On Self-Similar Finitely Generated Uniformly Discrete (SFU-) sets and Sphere Packings

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Abstract. This paper is first a brief survey on links between Geometry of Numbers and aperiodic crystals in Physics, viewed from the mathematical side. In a second part, we prove the existence of a canonical cut-and-project scheme above a (ssfgud set) self-similar finitely generated packing of (equal) spheres Λ in $\mathbb{R}^n$ and investigate its consequences, in particular the role played by the Euclidean and inhomogeneous minima of the algebraic number field generated by the self-similarity on the Delone constant of the sphere packing. We discuss the isolation phenomenon. The degree d of this field divides the $\mathbb{Z}$-rank of $\mathbb{Z}[\Lambda - \Lambda]$. We give a lower bound of the Delone constant of a k-thin ssfgud sphere packing which arises from a model set or a Meyer set when d is large enough.

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Contents

1 Introduction

The mathematics of uniformly discrete point sets and Delone sets developped recently has at least four different origins: (i) the experimental evidence of nonperiodic states of matter in condensed matter physics, so-called aperiodic crystals, like quasicrystals [4] [51] [58] [63] [97] incommensurate modulated crystals phases [62] [64] and their geometric modelization (cf Appendix), (ii) some works of Delone (Delone) [33] [34] [39] [91] on geometric crystallography (comparatively, see [54] [78] [84] [95] for a classical mathematical approach of periodic crystals), (iii) some works of Meyer on now called cut-and-project sets and Meyer sets [75] [76] [77] [86] (for a modern language of Meyer sets in locally compact Abelian groups: [79]), (iv) the theory of self-similar tilings [8] [70] [102] and the use of ergodic theory to understand diffractivity [5] [92] [102]. In particular, the impact on mathematics of the discovery of quasicrystals in 1984 [97], as long-range ordered phases, was outlined by Lagarias [66]. The term *mathematical quasicrystals* [6] [67] was proposed to name these Delone sets which are used as discrete geometrical models of these new states

of matter which have particular spectral or diffraction properties; in particular *crystals* those for which the spectrum is essentially pure point (see [61] [96] and the Appendix for the new definition of what is a *crystal*, and [31] [53] [59] for spectral/diffraction theory). Delone sets are conceived as natural generalizations of lattices in modern crystallography.

In this note we will briefly review these notions (Section 2) and will consider more generally uniformly discrete sets of $\mathbb{R}^n$, in particular (ssfgud) self-similar finitely generated uniformly discrete sets (Definition 2.19). A uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$ is a packing of (equal) spheres of $\mathbb{R}^n$ of (common) radius $r/2$. There are several advantages to consider uniformly discrete sets instead of Delone sets only: their $\mathbb{R}$-spans may take arbitrary dimensions between 0 and n, while that of a Delone set is only n, they can be finite sets which is forbidden for Delone sets, they may exhibit (spherical) holes of arbitrary size at infinity whereas the size of holes in Delone sets is limited by the Delone constant. A classification of uniformly discrete sets, hence of Delone sets, which extends that given in [65], is proposed in Subsection 2.3. Finitely generated uniformly discrete sets of $\mathbb{R}^n$ constitute the largest class on which an address map (Subsection 2.3) can be defined.

The theory of ssfgud sets complements that of lattice packings of (equal) spheres of $\mathbb{R}^n$ [19] [22] [26] [52] [72] [107] since a lattice is already a ssfgud set itself (integers are self-similarities), where lattices are or not $\mathcal{O}_F$-lattices for F an algebraic number field with involution [13] [30], and makes use of algebraic integers of certain types (Subsection 2.4). Self-similar Meyer sets only admit self-similarities which are Pisot or Salem numbers [75], while self-similar finitely generated Delone sets only provide Perron or Lind numbers as self-similarities [65]. It is an open problem to find a criterium which ensures that a given uniformly discrete set admits at least one self-similarity. For a general Delone set symmetries and in particular inflation symmetries are expected to be rare, especially when the dimension of the ambient space is large, probably more frequently than for lattices; a fortiori for uniformly discrete sets. For lattices, Bannai [9] has shown the existence of many unimodular $\mathbb{Z}$-lattices with trivial automorphism group in a given genus of positive definite unimodular $\mathbb{Z}$-lattices of sufficiently large rank (see also [25]).

The existence of cut-and-project schemes above Delone sets is useful to characterize the set of its self-similarities, inflation centers, local clustering, etc [27] [28] [49] [73]. Given an arbitrary uniformly discrete set it is an open problem whether a cut-and-project scheme lies above it (Subsection 2.1.2). Theorem 1.1 answers in full generality to this problem for a given ssfgud set $\Lambda \subset \mathbb{R}^n$ (with $\Lambda \in \mathcal{UD}_{fg}$, see Subsection 2.3) with self-similarity λ. The constructions use the Archimedean embeddings of the number field $K := \mathbb{Q}(\lambda)$ generated by the self-similarity λ (Section 3) in a vectorial way, as a product of copies of the étale $\mathbb{R}$-algebra $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$. Denote by $\Sigma : K \to K_{\mathbb{R}}$ the canonical map. The structure of the lattice in the cut-and-project scheme

above Λ arises as a consequence of the Jordan invariants of $\mathbb{R}^{\mathrm{rk}\,\Lambda}$ as a $K[X]$-module (from (ii) in Theorem 1.1).

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a uniformly discrete set such that $m := rk\ \mathbb{Z}[\Lambda - \Lambda] < +\infty$ with $m \geq 1$. Let $\lambda > 1$ be a (affine) self-similarity of Λ, i.e. a real number > 1 such that $\lambda(\Lambda - c) \subset \Lambda - c$ for a certain $c \in \mathbb{R}^n$. Then*

(i) *λ is a real algebraic integer of degree $d \geq 1$ and d divides m,*

(ii) *there exist $r = m/d$ $\mathbb{Q}$-linearly independent vectors $w_1, w_2, \ldots, w_r$ in the $\mathbb{Q}(\lambda)$-vector space $\mathbb{Q}[\Lambda - \Lambda]$ such that $\mathbb{Z}[\Lambda - \Lambda]$ is a rank m $\mathbb{Z}$-submodule of the $\mathbb{Z}$-module:*
$\mathbb{Z}[w_1, \lambda w_1, \ldots, \lambda^{d-1} w_1, w_2, \lambda w_2, \ldots, \lambda^{d-1} w_2, \ldots, w_r, \lambda w_r, \ldots, \lambda^{d-1} w_r]$,

(iii) *for every $\mathbb{Z}$-basis $\{v_1, v_2, \ldots, v_m\}$ of $\mathbb{Z}[\Lambda - \Lambda]$, a matrix relation: $\lambda V = MV$ holds, where $V = {}^t[v_1, \ldots, v_m]$ and M is an invertible integral $m \times m$ matrix with characteristic polynomial $det(X\, I - M) = (\varphi(X))^{m/d}$ in which $\varphi(X)$ is the minimal polynomial of λ; in particular, $det\, M = N_{K/\mathbb{Q}}(\lambda)^{m/d}$, where $N_{K/\mathbb{Q}}(\lambda)$ is the algebraic norm of λ,*

(iv) *there exists a cut-and-project scheme above Λ:*

$$\Big(\prod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|} \ \simeq\ H \times \mathbb{R}[\Lambda],\ L,\ \pi,\ \mathrm{pr}_1 \Big)$$

where the lattice $L = \coprod_{i=1}^{r} \Sigma\big(\mathbb{Z}[\lambda]\big) \frac{w_i}{\|w_i\|}$ is such that $pr_1(L) \supset \mathbb{Z}[\Lambda - \Lambda]$, whose internal space H is the product of two spaces:

$$H = \big(\ R_K \setminus \mathbb{R}[\Lambda] \ \big) \times \overline{G}$$

where R_K is the image of $\mathbb{R}[\Lambda]$ in $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ by the real and imaginary embeddings of K, and $\overline{G}$ the closure in $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ of the image by Σ of the space of relations over K between the generators $w_1, \ldots, w_r$. The space $R_K \setminus \mathbb{R}[\Lambda]$ is called the shadow space of Λ. This cut-and-project scheme is endowed with an Euclidean structure given by a real Trace-like symmetric bilinear form for which R_K and $\overline{G}$ are orthogonal.

The *central cluster* of the basis $(\lambda^j w_i)_{i=1,\ldots,r, j=0,\ldots,d-1}$ is by definition the set $\{w_1, w_2, \ldots, w_r\}$. Note that some vectors in a central cluster may be $\mathbb{R}$-linearly dependent. When $w_1, w_2, \ldots, w_r$ have identical norms and constitute orbits (i.e. F-clusters) under the action of a finite group, say F, constructions in (iv) in Theorem 1.1 can be deduced from [29]. It is easy to check that

$$r = 1 \text{ in Theorem 1.1} \implies \text{the } \mathbb{R}\text{-span of } \Lambda \text{ is one-dimensional .}$$

The converse is generally wrong and Subsection 2.5 gives the example of sets $\mathbb{Z}_\beta$ of beta-integers [10] on the line for which open problems exist.

Corollary 1.2. *If Λ is a self-similar finitely generated sphere packing in $\mathbb{R}^n$, with self-similarity λ, such that $r = 1$, i.e. for which the degree d of λ equals the rank m of $\mathbb{Z}[\Lambda - \Lambda]$, then $\mathbb{Z}[\Lambda - \Lambda]$ is the projection of a sublattice of finite index of an Arakelov divisor of $K = \mathbb{Q}(\lambda)$ in the cut-and-project scheme above Λ. This index is an integer multiple of $(\mathcal{O}_K : \mathbb{Z}[\lambda])$.*

Theorem 1.1 gives a framework for constructing aperiodic (equal) sphere packings $\mathcal{B}(\Lambda)$ for which local arrangements, for instance like t-designs [7], can be computed from a lattice in higher dimension above Λ.

Dense sphere packings of $\mathbb{R}^n$ are of general interest [17] [22] [26] [50] [52] [81] [107]. For a sphere packing whose set of centers is a Delone set Λ which is a uniformly discrete of constant $r > 0$, of Delone constant $R(\Lambda) := \sup_{z\in\mathbb{R}^n} \inf_{\mu\in\Lambda} \|z - \mu\|$, the density $\delta(\mathcal{B}(\Lambda))$ of $\mathcal{B}(\Lambda)$ satisfies [81]: $\delta(\mathcal{B}(\Lambda)) \geq (\,2R(\Lambda)/r\,)^{-n}$. If Λ is only a uniformly discrete set, no equivalent formula for bounding from below the density $\delta(\mathcal{B}(\Lambda))$ exists in general. However, it is not the case for Delone-like uniformly discrete sets of $\mathbb{R}^n$, which behave in some sense like Delone sets (Subsection 5.1).

Theorem 1.3. *Let Λ be a uniformly discrete set of $\mathbb{R}^n$, of constant $r > 0$, which is Delone-like of pseudo-Delone constant $R(\Lambda)$. Then the density of the sphere packing $\mathcal{B}(\Lambda)$ (of common radius $r/2$) whose set of centers is Λ satisfies:*

$$\delta(\mathcal{B}(\Lambda)) \geq \left(\frac{2R(\Lambda)}{r}\right)^{-n}. \tag{1.1}$$

Therefore it is crucial to obtain interesting lower bounds of the Delone constant (or pseudo-Delone constant) $R(\Lambda)$ to control dense sphere packings, in particular, sphere packings whose set of centers is a ssfgud set.

In Section 6 we comment on the two origins of the (pseudo-) Delone constant of a sphere packing whose set of centers is a ssfgud set: the first one lies in the geometrical properties of the central cluster $\{w_1, w_2, \ldots, w_r\}$ as given by Theorem 1.1 (ii), the second one is of purely arithmetical nature; it comes from the Euclidean and inhomogeneous minima associated with a sublattice of a product of ideal lattices [12] [23] [24] in bijection with $\mathbb{Z}[\Lambda - \Lambda]$ in the cut-and-project scheme given by Theorem 1.1 (iv). Only the case $r = 1$ is reported in Section 6.

Theorem 1.4. *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a ssfgud set which is either a model set or a Meyer set in the cut-and-project scheme defined by Theorem 1.1 (iv) with $r = 1$, Ω as window and lattice L' such that $pr_1(L') = \mathbb{Z}[\Lambda - \Lambda]$.*

Assume that the self-similarity λ is of degree $d\,(\geq 3)$ large enough, that $K = \mathbb{Q}(\lambda)$ has a unit rank > 1 and is not a CM-field. Then, if Λ is k-thin, $k \geq 2$, its Delone constant $R(\Lambda)$ satisfies:

$$R(\Lambda) \;\geq\; \sqrt{d}\left(M(K)^{2/d} - M_k(K)^{2/d}\right)^{\frac{1}{2}} \;> 0, \tag{1.2}$$

where $M(K)$, resp. $M_k(K)$, is the Euclidean minimum, resp. the k-th Euclidean minimum, of K.

The $75^{ièmes}$ *Rencontres between Mathematicians and Physicists* held at IRMA - Strasbourg on the Thema "Number Theory and Physics" have offered to the author the opportunity of writing this brief note, initially conceived as a short survey, on the relationships between sphere packings, the mathematics of aperiodic crystals, algebraic number theory and numeration in base an algebraic integer > 1.

2 Uniformly discrete sets and Delone sets

2.1 Definitions and Topology

Let us define *uniformly discrete sets* and *Delone sets* in two different contexts: in the metric case when the ambient space is a metric space which is σ-compact and locally compact, like $\mathbb{R}^n$, and when the ambient space is $\mathbb{R}^n$ with a cut-and-project scheme that lies above it with a locally compact abelian group as internal space.

2.1.1 Metric Case Let (H, δ) be a σ-compact and locally compact metric space with infinite diameter (for δ). A discrete subset Λ of H is said to be *uniformly discrete* if there exists a real number $r > 0$ such that

$$x, y \in \Lambda, x \neq y \text{ implies } \delta(x, y) \geq r.$$

A uniformly discrete set is either the empty set, or a subset $\{x\}$ of H reduced to one element, or, if it contains at least two points, they satisfy such an inequality. If r is equal to the minimal interpoint distance

$$\inf\{\delta(x, y) \mid x, y \in H, x \neq y\}$$

(when $\mathrm{Card}(\Lambda) \geq 2$) Λ is said to be a uniformly discrete set *of constant* r. The space of uniformly discrete sets of constant $r > 0$ of (H, δ) is denoted by $\mathcal{UD}(H, \delta)_r$. It is the space $SS(H, \delta)_r$ of systems of equal spheres (or space of sphere packings) of radius $r/2$ of (H, δ): $\Lambda = (a_i)_{i \in \mathbb{N}} \in \mathcal{UD}(H, \delta)_r$ is the set of sphere centers of

$$\mathcal{B}(\Lambda) = \{B(a_i, r/2) \mid i \in \mathbb{N}\} \in SS(H, \delta)_r$$

where $B(z, t)$ denotes generically the closed ball centered at $z \in H$ of radius $t > 0$.

An element $\Lambda \in \cup_{r>0} \mathcal{UD}(H,\delta)_r$ is said to be a *Delone set* if there exists $R > 0$ such that, for all $z \in H$, there exists an element $\lambda \in \Lambda$ such that $\delta(z,\lambda) \leq R$ (relative denseness property). Then a Delone set is never empty. If Λ is a Delone set, then

$$R(\Lambda) := \sup_{z\in H} \inf_{\lambda\in\Lambda} \delta(z,\lambda) \tag{2.1}$$

is called the *Delone constant* of Λ. In [81] the range of values of the ratio $R(\Lambda)/r$ in the case $H = \mathbb{R}^n, n \geq 1$, is shown to be the continuum

$$\Big[\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}, +\infty\Big). \tag{2.2}$$

In the context of lattices which are $\mathcal{O}_K$-modules, with K a number field, the "ambient space" is obtained in a canonical way via the real and complex embeddings of K and the Delone constant is reminicent of the Euclidean minimum or inhomogeneous minimum, with possible isolated values instead of the continuum (2.2) (Section 6).

The Delone constant of Λ is the maximal circumradius of all its Voronoi cells. If $\Lambda \in \mathcal{UD}(H,\delta)_r$ is a Delone set of Delone constant R, the discrete set Λ is also called a (r,R)-system [91]. Let $X(H,\delta)_{r,R} \subset \mathcal{UD}(H,\delta)_r$ be the subset of uniformly discrete sets of constant r which are Delone sets of H of Delone constant $\leq R$.

Theorem 2.1. *Let (H,δ) be a σ-compact and locally compact metric space for which* $\mathrm{diam}(H)$ *is infinite. Then, for all $r > 0$, $\mathcal{UD}(H,\delta)_r$ can be endowed with a metric d such that the topological space $(\mathcal{UD}(H,\delta)_r, d)$ is compact and such that the Hausdorff metric on $\mathcal{UD}(H,\delta)_{r,f}$ is compatible with the restriction of the topology of $(\mathcal{UD}(H,\delta)_r, d)$ to $\mathcal{UD}(H,\delta)_{r,f}$. For all $R > 0$ the subspace $X(H,\delta)_{r,R}$ is closed.*

In [82] several (classes of equivalent) metrics on H are constructed. In such constructions a base point, say $\alpha \in \mathbb{R}^n$, is required. When $H = \mathbb{R}^n$, endowed with the Euclidean norm $\|\cdot\|$, the topology on $\mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r, r > 0$, is expressed by "unique local pairings of points in big balls centered at the base point α", as follows (Proposition 3.6 in [82]). Let $r = 1$, the general case being the same.

Proposition 2.2. *Let $\Lambda, \Lambda' \in \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_1$ with Λ and Λ' nonempty. Let $l = \inf\{\|t-\alpha\| \mid t \in \Lambda\} < +\infty$ and $\epsilon \in (0, (1+2l)^{-1})$. Assume $d(\Lambda,\Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda - \alpha\| < \frac{1-\epsilon}{2\epsilon}$,*

(i) *there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda - \lambda'\| < \frac{1}{2}$,*

(ii) *this pairing (λ,λ') satisfies the inequality: $\|\lambda - \lambda'\| \leq (\frac{1}{2} + \|\lambda - \alpha\|)\epsilon$.*

2.1.2 Cut-and-Project Schemes Above Uniformly Discrete Sets A locally compact abelian (lca) group is an abelian group G endowed with a topology for which G is a Hausdorff space, each point admits a compact neighbourhood, and such that the mapping $G \times G \to G, (x, y) \to x - y$ is continuous. In the sequel we will denote additively the additive law of G so that 0 is the neutral element of G.

Definition 2.3. Let G be a lca group.

(i) A subset Λ of G is *uniformly discrete* if there exists an open neighbourhood W of 0 so that $(\Lambda - \Lambda) \cap W = \{0\}$,

(ii) a subset Λ of G is *relatively dense* if there exists a compact subset K of G such that $G = \Lambda + K$,

(iii) a Delone set of G is a subset Λ of G which is relatively dense and uniformly discrete.

Definition 2.4. A lattice of $\mathbb{R}^n, n \geq 1$, is a discrete $\mathbb{Z}$-module of rank n. A lattice in a lca group G is a subgroup L of G such that:

(i) L is discrete, i.e. the topology on L induced by that of G is the discrete topology,

(ii) L is cocompact, i.e. G/L is compact.

In the sequel we will only define cut-and-project schemes over uniformly discrete sets Λ which lie in finitely dimensional Euclidean spaces $\mathbb{R}^n$, leaving aside the general case where the ambient space of Λ is a lca group. Such more general constructions can be found in [75], Chap. II, and in [94]. Denote by $\mathcal{L}_n$ the space of (affine) lattices of $\mathbb{R}^n, n \geq 1$.

Definition 2.5. A cut-and-project scheme (over $\mathbb{R}^n$) is given by a 4-tuple $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ where:

(i) $G \times \mathbb{R}^n$ is the direct product of a lca group G and the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$,

(ii) L is a lattice in $G \times \mathbb{R}^n$,

such that the natural projections $\pi_1 : G \times \mathbb{R}^n \to G$ and $\pi_2 : G \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy

(1) the restriction $\pi_2|_L$ of π_2 to L is a bijection from L to $\pi_2(L)$,

(2) the image $\pi_1(L)$ is dense in G.

G is called the *internal space*.

Definition 2.6. Let Λ be a uniformly discrete set in the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. A cut-and-project scheme given by the 4-tuple $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ is said to lie *above* Λ if there exists $t \in \mathbb{R}^n$ such that

$$\Lambda - t \ \subset \ \pi_2(L).$$

Remark 2.7. The need to introduce the translation t in the last definition comes from the fact that the uniformly discrete set Λ does not necessarily contain the base point of the cut-and-project scheme, which is the origin of $\mathbb{R}^n$ and at the same time the origin of G. Being a uniformly discrete set, or a Delone set, is an affine notion in the ambient space $\mathbb{R}^n$, while cut-and-project schemes priviledge a base point. For instance the lattice $u + \mathbb{Z}, u = 1/2$, of $\mathbb{R}$ admits $(\{0\} \times \mathbb{R}, \mathbb{Z}, 0, Id)$ as cut-and-project scheme above it; the translation t being $1/2$ in this case. If Λ is a Delone set, the translation t can be chosen such that: $\|t\| \leq R(\Lambda)$ the Delone constant of Λ.

In the last definition, the image $\pi_2(L)$ of the discrete subgroup $L \subset G \times \mathbb{R}^n$ is a $\mathbb{Z}$-module in $\mathbb{R}^n$ (the classical structure of $\mathbb{Z}$-modules in $\mathbb{R}^n$ is given for instance in [35], Theorem 2.3.7).

Cut-and-project sets, also called model sets, of $\mathbb{R}^n$ form a particular class of Delone sets.

Definition 2.8. A discrete subset Λ of $\mathbb{R}^n, n \geq 1$, is a cut-and-project set, or model set, if there exists a cut-and-project scheme $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ over Λ, with G a lca group, and a relatively compact subset Ω of the internal space G, with nonempty interior, such that:

$$\Lambda - t = \{\pi_2(\omega) \mid \pi_1(\omega) \in \Omega\},$$

for a certain $t \in \mathbb{R}^n$. The set Ω is called the *window* of the cut-and-project set $\Lambda = \Lambda(\Omega)$.

Model sets which arise from cut-and-schemes $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ with a lca group G as internal space do not differ too much from model sets that come from cut-and-project sets where the internal space is $\mathbb{R}^m$, for a certain m, by the following proposition.

Proposition 2.9. *Let $\Lambda(\Omega)$ be a cut-and-project set in the cut-and-project scheme $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ where G is a lca group. Then there exists a subgroup of G isomorphic to $\mathbb{R}^m$, for a certain $m \geq 0$, and a model set $\Lambda' \subset \mathbb{R}^n$ having $(\mathbb{R}^m \times \mathbb{R}^n, L', \pi_1, \pi_2)$ as cut-and-project scheme above it such that $\Lambda(\Omega)$ is contained in a finite number of translates of Λ'.*

Proof. Proposition 2.7 in [79]. □

Proposition 2.10. *Let $\Lambda = \Lambda(\Omega)$ be a model set in a cut-and-project scheme $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ where G is a lca group. Then*

(i) *Λ is Delone set of $\mathbb{R}^n$,*

(ii) *if $\Omega \subset \overline{\text{int}(\Omega)}$ (adherence of its interior) and Ω generates G as a group, the following equality holds:*

$$\mathbb{Z}[\Lambda - \Lambda] \;=\; \pi_2(L).$$

Proof. Proposition 2.6 in [79]. □

Remark 2.11. In [79] and [65], the origin implicitely belongs to the Delone set Λ, whereas in the present note we do not assume this minor fact. That is why we refer to $\mathbb{Z}[\Lambda - \Lambda]$ everywhere instead of $\mathbb{Z}[\Lambda]$, as in Theorem 1.1 or in Proposition 2.10 for instance.

2.2 Continuity of sphere packings arising from model sets

Let Λ be a model set in $\mathbb{R}^n$, viewed as set of centers of a sphere packing, and consider the cut-and-project scheme $(\mathbb{R}^k \times \mathbb{R}^n, L, \pi_1, \pi_2)$ above Λ which allows the construction of Λ by means of a window $\Omega \subset \mathbb{R}^k$. Let us fix the direct product $\mathbb{R}^k \times \mathbb{R}^n$. Let us write $\Lambda = \Lambda_L(\Omega)$ and consider how the model set $\Lambda_L(\Omega)$ varies when Ω and L vary continuously.

Let $\mathcal{W}(\mathbb{R}^k)$ be the uniform space of nonempty open relatively compact subsets of $\mathbb{R}^k$ (set of acceptance windows in $\mathbb{R}^k$) whose affine hull is $\mathbb{R}^k$, endowed with the pseudo-metric

$$\Delta_{\mathcal{W}}(\Omega_1, \Omega_2) := \Delta(\overline{\Omega_1}, \overline{\Omega_2})$$

where Δ is the Hausdorff metric on the space of nonempty closed subsets of $\mathbb{R}^k$. The space of lattices $\mathcal{L}_{n+k}$ in $\mathbb{R}^{n+k}$ is equipped with the quotient topology of $GL(n+k,\mathbb{R})/GL(n+k,\mathbb{Z})$. The metric d built on $\bigcup_{r>0} \mathcal{UD}(\mathbb{R}^{n+k}, \|\cdot\|)_r$ with 0 as base point [82] is compatible with the quotient topology of $GL(n+k,\mathbb{R})/GL(n+k,\mathbb{Z})$. Let $\mathcal{UD} = \bigcup_{r>0} \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r$ be the space of uniformly discrete subsets of $\mathbb{R}^n$, endowed with the metric d where here an arbitrary base point $\alpha \in \mathbb{R}^n$ is taken (see [82], Theorem 2.1 and Proposition 2.2). Denote by d_α the metric d in this paragraph only. The two origins, of the cut-and-project scheme and of $\mathbb{R}^n$ for the construction of the metric d_α on $\mathcal{UD}$, are taken a priori different.

Theorem 2.12. *For any base point $\alpha \in \mathbb{R}^n$, the mapping*

$$\mathcal{W}(\mathbb{R}^k) \times \mathcal{L}_{n+k} \to (\mathcal{UD}, d_\alpha) \quad : \quad (\Omega, L) \to \Lambda_L(\Omega)$$

is continuous.

Proof. Let $\epsilon > 0$. Let $L_0 \in \mathcal{L}_{n+k}$ and $\Omega_0 \in \mathcal{W}(\mathbb{R}^k)$. Let us show the continuity at (Ω_0, L_0). Let $t = \|\alpha\| + \frac{1-\epsilon/2}{\epsilon}$. Since Ω_0 is open, there exists $\eta_1 > 0$ such that all the sets $\{x \in L \mid \pi_1(x) \in \Omega_0, \|\pi_2(x)\| \leq t\}$ have the same cardinality if L belongs to the open set $\{L \mid d(L, L_0) < \eta_1\}$. Since π_2 is continuous and $\pi_{2_{|L_0}}$ is assumed to be a bijection from L_0 onto $\pi_2(L_0)$, then $\pi_{2_{|L}}$ is also a bijection from L onto $\pi_2(L)$ as soon as $d(L, L_0)$ is small enough. Then, using Proposition 2.2 and invoking the continuity of π_2, there exists $\eta' \leq \eta_1$ such that $d(L, L_0) < \eta'$ implies $d_\alpha(\Lambda_L(\Omega_0), \Lambda_{L_0}(\Omega_0)) < \epsilon/2$.

The subset $\{x \in L \mid \pi_1(x) \in \Omega, \|\pi_2(x)\| \leq t\}$ of L is such that its projection by π_1 is made of a finite collection of points which lie inside Ω (which is open), and its projection by π_2 is a finite subset of $\pi_2(L)$ which contains $\Lambda_L(\Omega_0) \cap B(\alpha, \frac{1-\epsilon/2}{\epsilon})$ (see Proposition 2.2). Since the projection mappings π_1 and π_2 are continuous and that $\pi_{2_{|_L}}$ is a bijection from L onto $\pi_2(L)$, the mapping $L \to \pi_1 \circ (\pi_{2_{|_L}})^{-1}$ is continuous on the open set $\{L \mid d(L, L_0) < \eta'\}$. Then there exists $\eta'' > 0$ such that $\Delta_{\mathcal{W}}(\Omega, \Omega_0) < \eta''$ implies $d_\alpha(\Lambda_L(\Omega), \Lambda_L(\Omega_0)) < \epsilon/2$ (the value of t is chosen according to this last inequality and Proposition 2.2).

Then, as soon as $\Delta_{\mathcal{W}}(\Omega, \Omega_0) < \eta''$ and $d(L, L_0) < \eta'$ hold, we have:

$$d_\alpha(\Lambda_L(\Omega), \Lambda_{L_0}(\Omega_0)) \leq d_\alpha(\Lambda_L(\Omega), \Lambda_L(\Omega_0)) + d_\alpha(\Lambda_L(\Omega_0), \Lambda_{L_0}(\Omega_0)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We deduce the claim. □

Note that the assumption "open" for windows in $\mathcal{W}(\mathbb{R}^k)$ is essential to obtain the continuity in Theorem 2.12. If we consider a collection of model sets parametrized by a sequence of windows which are not necessarily open, but with nonempty interiors, then Theorem 2.12 should be applied with the collections of the interiors of the windows.

2.3 A Classification

Classes of Uniformly Discrete Sets and Delone sets in $\mathbb{R}^n$, and their relative inclusions, are given in Theorem 2.16, following Lagarias [65] for Delone sets. We first define some classes of uniformly discrete sets intrinsically, i.e. without any cut-and-project scheme formalism above them. Then we indicate the definitions of point sets which invoke cut-and-project schemes.

Definition 2.13. Let Λ be a nonempty uniformly discrete set of $\mathbb{R}^n$.

(i) Λ is *finitely generated* if the $\mathbb{Z}$-module

$$\mathbb{Z}[\Lambda - \Lambda] := \Big\{ \sum_{\text{finite}} \alpha_i (x_i - y_i) \mid \alpha_i \in \mathbb{Z}, x_i, y_i \in \Lambda \Big\}$$

is finitely generated, i.e. $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \mathbb{Z}[\Lambda - \Lambda] < +\infty$,

(ii) Λ is *of finite type* if, for all $t > 0$, the intersection

$$(\Lambda - \Lambda) \cap B(0, t)$$

is a finite set.

If Λ is a nonempty finitely generated uniformly discrete set, the *rank of* Λ, denoted by rk Λ, is by definition the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes \mathbb{Z}[\Lambda - \Lambda] = \mathbb{Q}[\Lambda - \Lambda]$. The rank rk Λ is an invariant of Λ. Let $c \in \mathbb{R}^n$. The rank of $\mathbb{Z}[\Lambda - c]$ varies with c and may be different of that of $\mathbb{Z}[\Lambda - \Lambda]$.

For instance, with $c = 0$ and $\Lambda = \sqrt{2} + \mathbb{Z}$ in $\mathbb{R}$ we have: rk $\Lambda = 1$ while the rank of $\mathbb{Z}[\Lambda] = \mathbb{Z}[\Lambda - c]$ equals 2 (the notations rk Λ and rk $\mathbb{Z}[\Lambda]$ should not be confused); moreover $\Lambda = \Lambda - c$ and $\mathbb{Z}[\Lambda - \Lambda] = \mathbb{Z}$ are disjoint.

Theorem 2.14 (Lagarias). *Let Λ be a Delone set of finite type of $\mathbb{R}^n, n \geq 1$. Then*

$$rk\ \Lambda \leq Card\,((\Lambda - \Lambda) \cap B(0, 2R(\Lambda))) \ < \ +\infty \tag{2.3}$$

where $R(\Lambda)$ is the Delone constant of Λ.

Proof. Theorem 2.1 in [65]. □

Definition 2.15. Let Λ be a relatively dense discrete subset of $\mathbb{R}^n$. Λ is a Meyer set if one of the following equivalent assertions is satisfied:

(i) $\Lambda - \Lambda$ is uniformly discrete,

(ii) Λ is a Delone set and there exists a finite set $F \subset \mathbb{R}^n$ such that

$$\Lambda - \Lambda \subset \Lambda + F, \tag{2.4}$$

(iii) Λ is a subset of a model set.

Proof. Theorem 9.1 and Proposition 9.2 in [79]. □

Conditions (i) and (ii) in the definition of Meyer sets are given independently of any "cut-and-project scheme above Λ" consideration while condition (iii) asserts the existence of such a cut-and-project scheme above it. In a similar way a (affine) lattice $L \in \mathcal{L}_n$ in $\mathbb{R}^n$ is intrinsically defined in $\mathbb{R}^n$, without any help of cut-and-project schemes, admits also $(\{0\} \times \mathbb{R}^n, L, 0, \pi_2)$ as cut-and-project scheme above it and is a model set in this cut-and-project scheme. The objectives of Theorem 1.1 consist in showing the existence of general constructions of cut-and-project schemes above ssfgud sets in $\mathbb{R}^n$.

Let $n \geq 1$. Denote:

$\mathrm{M}^{(\mathbb{R})}$:= { Model sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a m-dimensional Euclidean space $\mathbb{R}^m$ as internal space }

$\mathrm{M}^{(\mathrm{lcag})}$:= { Model sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a lca group G as internal space }

$\mathcal{M}^{(\mathbb{R})}$:= { Meyer sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a m-dimensional Euclidean space $\mathbb{R}^m$ as internal space }

$\mathcal{M}^{(\mathrm{lcag})}$:= { Meyer sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a lca group G as internal space }

$\mathcal{UD}$:= { Uniformly discrete sets in $\mathbb{R}^n$ }

$\mathcal{UD}_{\mathrm{fg}}$:= { Finitely generated uniformly discrete sets in $\mathbb{R}^n$ }

$$\begin{array}{lll} & \subset & \cup_{r>0}\,\mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r \\ \mathcal{UD}_{\rm ft} & := & \{\text{ Uniformly discrete sets of finite type in } \mathbb{R}^n \ \} \\ X_{\rm fg} & := & \{\text{ Finitely generated Delone sets in } \mathbb{R}^n \ \} \\ X_{\rm ft} & := & \{\text{ Delone sets of finite type in } \mathbb{R}^n \ \} \end{array}$$

Theorem 2.16. *The following inclusions hold:*

$$\begin{array}{ccccccccc} & & & & & & \mathcal{UD}_{\rm ft} & \subset & \mathcal{UD}_{\rm fg} \\ & & & & & & \cup & & \cup \\ \mathcal{M}^{(\rm lcag)} & \supset & \mathrm{M}^{(\rm lcag)} & \supset & \mathcal{L}_n & \subset \mathrm{M}^{(\mathbb{R})} \subset & \mathcal{M}^{(\mathbb{R})} \subset & X_{\rm ft} & \subset \quad X_{\rm fg} \end{array} \tag{2.5}$$

Proof. Theorem 9.1 in [79], Theorem 2.1 and Theorem 3.1 in [65]. □

Definition 2.17. Let $\Lambda \in \mathcal{UD}_{fg}$. If $\{e_1, e_2, \ldots, e_v\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\Lambda]$, i.e. $\mathbb{Z}[\Lambda] = \mathbb{Z}[e_1, e_2, \ldots, e_v]$, then the address map $\varphi : \mathbb{Z}[\Lambda] \to \mathbb{Z}^v$ of Λ associated to this basis is by definition

$$\varphi\Big(\sum_{i=1}^{v} m_i e_i\Big) = (m_1, m_2, \ldots, m_v).$$

In Section 3 we will mainly use address maps of difference sets $\Lambda - \Lambda$ for the elements Λ of $\mathcal{UD}_{fg}$.

2.4 Algebraic integers, inflation centers and self-similarities

Given Λ a uniformly discrete set of $\mathbb{R}^n$, a (affine) *self-similarity* of Λ is by definition a real number $\lambda > 1$ such that

$$\lambda(\Lambda - c) \quad \subset \quad \Lambda - c \tag{2.6}$$

for a certain point c in $\mathbb{R}^n$ (note that c belongs or not to Λ). A point $c \in \mathbb{R}^n$ for which (2.6) occurs for a certain $\lambda > 1$ is called an *inflation center* of Λ. The concept of self-similarity is an affine notion and λ depends upon c. Denote by

$$\mathcal{C}(\Lambda) := \{c \mid \exists \lambda > 1 \text{ such that } \lambda(\Lambda - c) \quad \subset \quad \Lambda - c\} \tag{2.7}$$

the set of inflation centers of Λ and by

$$\mathcal{S}(c) := \{\lambda > 1 \mid \lambda(\Lambda - c) \quad \subset \quad \Lambda - c\}, \qquad \text{for } c \in \mathcal{C}(\Lambda), \tag{2.8}$$

the set of self-similarities associated with the point c.

Proposition 2.18. *Let* $\Lambda = t+\mathbb{Z}$ *be a (affine) lattice of* $\mathbb{R}$ *of period* 1 *with* $t \in [0,1)$. *Then*

(i) $\mathcal{C}(\Lambda) = t + \mathbb{Q}$,

(ii) $\mathcal{S}(c) = \begin{cases} \mathbb{N}\setminus\{0\} & \text{if } c \in t+\mathbb{Z}, \\ (1-q\mathbb{Z})\cap\mathbb{N}\setminus\{0\} & \text{if } c \in \mathcal{C}(\Lambda),\ c = t+\frac{p}{q},\ \text{with } p,q \\ & \text{relatively prime } (p\in\mathbb{Z}, q\geq 2). \end{cases}$

The set of inflation centers of Λ *of given (point) density* $1/(2q)$ *of self-similarities, with* q *an integer* ≥ 2, *is exactly the uniformly discrete set*

$$t + \{\frac{p}{q} \mid p\in\mathbb{Z}, gcd(p,q) = \pm 1\}.$$

Proof. Routine, with the following definition of the *(point) density of self-similarities* of an inflation center $c \in \mathcal{C}(\Lambda)$:

$$\mathrm{dens}(\mathcal{S}(c)) = \limsup_{t\to\infty} \frac{1}{2t} \#\left(\mathcal{S}(c)\cap(1,t]\right). \tag{2.9}$$

□

In the general case, given a uniformly discrete set, the characterization of $\mathcal{C}(\Lambda)$ and $\mathcal{S}(c)$ with $c \in \mathcal{C}(\Lambda)$ remains an open problem, even for Delone sets; see [27], [28] and [73] for Penrose tilings and sets $\mathbb{Z}_\beta$ of β-integers [49], with β a quadratic Pisot number. At least, $\mathcal{C}(\Lambda)$ is expected to be far from being everywhere dense as in Proposition 2.18.

Definition 2.19. A uniformly discrete of $\mathbb{R}^n, n \geq 1$, is a (ssfgud set) self-similar finitely generated uniformly discrete set if it is finitely generated and admits at least one (affine) self-similarity.

Although a uniformly discrete set of $\mathbb{R}^n, n \geq 1$, may be finite, let us observe that a nonempty (ssfgud) self-similar finitely generated uniformly discrete set in $\mathbb{R}^n, n \geq 1$, is always infinite.

Definition 2.20. Let $\lambda > 1$ be a real algebraic integer. Denote by $\lambda^{(i)}$ its conjugates. We say that λ is

(i) a Pisot number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| < 1$,

(ii) a Salem number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| \leq 1$, with at least one on the unit circle,

(iii) a Perron number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| < \lambda$,

(iv) a Lind number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| \leq \lambda$, with at least one on the circle $\{|z| = \lambda\}$.

Theorem 2.21 (Meyer). *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a Meyer set. If Λ is a ssfgud set, then all its self-similarities are Pisot or Salem numbers.*

If $\beta > 1$ is a Pisot number of a Parry number, then the sets $\mathbb{Z}_\beta$ are Meyer sets that admit by construction the self-similarity β [49]. However, there exist many Meyer sets which have no self-similarities at all.

Theorem 2.22 (Lagarias). *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a Delone set. If Λ is a ssfgud set, then all its (affine) self-similarities λ are algebraic integers such that* degree(λ) *divides* rk Λ. *Moreover, if Λ is of finite type, then all the self-similarities are Perron or Lind numbers.*

The concept of (affine) self-similarity is extended as follows in a natural way [79].

Definition 2.23. Let Λ be a nonempty uniformly discrete set of $\mathbb{R}^n$. A self-similarity of Λ is given by a triple (c, λ, Q) where λ is a real number > 1, Q an element of the orthogonal group $O(n, \mathbb{R})$ such that

$$\lambda Q(\Lambda - c) \quad \subset \quad \Lambda - c \tag{2.10}$$

for a certain $c \in \mathbb{R}^n$ (note that c belongs or not to Λ).

A point c for which (2.10) occurs for certain couple (λ, Q) is called an *inflation center* of Λ, as in the affine case. Problems on self-similar sets are reported in [87].

2.5 Sets $\mathbb{Z}_\beta$ of beta-integers and Rauzy fractals

Meyer sets $\mathbb{Z}_\beta \subset \mathbb{R}$ of β-integers with β a Pisot number, and their vectorial extension to $\mathbb{R}^n$ - so-called β-grids -, are useful tools for modeling quasicrystals in physics [36] [37] [47] [48]. Indeed, Penrose tilings in the plane and in space play a fundamental role in this modelling process, with suitable positioning of atoms in the tiles. Gazeau [48] has observed that Penrose tilings can easily be deduced from τ-grids, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean, quadratic Pisot number. Therefore it is natural to extend the constructions of Penrose tilings and β-grids with Pisot numbers β (or more generally with algebraic integers) of higher degree which could be used in the objective of providing possibly new models of aperiodic crystals in crystallography to physicists.

Let us recall the mathematical construction of $\mathbb{Z}_\beta$ on the line and its properties when $\beta > 1$ is a real number, in a general way, and some open questions related to them when β is in particular an algebraic integer. We refer to [44] [45] [85] [89] and [10] for an overview on recent studies on Numeration and its applications.

2.5.1 Construction and Properties For a real number $x \in \mathbb{R}$, the integer part of x will be denoted by $\lfloor x \rfloor$ and its fractional part by $\{x\} = x - \lfloor x \rfloor$. The smallest integer larger than or equal to x will be denoted by $\lceil x \rceil$. For $\beta > 1$ a real number and $z \in [0,1]$ we denote by $T_\beta(z) = \beta z \pmod 1$ the β-transform on $[0,1]$ associated with β, and iteratively, for all integers $j \geq 0$, $T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z))$, where by convention $T_\beta^0 = Id$.

Let $\beta > 1$ be a real number. A beta-representation (or β-representation, or representation in base β) of a real number $x \geq 0$ is given by an infinite sequence $(x_i)_{i \geq 0}$ and an integer $k \in \mathbb{Z}$ such that $x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k}$, where the digits x_i belong to a given alphabet ($\subset \mathbb{N}$). Among all the beta-representations of a real number $x \geq 0, x \neq 1$, there exists a particular one called Rényi β-expansion, which is obtained through the greedy algorithm [44] [45]: in this case, k satisfies $\beta^k \leq x < \beta^{k+1}$ and the digits

$$x_i := \lfloor \beta T_\beta^i(\frac{x}{\beta^{k+1}}) \rfloor \qquad\qquad i = 0, 1, 2, \ldots, \tag{2.11}$$

belong to the finite canonical alphabet $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. If β is an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \beta - 1\}$; if β is not an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. We denote by

$$\langle x \rangle_\beta := x_0 x_1 x_2 \ldots x_k \, . \, x_{k+1} x_{k+2} \ldots \tag{2.12}$$

the couple formed by the string of digits $x_0 x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots$ and the position of the dot, which is at the kth position (between x_k and x_{k+1}). By definition the integer part (in base β) of x is $\sum_{i=0}^{k} x_i \beta^{-i+k}$ and its fractional part (in base β) is $\sum_{i=k+1}^{+\infty} x_i \beta^{-i+k}$. If a Rényi β-expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros).

There is a particular Rényi β-expansion which plays an important role in the theory, which is the Rényi β-expansion of 1, denoted by $d_\beta(1)$ and defined as follows: since $\beta^0 \leq 1 < \beta$, the value $T_\beta(1/\beta)$ is here set (by convention) to 1. Then using (2.11) for all $i \geq 1$, we obtain: $t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{\beta\} \rfloor, t_3 = \lfloor \beta \{\beta \{\beta\}\} \rfloor$, etc. The equality $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$ corresponds to $1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}$. By definition, a real number $\beta > 1$ such that $d_\beta(1)$ is finite or eventually periodic is called a beta-number or more recently *a Parry number* (this new name appears in [37]). In particular, it is called a simple beta-number or *a simple Parry number* (after [37]) when $d_\beta(1)$ is finite. Beta-numbers (Parry numbers) are algebraic integers [85] and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [41] [102]. The conjugates of Parry numbers are all bounded above in modulus by the golden mean $\frac{1}{2}(1 + \sqrt{5})$ [41] [102].

Definition 2.24. The set

$$\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid |x| \text{ is equal to its integer part in base } \beta\}$$

is called set of beta-integers, or set of β-integers, or set of integers in base β.

By construction, the set $\mathbb{Z}_\beta$ is discrete, relatively dense and locally finite (its intersection with any interval of the line is finite), self-similar, with β as self-similarity (with inflation center the origin), and symmetrical with respect to the origin: $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$, $\mathbb{Z}_\beta = -\mathbb{Z}_\beta$. Its complete set of self-similarities is unknown. Thurston [104] has shown that it is uniformly discrete, hence a Delone set, when β is a Pisot number. From [49], for β a Pisot number, the relatively dense set $\mathbb{Z}_\beta \cap \mathbb{R}^+$ is finitely generated over $\mathbb{N}$.

Theorem 2.25. *If β is a Pisot number, the Delone set $\mathbb{Z}_\beta$ is a Meyer set which is a ssfgud set in $\mathbb{R}$.*

Proof. [21], [49]. For the definition of a ssfgud set, see Definition 2.19. It is a ssfgud set since $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. □

Open problem ($\mathbf{P}_1$).— What is the class of real numbers $\beta > 1$ for which $\mathbb{Z}_\beta$ is uniformly discrete, equivalently a Delone set ?

We know that this class contains Pisot numbers (see [18] [49]) and Parry numbers. Whether it contains Salem numbers or Perron numbers, except a few cases [20], is unknown [105]. Problem (P_1) is equivalent to knowing whether the β-shift is specified [18] [49] [105].

The set $\mathbb{Z}_\beta$ contains $\{0, \pm 1\}$ and all the polynomials in β for which the coefficients are given by the equations (2.11). Parry [85] has shown that the knowledge of $d_\beta(1)$ suffices to exhaust all the possibilities of such polynomials by the so-called "Conditions of Parry (CP_β)". Let us recall them. Let $(c_i)_{i\geq 1} \in \mathbb{A}_\beta^{\mathbb{N}}$ be the following sequence:

$$c_1c_2c_3\cdots = \begin{cases} t_1t_2t_3\cdots & \text{if } d_\beta(1) = 0.t_1t_2\cdots \text{ is infinite,} \\ (t_1t_2\cdots t_{m-1}(t_m-1))^\omega & \text{if } d_\beta(1) \text{ is finite and} \\ & \text{equal to } 0.t_1t_2\cdots t_m, \end{cases} \tag{2.13}$$

where $(\ )^\omega$ means that the word within $(\ )$ is indefinitely repeated. When the degree of β is ≥ 2, we have $c_1 = t_1 = \lfloor \beta \rfloor$. Then the polynomial $\sum_{i=0}^{v} y_i\beta^{-i+v} \geq 0$, with $v \geq 0, y_i \in \mathbb{Z}$ arbitrary, belongs to $\mathbb{Z}_\beta^+ := \mathbb{Z}_\beta \cap \mathbb{R}^+$ if and only if $y_i \in \mathbb{A}_\beta$ and the following $v+1$ inequalities are satisfied:

$$(\mathrm{CP}_\beta): \quad (y_j, y_{j+1}, y_{j+2}, \ldots, y_{v-1}, y_v, 0, 0, 0, \ldots) \prec (c_1, c_2, c_3, \ldots), \quad \text{for all} j = 0, 1, 2, \ldots, v, \tag{2.14}$$

where "$\prec$" means lexicographical smaller. For a negative polynomial, we consider the above criterium applied to its opposite. Conditions of Parry (CP_β) sieve the elements of the ring $\mathbb{Z}[\beta]$ in the number field $\mathbb{Q}(\beta)$.

The set $\mathbb{Z}_\beta$ can be viewed as the set of vertices of the tiling $\mathcal{T}_\beta$ of the real line for which the tiles are the closed intervals whose extremities are two successive β-integers. When β is a Pisot number, the number of (non-congruent) tiles in $\mathcal{T}_\beta$ is finite [104]. If V is a tile of $\mathcal{T}_\beta$ we denote by $l(V)$ its length. If β is a Pisot number and $d_\beta(1)$ is finite, say $d_\beta(1) = 0.t_1t_2\dots t_m$, then the set of the lengths of the tiles of $\mathcal{T}_\beta$ is exactly $\{T_\beta^i(1) \mid 0 \leq i \leq m-1\} = \{1, \beta - t_1, \beta^2 - t_1\beta - t_2, \dots, \beta^{m-1} - t_1\beta^{m-2} - t_2\beta^{m-3} - \dots - t_{m-1}\}$. If β is a Pisot number with $d_\beta(1)$ eventually periodic, say $d_\beta(1) = 0.t_1t_2\dots t_m(t_{m+1}t_{m+2}\dots t_{m+p})^\omega$, then the set of the lengths of the tiles of $\mathcal{T}_\beta$ is exactly $\{T_\beta^i(1) \mid 0 \leq i \leq m+p-1\} = \{1, \beta - t_1, \beta^2 - t_1\beta - t_2, \dots, \beta^{m-1} - t_1\beta^{m-2} - t_2\beta^{m-3} - \dots - t_{m-1}, \beta^m - t_1\beta^{m-1} - t_2\beta^{m-2} - \dots - t_m, \dots, \beta^{m+p-1} - t_1\beta^{m+p-2} - t_2\beta^{m+p-3} - \dots - t_{m+p-1}\}$. Hence, when β is a Pisot number, the set $\mathbb{Z}_\beta$ is a Delone set of (sharp) constants (r, R) with $r = \min\{l(V) \mid V \in \mathcal{T}_\beta\} > 0$ and $R = \frac{1}{2}\max\{l(V) \mid V \in \mathcal{T}_\beta\} = \frac{1}{2}$. The tiling $\mathcal{T}_\beta$ can be obtained directly from a substitution system on a finite alphabet which is associated to β in a canonical way [38] [42] [45].

2.5.2 Rauzy fractals and Meyer sets of beta-integers for β a Pisot number Rauzy fractals were introduced by Rauzy [1] [2], [42] (Chapter 7), [74] [88] [98] [99] to provide geometric interpretations and geometric representations of symbolic dynamical systems, in the general objective of understanding whether substitutive dynamical systems are isomorphic to already known dynamical systems or if they are new. Rauzy [88] generalized the dynamical properties of the Fibonacci substitution [42] to a three-letter alphabet substitution, called Tribonacci substitution or Rauzy substitution, defined by: $1 \to 12, 2 \to 13, 3 \to 1$. The incidence matrix of this substitution is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

its characteristic polynomial is $X^3 - X^2 - X - 1$ with $\beta > 1$ a Pisot number as dominant root, and two complex conjugates roots α and $\overline{\alpha}$ in the unit disc. The incidence matrix admits as eigenspaces in $\mathbb{R}^3$ an expanding one-dimensional direction and a contracting plane [49].

Theorem 2.26 (Rauzy). *The Rauzy fractal generates a self-similar periodic tiling of the plane. The symbolic dynamical system generated by the Tribonacci substitution is measure-theoretically isomorphic to a toral substitution. The Tribonacci substitutive dynamical system has a purely discrete spectrum.*

Properties of the Rauzy fractal (connectedness, interiors, boundary, etc) were obtained in [2] [32] [60] [74] [98] [99] [100] [101]. Gazeau and Verger-Gaugry [49] proved that the set $\mathbb{Z}_\beta$ of integers in base β are in close relation with the Rauzy fractal, within the framework of a canonical cut-and-project scheme $(\mathbb{R}^3 = G \times E, L = \mathbb{Z}^3, \pi_1, \pi_2)$ above $\mathbb{Z}_\beta$, where E is a line in $\mathbb{R}^3$ and G the corresponding internal space (hyperplane): the Rauzy fractal is the adherence of the image of $\mathbb{Z}_\beta$ by the map $\pi_1 \circ (\pi_{2_{|_L}})^{-1}$ in the internal space G. This situation is quite general for Pisot numbers β and the Rauzy fractal appears as a compact canonical window [49]. However, all the points of L are not selected by this window and only some of them which satisfy the conditions of Parry are projected on E, $\mathbb{Z}_\beta$ being a Meyer set [21] [49].

For all Perron numbers β, the construction of the cut-and-project scheme $(\mathbb{R}^d = G \times E, L = \mathbb{Z}^d, \pi_1, \pi_2)$ over $\mathbb{Z}_\beta$, where d is the degree of β, E a line in $\mathbb{R}^d$ and G the corresponding internal space (hyperplane), is canonical, and does not use the fact that $\mathbb{Z}_\beta$ should be uniformly discrete [49]. If β is a Pisot number then the image of L by $\pi_1 \circ (\pi_{2_{|_L}})^{-1}$ is relatively compact and its adherence is the (geometric) Rauzy fractal. Whether this image is relatively compact for β a general Salem number is not known.

The $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{Z}_\beta - \mathbb{Z}_\beta]$ is finitely generated for β a Pisot number, but it is not known whether it is the case for Perron numbers in general (which are not Pisot numbers).

Open problem ($\mathbf{P}_2$).— What is the class of real numbers $\beta > 1$ for which $\mathbb{Z}_\beta$ is uniformly discrete and is not finitely generated, i.e. for which

$$\operatorname{rank} \mathbb{Z}[\mathbb{Z}_\beta - \mathbb{Z}_\beta] = +\infty?$$

3 Proof of Theorem 1.1

(i) (same proof as [65] Theorem 4.1 (i)) Let $s = \dim_{\mathbb{R}} \mathbb{R}[\Lambda]$ be the dimension of the $\mathbb{R}$-span of Λ (by $\mathbb{R}$-span of Λ, we mean the intersection of all the real affine subspaces of $\mathbb{R}^n$ which contain Λ). Then $1 \leq s \leq n$ and $m := \operatorname{rk} \Lambda \geq s$. By definition the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda - \Lambda] := \{\sum_{\text{finite}} a_i(x_i - x_j) \mid a_i \in \mathbb{Z}, x_i, x_j \in \Lambda\}$ admits a set of m generators, say $\{v_1, v_2, \ldots, v_m\}$, which are $\mathbb{Q}$-linearly independent (nonzero) vectors of $\mathbb{R}^n$. Then

$$\mathbb{Z}[\Lambda - \Lambda] = \mathbb{Z}[v_1, v_2, \ldots, v_m].$$

If $\lambda > 1$ is a self-similarity of Λ then there exists $c \in \mathbb{R}^n$ such that $\lambda(\Lambda - c) \subset \Lambda - c$. Since $\Lambda - \Lambda = \Lambda - c - (\Lambda - c)$, this implies

$$\lambda\mathbb{Z}[\Lambda - \Lambda] \subset \mathbb{Z}[\Lambda - \Lambda]. \tag{3.1}$$

We deduce that there exist integers $a_{i,j} \in \mathbb{Z}$ such that

$$\lambda v_i = \alpha_{i,1}v_1 + \alpha_{i,2}v_2 + \ldots + \alpha_{i,m}v_m \qquad i = 1, 2, \ldots, m$$

with $M = \left(\alpha_{i,j}\right)_{i,j} \in \mathrm{Mat}_m(\mathbb{Z})$ the space of $m \times m$ integral matrices. Hence

$$\lambda V = MV\ , \qquad \text{with} \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.2}$$

Equivalently the transposed matrix ${}^tM$ is the matrix associated with the $\mathbb{Q}$-linear map which sends $\{v_1, v_2, \ldots, v_m\}$ to the system $\{\lambda v_1, \lambda v_2, \ldots, \lambda v_m\}$ with respect to the $\mathbb{Q}$-free system $\{v_1, v_2, \ldots, v_m\}$. Since the polynomial $h(X) := \det(X\,\mathrm{I} - M) \in \mathbb{Z}[X]$ is monic and cancels at λ, the real number λ is an algebraic integer of degree less than m.

Let d be the degree of λ and

$$\varphi(X) = X^d + a_1X^{d-1} + a_2X^{d-2} + \ldots + a_d, \qquad \text{with } a_i \in \mathbb{Z}, a_d \neq 0,$$

be the minimal polynomial of λ. From (3.2) we deduce $\lambda^j V = M^j V$ for all $j \in \mathbb{N}$. Hence, since $\varphi(\lambda) = 0$,

$$\varphi(M)V = (M^d + a_1M^{d-1} + a_2M^{d-2} + \ldots + a_d)V = 0. \tag{3.3}$$

Since $\varphi(M) \in \mathrm{Mat}_m(\mathbb{Z})$ and that the vectors $v_1, v_2, \ldots, v_m$ are $\mathbb{Q}$-linearly independent, we deduce $\varphi(M) = \varphi({}^tM) = 0$. Hence the minimal polynomial $\psi(X) \in \mathbb{Z}[X]$ of the matrix ${}^tM$ divides $\varphi(X)$ in $\mathbb{Z}[X]$. Since $\varphi(X)$ is irreducible over $\mathbb{Q}$, there is equality: $\psi(X) = \varphi(X)$.

Denote by K the number field $\mathbb{Q}(\lambda)$. Equation (3.1) implies that $\mathbb{Z}[\Lambda - \Lambda]$ is a module over the ring $\mathbb{Z}[\lambda]$ and that $\mathbb{Q}[\Lambda - \Lambda]$ is a K-vector space. The ring $\mathbb{Z}[\lambda]$ is a subring of finite index of the ring of integers $\mathcal{O}_K$ of K. The $m \times m$ integral matrix ${}^tM$ corresponds to an endomorphism of $\mathbb{R}^m$, say u, expressed in the canonical basis $\{e_1, e_2, \ldots, e_m\}$. Since $v_1, v_2, \ldots, v_m$ are $\mathbb{Q}$-linearly independent, the base $\{e_1, e_2, \ldots, e_m\}$ of $\mathbb{R}^m$ and the system $\{v_1, v_2, \ldots, v_m\}$ of $\mathbb{R}^n$ can be identified as well as the two $\mathbb{Q}$- vector spaces $\oplus_{i=1}^m \mathbb{Q}e_i$ and $\oplus_{i=1}^m \mathbb{Q}v_i$. There are two cases: $m = 1$ and $m > 1$. When $m = 1$, then necessarily λ is an integer > 1 and $d = 1$. When $m > 1$ and $d = 1$, then the matrix M is the diagonal matrix $\lambda\mathrm{I}$ and λ is an integer > 1. This case occurs for instance for (affine) lattices Λ of $\mathbb{R}^n$. Now, if $m > 1$ and $d \geq 2$, then the endomorphism u induces a Jordan decomposition of $\mathbb{R}^m$ as $K[X]$-module as follows (for instance

[43] pp 295-301). We assume $d \geq 2$ in the sequel. Let

$$A(\psi) = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_d \\ 1 & \ddots & 0 & \vdots & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & -a_2 \\ 0 & \dots & & 1 & -a_1 \end{pmatrix}, \text{ resp. } U_d := \begin{pmatrix} 0 & \dots & 1 \\ \vdots & 0 & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

be the $d \times d$ integral matrix whose terms are 0 except the last column which is composed of the coefficients of $\psi(X)$ (up to sign) and the diagonal under the main diagonal, which is composed with 1, respectively the $d \times d$ matrix whose terms are 0 except the term of the first row and the last column, which is 1 (this makes sense since $d > 1$). Then there exists a basis of $\mathbb{R}^m$, say $\{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$, in which the matrix of the endomorphism u takes the diagonal form

$$\mathcal{J} := \begin{pmatrix} J_{i_1} & 0 & \dots & 0 \\ 0 & J_{i_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{i_r} \end{pmatrix}$$

where $i_1 + i_2 + \ldots + i_r = m$, $i_1 \geq i_2 \geq \ldots \geq i_r \geq d$, and $J_{i_q}, 1 \leq q \leq r$, is the $i_q \times i_q$ integral matrix given by

$$J_{i_q} := \begin{pmatrix} A(\psi) & 0 & \dots & 0 \\ U_d & A(\psi) & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & U_d & A(\psi) \end{pmatrix}$$

with 0 everywhere except $A(\psi)$ on the main diagonal and U_d on the diagonal under the main diagonal. Since all the diagonal terms of the matrix J_{i_q} are $A(\psi)$, they are identical, and therefore d divides i_q. Consequently d divides $\sum_{q=1}^{r} i_q = m$.

(ii) Let us transcript back this Jordan decomposition to the ambient space $\mathbb{R}^n$ of the uniformly discrete set Λ, block by block. Let us consider the first block J_{i_1}, the situation being the same for the others. The system $\{\epsilon_1, \epsilon_2, \dots, \epsilon_{i_1}\}$ satisfies the following relations:

- for $1 \leq \beta < d$, $0 \leq \alpha \leq \frac{i_1}{d} - 1$,

$$u(\epsilon_{\alpha d+\beta}) = \epsilon_{\alpha d+\beta+1}, \tag{3.4}$$

- for $\beta = d$, $0 \leq \alpha < \frac{i_1}{d} - 1$,

$$u(\epsilon_{(\alpha+1)d}) = \epsilon_{(\alpha+1)d+1} - a_1\epsilon_{(\alpha+1)d} - a_2\epsilon_{(\alpha+1)d-1} - \ldots - a_d\epsilon_{\alpha d+1}, \tag{3.5}$$

- for $\beta = d$, $\alpha = \frac{i_1}{d} - 1$,

$$u(\epsilon_{i_1}) = -a_1\epsilon_{i_1} - a_2\epsilon_{i_1-1} - \ldots - a_d\epsilon_{i_1-d+1}. \tag{3.6}$$

Now the matrices ${}^tM$ and $\mathcal{J}$ have coefficients in $\mathbb{Q}$ and are such that there exist a $m \times m$ invertible matrix $C \in GL(m, \mathbb{R})$ such that ${}^tM = CJC^{-1}$. Then (Corollary 2 in [68], Chap. XV, §3) there exists a $m \times m$ invertible matrix C' in $\mathbb{Q}$ such that:

$${}^tM = C'JC'^{-1}.$$

The matrix C' is the matrix associated with the linear map which sends $\{e_1, e_2, \ldots, e_m\}$ to $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ with respect to the basis $\{e_1, e_2, \ldots, e_m\}$. Let

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} = {}^tC'V \ , \qquad \text{with} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.7}$$

The $\mathbb{Q}$-free system $\{f_1, f_2, \ldots, f_m\}$ of nonzero vectors of $\mathbb{R}^n$, identified with the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ of $\mathbb{R}^m$, admits the following structure: from (3.2)

$$\lambda V = MV = {}^t(C'JC'^{-1}) \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \Leftrightarrow {}^tC'^{-1}(\lambda \mathrm{I} - {}^tJ){}^tC' \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = 0; \tag{3.8}$$

hence

$$\lambda \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = {}^tJ \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}. \tag{3.9}$$

From (3.9), considering the first block J_{i_1}, the situation being the same with the other blocks $J_{i_q}, 1 \leq q \leq r$, we deduce (see (3.4), (3.5)):

- for $1 \leq \beta < d$, $0 \leq \alpha \leq \frac{i_1}{d} - 1$,

$$f_{\alpha d+\beta+1} = \lambda f_{\alpha d+\beta} = \lambda^\beta f_{\alpha d+1}, \tag{3.10}$$

- for $\beta = d$, $0 \leq \alpha < \frac{i_1}{d} - 1$,

$$f_{(\alpha+1)d+1} = \lambda f_{(\alpha+1)d} + a_1 f_{(\alpha+1)d} + a_2 f_{(\alpha+1)d-1} + \ldots + a_d f_{\alpha d+1}. \tag{3.11}$$

Let us show that the assumption $d < i_1$ leads to a contradiction. Assume $d < i_1$. Then we would have $f_{d+1} \neq 0$ from (3.7) since C' is invertible. But,

from (3.10) and (3.11), with $\alpha = 0$,

$$\begin{array}{rcl} f_{d+1} & = & (\lambda + a_1)\, f_d + a_2\, f_{d-1} + \ldots + a_d\, f_1 \\ & = & (\lambda + a_1)\, \lambda^{d-1}\, f_1 + a_2\, \lambda^{d-2}\, f_1 + \ldots + a_d\, f_1 \quad = \quad \varphi(\lambda)\, f_1 = \quad 0. \end{array}$$

Contradiction. Hence $d = i_1$. Proceeding now with the other blocks in the same way leads to the equalities $d = i_1 = i_2 = \ldots = i_r$.

The matrix C' belongs to $GL(m, \mathbb{Q})$. If $C' \in GL(m, \mathbb{Z})$ we take

$$w_1 = f_1, w_2 = f_{d+1}, w_3 = f_{2d+1}, \ldots, w_r = f_{(r-1)d+1}.$$

Then we deduce from (3.7) and (3.10) (and its analogs for the other blocks) that $\oplus_{i=1}^{m} \mathbb{Z}\, v_i$ and $\oplus_{q=1}^{r} \oplus_{i=1}^{m} \mathbb{Z} \lambda^i\, w_q$ are isomorphic as $\mathbb{Z}$-modules. We deduce the result in this case. If $C' \in GL(m, \mathbb{Q}) \setminus GL(m, \mathbb{Z})$, let us denote by μ the lcm of the m^2 denominators of the coefficients of C'^{-1} and take

$$w_1 = f_1/\mu, w_2 = f_{d+1}/\mu, w_3 = f_{2d+1}/\mu, \ldots, w_r = f_{(r-1)d+1}/\mu.$$

The coefficients of $D := \mu\, {}^t C'^{-1}$ are in $\mathbb{Z}$ and relatively prime so that

$$D \begin{bmatrix} f_1/\mu \\ \lambda f_1/\mu \\ \vdots \\ \lambda^{d-1} f_1/\mu \\ \vdots \\ f_r/\mu \\ \lambda f_r/\mu \\ \vdots \\ \lambda^{d-1} f_r/\mu \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.12}$$

From (3.12), the $\mathbb{Z}$-module $\oplus_{i=1}^{m} \mathbb{Z}\, v_i$ is a $\mathbb{Z}$-submodule of $\oplus_{q=1}^{r} \oplus_{i=0}^{d-1} \mathbb{Z} \lambda^i\, w_q$. Hence the result.

(iii) Since $d = i_1 = i_2 = \ldots = i_r$ and $r = \frac{m}{d}$ by (ii), we deduce that $\mathcal{J}$ is the $m \times m$ diagonal matrix for which the diagonal terms are all identical and equal to $A(\psi)$:

$$\mathcal{J} = \begin{pmatrix} A(\psi) & 0 & \ldots & 0 \\ 0 & A(\psi) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & A(\psi) \end{pmatrix}.$$

Since $\det M = \det \mathcal{J}$, we obtain the characteristic polynomial of M:

$$\det(X\, \mathrm{I}_m - M) = \det(X\, \mathrm{I}_d - A(\psi))^{m/d} = (\varphi(X))^{m/d};$$

in particular, $\det M = (\det A(\psi))^{m/d} = N(\lambda)^{m/d}$, where $N(\lambda) = (-1)^d a_d$, product of the conjugates of λ, is the algebraic norm of λ. This formula is reminiscent of the algebraic norm of an element in a ring extension [68].

(iv) There are two cases: either (iv-1) $w_1, w_2, \ldots, w_r$ are K-linearly independent, or (iv-2) they are K-linearly dependent. In the first case the cut-and-project scheme above Λ will admit an internal space reduced to its *shadow space* (see below and [65]), while, in the second case, the internal space will come from the shadow space and the space of relations over K between the vectors w_i.

First let us fix some notations. Denote by $\mathcal{C}$ the finite set $\{w_1, w_2, \ldots, w_r\}$ and call it the *central cluster* of the basis $(\ \lambda^j w_i\ )_{i=1,\ldots,r, j=0,1,\ldots,d-1}$. For $i = 1, 2, \ldots, r$, let $\widetilde{w_i} := \|w_i\|^{-1} w_i$. Let $\widetilde{\mathcal{C}} := \{\widetilde{w_1}, \ldots, \widetilde{w_r}\}$ the image of $\mathcal{C}$ on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. We have $\mathrm{Card}(\widetilde{\mathcal{C}}) \leq r$. We assume that the signature of the field $K = \mathbb{Q}(\lambda)$, of degree d, is (r_1, r_2). Then $d = r_1 + 2r_2$. Denote by $\sigma_j, 1 \leq j \leq r_1$, the real embeddings of K in $\mathbb{R}$, and by $\sigma_j, \sigma_{r_2+j} = \overline{\sigma_j}$, where $r_1 + 1 \leq j \leq r_1 + r_2$, the imaginary embeddings of K in $\mathbb{C}$. Assume $\sigma_1(\lambda) = \lambda$. Let Σ be the embedding of K in $\mathbb{R}^{r_1} \times \mathbb{C}^{2r_2}$ defined by

$$\forall \xi \in K, \quad \Sigma(\xi) = (\sigma_1(\xi), \sigma_2(\xi), \ldots, \sigma_{r_1+2r_2}(\xi))\,.$$

Let $(g_j)_{1 \leq j \leq d}$ be a $\mathbb{Z}$-basis of the ring of integers $\mathcal{O}_K$ of K. We identify the field K to $\mathbb{Q}^d$ via the mapping Ψ defined by $\Psi(z) = \sum_{i=1}^d z_i g_i$ if $z \in \mathbb{Q}^d$, resp. $\mathcal{O}_K$ to $\mathbb{Z}^d$ if $z \in \mathbb{Z}^d$. The composed mapping $\Phi := \Sigma \circ \Psi$ is then extended in a continuous way from $\mathbb{Q}^d$ to $\mathbb{R}^d$, and denoted in the same way:

$$\forall z \in \mathbb{R}^d, \quad \Phi(z) := \left(\sum_{i=1}^d z_i \sigma_1(g_i), \sum_{i=1}^d z_i \sigma_2(g_i), \ldots, \sum_{i=1}^d z_i \sigma_{r_1+2r_2}(g_i) \right).$$

Σ is an injective homorphism for the ring structures while Ψ is $\mathbb{Q}$-vector space isomorphism. Thus Φ is a $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^d$ onto the étale $\mathbb{R}$-vector space

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \{z \in \mathbb{C}^{2r_2} \mid z_{r_2+j} = \overline{z_j} \text{ for all } j = 1, 2, \ldots, r_2\}.$$

The $\mathbb{R}$-subspace $\Sigma(\mathcal{O}_K)$ of $K_{\mathbb{R}}$ is a lattice. Let us extend Φ to $\mathbb{C}^d$ as a $\mathbb{C}$-endomorphism, keeping the same notation, by (with $\mathrm{I} = \sqrt{-1}$):

$$\Phi(x + \mathrm{I}\ y) = \Phi(x) + \mathrm{I}\ \Phi(y), \quad \text{for all } \ x, y \in \mathbb{R}^d.$$

Let us construct the cut-and-project scheme above Λ. Let $x \in \mathbb{R}[\Lambda] \subset \mathbb{R}^n$ be in the $\mathbb{R}$-span of Λ. For $i = 1, 2, \ldots, r$, denote by $p_i(x)$ the orthogonal projection of x onto the line $\mathbb{R}w_i$ and $\widetilde{w_i} := \|w_i\|^{-1} w_i$. Then $p_i(x)$ can be written

$$p_i(x) = \frac{\langle x, w_i \rangle}{\langle w_i, w_i \rangle} w_i = \langle x, \widetilde{w_i} \rangle \widetilde{w_i},$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. The point $x \in \mathbb{R}^n$ will be said *K-rational* if the r coefficients $\langle x, \widetilde{w_i} \rangle$ belongs to K. In this case, for all $i = 1, \dots, r$, $\langle x, \widetilde{w_i} \rangle = \sum_{j=1}^{d} \alpha_{i,j}(x) g_j$ with all coefficients $\alpha_{i,j}(x) \in \mathbb{Q}$. Let us define $\Sigma_i : K \to K_{\mathbb{R}} \widetilde{w_i}$ by $\Sigma_i(\xi) = \Sigma(\xi)\widetilde{w_i}$, for all $i = 1, \dots, r$, and $\Phi_i = \Sigma_i \circ \Psi$ the $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^d$ onto the $\mathbb{R}$-vector space

$$K_{\mathbb{R}} \widetilde{w_i} := (\mathbb{R}\widetilde{w_i})^{r_1} \times \{\{z_j \widetilde{w_i}\} \mid z = (z_j) \in \mathbb{C}^{2r_2}, z_{r_2+j} = \overline{z_j} \text{ for all } j = 1, 2, \dots, r_2\}.$$

The $\mathbb{R}$-subspace $\Sigma_i(\mathcal{O}_K)$ of $K_{\mathbb{R}} \widetilde{w_i}$ is a lattice. For any set A, denote by $\mathrm{pr}_k : A^d \to A$ the k-th projection, so that $\mathrm{pr}_k(K_{\mathbb{R}} \widetilde{w_i}) = \sigma_k(K) \widetilde{w_i}$ for all $i = 1, \dots, r$ and $k = 1, \dots, d$. Since the mapping

$$\mathbb{R}[\Lambda] \to \coprod_{i=1}^{r} \mathrm{pr}_1 \left(K_{\mathbb{R}} \widetilde{w_i} \right), \qquad x \to \left(\langle x, \widetilde{w_i} \rangle \right)_i \tag{3.13}$$

is injective, as $\mathbb{R}$-morphism of vector spaces, the $\mathbb{R}$-span $\mathbb{R}[\Lambda]$ of Λ is identified by (3.13) with a s-dimensional $\mathbb{R}$-subspace of the first component $\coprod_{i=1}^{r} K\widetilde{w_i} = \coprod_{i=1}^{r} \mathrm{pr}_1(K_{\mathbb{R}} \widetilde{w_i})$. Denote by R_K the subspace of $\coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i}$ which is the closure of

$$\{(\Sigma_i(\langle x, \widetilde{w_i} \rangle))_i \mid \ x \in \mathbb{R}[\Lambda] \text{ is } K\text{- rational}\}.$$

It is a product of r_1 copies of $\mathbb{R}[\Lambda]$ and r_2 copies of $\mathbb{C}[\Lambda]$, with $\mathrm{pr}_1(R_K) = \mathbb{R}[\Lambda]$.

The space $\coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i}$ is a K-vector space, the external law being given by

$$\begin{array}{ccc} K \times \coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i} & \to & \coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i} \\ (\mu, u) & & \Sigma(\mu) \cdot u \end{array} \tag{3.14}$$

with componentwise multiplication, where $u = (u_k)_{k=1,\dots,d}$, so that the external law, on the k-th component, is given by:

$$\begin{array}{ccc} K \times \coprod_{i=1}^{r} \mathrm{pr}_k \left(K_{\mathbb{R}} \widetilde{w_i} \right) & \to & \coprod_{i=1}^{r} \mathrm{pr}_k \left(K_{\mathbb{R}} \widetilde{w_i} \right). \\ (\mu, u_k) & & \sigma_k(\mu) \cdot u_k \end{array} \tag{3.15}$$

The actions (3.14) and (3.15) are extended from K to $\mathbb{R}$ by continuity. Thus, by (3.15) and since the conjugate fields $\sigma_j(K), \sigma_{r_2+j}(K)$ are not subfields of $\mathbb{R}$ for $j = 1, 2, \dots, r_2$ (if $r_2 \neq 0$), the usual scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ should be considered as the restriction to $\mathbb{R}^n$ of the standard hermitian form on $\mathbb{C}^n$; in particular, it is anti-linear for the second variable.

We now construct a real positive definite symmetric bilinear form on $\coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i}$. Since

$$\coprod_{i=1}^{r} K_{\mathbb{R}} \widetilde{w_i} = \prod_{k=1}^{d} \left(\coprod_{i=1}^{r} \mathrm{pr}_k \left(K_{\mathbb{R}} \right) \widetilde{w_i} \right)$$

it suffices to construct it on the k-th component $\coprod_{i=1}^{r} \sigma_k(K)\,\widetilde{w_i}$. Let us define $q_k : \coprod_{i=1}^{r} \sigma_k(K)\widetilde{w_i} \times \coprod_{i=1}^{r} \sigma_k(K)\widetilde{w_i} \to \mathbb{R}$ by

$$q_k(U,V) := \begin{cases} \sum_{i=1}^{r} u_i v_i & k = 1,2,\ldots,r_1, \\ \sum_{i=1}^{r} (\overline{u_i} v_i + u_i \overline{v_i}) & k = r_1+1,\ldots,r_1+r_2. \end{cases} \tag{3.16}$$

where $U = (u_i\widetilde{w_i})_i$ and $V = (v_i\widetilde{w_i})_i$. Then we define

$$q : \prod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} \times \prod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} \to \mathbb{R}, \quad q(U,V) := \sum_{k=1}^{r_1+r_2} q_k(\mathrm{pr}_k(U), \mathrm{pr}_k(V)). \tag{3.17}$$

Let

$$\mathcal{G} := \{\varphi \in \mathbb{R}^{\mathcal{C}} \mid \frac{\varphi(w)}{\|w\|} \in K \text{ for all } w \in \mathcal{C},\ \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i} = 0\ \}.$$

We call $\mathcal{G}$ the *space of relations over K between the generators $w_1,\ldots,w_r$*. The space $\mathcal{G}$ can be identified with a subspace of $K^{\mathcal{C}} \simeq K^r$, therefore with a subspace of $\coprod_{i=1}^{r} K\widetilde{w_i}$. Denote by G its image by $\coprod_{i=1}^{r} \Sigma_i$ in $\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}$ and by $\overline{G}$, resp. $\overline{\mathcal{G}}$, the closure of G, resp. of $\mathcal{G}$. For all $x \in \mathbb{R}[\Lambda]$ and all $\varphi \in \mathcal{G}$,

$$\langle x, \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i}\rangle = \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\langle x, \widetilde{w_i}\rangle = 0. \tag{3.18}$$

(3.18) implies that $q_1(\ (\|w_i\|^{-1}\varphi(w_i)\widetilde{w_i}\ )_i,\ (\langle x,\widetilde{w_i}\rangle\widetilde{w_i}\ )_i) = 0$. Then the two subspaces $\mathbb{R}[\lambda]$ and $\overline{\mathcal{G}}$ are orthogonal and complementary in $\coprod_{i=1}^{r} \mathbb{R}\,\widetilde{w_i} \simeq \mathbb{R}^r$, of respective dimensions s and $r-s$.

Let us prove that R_K and $\overline{G}$ are orthogonal and complementary, of respective dimension sd and $(r-s)d$, in $\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}$. It suffices to prove $q_k(U,V) = 0$ with $U = (\sigma_k(\|w_i\|^{-1}\varphi(w_i))\widetilde{w_i}\ )_i$ and $V = (\sigma_k(\langle x,\widetilde{w_i}\rangle)\,\widetilde{w_i}\ )_i$, for all $k = 2,3,\ldots,d$, $x \in \mathbb{R}[\Lambda]$ K-rational and $\varphi \in \mathcal{G}$. We have

$$\frac{1}{2}\,q_k(U,V) = Re\left[\sum_{i=1}^{r} \overline{\sigma_k\left(\|w_i\|^{-1}\varphi(w_i)\right)}\,\sigma_k\left(\langle x,\widetilde{w_i}\rangle\right)\right] =$$

$$Re\left[\sigma_k\left(\sum_{i=1}^{r} \overline{\|w_i\|^{-1}\varphi(w_i)}\,\langle x,\widetilde{w_i}\rangle\right)\right] = Re\left[\sigma_k\left(\langle x, \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i}\rangle\right)\right] = 0.$$

We deduce the claim.

The cut-and-project scheme above Λ we have constructed is the following:

$$\left(\prod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}\ \simeq\ \overline{G} \times R_K\ \simeq\ H \times \mathbb{R}[\Lambda], L, \pi, \mathrm{pr}_1\right)$$

where $L = \coprod_{i=1}^{r} \Sigma_i(\mathcal{O}_K)$ is a lattice in $\overline{G} \times R_K$ and pr_1 such that $\mathrm{pr}_1(R_K) = \mathbb{R}[\Lambda]$, $\mathrm{pr}_1(\overline{G}) = 0$. Because of the structure of the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda - \Lambda]$ given by

(ii), it suffices to take $L = \coprod_{i=1}^{r} \Sigma_i(\mathbb{Z}[\lambda])$. The projection mapping π is $\mathrm{Id}-\mathrm{pr}_1$. The internal space, say H, is $\overline{G}\times(R_K\setminus\mathbb{R}[\Lambda])$. By construction, $\pi(L)$ is dense in H and pr_1 is one-to-one on L, onto $\mathrm{pr}_1(L) = \mathbb{Z}[\lambda][w_1, w_2, \ldots, w_r] \supset \mathbb{Z}[\Lambda - \Lambda]$. Note that $\mathbb{Z}[\Lambda - \Lambda]$ is not necessarily a free $\mathbb{Z}[\lambda]$-module, but it is of finite index in $\mathrm{pr}_1(L) = \mathbb{Z}[\lambda][w_1, w_2, \ldots, w_r]$. The Euclidean structure on the cut-and-project scheme given by q is such that $\mathbb{R}[\Lambda]$ and H are orthogonal. The component $R_K \setminus \mathbb{R}[\Lambda]$ of the internal space H is called the *shadow space* in [65].

If the vectors $w_1, w_2, \ldots, w_r$ are K-linearly independent (case (iv-1)) then $\overline{G}$ is trivial and the internal space H is $R_K \setminus \mathbb{R}[\Lambda]$.

Now, this cut-and-project scheme lies above Λ since, for all $\nu \in \Lambda$, $\Lambda - \nu \subset \Lambda - \Lambda \subset \mathbb{Z}[\Lambda - \Lambda] \subset \mathrm{pr}_1(L)$. We deduce the claim.

4 Ideal lattices and proof of Corollary 1.2

The objectives of this section are the following: (i) to recall some definitions concerning ideal lattices, referring to [11] [12] [13] [14] [15], (ii) to show that the sublattice (L', q) of (L, q) such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ in the cut-and-project scheme above the ssfgud set Λ given by Theorem 1.1 (iv) is a sublattice of an ideal lattice.

It will suffice to show that the canonical bilinear form q defined by (3.17) has suitable properties.

The *canonical involution* (or complex conjugation) of the algebraic number field K generated by the self-similarity λ is the involution $^- : K_{\mathbb{R}} \to K_{\mathbb{R}}$ that is the identity on $\mathbb{R}^{r_1}$ and complex conjugation on $\mathbb{C}^{r_2}$. Let $\mathcal{P} := \{\alpha \in K_{\mathbb{R}} \mid \overline{\alpha} = \alpha$ and all components of α are $> 0\}$. Let us denote by $\mathrm{Tr} : K_{\mathbb{R}} \to \mathbb{R}$ the trace map, i.e. $Tr(x_1, x_2, \ldots, x_d) = x_1 + \ldots + x_d$.

A *generalized ideal* will be by definition a sub $\mathcal{O}_K$-module of K-rank one of $K_{\mathbb{R}}$. As examples, fractional ideals of K are generalized ideals; ideals of the type uI where I is an $\mathcal{O}_K$-ideal and $u \in K_{\mathbb{R}}$ are also generalized ideals.

Proposition 4.1. *Let $b : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$ be a symmetric bilinear form. The following statements are equivalent:*

(i) *there exists $\alpha \in K_{\mathbb{R}}$ with $\alpha = \overline{\alpha}$ such that*

$$b(x, y) \;=\; Tr(\alpha\, x\overline{y})$$

for all $x, y \in K_{\mathbb{R}}$,

(ii) *the identity*

$$b(\mu x, y) \;=\; b(x, \overline{\mu} y)$$

holds for all $x, y, \mu \in K_{\mathbb{R}}$.

Proof. [11] Proposition 2.1, [12] Proposition 1. □

An *ideal lattice* is a lattice (I, b) where I is a generalized ideal, and $b : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$ which satisfies the equivalent conditions of Proposition 4.1 with $\alpha \in \mathcal{P}$. Ideal lattices with respect to the canonical involution correspond bijectively to Arakelov divisors of the number field K [11] [12].

It is easy to check that Proposition 4.1 is satisfied by the real symmetric bilinear form q defined by (3.16) and (3.17) in Section 3, where $\alpha = (\alpha_i)_{1\leq i\leq d}$ with $\alpha_i = 1$ for all i. We assume $r = 1$, i.e. that the degree of the self-similarity λ is equal to the rank rk Λ. The lattice L' given by Theorem 1.1 (ii) such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ is of finite index in the $\mathcal{O}_K$-module $\Sigma(\mathcal{O}_K)\widetilde{w_1}$, of K-rank one in $K_{\mathbb{R}}\widetilde{w_1}$. We have:

$$q(\mu\, U, V) = q(U, \overline{\mu} V) \qquad \text{forall } U, V \in K_{\mathbb{R}}\widetilde{w_1} \quad \text{and all } \mu \in K_{\mathbb{R}}.$$

Then, by Proposition 4.1, the bilinear form q has the following expression:

$$q(U, V) = \mathrm{Tr}(\alpha\, U\overline{V}) \qquad \text{forall } U, V \in K_{\mathbb{R}}\widetilde{w_1}. \tag{4.1}$$

Therefore (L', q) is a sublattice of finite index of an Arakelov divisor of K in bijection with $\mathbb{Z}[\Lambda - \Lambda]$ by the projection mapping $\mathrm{pr}_{1_{|_L}}$. We deduce Corollary 1.2.

The construction of the real bilinear form q in Theorem 1.1 (iv) which provides the Euclidean structure to the cut-and-project scheme is obtained with $\alpha = (1)_{1\leq i\leq d}$ in Proposition 4.1. Other choices of α are possible and the parametrization of the set of possible constants $\alpha = \alpha(\widetilde{w_1})$ in q in (4.1) is studied for instance in Schoof ([93] and related works).

5 Lower bounds of densities and pseudo-Delone constants

5.1 Pseudo-Delone sphere packings

The following definition is inspired by the "empty sphere" method of Delone [34]. If $A \subset \mathbb{R}^n$ is any nonempty subset of $\mathbb{R}^n$ and Λ is a uniformly discrete set of constant $r > 0$, we define the density of $\mathcal{B}(\Lambda)$ in A by

$$\delta_A(\mathcal{B}(\Lambda)) := \limsup_{t\to+\infty} \frac{\mathrm{vol}(\bigcup_{z_i\in\Lambda, \|z_i\|\leq t} B(z_i, r/2) \cap A)}{\mathrm{vol}(B(0,t) \cap A)}. \tag{5.1}$$

We omit the subscript "$\mathbb{R}^n$" when $A = \mathbb{R}^n$.

Definition 5.1. A uniformly discrete set Λ of $\mathbb{R}^n, n \geq 1$, of constant $r > 0$ is *Delone-like* of constant $R_\xi > 0$ when there exists a sequence $\xi := (x_i, T_i)_i$ where $(x_i)_i$ is a sequence of points of $\mathbb{R}^n$ and $(T_i)_i$ a sequence of real numbers such that, with the notation $A_\xi := \mathbb{R}^n \setminus \bigcup_i \overset{\circ}{B}(x_i, T_i)$:

(i) $\forall i,\ T_{i+1} \geq T_i$, with $T_i \geq r/2$,

(ii) $\forall i,\ \overset{\circ}{B}(x_i, T_i) \cap \mathcal{B}(\Lambda) = \emptyset$,

(iii) $\forall x \in A_\xi$, $\exists \lambda \in \Lambda$ such that $\|x - \lambda\| \leq R_\xi$,

(iv) $\lim_{T \to +\infty} \frac{\mathrm{vol}(A_\xi \cap B(0,T))}{\mathrm{vol}(B(0,T))} = 1$.

If it is finite, the infimum $\inf\{R_\xi\}$ over all possibilites of point sets $(x_i)_i$ in $\mathbb{R}^n$ and collections of radii $(T_i)_i$, such that (i) to (iv) are satisfied, is called the *pseudo-Delone constant* of Λ and denoted by $R(\Lambda)$. Let us call *optimal* a collection ξ such that R_ξ is equal to $R(\Lambda)$.

By Zorn's Lemma, optimal collections exist. The portion of space $\cup_i \overset{\circ}{B}(x_i, T_i)$ defined by an optimal collection is an invariant, independent of the optimal collection used for defining it. Definition 5.1 means that we can remove the portion of ambient space which does not intervene at infinity for the computation of the density of Λ (in $\mathbb{R}^n$). Note that, for a ssfgud set of $\mathbb{R}^n$, this portion of space does not contribute to the determination of the generators w_i in Theorem 1.1 since $\bigcup_i \overset{\circ}{B}(x_i, T_i)$ contains no point of Λ, and it is legitimate to remove it.

5.2 Proof of Theorem 1.3

Let $R_c := \inf\{R(\Lambda) \mid \Lambda \text{ is uniformly discrete of } \mathbb{R}^n \text{ of constant } 1\}$ be the infimum of possible Delone constants over sphere packings of common radius $1/2$. R_c is only a function of n. Then, for all $r > 0$, rR_c is the infimum of Delone constants of uniformly discrete sets of constant r.

Let $r > 0$ and ω_n be the volume of the unit ball of $\mathbb{R}^n$. Let $R \geq rR_c$ and $T > R$ be a real number. Let Λ be a uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$ which is pseudo-Delone of pseudo-Delone constant R. Let $\xi := (x_i, T_i)_i$ be an optimal sequence and $A_\xi := \mathbb{R}^n \setminus \cup_i \overset{\circ}{B}(x_i, T_i)_i$. For all $\epsilon > 0$, the pseudo-Delone constant of Λ in A_ξ is smaller than $R + \epsilon$. Then $(B(0, R+\epsilon) + \Lambda) \cap B(0,T)$ covers the set $B(0, T - R - \epsilon) \cap A_\xi$. The number of elements of $\Lambda \cap B(0,T)$ is equal to the number of elements of $\Lambda \cap B(0,T) \cap A_\xi$. This number is at least

$$\frac{\omega_n (T - R - \epsilon)^n - \mathrm{vol}((\mathbb{R}^n \setminus A_\xi) \cap B(0, T - R - \epsilon))}{\omega_n (R + \epsilon)^n}$$

$$= \frac{(T-R-\epsilon)^n}{(R+\epsilon)^n}\left(1-\frac{\mathrm{vol}((\mathbb{R}^n\setminus A_\xi)\cap B(0,T-R-\epsilon))}{\omega_n(T-R-\epsilon)^n}\right).$$

On the other hand, since all the balls of radius $r/2$ centered at the elements of $\Lambda\cap B(0,T)$ lie within $B(0,T+r/2)$ and also within A_ξ, the proportion of space they occupy in $B(0,T+r/2)\cap A_\xi$ is at least

$$\left(\frac{T-R-\epsilon}{R+\epsilon}\right)^n\left(1-\frac{\mathrm{vol}((\mathbb{R}^n\setminus A_\xi)\cap B(0,T-R-\epsilon))}{\omega_n(T-R-\epsilon)^n}\right)\frac{\mathrm{vol}(B(0,r/2))}{\mathrm{vol}(B(0,T+r/2)\cap A_\xi)} \tag{5.2}$$

But, for all $\epsilon>0$,

$$\lim_{T\to+\infty}\frac{\mathrm{vol}((\mathbb{R}^n\setminus A_\xi)\cap B(0,T-R-\epsilon))}{\omega_n(T-R-\epsilon)^n}=0$$

$$\text{and}\qquad \lim_{T\to+\infty}\frac{\mathrm{vol}(B(0,T+r/2))}{\mathrm{vol}(B(0,T+r/2)\cap A_\xi)}=1.$$

Hence, if T is large enough, the quantity (5.2) is greater than

$$\left(\frac{r(T-R-\epsilon)}{2(R+\epsilon)(T+r/2)}\right)^n.$$

When T tends to infinity, this quantity tends to $(2(R+\epsilon)/r)^{-n}$, for all $\epsilon>0$, which is a lower bound of $\delta(\mathcal{B}(\Lambda))$. We deduce the claim.

6 Lower bounds of the Delone constant of a ssfgud set

The field $K=\mathbb{Q}(\lambda)$ generated by the self-similarity λ of the ssfgud set Λ in Theorem 1.1 has its own Euclidean spectrum [23] [24] which leads to specific geometric properties of the Voronoi cell of the lattice L' [30] of the cut-and-project scheme above Λ, where L' such that $\mathrm{pr}_1(L')=\mathbb{Z}[\Lambda-\Lambda]$. By projection by pr_1, in this cut-and-project scheme above Λ, the Delone (or pseudo-Delone) constant of Λ, whatever the occupation of the elements of $\Lambda-\Lambda$ in $\mathbb{Z}[\Lambda-\Lambda]$, reflects the arithmetical features of K (Euclidean minimum, Euclidean spectrum ... [23] [24]) as well as the geometrical characteristics of the central cluster $\{w_1,\ldots,w_r\}$; in particular if Λ is a model set (see Proposition 2.10) or a Meyer set (see Definition 2.15 (iii)), since, in both cases, a window in the internal space controls the thickness of the band around the $\mathbb{R}$-span of Λ which is used for selecting the points of the lattice L'. Recall that the Delone (or pseudo-Delone) constant $R(\Lambda)$ of the ssfgud set Λ, if finite, "measures" the maximal size of (spherical) holes in Λ with respect to the portion of space

where the density is computed (see §2.1.1, §5 and [81]) In the sequel, we recall these notions and refer to [11] [14] [15] [23] [24] [30] [69].

In the following we will only consider the case $r = 1$, i.e. the case where the degree of the self-similarity λ is equal to the rank of $\mathbb{Z}[\Lambda - \Lambda]$, leaving aside the case $r > 1$. Theorem 6.1, resp. Theorem 6.2, Corollary 6.3 and Theorem 6.4, is a reformulation in the present context of Theorem 3, resp. Theorem 5, Corollary 6 and Theorem 4 (Remark 2), obtained by Cerri [23]. Recall that, for all $\xi \in K$, $\Sigma_1(\xi) = \Sigma(\xi)\widetilde{w_1}$, with $\widetilde{w_1} = \|w_1\|^{-1} w_1$ the unit vector.

6.1 Euclidean and inhomogeneous spectra of the number field generated by the self-similarity

Let $N_{K/\mathbb{Q}}$ be the norm defined on K by

$$\forall \xi \in K, \quad N_{K/\mathbb{Q}}(\xi) = \prod_{i=1}^{d} \sigma_i(\xi) = \prod_{i=1}^{r_1} \sigma_i(\xi) \prod_{i=r_1+1}^{r_1+r_2} |\sigma_i(\xi)|^2 . \tag{6.1}$$

The field K is said to be *norm-Euclidean* if:

$$\forall \xi \in K, \ \exists y \in \mathcal{O}_K \text{ such that } \left|N_{K/\mathbb{Q}}(\xi - y)\right| < 1.$$

Following the notations of Section 3 and [23] [24], we extend $N_{K/\mathbb{Q}} \circ \Psi$ from $\mathbb{Q}^d$ to $\mathbb{R}^d$ by the map denoted by $\mathcal{N}$ as follows:

$$\forall x \in \mathbb{R}^d, \quad \mathcal{N}(x) = \prod_{i=1}^{d} \left(\sum_{j=1}^{d} x_j \sigma_i(g_j) \right). \tag{6.2}$$

Let $\xi \in K$. The *Euclidean minimum of* ξ (relatively to the norm $N_{K/\mathbb{Q}}$) is the real number $m_K(\xi) := \inf\{ |N_{K/\mathbb{Q}}(\xi - y)| \ | \ y \in \mathcal{O}_K\}$. The *Euclidean minimum of* K (for the norm $N_{K/\mathbb{Q}}$) is denoted by $M(K)$ and is by definition:

$$M(K) := \sup_{\xi \in K} m_K(\xi). \tag{6.3}$$

The mapping $m_K \circ \Psi$ defined on $\mathbb{Q}^d$ is extended to $\mathbb{R}^d$ and is denoted by m:

$$m(z) := \inf\{ |\mathcal{N}(z - l)| \ | \ l \in \mathbb{Z}^d\} \qquad \text{for } z \in \mathbb{R}^d.$$

The *inhomogeneous minimum of* K is denoted by $M(\overline{K})$ and is defined by

$$M(\overline{K}) := \sup_{x \in \mathbb{R}^d} m(x). \tag{6.4}$$

The mapping $m_K \circ \Sigma_1^{-1}$ is extended to $K_{\mathbb{R}}\widetilde{w_1}$ and is denoted by $m_{\overline{K}}$:

$$m_{\overline{K}}(U) := \inf \left\{ \left|\prod_{i=1}^{d} (U_i - Z_i)\right| \ | \ Z = (Z_i)_i \widetilde{w_1} \in \Sigma(\mathcal{O}_k)\widetilde{w_1} \right\}$$

for $U = (U_i)_i\widetilde{w_1} \in K_{\mathbb{R}}\widetilde{w_1}$. The set of values of m_K, resp. of m, is called the *Euclidean spectrum*, resp. the *inhomogeneous spectrum of* K. Successive minima are enumerated: the *second inhomogeneous minimum of* K is defined by

$$M_2(\overline{K}) := \sup_{\substack{x\in\mathbb{R}^d \\ m(x)<M(\overline{K})}} m(x),$$

and the *second Euclidean minimum of* K by

$$M_2(K) := \sup_{\substack{\xi\in K \\ m_K(\xi)<M(K)}} m_K(\xi).$$

Iteratively we define ($p \geq 2$):

$$M_{p+1}(\overline{K}) := \sup_{\substack{x\in\mathbb{R}^d \\ m(x)<M_p(\overline{K})}} m(x),$$

and

$$M_{p+1}(K) := \sup_{\substack{\xi\in K \\ m_K(\xi)<M_p(K)}} m_K(\xi).$$

The inhomogeneous minimum $M(\overline{K})$ of K is said to be *isolated* if

$$M_2(\overline{K}) < M(\overline{K}).$$

This isolation phenomenon has been conjectured for $d = 2$ and K totally real by Barnes and Swinnerton-Dyer. Corollary 6.3 below shows that it occurs frequently.

If the inhomogeneous minimum $M(\overline{K})$ of K satisfies the following property:

$$\forall x \in \mathbb{R}^d,\ \exists l \in \mathbb{Z}^d \text{ such that } |\mathcal{N}(x-l)| \leq M(\overline{K}), \tag{6.5}$$

we will say that $M(\overline{K})$ is *attained*. Note that (6.5) is not verified for the quadratic field $K = \mathbb{Q}(\sqrt{13})$ [69].

Theorem 6.1. *Assume that the degree d of the field K generated by the self-similarity λ of the ssfgud sphere packing Λ is ≥ 3 and is equal to the rank of $\mathbb{Z}[\Lambda-\Lambda]$. If the unit rank r_1+r_2-1 of K is >1, in particular if K is totally real, then*

(i) *there exists $\xi \in K$ such that*

$$M(\overline{K}) \ = \ m_{\overline{K}}(\Sigma_1(\xi)),$$

(ii)

$$M(K) \ = \ M(\overline{K}) \in \mathbb{Q}.$$

Proof. Theorem 3 in [23]. □

The question whether ξ is unique under some assumptions is not clear [23] [24].

Theorem 6.2. *Assume that the degree d of the field K generated by the self-similarity λ of the ssfgud sphere packing Λ is ≥ 3 and is equal to the rank of $\mathbb{Z}[\Lambda-\Lambda]$. If the unit rank r_1+r_2-1 of K is >1 and if K is not a CM-field, in particular if K is totally real, there exists a strictly decreasing sequence $(y_p)_{p\geq 1}$ of positive rational integers, which satisfies:*

(i) $\lim_{p\to+\infty} y_p = 0$,

(ii) $m(\mathbb{R}^d) = \bigcup_{p\geq 1}\{y_p\}$,

(iii) *for each $p\geq 1$, the set $\{x+\mathbb{Z}^d \mid m(x)=y_p\}$ of classes modulo the lattice $\mathbb{Z}^d$ is finite and lifts up to points of $\mathbb{Q}^d$, i.e. $m(x)=0$ for all $x\notin\mathbb{Q}^d$.*

Proof. Theorem 5 in [23]. □

From the definitions, the inequality $M(K) \leq M(\overline{K})$ holds for an arbitrary number field, with equality if $d = 2$ (Barnes and Swinnerton-Dyer [69]). Recently Cerri [23] (Corollary 3 of Theorem 3) proved that the equality $M(K) = M(\overline{K})$ does hold true for every number field.

Corollary 6.3. *Under the same hypotheses $M(\overline{K})$ is attained and*

(i) $M_p(K) = M_p(\overline{K})$ *for all $p>1$,*

(ii) $M_2(\overline{K}) < M(\overline{K})$ *($M(\overline{K})$ is isolated),*

(iii) $\forall p>1,\ M_{p+1}(\overline{K}) < M_p(\overline{K})$ *and* $\lim_{p\to+\infty} M_p(\overline{K}) = 0$.

What are the possible fundamental regions of the sublattices L' of $L = \Sigma_1(\mathbb{Z}[\lambda])$ in $K_{\mathbb{R}}\widetilde{w_1}$?

Theorem 6.4. *Denote, for all $t>0$,*

$$\mathcal{A}_t := \{U=(U_i)\widetilde{w_1} \in K_{\mathbb{R}}\widetilde{w_1} \mid \ \Big|\prod_{i=1}^{d} U_i\Big| \leq t\}.$$

If (unit rank) $r_1+r_2-1>1$, then

$$K \text{ is norm-Euclidean} \iff \exists t\in(0,1) \text{ such that } \Sigma_1(\mathcal{O}_K)+\mathcal{A}_t = K_{\mathbb{R}}\widetilde{w_1}.$$

Proof. Remark 2 after Theorem 4 in [23]. □

Then, if the unit rank $r_1 + r_2 - 1$ of K is strictly greater than 1 and K is norm-Euclidean, there exists $t \in (0,1)$ such that the number of copies of $\mathcal{A}_t$ to be considered for obtaining the fundamental region of L' is equal to the index of L' in $\Sigma_1(\mathcal{O}_K)\widetilde{w_1}$, an integer multiple of $(\mathcal{O}_K : \mathbb{Z}[\lambda])$. Recall that ([23] Proposition 4):

(i) $M(K) < 1 \implies K$ is norm-Euclidean,

(ii) $M(K) > 1 \implies K$ is not norm-Euclidean,

If $M(K) = 1$, it is not possible to conclude except if there exists $\xi \in K$ such that $M(K) = m_K(\xi)$; in this case K is not norm-Euclidean. See [24] for computations of $M_p(K), p \geq 1$, with the conventions: $M(K) = M_1(K)$, $M(\overline{K}) = M_1(\overline{K})$.

6.2 Proof of Theorem 1.4

Let us now deduce lower bounds of the Delone constant of the ssfgud set $\Lambda \subset \mathbb{R}^n, n \geq 1$, that we will assume either a model set or a Meyer set, subset of a model set, defined by a window Ω in the internal space $\mathbb{R}_K \setminus \mathbb{R}[\Lambda]$ of the cut-and-project scheme

$$\big(K_{\mathbb{R}}\widetilde{w_1} = (\,\mathbb{R}_K \setminus \mathbb{R}[\Lambda]\,) \times \mathbb{R}[\Lambda], L', \pi, \mathrm{pr}_1 \big) \tag{6.6}$$

given by Theorem 1.1 (iv) with L' such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ and the window Ω nonempty, open and relatively compact such that

$$\Lambda \ \subset \ \nu + \{\mathrm{pr}_1(U) \mid U \in L', \pi(U) \in \Omega\} \ \subset \ \nu \ + \ \mathbb{R}\,\widetilde{w_1} \qquad \text{with } \nu \in \Lambda. \tag{6.7}$$

Λ is a Delone set. The central cluster is $\{w_1\}$. Denote by $\mathbb{R}[\Lambda] + \Omega$ the band $\{P = (U,V) \in K_{\mathbb{R}}\widetilde{w_1} \mid U = \mathrm{pr}_1(P) \in \mathbb{R}[\Lambda], V = \pi(P) \in \Omega\}$ parallel to the one-dimensional $\mathbb{R}$-span $\mathbb{R}[\Lambda]$ of Λ. Note that $\mathbb{R}[\Lambda] = \nu + \mathbb{R}\,\widetilde{w_1}$ in $\mathbb{R}^n$.

In the sequel we will use the notations of §6.1 and the assumptions of Theorem 6.2.

Definition 6.5. Let $k \geq 2$ be an integer. The self-similar finitely generated Delone set Λ defined by (6.6) and (6.7) is called

(i) *thin* if the following condition on Ω and L' holds:

$$0 \ < \ \|\pi\big(\Sigma_1(x-t)\big)\| \ < \ d\,M(K)^{2/d}\,, \tag{6.8}$$

for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$, and all $x \in m_K^{-1}(M(K))$,

(ii) *k-thin* if

$$d\,(M_{k+1}(K))^{2/d} \leq \|\pi\big(\Sigma_1(x-t)\big)\| \ < \ d\,(M_k(K))^{2/d} \tag{6.9}$$

for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$ and $x \in \bigcup_{p=1}^{k-1} m_K^{-1}(M_p(K))$.

This definition is consistent with the following facts:

(a) the values of m constitute a strictly decreasing sequence of positive rational integers (Theorem 6.2 (ii))which are reached a finite number of times modulo L' (Theorem 6.2 (iii)),

(b) the infinite (band) cylinder $\mathbb{R}[\Lambda] + \Omega$, parallel to the one-dimensional space $\mathbb{R}[\Lambda]$, can be made sufficiently narrow in order to avoid the set of points z of $\mathbb{Q}^d$ such that $m_K \circ \Psi(z) \in \bigcup_{p=1}^{k-1} M_p(K)$, for all $k \geq 2$, which is a finite union of translates of L',

(c) Assertion (b) is possible since *free planes*, a fortiori free lines, do exist in any lattice (equal) sphere packings in $\mathbb{R}^d$ once d is large enough: Henk [57] has proved the existence of an $\frac{d}{\log_2(d)}$-dimensional affine plane (called free plane) which does not meet any of the spheres in their interiors. Hence, provided d is large enough, free lines exist in the lattice sphere packing $\mathcal{B}(L')$ corresponding to L' in $K_{\mathbb{R}}\widetilde{w_1}$. Equivalently narrow bands, with section a nonempty open set, about free lines, exist in $K_{\mathbb{R}}\widetilde{w_1}$ which do not intersect L'. By continuity, there exist narrow bands, with nonempty open cross-sections, about free lines, which do not intersect any finite union of translates of L'.

For all $x \in K, t \in \mathcal{O}_K$, by the geometric mean inequality, we have:

$$|N_{K/\mathbb{Q}}(x)|^2 = |N_{K/\mathbb{Q}}(x-t)|^2 = \prod_{i=1}^{r_1} |\sigma_i(x-t)|^2 \prod_{i=r_1+1}^{r_1+r_2} |\sigma_i(x-t)|^4$$

$$\leq \left(\frac{1}{d} \sum_{i=1}^{r_1} |\sigma_i(x-t)|^2 + \frac{2}{d} \sum_{i=r_1+1}^{r_1+r_2} |\sigma_i(x-t)|^2 \right)^d = \left(\frac{1}{d}\, q\big(\Sigma_1(x-t), \Sigma_1(x-t)\big) \right)^d \tag{6.10}$$

where $q\big(\Sigma_1(x-t), \Sigma_1(x-t)\big) = \|\mathrm{pr}_1(\Sigma_1(x-t))\|^2 + \|\pi(\Sigma_1(x-t))\|^2$ with the notations of Section 3. We have:

$$\|\mathrm{pr}_1(\Sigma_1(x-t))\|^2 = \|(x-t)\widetilde{w_1}\|^2 = \|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2.$$

The set $m_K^{-1}(M(K)) = \{x \in K \mid m_K(x) = M(K) = M(\overline{K})\}$ is such that $\Sigma_1\big(m_K^{-1}(M(K))\big) = \{\Sigma_1(x) \mid m_K(x) = M(K) = M(\overline{K})\}$ is finite modulo L' by Theorem 6.1 and Theorem 6.2.

Let us take x in $m_K^{-1}(M(K))$. Then, from (6.10), for all $t \in \mathcal{O}_K$,

$$M(K)^2 \leq \left(\frac{1}{d} \left(\|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2 + \|\pi(\Sigma_1(x-t))\|^2 \right) \right)^d.$$

Hence, for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$,

$$\|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2 \geq d\, M(K)^{2/d} - \|\pi(\Sigma_1(x-t))\|^2.$$

Since by hypothesis $\nu + x\widetilde{w_1}$ does not belong to Λ and that we consider the elements $t \in \mathcal{O}_k$ for which $\nu + t\widetilde{w_1}$ belongs to Λ (with $\Sigma_1(t) \in L' \cap (\mathbb{R}[\Lambda] + \Omega)$), we have:

$$R(\Lambda) \ \geq \ \|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|.$$

Hence

$$R(\Lambda)^2 \ \geq \ d\, M(K)^{2/d} - \sup \|\pi(\Sigma_1(x-t))\|^2 > 0 \qquad (6.11)$$

where the supremum is taken over all $x \in m_K^{-1}(M(K))$ and all $t \in \mathcal{O}_k$ for which $\nu + t\widetilde{w_1}$ belongs to Λ, with $\Sigma_1(t) \in L' \cap (\mathbb{R}[\Lambda] + \Omega)$.

Let us assume that Λ is k-thin and take x in $\bigcup_{p=1}^{k-1} m_K^{-1}(M_p(K))$. Then the supremum in (6.11) is bounded from above by $d\, M_k(K)^{\frac{2}{d}}$ which allows to deduce the claim.

Using Theorem 1.3 or [81] we deduce a lower bound of the density of the ssfgud set Λ.

The isolation phenomenon which frequently occurs (Corollary 6.3 (ii) of Theorem 6.2) in higher dimension is likely to occur in $\mathbb{R}^n$ as well by projection for Λ.

Appendix.— Crystallography of Aperiodic Crystals and Delone sets

New states of matter call for mathematical idealizations of packings of atoms and consequently a deep understanding of the mathematics which lies behind as far as they are characterized by experimental techniques described by a mathematical formalism which has to be mathematically settled: diffraction (X-rays, electrons, neutrons, synchrotron radiation, etc) and inverse problems (crystal reconstruction with satisfying local atom clustering, long-range order and self-similarities, etc). Indeed, the situation is well-known for (periodic) crystals [31] [53] [54] [63] [96] but fairly unknown, or at least badly understood for nonperiodic crystals. Quasicrystals and modulated crystals constitute exceptions since the use of cut-and-project sets allows periodization in higher dimension [3] [4] [51] [62] [64] [96]. The parts of mathematics concerned with the crystallography of aperiodic crystals are mainly Geometry of Numbers and Discrete Geometry [22] [52] [107], N-dimensional crystallography when periodization in higher dimension is concerned [78] [84] [95], Spectral Theory, Ergodic Theory and Fourier Transform of Delone sets as far as diffraction is concerned [5] [59], Harmonic Analysis as far as density is concerned (as an asymptotic measure). Atoms are viewed as *hard spheres* and aperiodic crys-

tals as Delone sets (sphere centers). Implicitely this means that atoms behave like spheres, that is do have a spherical potential. This is far from covering the large variety of possibilities of chemical boundings between atomic species (see [106] for quasicrystalline models of pure Boron for instance). This provides first-order crystalline models from which the computation of the electron density is made possible. Then comparison with experimental data (densities, physical properties, ...) leads to refine the models.

Looking for a fine hierarchy of Delone sets, from the mathematical side, either from arithmetics [37] [47] [65] or from tiling theory [6] [8] [80] [102], leads to interesting and new questions concerning crystals, without knowing whether these crystals will exist or not. To finish up let us recall the new definition of a crystal (in $\mathbb{R}^3$) which was recently chosen by the International Union of Crystallography [61] and the former one [95].

Definition 6.6 (former definition). Any solid for which the set of atom positions is a finite union of orbits under the action of a crystallographic group.

Definition 6.7 (new definition). Any solid having an essentially discrete diffraction diagram.

Definition 6.7 covers all cases of solids defined by Definition 6.6 by Poisson formula (see [67] for a proof).

References

[1] P. Arnoux, V. Berthé, H. Ei and S. Ito, Tilings, quasicrystals, discrete planes, generalized substitutions and multidimensional continued fractions, *Disc. Math. and Theor. Comp. Sci. Proc. AA (DM-CCG)*, (2001), 59–78.

[2] P. Arnoux and S. Ito, Pisot substitutions and Rauzy fractals, *Bull. Belg. Math. Soc.* 8 (2001), 181–207.

[3] F. Axel, F. Dénoyer and J.-P. Gazeau (Eds.), *From Quasicrystals to more complex systems*, EDP-Springer-Verlag, Berlin (2000).

[4] F. Axel and D. Gratias (Eds.), *Beyond Quasicrystals*, EDP-Springer-Verlag, Berlin (1995).

[5] M. Baake and D. Lenz, Dynamical systems on translation bounded measures: pure point dynamical and diffraction spectra, *Ergod. Theory Dynam. Systems* 24 (2004), 1867–1893.

[6] M. Baake and R.V. Moody (Eds.), *Directions in Mathematical Quasicrystals*, CRM Monographs Series Vol 13, AMS, Providence, (2000).

[7] C. Bachoc and B. Venkov, Modular forms, lattices and spherical designs, Réseaux Euclidiens, designs sphériques et formes modulaires, *Monogr. Enseign. Math.* 37 (2001), 87–111.

[8] C. Bandt, Self-similar tilings and patterns described by mappings, in [80], 45–83.

[9] E. Bannai, Positive definite unimodular lattice with trivial automorphism groups, *Mem. Amer. Math. Soc.* 85 (1990), n^o 429.

[10] G. Barat, V. Berthé and P. Liardet, Directions in Numeration, *Ann. Institut Fourier*, (2006), Special Issue on *Numeration and its Applications*, submitted.

[11] E. Bayer-Fluckiger, Upper bounds for Euclidean minima of algebraic number fields, preprint.

[12] E. Bayer-Fluckiger, Ideal Lattices, in *A panorama of number theory or the view from Baker's garden*, (Zürich 1999), G. Wüstholz (Ed.), Cambridge University Press, Cambridge, (2002), 168–184.

[13] E. Bayer-Fluckiger, Lattices and number fields, *Contemporary Math.* 241 (1999) 69–84.

[14] E. Bayer-Fluckiger and G. Nebe, On the Euclidean minimum of some real number fields, preprint.

[15] E. Bayer-Fluckiger and I. Suarez, Ideal lattices over totally real number fields and Euclidean minima, preprint.

[16] J. Bernat, Arithmetics in β-numeration, preprint, (2005).

[17] K. Bezdek, Compact packings in the Euclidean space, *Beiträge Algebra Geom.* 25 (1987), 79–84.

[18] F. Blanchard, β-expansions and Symbolic Dynamics, *Theoret. Comput. Sci.* 65, (1989), 131–141.

[19] K. Böröczky, Jr. *Finite packing and covering*, Cambridge Tracts in Mathematics, 154, Cambridge University Press, (2004).

[20] D.W. Boyd, The beta expansion for Salem numbers, in *Organic Mathematics, Canadian Math. Soc. Conf. Proc.* 20, (1997), AMS, Providence, 117–131.

[21] C. Burdik, C. Frougny, J.-P. Gazeau and R. Krejcar, beta-integers as natural counting systems for quasicrystals, *J. Phys. A: Math. Gen.* 31 (1998), 6449–6472.

[22] J.W.S. Cassels, *An introduction to the Geometry of Numbers*, Springer Verlag, Berlin, (1959).

[23] J.-P. Cerri, Inhomogeneous and Euclidean spectra of number fields with unit rank greater than 1, preprint.

[24] J.-P. Cerri, Euclidean minima of totally real number fields. Algorithmic determination, preprint.

[25] G. Collinet, Homologie rationelle du groupe $\mathrm{O}_n(\mathbb{Z}[\frac{1}{2}])$ et automorphismes des réseaux unimodulaires, *C. R. Math. Acad. Sci. Paris* 335 (2002), n^o 2, 127–132.

[26] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Springer-Verlag, Berlin (1988).

[27] N. Cotfas, G-Model Sets and Their Self-Similarities, *J. Phys. A* 32 (1999), 8079–8093.

[28] N. Cotfas, On the Self-Similarities of a Model Set, *J. Phys. A* 32 (1999), L165–L168.

[29] N. Cotfas and J.-L.Verger-Gaugry, A Mathematical Construction of n-dimensional Quasicrystals Starting from G-clusters, *J. Phys.A: Math. Gen.* 30 (1997), 4283–4291.

[30] R. Coulangeon, Voronoi theory over algebraic number fields, in *Réseaux Euclidiens, designs sphériques et formes modulaires*, Monogr. Enseign. Math. 37 (2001), 147–162.

[31] J.M. Cowley, *Diffraction Physics*, North-Holland, Amsterdam, 2nd Edition (1986).

[32] F.M. Dekking, Recurrent sets, *Adv. in Math.* 44 (1982), 78–104.

[33] B.N. Delone, Neue Darstellung des Geometrische Kristallographie, *Z. Kristallographie* 84 (1932), 109–149.

[34] B.N. Delone, Sur la Sphère Vide, *Bull. Acad. Sci. URSS*, VII Ser. 1934, N^o 6, (1934), 793–800.

[35] D. Descombes, *Eléments de Théorie des Nombres*, PUF, Paris (1986), p 57.

[36] M. Duneau and M. Audier, Approximant phases of quasicrystals, in [58], pp 283–333.

[37] A. Elkharrat, C. Frougny, J.-P. Gazeau and J.-L. Verger-Gaugry, Symmetry groups for beta-lattices, *Theoret. Comput. Sci.* 319 (2004), 281–305.

[38] S. Fabre, Substitutions et β-systèmes de numération, *Theoret. Comput. Sci.* 137 (1995), 219–236.

[39] D.K. Fadeev, N.P. Dolbilin, S.S. Ryshkov and M.I. Shtogrin, Boris Nikolaevich Delone (On his Life and Creative Work), Proc. Stekl. Inst. Math. *Discrete Geometry and Topology, On the 100th Anniversary of the Birth of Boris Nikolaevich Delone* 196, Issue 4 of 4 (1992), 1–9.

[40] G. Fejes-Tóth, Multiple packing and covering of spheres, *Acta Math. Acad. Sci. Hungar.* 34 (1979), 165–176.

[41] L. Flatto, J. Lagarias and B. Poonen, The zeta function of the beta transformation, *Ergod. Th. and Dynam. Sys.* 14, (1994), 237–266.

[42] N. Pythéas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lect. Notes Math., vol 1794, Springer-Verlag, (2003).

[43] J. Frenkel, *Géométrie pour l'élève-professeur*, Hermann, Paris (1973), Appendice I.

[44] C. Frougny, Number Representation and Finite Automata, *London Math. Soc. Lect. Notes Ser.* 279, (2000), 207–228.

[45] C. Frougny, Numeration Systems, Chapter 7 in *Algebraic Combinatorics on Words*, M. Lothaire (Ed.), Cambridge University Press, (2003).

[46] F. Gaehler, P. Kramer, H.-R. Trebin and Urban (Eds.), Proc. of the 7th Int. Conf. on Quasicrystals, *Mat. Sci. & Eng. A* (2000).

[47] J.-P. Gazeau, Pisot-cyclotomic integers for quasilattices, in [80], 175–198.

[48] J.-P. Gazeau, Counting Systems with Irrational Basis for Quasicrystals, in [3], 195–217.

[49] J.-P. Gazeau and J.-L. Verger-Gaugry, Geometric Study of the Beta-integers with β a Perron Number and Mathematical Quasicrystals, *J. Théorie Nombres Bordeaux* 16 (2004), 125–149.

[50] J.E. Goodman and J.O'Rourke, *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton (1997).

[51] D. Gratias and L. Michel (Eds.), *Int. Workshop on Aperiodic Crystals*, Les Houches, *J. de Physique France*, Colloque C3-1986, fasc. 7.

[52] P.M. Gruber and C.G. Lerkkerkerker, *Geometry of Numbers*, North-Holland, Amsterdam (1987).

[53] A. Guinier, *Theory and techniques for X-ray crystallography*, Dunod, Paris (1964).

[54] T. Hahn (Ed.), *International Tables for Crystallography*, Volume A, Reidel Publ. Co., Dordrecht, (1983).

[55] T.C. Hales, Sphere packings I, *Disc. Comp. Geom.* 17 (1997), 1–51.

[56] T.C. Hales, Sphere packings II, *Disc. Comp. Geom.* 18 (1997), 135–149.

[57] M. Henk, Free planes in lattice sphere packings, *Adv. in Geometry*, 5 (1) (2005), 137–144.

[58] F. Hippert and D. Gratias (Eds.), *Lectures on Quasicrystals*, Les Editions de Physique, Les Ulis, (1994).

[59] A. Hof, Diffraction by aperiodic structures, in [80], 239–268.

[60] S. Ito and M. Kimura, On Rauzy fractal, *Japan J. Indust. Appl. Math.* 8 (1991), 461–486.

[61] International Union of Crystallography, Report of the Executive Committee for 1991, *Acta Cryst.* A 48 (1992), 922–946.

[62] A. Janner and T. Janssen, Superspace Groups, *Physica* 99A (1979), 47–76.

[63] C. Janot, *Quasicrystals, A Primer* (2nd Edition), Clarendon Press, Oxford (1997).

[64] T. Janssen, From Quasiperiodic to more Complex Systems, in [4], 75–140.

[65] J.C. Lagarias, Geometric Models for Quasicrystals I. Delone sets of Finite Type, *Disc. Comp. Geom.* 21 (1999), 161–192.

[66] J.C. Lagarias, The Impact of Aperiodic Order on Mathematics, *Mat. Sci. & Eng.* 294–296 (2000), 186–191.

[67] J.C. Lagarias, Mathematical Quasicrystals and the Problem of Diffraction, in [6], 61–93.

[68] S. Lang, *Algebra*, Addison-Wesley, (1965).

[69] F. Lemmermeyer, The Euclidean algorithm in algebraic number fields, *Expo. Math.* 13 (1995), 385–416.

[70] J.-M. Luck, C. Godrèche, A. Janner and T. Janssen, The nature of the atomic surfaces of quasiperiodic self-similar structures, *J. Phys. A* 26 (1993), 1951–1999.

[71] J.M. Luck, P. Moussa and M. Waldschmidt, *Number Theory and Physics*, Springer Proceedings in Physics **47**, Springer-Verlag, Berlin, (1990).

[72] J. Martinet, *Les Réseaux Parfaits des Espaces Euclidiens*, Masson, Paris (1996).

[73] Z. Masákovà, J. Patera and E. Pelantovà, Inflation Centers of the Cut-and-Project Quasicrystals, *J. Phys. A:Math.Gen.* 31 (1998), 1443–1453.

[74] A. Messaoudi, Propriétés arithmétiques et dynamiques du fractal de Rauzy, *J. Théor. Nombres Bordeaux* 10, (1998), 135–162.

[75] Y. Meyer, *Nombres de Pisot, Nombres de Salem et Analyse Harmonique*, Lecture Notes in Math. **117**, (1970), Springer-Verlag, Berlin.

[76] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam (1972).

[77] Y. Meyer, Quasicrystals, Diophantine Approximation and Algebraic Numbers, in [4], 3–16.

[78] L. Michel and J. Morzrzymas, *N-Dimensional Crystallography*, IHES Workshop on Mathematical Crystallography, IHES/P/85/47, (August 1985).

[79] R.V. Moody, Meyer sets and their Duals, in [80], 403–442.

[80] R.V. Moody (Ed.), *The Mathematics of Long-Range Aperiodic Order*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences 489, Kluwer Academic Publishers, Dordrecht (1997).

[81] G. Muraz and J.-L. Verger-Gaugry, On Lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings, *Exp. Math.* 14:1 (2005), 47–57.

[82] G. Muraz and J.-L. Verger-Gaugry, On a Generalization of the Selection Theorem of Mahler, *J. Théorie Nombres Bordeaux* 17 (2005), 237–269.

[83] J. Oesterlé, Empilements de Sphères, *Séminaire Bourbaki* n 727 (1989-90), 375–397; *SMF Astérisque* 189–190 (1990).

[84] W. Opechowski, *Crystallographic and Metacrystallographic Groups*, North-Holland, Amsterdam (1986).

[85] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hung.* 11 (1960), 401–416.

[86] J. Patera (Ed.), *Quasicrystals and Discrete Geometry*, Proc. of the Fall Programme held at the University of Toronto (1995), Fields Institute Monographs 10, Amer. Math. Soc., Providence (1998).

[87] Y. Peres and B. Solomyak, Problems on self-similar sets and self-affine sets: An update, in Fractals and Stochastic II, Proc. of the University of Greifswald 1998 Conference, C. Bandt, S. Graf and M. Zaehle (Eds.), *Progress in Probability* 46 (2000), Birkhaüser, 95–106.

[88] G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. France* 110, (1982), 147–178.

[89] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hung.* 8, (1957), 477–493.

[90] C. A. Rogers, *Packing and covering*, Cambridge University Press, Cambridge (1964).

[91] S.S. Ryshkov, Density of an (r, R)-system, *Math. Notes* 16 (1975), 855–858.

[92] M. Schlottmann, Generalized Model Sets and Dynamical Systems, in [6], 143–159.

[93] R. Schoof, Computing Arakelov class groups, preprint.

[94] J.P. Schreiber, Approximations diophantiennes et problèmes additifs dans les groupes abéliens localement compacts, *Bull. Soc. Math. France* 101 (1973), 297–332.

[95] R.L.E. Schwarzenberger, *N-Dimensional Crystallography*, Pitman, London (1980).

[96] M. Sénéchal, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge (1995).

[97] D. Shechtman, I. Blech, D. Gratias and J. Cahn, Metallic Phase with Long-Range Orientational Order and No Translational Symmetry, *Phys. Rev. Lett.* 53 (1984), 1951–1953.

[98] A. Siegel, Représentations des systèmes dynamiques substitutifs non unimodulaires, *Ergod. Th. Dyn. Sys.* 23 (2003), 1247–1273.

[99] A. Siegel, Spectral theory and geometric representations of substitutions, in [42], 199–252.

[100] V.F. Sirvent, Identifications and dimension of the Rauzy fractal, *Fractals* 5 (1997), 281–294.

[101] V.F. Sirvent and Y. Wang, Self-affine tiling via substitution dynamical systems and Rauzy fractals, *Pacific J. Math.* 206 (2002), 465–485.

[102] B. Solomyak, Conjugates of beta-numbers and the zero-free domain for a class of analytic functions, *Proc. London Math. Soc.* (3) 68, (1993), 477-498.

[103] B. Solomyak, Dynamics of Self-Similar Tilings, *Ergod. Th. Dyn. Sys.* 17 (1997), 695–738.

[104] W. Thurston, Groups, Tilings and Finite State Automata, preprint (1989).

[105] J.-L. Verger-Gaugry, On lacunary Rényi β-expansions of 1 with $\beta > 1$ a real algebraic number, Perron numbers and a classification problem, preprint Institut Fourier 648, (2004).

[106] D. Weygand and J.-L. Verger-Gaugry, Modèle d'un quasicristal covalent de Bore pur, C. R. Acad. Sci. Paris 320 (1994), 253–257.

[107] C. Zong *Sphere packings*, Springer-Verlag, Berlin (1999).