(q-1)} \|\nabla\phi\|_\infty^q \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} |\nabla u|^q \right), \end{aligned}$$

using the Poincaré-Wiertringer inequality in  $\widetilde{\mathcal{M}}(\widetilde{Y}^0)$ :

$$\int_{\widetilde{\mathcal{M}}(\widetilde{Y}^0)} |u - \langle u \rangle|^q \leq C_q a^{nq} \int_{\widetilde{\mathcal{M}}(\widetilde{Y}^0)} |\nabla u|^q,$$

and where the constants  $c_q$  and  $C_q$  do not depend on  $n$ . From this and Lebesgue theorem, we see that  $\lim_{n \rightarrow \infty} \|u - u_n\|_V = 0$ .

**Step 3** We now aim at proving (37) and (38) in the general case, i.e. for  $Y^0$  satisfying the assumptions made in § 2. For that, we are going to construct a self-similar piecewise linear map from the domain  $\Omega$  to  $\widetilde{\Omega}$ . It is easy to see that there exist

1. a partition  $\mathcal{T}$  of  $Y^0$  into  $p$  non-overlapping triangles, such that  $\Gamma^0$ ,  $F_1(\Gamma^0)$ ,  $F_2(\Gamma^0)$  are the whole side of a triangle in the partition.
2. a partition  $\widetilde{\mathcal{T}}$  of  $\widetilde{Y}^0$  into  $p$  non-overlapping triangles, such that  $\widetilde{\Gamma}^0$ ,  $\widetilde{F}_1(\widetilde{\Gamma}^0)$ ,  $\widetilde{F}_2(\widetilde{\Gamma}^0)$  are the whole side of a triangle in the partition.
3. a continuous, one to one and piecewise linear function  $\psi$  from  $Y^0$  onto  $\widetilde{Y}^0$ , such that
  - its restriction to the triangles in  $\mathcal{T}$  is linear,
  - $\psi$  maps each triangle in  $\mathcal{T}$  to a triangle in  $\widetilde{\mathcal{T}}$ ,
  - the restriction of  $\psi$  to  $\Gamma^0$  is the identity,
  - for  $i = 1, 2$ ,

$$F_i(\psi^{-1}(x)) = \psi^{-1}(\widetilde{F}_i(x)), \quad \forall x \in \widetilde{\Gamma}^0.$$

An example of such a construction is shown in Figure 6.

This construction allows for the definition of the continuous linear operator

$$\begin{aligned} \Psi : W^{1,q}(\Omega) &\rightarrow W^{1,q}(\widetilde{\Omega}), \\ \Psi(u) = \widetilde{u}, \quad \widetilde{u} \text{ defined by } \begin{cases} \widetilde{u}|_{\widetilde{Y}^0} &= u|_{Y^0} \circ \psi^{-1}, \\ \widetilde{u}|_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} &= u|_{\mathcal{M}_\sigma(Y^0)} \circ \mathcal{M}_\sigma \circ \psi^{-1} \circ (\widetilde{\mathcal{M}}_\sigma)^{-1}, \end{cases} \quad \sigma \in \mathcal{A}_n, n \geq 1. \end{aligned} \quad (43)$$

Using the same argument as in the proof of Theorem 4 in [2], we have that

$$\|\ell^\infty(v)\|_{L_\mu^q} \lesssim \|\widetilde{\ell}^\infty(\Psi(v))\|_{L_\mu^q}, \quad \forall v \in \mathcal{V}_q(\Omega). \quad (44)$$

From (44) and (41), we see that

$$\|\ell^\infty(v)\|_{L_\mu^q}^q \lesssim \int_{\widetilde{Y}^0} |\nabla(\Psi(v))|^q + \sum_{n=1}^{\infty} \rho^{-n} (2a^2)^{n(q-1)} \sum_{\sigma \in \mathcal{A}_n} \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0)} |\nabla(\Psi(v))|^q, \quad \forall v \in \mathcal{V}_q(\Omega). \quad (45)$$

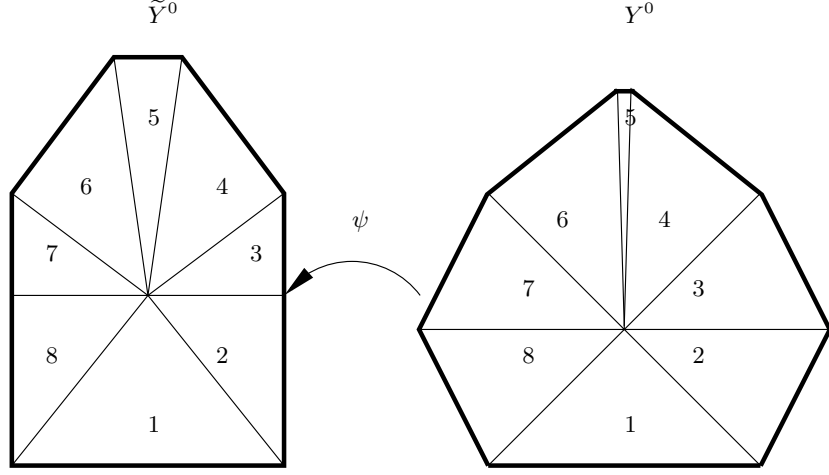


Figure 6: The domains  $Y^0$  and  $\tilde{Y}^0$  are partitioned into eight triangles

The inverse change of variables yields (37).

Similarly, the proof of (38) follows from the observation that  $P_0(\ell^\infty(v)) = \tilde{P}_0(\tilde{\ell}^\infty(\Psi(v)))$  and from (42).

**The case  $a = 1/2$ .** See Appendix A. ■

**Proof of Theorem 7** Call  $f = \ell^\infty(v)$ . Recall that

$$\|f\|_{JLip(1-\frac{2-d}{q}, q, q; 0; \Gamma^\infty)}^q = \|f\|_{L_\mu^q}^q + \int_{\Gamma^\infty} |f - P_0 f|^q d\mu + \sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu. \quad (46)$$

We have to prove that the right side of (46) is finite. A change of variables leads to

$$\begin{aligned} & \sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \\ &= \sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} 2^{-n} \int_{\Gamma^\infty} |f \circ \mathcal{M}_\sigma - P_0(f \circ \mathcal{M}_\sigma)|^q d\mu. \end{aligned}$$

This and Theorem 8 imply that

$$\begin{aligned} & \sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \\ & \lesssim \sum_{n=1}^{\infty} a^{n(2-q)} \sum_{\sigma \in \mathcal{A}_n} \left( \|\nabla(v \circ \mathcal{M}_\sigma)\|_{L^q(Y^0)}^q + \sum_{p=1}^{\infty} \rho^{-p} (2a^2)^{p(q-1)} \sum_{\eta \in \mathcal{A}_p} \|\nabla(v \circ \mathcal{M}_\sigma)\|_{L^q(\mathcal{M}_\eta(Y^0))}^q \right) \\ & = \sum_{n=1}^{\infty} a^{n(2-q)} \sum_{\sigma \in \mathcal{A}_n} \left( a^{n(q-2)} \left( \|\nabla v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q + \sum_{p=1}^{\infty} \rho^{-p} (2a^2)^{p(q-1)} \sum_{\eta \in \mathcal{A}_p} \|\nabla v\|_{L^q(\mathcal{M}_{\sigma \circ \eta}(Y^0))}^q \right) \right) \end{aligned}$$

where  $\rho$  is such that  $(2a^2)^{q-1} < \rho < 1$ . Thus,

$$\sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} |f - P_\sigma f|^q d\mu \lesssim \sum_{i=1}^{\infty} \rho^{-i} (2a^2)^{i(q-1)} \left( \sum_{n=1}^i \rho^n (2a^2)^{n(1-q)} \right) \sum_{\mu \in \mathcal{A}_i} \|\nabla v\|_{L^q(\mathcal{M}_\mu(Y^0))}^q.$$

From  $\rho(2a^2)^{1-q} > 1$ , we deduce that  $\sum_{n=1}^i \rho^n (2a^2)^{n(1-q)} \lesssim \rho^i (2a^2)^{i(1-q)}$ . Thus,

$$\sum_{n=1}^{\infty} (2a^{2-q})^n \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_{\sigma}(\Gamma^{\infty})} |f - P_{\sigma}f|^q d\mu \lesssim \sum_{i=1}^{\infty} \sum_{\mu \in \mathcal{A}_i} \|\nabla v\|_{L^q(\mathcal{M}_{\mu}(Y^0))}^q,$$

and the desired result follows easily. ■

## 6 Traces of functions belonging to $W^{m+1,q}(\Omega)$ for a positive integer $m$ .

We now assume that  $a \geq 1/2$  and that  $\Gamma^{\infty}$  is not contained in a straight line. We use the notations defined in § 4.2. To keep the length of the paper reasonable, some proofs will be omitted or just sketched.

**Proposition 6** *For all real number  $q \geq 1$ , there exists a constant  $C$  such that for all  $f \in J\text{Lip}(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^{\infty})$ , there exists  $\tilde{f} \in W^{m+1,q}(\Omega)$  with the properties  $\tilde{f}|_{Y^0} = 0$ ,  $\ell^{\infty}(\tilde{f}) = f$  and*

$$\|\tilde{f}\|_{W^{m+1,q}(\Omega)}^q \approx \int_{\Omega} |D^{m+1}\tilde{f}|^q \leq C \|f\|_{J\text{Lip}(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^{\infty})}^q. \quad (47)$$

**Proof.** The proof resembles very much that of Proposition 5. It mainly consists of constructing functions  $\tilde{g}^{0,i} \in W^{m+1,q}(\Omega)$ ,  $i = 1, \dots, M_0$  such that

- $\tilde{g}^{0,i}|_{Y^0} = 0$ ,
- $\tilde{g}^{0,i}|_{\mathcal{M}_{\sigma}(\Omega)}$  belongs to  $\mathcal{P}_m$ , for all  $\sigma \in \mathcal{A}_2$ ,
- $\ell^{\infty}(\tilde{g}^{0,i}) = g^{0,i}$ .

For a positive integer  $n$  and  $\sigma \in \mathcal{A}_n$ , let us define the function  $\tilde{g}^{\sigma,i}$  by:

$$\tilde{g}^{\sigma,i}|_{\Omega^{\sigma}} = 2^{n/2} \tilde{g}^{0,i} \circ \mathcal{M}_{\sigma}^{-1}, \quad (48)$$

$$\tilde{g}^{\sigma,i}|_{\Omega \setminus \Omega^{\sigma}} = 0. \quad (49)$$

From the definition of  $g^{\sigma,i}$ , we see that  $\ell^{\infty}(\tilde{g}^{\sigma,i}) = g^{\sigma,i}$ . For a fixed  $i \in \{1, \dots, M_0\}$ ,

- the supports of  $D^{m+1}\tilde{g}^{\sigma,i}$  and  $D^{m+1}\tilde{g}^{0,i}$  are disjoint,
- if  $(n, \sigma) \neq (n', \sigma')$  then the supports of  $D^{m+1}\tilde{g}^{\sigma,i}$  and  $D^{m+1}\tilde{g}^{\sigma',i}$  are disjoint.

Consider also  $M_0$  functions  $\tilde{\phi}_i \in W^{m+1,q}(\Omega)$  such that

- $\tilde{\phi}_i|_{Y^0} = 0$ ,
- $\ell^{\infty}(\tilde{\phi}_i) = \phi^i$ .

For a function  $f \in J\text{Lip}(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^{\infty})$  given by the expansion (24), it can be shown as in the proof of Proposition 5 that the function  $\tilde{f}$  given by

$$\tilde{f} = \sum_{i=1}^{M_0} \lambda_i \tilde{\phi}_i + \sum_{i=1}^{M_0} \beta_{0,i} \tilde{g}^{0,i} + \sum_{i=1}^{M_0} \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_{n,\sigma,i} \tilde{g}^{\sigma,i}$$

has the desired properties. ■

We have the analogue of Theorem 8:

**Theorem 9** For all  $q$ ,  $1 < q < \infty$  and all real number  $\rho$  satisfying (36), there exists a constant  $C$  such that, for all  $v \in W^{1,q}(\Omega)$ ,

$$\|\ell^\infty(v) - P_0^{(m)}(\ell^\infty(v))\|_{L_\mu^q}^q \leq C \left( \|D^{m+1}v\|_{L^q(Y^0)}^q + \sum_{n=1}^{\infty} \rho^{-n(m+1)} (2a^2)^{n((m+1)q-1)} \sum_{\sigma \in \mathcal{A}_n} \|D^{m+1}v\|_{L^q(\mathcal{M}_\sigma(Y^0))}^q \right). \quad (50)$$

**Proof.** We skip the proof, because it is long and resembles that of Theorem 8. ■

Theorem 9 makes it possible to prove the analogue of Theorem 7:

**Theorem 10** For all  $q$ ,  $1 < q < \infty$  and  $v \in W^{m+1,q}(\Omega)$ ,  $\ell^\infty(v)$  belongs to  $JLip(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^\infty)$  and there exists a constant  $c$  such that

$$\|\ell^\infty(v)\|_{JLip(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^\infty)} \leq c \|v\|_{W^{m+1,q}(\Omega)}, \quad \forall v \in W^{m+1,q}(\Omega). \quad (51)$$

**Proof.** The proof is very similar to that of Theorem 7. ■

From Proposition 6 and Theorem 10, we obtain that for all  $q$ ,  $1 < q < \infty$ ,

$$\ell^\infty(W^{m+1,q}(\Omega)) = JLip(m+1 - \frac{2-d}{q}, q, q; m; \Gamma^\infty). \quad (52)$$

## Appendix

### A Some details in the proof of Theorem 8 in the case when $a = 1/2$

The proof of Theorem 8 is similar to that given in the case  $a > 1/2$ , except that the set  $\tilde{\Omega}$  and  $\hat{\Omega}$  are defined in a slightly different manner. The set  $\tilde{\Omega}$  is defined in § 2.2.1 and is displayed in Figure 1, so the similitudes are the affine maps ( $\theta = 0$ ) given by (7) with  $a = 1/2$ . The polygonal cell  $\tilde{Y}^0$  is the convex hull of the points  $A = (-1, 0)$ ,  $B = (1, 0)$ ,  $F_2(B)$ , and  $F_1(A)$ .

**Step 1** We limit ourselves to describing the set  $\hat{\Omega}$ . It is possible to define a piecewise affine one to one and measure preserving map  $H$  from  $\tilde{Y}^0$  onto the fractured set

$$\hat{Y}^0 = \text{Interior}((( -1, 1) \times (0, h)) \setminus (\{0\} \times (h', h))),$$

where  $h = \frac{\beta}{2}(\alpha + \frac{3}{2})$ , and  $h'$  is a real number smaller than  $h$ . The map  $H$  is such that

- the restriction of  $H$  to  $\Gamma^0$  is the identity,
- for  $i = 1, 2$ ,  $H$  induces a one to one map from  $F_1(\Gamma^0)$  onto  $[-1, 0] \times \{h\}$  and from  $F_2(\Gamma^0)$  onto  $[0, 1] \times \{h\}$ .

The domains  $\tilde{Y}^0$  and  $\hat{Y}^0$  are shown in Figure 7.

In Figure 7, we also show a partition of  $\tilde{Y}^0$  into triangles corresponding to a possible map  $H$  with  $H$  linear on the triangles. The images of the triangles by  $H$  are triangles which form a partition of  $\hat{Y}^0$ . The points on the figure are  $I = (1 + \frac{1}{8}(\alpha - \frac{1}{2}), \frac{\beta}{8})$ ,  $J = (0, \frac{2\beta}{3})$ ,  $K =$

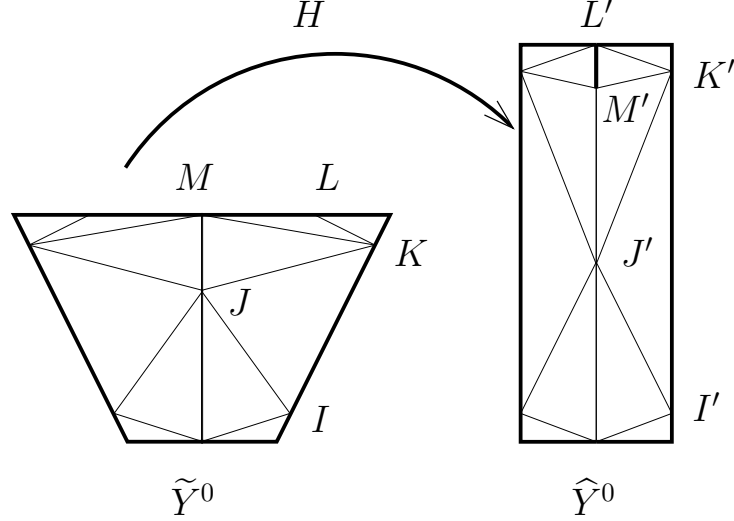


Figure 7: The domains  $\tilde{Y}^0$  (left) and  $\hat{Y}^0$  (right). The restriction of the map  $H$  to the triangles covering  $\tilde{Y}^0$  is linear.

$\left(1 + \frac{7}{8}(\alpha - \frac{1}{2}), \frac{7\beta}{8}\right)$ ,  $L = (\alpha - \frac{1}{2}, \beta)$ ,  $I' = \left(1, \frac{\beta}{8}\right)$ ,  $J' = \left(0, \frac{2\beta}{3}(1 + \frac{1}{8}(\alpha - \frac{1}{2}))\right)$ ,  $K' = \left(1, h - \frac{\beta}{8}\right)$ ,  $M' = \left(0, h - \frac{\beta}{8}(\alpha - \frac{1}{2})\right)$ . In this construction,  $h'$  is  $h' = h - \frac{\beta}{8}(\alpha - \frac{1}{2}) = \beta \left(\frac{3\alpha}{8} + \frac{13}{16}\right)$ .

Let us introduce  $\hat{\mathcal{Q}}^0 = ((-1, 1) \times (0, h]) \setminus (\{0\} \times [h', h])$ . Let  $G_1$  and  $G_2$  be the maps in  $\mathbb{R}^2$  defined by

$$\begin{aligned} G_1(x_1, x_2) &= \left(\frac{1}{2}(x_1 - 1), h + \frac{x_2}{2}\right), \\ G_2(x_1, x_2) &= \left(\frac{1}{2}(x_1 + 1), h + \frac{x_2}{2}\right). \end{aligned}$$

We define

$$\widehat{\mathcal{M}}_\sigma = G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n,$$

the sets

$$\begin{aligned} \hat{Y}^N &= \text{Interior} \left( \hat{\mathcal{Q}}^0 \cup \bigcup_{n=1}^N \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\hat{\mathcal{Q}}^0) \right), \\ \hat{\Omega} &= \text{Interior} \left( \overline{\hat{\mathcal{Q}}^0} \cup \bigcup_{n=1}^\infty \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\hat{\mathcal{Q}}^0) \right), \end{aligned}$$

see Figure 8, and the one to one mapping

$$\chi^N : \begin{cases} \hat{Y}^N \rightarrow \tilde{Y}^N, \\ x \mapsto \mathcal{M}_\sigma \circ H^{-1} \circ \widehat{\mathcal{M}}_\sigma^{-1}(x) \quad \text{if } x \in \widehat{\mathcal{M}}_\sigma(\hat{Y}^0). \end{cases}$$

The set  $\hat{\Omega}$  is obtained by removing an infinite number of vertical segments from the rectangle  $(-1, 1) \times (0, 2h)$ . It is constructed by assembling translated/dilated copies of the set  $\hat{\mathcal{Q}}^0$  displayed on the left of Figure 8. Note that  $\chi^N$  is a piecewise affine function and that the Jacobian of  $\chi^N$  is almost everywhere 1. Similarly, we can construct a one to one map  $\chi^\infty$  from  $\hat{\Omega}$  to  $\tilde{\Omega}$ , which preserves the measure.

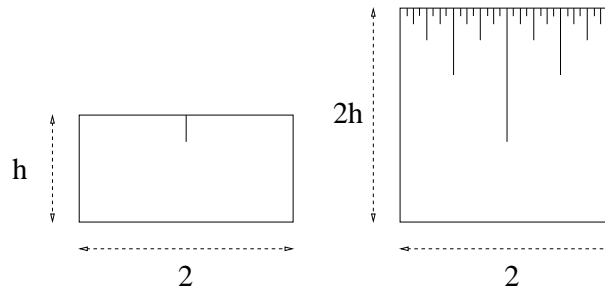


Figure 8: Left: the set  $\widehat{\mathcal{Q}}^0$ . Right: the open set  $\widehat{\Omega}$  (only the longest fractures are displayed)

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