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The Fate of the Landau Levels under Perturbations of Constant Sign

January 13, 2009

FRÉDÉRIC KLOPP, GEORGI RAIKOV

Abstract

We show that the Landau levels cease to be eigenvalues if we perturb the 2D Schrödinger operator with constant magnetic field, by bounded electric potentials of fixed sign. We also show that, if the perturbation is not of fixed sign, then any Landau level may be an eigenvalue of arbitrary, finite or infinite, multiplicity of the perturbed problem.

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1 Introduction. Main results

In this note we consider the Landau Hamiltonian H_0 , i.e. the 2D Schrödinger operator with constant magnetic field. It is well-known that the spectrum of H_0 consists of an arithmetic progression of eigenvalues called *Landau levels* of infinite multiplicity. In Theorem 1 we show that under perturbations by fairly general electric potentials of constant sign, the Landau levels cease to be eigenvalues of the perturbed operator. Moreover, in Theorem 2 we show that for each fixed Landau level there exist non-constant-sign electric potentials such that the Landau level is still an eigenvalue of the perturbed operator.

Let

$$H_0 := \left(-i \frac{\partial}{\partial x} + \frac{by}{2} \right)^2 + \left(-i \frac{\partial}{\partial y} - \frac{bx}{2} \right)^2 - b$$

be the Landau Hamiltonian shifted by the value $b > 0$ of the constant magnetic field. The operator H_0 is self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Note that $C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ is a form core for the operator H_0 . It is well-known (see [3, 6, 1]) that the spectrum $\sigma(H_0)$ of the operator H_0 consists of the so-called Landau levels $2bq$, $q \in \mathbb{N} := \{0, 1, 2, \dots\}$, which are eigenvalues of H_0 of infinite multiplicity. Let $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$. We will suppose that

$$c\chi(x) \leq V(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2, \quad (1.1)$$

where $c > 0$ is a constant and χ is the characteristic function of a disk of radius $r > 0$ in \mathbb{R}^2 , and

$$\|V\|_{L^\infty(\mathbb{R}^2)} < 2b. \quad (1.2)$$

Set $H_\pm := H_0 \pm V$. The main result of the note is the following

Theorem 1. *Fix $q \in \mathbb{N}$.*

(i) *Assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfies (1.1); if $q \geq 1$, suppose in addition that (1.2) holds true. Then we have*

$$\text{Ker}(H_+ - 2bq) = \{0\}. \quad (1.3)$$

(ii) *Assume that V satisfies (1.1) and (1.2). Then we have*

$$\text{Ker}(H_- - 2bq) = \{0\}. \quad (1.4)$$

The proof of Theorem 1 can be found in Section 2.

The asymptotic distribution of the discrete spectrum near the Landau levels of various perturbations of the Landau Hamiltonian and its generalizations has been considered by numerous authors (see [10, 4, 11, 7, 2, 9, 14, 13, 8]). However, to the authors' best knowledge the fate of the Landau levels themselves has never been addressed in the mathematical literature.

The fact that V has a fixed sign plays a crucial role in our result as shows the following

Theorem 2. *Fix $q \geq 0$ and $n \in \mathbb{N}^* \cup \{+\infty\}$. Then, there exists a bounded compactly supported potential V such that $\|V\|_{L^\infty(\mathbb{R}^2)} < 2b$ and*

$$\dim \text{Ker}(H_0 + V - 2bq) = n.$$

As can be checked from the proof of Theorem 2 contained in Section 3, V can be taken arbitrarily small. The same method of proof can also be applied to construct a compactly supported non-constant-sign potential V such that an arbitrary number of Landau levels stay eigenvalues of $H_0 + V$ with arbitrary fixed multiplicities.

2 Proof of Theorem 1

Denote by Π_q , $q \in \mathbb{N}$, the orthogonal projection onto $\text{Ker}(H_0 - 2bq)$. Set

$$\Pi_q^+ := \sum_{j=q}^{\infty} \Pi_j, \quad \Pi_q^- := I - \Pi_q^+, \quad q \in \mathbb{N}.$$

In order to prove Theorem 1, we need a technical result concerning some Toeplitz-type operators of the form $\Pi_q V \Pi_q$.

Lemma 3. Let $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfy (1.1). Fix $q \in \mathbb{N}$. Then

$$\langle \Pi_q V \Pi_q u, u \rangle = 0, \quad u \in L^2(\mathbb{R}^2), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$, implies

$$\Pi_q u = 0. \quad (2.2)$$

Proof. By (1.1) and (2.1),

$$0 \leq c \langle \Pi_q \chi \Pi_q u, u \rangle \leq \langle \Pi_q V \Pi_q u, u \rangle = 0, \quad (2.3)$$

i. e.

$$\langle \Pi_q \chi \Pi_q u, u \rangle = 0. \quad (2.4)$$

Denote by $T := \Pi_q \chi \Pi_q$ the operator self-adjoint in the Hilbert space $\Pi_q L^2(\mathbb{R}^2)$. The operator T is positive and compact, and its eigenvalues can be calculated explicitly (see [11, Eq. (3.32)]). This explicit calculation implies that $\text{Ker } T = \{0\}$. Therefore, (2.2) follows from (2.4). \square

Proof of Theorem 1. First, we prove (1.3) in the case $q = 0$. Assume that there exists $u \in D(H_+) = D(H_0)$ such that $H_+ u = 0$. Hence,

$$\langle H_0 u, u \rangle + \langle V u, u \rangle = 0. \quad (2.5)$$

The two terms at the l.h.s. of (2.5) are non-negative, and therefore they both should be equal to zero. Since $\langle H_0 u, u \rangle = 0$, we have

$$u = \Pi_0 u. \quad (2.6)$$

Therefore, $\langle V u, u \rangle = \langle \Pi_0 V \Pi_0 u, u \rangle = 0$. By Lemma 3, we have $\Pi_0 u = 0$, and by (2.6) we conclude that $u = 0$.

Next, we prove (1.3) in the case $q \geq 1$. Assume that there exists $u \in D(H_0)$ such that

$$H_+ u = 2bqu. \quad (2.7)$$

Set $u_+ := \Pi_q^+ u$, $u_- := u - u_+$; evidently, $u_\pm \in D(H_0)$. Since H_0 commutes with the projections Π_q^\pm , (2.7) implies

$$H_0 u_+ - 2bqu_+ + \Pi_q^+ V \Pi_q^+ u_+ + \Pi_q^+ V \Pi_q^- u_- = 0, \quad (2.8)$$

$$H_0 u_- - 2bqu_- + \Pi_q^- V \Pi_q^- u_- + \Pi_q^- V \Pi_q^+ u_+ = 0. \quad (2.9)$$

Now note that the operator $H_0 + \Pi_q^- V \Pi_q^- - 2bq$ is boundedly invertible in $\Pi_q^- L^2(\mathbb{R}^2)$, and its inverse is a negative operator. Moreover, by (2.9) we have

$$u_- = - (H_0 + \Pi_q^- V \Pi_q^- - 2bq)^{-1} \Pi_q^- V \Pi_q^+ u_+, \quad (2.10)$$

which inserted into (2.8) implies

$$H_0 u_+ - 2bq u_+ + \Pi_q^+ V \Pi_q^+ u_+ - \Pi_q^+ V \Pi_q^- (H_0 + \Pi_q^- V \Pi_q^- - 2bq)^{-1} \Pi_q^- V \Pi_q^+ u_+ = 0,$$

and hence,

$$\begin{aligned} & \langle (H_0 - 2bq)u_+, u_+ \rangle + \langle \Pi_q^+ V \Pi_q^+ u_+, u_+ \rangle \\ & - \langle \Pi_q^+ V \Pi_q^- (H_0 + \Pi_q^- V \Pi_q^- - 2bq)^{-1} \Pi_q^- V \Pi_q^+ u_+, u_+ \rangle = 0. \end{aligned} \quad (2.11)$$

The three terms on the l.h.s. of (2.11) are non-negative, and hence they all should be equal to zero. Since $u_+ = \Pi_q^+ u_+$, the equality $\langle (H_0 - 2bq)u_+, u_+ \rangle = 0$ implies

$$u_+ = \Pi_q u_+. \quad (2.12)$$

Therefore, $\langle \Pi_q^+ V \Pi_q^+ u_+, u_+ \rangle = \langle \Pi_q V \Pi_q u_+, u_+ \rangle$, and $\langle \Pi_q^+ V \Pi_q^+ u_+, u_+ \rangle = 0$ is equivalent to $\langle \Pi_q V \Pi_q u_+, u_+ \rangle = 0$. Now by Lemma 3 we have $\Pi_q u_+ = 0$, by (2.12) we have $u_+ = 0$, and by (2.10) we have $u_- = 0$. Therefore, $u = 0$.

Finally, we prove (1.4). Assume that there exists $u \in D(H_0)$ such that $H_- u = 2bqu$. This time, set $u_+ := \Pi_{q+1}^+ u$, $u_- := u - u_+$; as before, $u_{\pm} \in D(H_0)$. Analogously to (2.8) - (2.9), we get

$$H_0 u_+ - 2bq u_+ - \Pi_{q+1}^+ V \Pi_{q+1}^+ u_+ - \Pi_{q+1}^+ V \Pi_{q+1}^- u_- = 0, \quad (2.13)$$

$$H_0 u_- - 2bq u_- - \Pi_{q+1}^- V \Pi_{q+1}^- u_- - \Pi_{q+1}^- V \Pi_{q+1}^+ u_+ = 0. \quad (2.14)$$

The operator $H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq$ is boundedly invertible in $\Pi_{q+1}^+ L^2(\mathbb{R}^2)$, its inverse is a positive operator, and by as for (2.10) we find that (2.13) implies

$$u_+ = (H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq)^{-1} \Pi_{q+1}^+ V \Pi_{q+1}^- u_-. \quad (2.15)$$

Further, analogously to (2.11), we find that (2.14) and (2.15) imply

$$\begin{aligned} & \langle (H_0 - 2bq)u_-, u_- \rangle - \langle \Pi_{q+1}^+ V \Pi_{q+1}^- u_-, u_- \rangle \\ & - \langle \Pi_{q+1}^- V \Pi_{q+1}^+ (H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq)^{-1} \Pi_{q+1}^+ V \Pi_{q+1}^- u_-, u_- \rangle = 0. \end{aligned} \quad (2.16)$$

The three terms on the l.h.s. of (2.16) are non-positive, and hence they should vanish. Since $u_- = \Pi_{q+1}^- u_-$, the equality $\langle (H_0 - 2bq)u_-, u_- \rangle = 0$ implies

$$u_- = \Pi_q u_-. \quad (2.17)$$

Therefore, $\langle \Pi_{q+1}^- V \Pi_{q+1}^- u_-, u_- \rangle = \langle \Pi_q V \Pi_q u_-, u_- \rangle$, and $\langle \Pi_{q+1}^- V \Pi_{q+1}^- u_-, u_- \rangle = 0$ is equivalent to $\langle \Pi_q V \Pi_q u_-, u_- \rangle = 0$. Now by Lemma 3 we have $\Pi_q u_- = 0$, by (2.17) we have $u_- = 0$, and by (2.15) we have $u_+ = 0$. Therefore, $u = 0$. This completes the proof of Theorem 1. \square

3 Proof of Theorem 2

Define the operators

$$H_0^{(m)} := -\frac{1}{\varrho} \frac{d}{d\varrho} \varrho \frac{d}{d\varrho} + \left(\frac{m}{\varrho} - b\varrho \right)^2 - b, \quad m \in \mathbb{Z},$$

self-adjoint in $L^2(\mathbb{R}_+; \varrho d\varrho)$, as the Friedrichs' extensions of the operators defined on $C_0^\infty(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0, \infty)$. Then, the operator H_0 is unitarily equivalent to the orthogonal sum $\bigoplus_{m \in \mathbb{Z}} H_0^{(m)}$ under the passage to polar coordinates (ϱ, ϕ) in \mathbb{R}^2 , and a subsequent decomposition into a Fourier series with respect to the angular variable ϕ . For any $m \in \mathbb{Z}$, we have

$$\sigma(H_0^{(m)}) = \bigcup_{q=m_-}^{\infty} \{2bq\}$$

where, as usual, $m_- := \max\{0, -m\}$ (see e.g. [1]). In contrast to the 2D Landau Hamiltonian H_0 however, we have $\dim \text{Ker}(H_0^{(m)} - 2bq) = 1$ for all $q \geq m_-$, $m \in \mathbb{Z}$. Further, assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and V is radially symmetric i.e.

$$V(x, y) = v\left(\sqrt{x^2 + y^2}\right), \quad (x, y) \in \mathbb{R}^2.$$

Then, the operator $H_0 + V$ is unitarily equivalent to the orthogonal sum $\bigoplus_{m \in \mathbb{Z}} (H_0^{(m)} + v)$. Thus,

$$\dim \text{Ker}(H_0 + V - 2bq) = \sum_{m \in \mathbb{Z}} \dim \text{Ker}(H_0^{(m)} + v - 2bq), \quad q \in \mathbb{N}. \quad (3.1)$$

If $\|V\|_{L^\infty(\mathbb{R}^2)} = \|v\|_{L^\infty(\mathbb{R}_+)} < b$, for all m , the q -th eigenvalue of $H_0^{(m)} + v$ that we denote by $E_q(v; m)$ stays in the interval $2bq + (-b, b)$; in particular, it stays simple. So, as a consequence of regular perturbation theory, see e.g. [5, 12], the eigenvalues $(E_q(v; m))_{q \geq 0}$ are real analytic functions of the potential v . Moreover, one computes

$$\frac{\partial}{\partial t} E_q(tv; m)|_{t=0} = \int_{\mathbb{R}_+} v(\varrho) \varphi_{q,m}(\varrho)^2 \varrho d\varrho \quad (3.2)$$

where

$$\varphi_{q,m}(\varrho) := \sqrt{\frac{q!}{\pi(q+m)!} \left(\frac{b}{2}\right)^{m+1}} \varrho^m L_q^{(m)}(b\varrho^2/2) e^{-b\varrho^2/4}, \quad \varrho \in \mathbb{R}_+, \quad q \in \mathbb{N},$$

are the normalized eigenfunctions of the operator $H_0^{(m)}$, $m \in \mathbb{N}$, and

$$L_q^{(m)}(s) := \sum_{l=0}^q \frac{(q+m)!}{(m+l)!(q-l)!} \frac{(-s)^l}{l!}, \quad s \in \mathbb{R},$$

are the generalized Laguerre polynomials.

Pick $t \in]-b/2, b/2[^{\mathbb{N}^*}$ and consider the potential

$$v_t(\rho) = \sum_{j \in \mathbb{N}^*} t_{2j} \mathbf{1}_{[x_{2j}^-, x_{2j}^+]}(\rho) - \sum_{j \in \mathbb{N}^*} t_{2j+1} \mathbf{1}_{[x_{2j+1}^-, x_{2j+1}^+]}(\rho) \quad (3.3)$$

where $x_j^- := e^{-\alpha_j}$, $x_j^+ := e^{-\beta_j}$, and

$$\alpha_{2j} := 2^{-N(j-1/2)^2}, \quad \beta_{2j} := 2^{-Nj^2}, \quad \alpha_{2j+1} := 2^{-N(j-1/2)^2+1}, \quad \beta_{2j+1} := 2^{-Nj^2+1}. \quad (3.4)$$

We will choose the large integer N later on.

As, for $j \geq 1$, one has

$$N(j-1)^2 < N(j-1/2)^2 - 1 < N(j-1/2)^2 < Nj^2 - 1 < Nj^2 < N(j+1/2)^2 - 1,$$

we note that, for N sufficiently large, one has:

- $\|v_t\|_{L^\infty(\mathbb{R}_+)} < b$ for $t \in]-b/2, b/2[^{\mathbb{N}^*}$;
- v_t vanishes identically if and only if the vector $(t_j)_j$ vanishes identically.

For $j \geq 0$, define $m_j = 2^{Nj^2} - 1$ and consider the mapping

$$\mathcal{E} : t \in]-b, b[^{\mathbb{N}^*} \mapsto (\mathcal{E}_{2j}(t), \mathcal{E}_{2j+1}(t))_{j \geq 0} = (\tilde{E}_q(v_t; m_j), t_{2j} + t_{2j+1})_{j \geq 0} \in]-r, r[^{\mathbb{N}^*}$$

where

$$\tilde{E}_q(v_t; m_j) = \frac{1}{C_j} \frac{(m_j!)^2}{(q + m_j)!} \left(\frac{2}{b}\right)^{m_j} (E_q(v_t; m_j) - 2bq)$$

The constants $(C_j)_j$ are going to be chosen later on.

The mapping is real analytic and we can compute its Jacobi matrix at $t = 0$. First, bearing in mind (3.2), (3.3), and (3.4), we easily find that

$$\begin{aligned} \partial_{t_{2j}} \mathcal{E}_{2l}(0) &= C_j^{-1} C_q (e^{-m_l \beta_{2j}} (1 + o(1)) - e^{-m_l \alpha_{2j}} (1 + o(1))) \\ &= \begin{cases} 1 & \text{if } j = l, \\ O(e^{-2^{N|l-j|}}) & \text{if } l > j, \\ O(2^{-N|j-l|}) & \text{if } l < j, \end{cases} \\ \partial_{t_{2j+1}} \mathcal{E}_{2l}(0) &= -C_j^{-1} C_q (e^{-m_l \beta_{2j+1}} (1 + o(1)) - e^{-m_l \alpha_{2j+1}} (1 + o(1))) \\ &= \begin{cases} -1 & \text{if } j = l, \\ O(e^{-2^{N|j-l|}}) & \text{if } l > j, \\ O(2^{-N|j-l|}) & \text{if } l < j, \end{cases} \end{aligned}$$

when one chooses C_j properly. In this formula, $o(1)$ refers to the behavior of the function when $N \rightarrow +\infty$ uniformly in l, j . Moreover, obviously,

$$\partial_{t_{2j}} \mathcal{E}_{2l+1}(0) = \partial_{t_{2j+1}} \mathcal{E}_{2l+1}(0) = \delta_{jl}.$$

Hence, the Jacobi matrix of the mapping \mathcal{E} can be written as $J + E$ where J is a block diagonal matrix made of the blocks $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and the error matrix E is a bounded operator from $l^\infty(\mathbb{N}^*)$ to itself with a norm bounded by $C2^{-N}$. So for N large enough this Jacobi matrix is invertible and, using the analytic inverse mapping theorem, we see that there exists a real analytic diffeomorphism φ on a ball of $l^\infty(\mathbb{N}^*)$ centered at 0, such that

$$\mathcal{E} \circ \varphi(u) = (u_{2j} - u_{2j+1}, u_{2j} + u_{2j+1})_{j \geq 0} \in]-r, r[{}^{\mathbb{N}^*},$$

and $\varphi(0) = 0$. To construct the potential v_t having the Landau level $2bq$ as an eigenvalue with prescribed multiplicity, it suffices to take $t = \varphi(u)$ with $u_{2j} = u_{2j+1} \neq 0$ for $j = 0, \dots, n-1$, and $u_{2j} \neq u_{2j+1}$ for $j \geq n$. This completes the proof of Theorem 2.

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