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# VALUATION DOMAINS WITH A MAXIMAL IMMEDIATE EXTENSION OF FINITE RANK.

FRANÇOIS COUCHOT

ABSTRACT. If  $R$  is a valuation domain of maximal ideal  $P$  with a maximal immediate extension of finite rank it is proven that there exists a finite sequence of prime ideals  $P = L_0 \supset L_1 \supset \cdots \supset L_m \supseteq 0$  such that  $R_{L_j}/L_{j+1}$  is almost maximal for each  $j$ ,  $0 \leq j \leq m-1$  and  $R_{L_m}$  is maximal if  $L_m \neq 0$ . Then we suppose that there is an integer  $n \geq 1$  such that each torsion-free  $R$ -module of finite rank is a direct sum of modules of rank at most  $n$ . By adapting Lady's methods, it is shown that  $n \leq 3$  if  $R$  is almost maximal, and the converse holds if  $R$  has a maximal immediate extension of rank  $\leq 2$ .

Let  $R$  be a valuation domain of maximal ideal  $P$ ,  $\widehat{R}$  a maximal immediate extension of  $R$ ,  $\widetilde{R}$  the completion of  $R$  in the  $R$ -topology, and  $Q, \widehat{Q}, \widetilde{Q}$  their respective fields of quotients. If  $L$  is a prime ideal of  $R$ , as in [5], we define the **total defect** at  $L$ ,  $d_R(L)$ , the **completion defect** at  $L$ ,  $c_R(L)$ , as the rank of the torsion-free  $R/L$ -module  $\widehat{(R/L)}$  and the rank of the torsion-free  $R/L$ -module  $\widetilde{(R/L)}$ , respectively. Recall that a local ring  $R$  is **Henselian** if each indecomposable module-finite  $R$ -algebra is local and a valuation domain is **strongly discrete** if it has no non-zero idempotent prime ideal. The aim of this paper is to study valuation domains  $R$  for which  $d_R(0) < \infty$ . The first example of a such valuation domain was given by Nagata [11]; it is a Henselian rank-one discrete valuation domain of characteristic  $p > 0$  for which  $d_R(0) = p$ . By using a generalization of Nagata's idea, Facchini and Zanardo gave other examples of characteristic  $p > 0$ , which are Henselian and strongly discrete. More precisely:

**Example 0.1.** [5, Example 6] *For each prime integer  $p$  and for each finite sequence of integers  $\ell(0) = 1, \ell(1), \dots, \ell(m)$  there exists a strongly discrete valuation domain  $R$  with prime ideals  $P = L_0 \supset L_1 \supset \cdots \supset L_m = 0$  such that  $c_R(L_i) = p^{\ell(i)}$ ,  $\forall i, 1 \leq i \leq m$ .*

So,  $d_R(0) = p^{(\sum_{i=1}^m \ell(i))}$  by [5, Corollary 4].

**Theorem 0.2.** [5, Theorem 8] *Let  $\alpha$  be an ordinal number,  $\ell : \alpha + 1 \rightarrow \mathbb{N} \cup \{\infty\}$  a mapping with  $\ell(0) = 1$  and  $p$  a prime integer. Then there exists a strongly discrete valuation domain  $R$  and an antiisomorphism  $\alpha + 1 \rightarrow \text{Spec}(R)$ ,  $\lambda \mapsto L_\lambda$ , such that  $c_R(L_\lambda) = p^{\ell(\lambda)}$ ,  $\forall \lambda \leq \alpha$ .*

So, if  $\ell(\lambda) = 0$ ,  $\forall \lambda \leq \alpha$ , except for a finite subset, then  $d_R(0) < \infty$  by [5, Corollary 4].

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In 1990, Vámos gave a complete characterization of non-Henselian valuation domains with a finite total defect, and examples of such rings. His results are summarized in the following:

**Theorem 0.3.** [12, Theorem 5] *Let  $R$  be a non-Henselian valuation domain and assume that  $d_R(0) < \infty$ . Then one of the following holds:*

- (rc)  $d_R(0) = 2$ ,  $R$  has characteristic zero,  $\widehat{Q}$  is algebraically closed and its cardinality  $|\widehat{Q}|^{\aleph_0} = |\widehat{Q}|$ . Further,  $Q$  is real-closed, the valuation on  $Q$  has exactly two extensions to  $\widehat{Q}$  and  $R$  is almost maximal.
- (y) There is a non-zero prime ideal  $L$  of  $R$  such that  $R_L$  is a maximal valuation ring, and  $R/P$  and its field of quotients satisfy (rc).

As Vámos, if  $R$  is a domain, we say that  $fr(R) \leq n$  (respectively  $fr^o(R) \leq n$ ) if every torsion-free module (respectively every submodule of a free module) of finite rank is a direct sum of modules of rank at most  $n$ . By [12, Theorem 3]  $fr(R) \geq d_R(0)$  if  $R$  is a valuation domain. So, the study of valuation domains  $R$  for which  $d_R(0) < \infty$  is motivated by the problem of the characterization of valuation domains  $R$  for which  $fr(R) < \infty$ .

When  $R$  is a valuation domain which is a  $\mathbb{Q}$ -algebra or not Henselian, then  $fr(R) < \infty$  if and only if  $fr(R) = d_R(0) \leq 2$  by [12, Theorem 10]. Moreover, if  $fr(R) = 2$ , either  $R$  is of type (rc) and  $fr^o(R) = 1$  or  $R$  is of type (y) and  $fr^o(R) = 2$ . When  $R$  is a rank-one discrete valuation domain, then  $fr(R) < \infty$  if and only if  $fr(R) = d_R(0) \leq 3$  by [13, Theorem 8] and [1, Theorem 2.6].

In this paper we complete Vámos's results. In Section 1, a description of valuation domains with a finite total defect is given by Theorem 1.7 and Proposition 1.8. In Section 2 we give some precisions on the structure of torsion-free  $R$ -modules of finite rank when  $R$  satisfies a condition weaker than  $d_R(0) < \infty$ . In Section 3 we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. If  $R$  is an almost maximal valuation domain, we prove that  $d_R(0) \leq 3$  if  $fr(R) < \infty$  and that  $fr(R) = d_R(0)$  if  $d_R(0) \leq 2$ .

For definitions and general facts about valuation rings and their modules we refer to the books by Fuchs and Salce [6] and [7].

## 1. MAXIMAL IMMEDIATE EXTENSION OF FINITE RANK

We recall some preliminary results needed to prove Theorem 1.7 which gives a description of valuation domains with a finite total defect.

Let  $M$  be a non-zero module over a valuation domain  $R$ . As in [7, p.338] we set  $M^\# = \{s \in R \mid sM \subset M\}$ . Then  $M^\#$  is a prime ideal of  $R$  and is called the **top prime ideal** associated with  $M$ .

**Proposition 1.1.** *Let  $A$  be a proper ideal of  $R$  and let  $L$  be a prime ideal such that  $A^\# \subseteq L$  and  $A$  is not isomorphic to  $L$ . Then  $R/A$  is complete in its ideal topology if and only if  $R_L/A$  is also complete in its ideal topology.*

*Proof.* Let  $(a_i + A_i)_{i \in I}$  be a family of cosets of  $R_L$  such that  $a_i \in a_j + A_j$  if  $A_i \subset A_j$  and such that  $A = \bigcap_{i \in I} A_i$ . We may assume that  $A_i \subseteq L$ ,  $\forall i \in I$ . So,  $a_i + L = a_j + L$ ,  $\forall i, j \in I$ . Let  $b \in a_i + L$ ,  $\forall i \in I$ . It follows that  $a_i - b \in L$ ,  $\forall i \in I$ . If  $R/A$  is complete in the  $R/A$ -topology,  $\exists c \in R$  such that  $c + b - a_i \in A_i$ ,  $\forall i \in I$ . Hence  $R_L/A$  is complete in the  $R_L/A$ -topology too.

Conversely let  $(a_i + A_i)_{i \in I}$  be a family of cosets of  $R$  such that  $a_i \in a_j + A_j$  if  $A_i \subset A_j$  and such that  $A = \bigcap_{i \in I} A_i$ . We may assume that  $A \subset A_i \subseteq L$ ,  $\forall i \in I$ . We put  $A'_i = (A_i)_L$ ,  $\forall i \in I$ . We know that  $A = \bigcap_{a \notin A} La$ . Consequently, if  $a \notin A$ , there exists  $i \in I$  such that  $A_i \subseteq La$ , whence  $A'_i \subseteq La$ . It follows that  $A = \bigcap_{i \in I} A'_i$ . Clearly,  $a_i \in a_j + A'_j$  if  $A'_i \subset A'_j$ . Then there exists  $c \in R_L$  such that  $c \in a_i + A'_i$ ,  $\forall i \in I$ . Since  $A'_i \subset R$ ,  $\forall i \in I$ ,  $c \in R$ . From  $A = \bigcap_{j \in I} A'_j$  and  $A \subset A_i$ ,  $\forall i \in I$  we deduce that  $\forall i \in I$ ,  $\exists j \in I$  such that  $A'_j \subset A_i$ . We get that  $c \in a_i + A_i$  because  $c - a_j \in A'_j \subseteq A_i$  and  $a_j - a_i \in A_i$ . So,  $R/A$  is complete in the  $R/A$ -topology.  $\square$

**Proposition 1.2.** [7, Exercise II.6.4] *Let  $R$  be a valuation ring and let  $L$  be a non-zero prime ideal. Then  $R$  is (almost) maximal if and only if  $R/L$  is maximal and  $R_L$  is (almost) maximal.*

*Proof.* If  $R$  is (almost) maximal, it is obvious that  $R/L$  is maximal and by Proposition 1.1  $R_L$  is (almost) maximal. Conversely let  $A$  be a non-zero ideal and  $J = A^\#$ . Suppose that either  $J \subset L$  or  $J = L$  and  $A$  is not isomorphic to  $L$ . Since  $R_L$  is (almost) maximal it follows that  $R_L/A$  is complete in its ideal topology. From Proposition 1.1 we deduce that  $R/A$  is complete in its ideal topology. Now, suppose that  $L \subset J$ . If  $A \subset L$  let  $t \in J \setminus L$ . Thus  $A \subset t^{-1}A$ . Let  $s \in t^{-1}A \setminus A$ . Therefore  $L \subset tR \subseteq s^{-1}A$ . So,  $R/s^{-1}A$  is complete in its ideal topology because  $R/L$  is maximal, whence  $R/A$  is complete too. Finally if  $A \cong L$  the result is obvious.  $\square$

**Proposition 1.3.** *Let  $(L_\lambda)_{\lambda \in \Lambda}$  be a non-empty family of prime ideals of  $R$  and let  $L = \bigcup_{\lambda \in \Lambda} L_\lambda$ . Then  $L$  is prime,  $R_L = \bigcap_{\lambda \in \Lambda} R_{L_\lambda}$  and  $R_L$  is maximal if and only if  $R_{L_\lambda}$  is maximal  $\forall \lambda \in \Lambda$ .*

*Proof.* It is obvious that  $L$  is prime. Let  $Q$  be the field of fractions of  $R$ . If  $x \in Q \setminus R_L$  then  $x = \frac{1}{s}$  where  $s \in L$ . Since  $L = \bigcup_{\lambda \in \Lambda} L_\lambda$ ,  $\exists \mu \in \Lambda$  such that  $s \in L_\mu$ . We deduce that  $x \notin R_{L_\mu}$  and  $R_L = \bigcap_{\lambda \in \Lambda} R_{L_\lambda}$ .

If  $R_L$  is maximal, we deduce that  $R_{L_\lambda}$  is maximal  $\forall \lambda \in \Lambda$  by Proposition 1.1. Conversely, by [14, proposition 4]  $R_L$  is linearly compact in the inverse limit topology. Since  $R_L$  is Hausdorff in this linear topology then every nonzero ideal is open and also closed. Hence  $R_L$  is linearly compact in the discrete topology.  $\square$

Recall that a valuation domain  $R$  is **Archimedean** if its maximal  $P$  is the only non-zero prime ideal and an ideal  $A$  is **Archimedean** if  $A^\# = P$ .

**Proposition 1.4.** [3, Corollary 9] *Let  $R$  be an Archimedean valuation domain. If  $d_R(0) < \infty$ , then  $R$  is almost maximal.*

From Propositions 1.1, 1.2, 1.3 and 1.4 we deduce the following:

**Proposition 1.5.** *Let  $R$  be a valuation domain such that  $d_R(0) < \infty$  and  $R/A$  is Hausdorff and complete in its ideal topology for each non-zero non-Archimedean ideal  $A$ . Then  $R$  is almost maximal.*

*Proof.* Let  $L, L'$  be prime ideals such that  $L' \subset L$ . Since  $\widehat{(R_L)}$  is a summand of  $\widehat{(R)}_L$  we have  $d_{R_L}(0) \leq d_R(0)$ . On the other hand, by tensoring a pure-composition series of  $\widehat{(R_L)}$  with  $R_L/L'$  we get a pure-composition series of  $\widehat{(R_L/L')}$ . So,  $d_{R_L}(L') \leq d_R(0)$ .

If  $R$  is Archimedean the result follows from Proposition 1.4. Suppose that  $R$  is not Archimedean, let  $J$  be a non-zero ideal and let  $(L_\lambda)_{\lambda \in \Lambda}$  be the family of prime ideals properly containing  $J$  and properly contained in  $P$ . If  $\Lambda = \emptyset$  we get that  $R$  is almost maximal by applying Propositions 1.4 and 1.2. Else, let  $L' = \cup_{\lambda \in \Lambda} L_\lambda$ . By Proposition 1.3,  $R_{L'}/J$  is maximal for each non-zero prime  $J$ . If  $L' \neq P$  then  $R/L'$  is maximal by Proposition 1.4 and it follows that  $R/J$  is maximal by Proposition 1.2. If the intersection  $K$  of all non-zero primes is zero then  $R$  is almost maximal. If  $K \neq 0$  then  $R_K$  is Archimedean. We conclude by using Propositions 1.4 and 1.2.  $\square$

Given a ring  $R$ , an  $R$ -module  $M$  and  $x \in M$ , the **content ideal**  $c(x)$  of  $x$  in  $M$ , is the intersection of all ideals  $A$  for which  $x \in AM$ . We say that  $M$  is a **content module** if  $x \in c(x)M$ ,  $\forall x \in M$ .

**Lemma 1.6.** *Let  $U$  be a torsion-free module such that  $U = PU$ . Then:*

- (1)  $\forall x \in U$ ,  $x \neq 0$ ,  $x \notin c(x)U$ ;
- (2) let  $0 \neq x, y \in U$  and  $t \in R$  such that  $x = ty$ . Then  $c(y) = t^{-1}c(x)$ ;
- (3) if  $U$  is uniserial then, for each  $x \in U$ ,  $x \neq 0$ ,  $c(x)^\# = U^\#$ .

*Proof.* (1). If  $x \in c(x)U$ , there exist  $a \in R$  and  $z \in U$  such that  $x = az$  and  $c(x) = Ra$ . But, since  $z \in PU$ , we get a contradiction.

(2). Let  $0 \neq x, y \in U$  such that  $x = ty$ . If  $s \notin c(y)$  then  $x = tsz$  for some  $z \in U$  and  $st \notin c(x)$ . So,  $s \notin t^{-1}c(x)$ . Conversely, if  $s \notin t^{-1}c(x)$  then  $st \notin c(x)$ . We have  $x = stz$  for some  $z \in U$ . We get that  $y = sz$ . So,  $s \notin c(y)$ .

(3). We put  $A = c(x)$  and  $L = A^\#$ . Let  $s \notin L$  and  $y \in U$  such that  $x = ty$  for some  $t \in R$ . Then  $c(y) = t^{-1}A$  and  $t \notin A$ . So,  $t^{-1}A \subseteq L$ . Consequently  $y \in sU$ . Let  $s \in L$ . If  $s \in A$  then  $x \notin sU$ . If  $s \in L \setminus A$  let  $t \in s^{-1}A \setminus A$ . There exists  $y \in U$  such that  $x = ty$ . Since  $c(y) = t^{-1}A$  and  $s \in t^{-1}A$  we deduce that  $y \notin sU$ .  $\square$

This lemma and the previous proposition allow us to show the following theorem.

**Theorem 1.7.** *Let  $R$  be a valuation domain such that  $d_R(0) < \infty$ . Then there exists a finite family of prime ideals  $P = L_0 \supset L_1 \supset \dots \supset L_{m-1} \supset L_m \supseteq 0$  such that  $R_{L_k}/L_{k+1}$  is almost maximal,  $\forall k$ ,  $0 \leq k \leq m-1$  and  $R_{L_m}$  is maximal if  $L_m \neq 0$  (or equivalently, for each proper ideal  $A \not\supseteq L_k$ ,  $\forall k$ ,  $0 \leq k \leq m$ ,  $R/A$  is Hausdorff and complete in its ideal topology). Moreover,  $d_R(0) = \prod_{k=1}^{k=m} c_R(L_k)$ .*

*Proof.* Let  $n = d_R(0)$ . Then  $\widehat{R}$  has a pure-composition series

$$0 = G_0 \subset R = G_1 \subset \dots \subset G_{n-1} \subset G_n = \widehat{R}$$

such that,  $\forall k$ ,  $1 \leq k \leq n$ ,  $U_k = G_k/G_{k-1}$  is a uniserial torsion-free module. The family  $(L_0, \dots, L_m)$  is defined in the following way:  $\forall j$ ,  $0 \leq j \leq m$ , there exists  $k$ ,  $1 \leq k \leq n$  such that  $L_j = U_k^\#$ .

Now, let  $A$  be a proper ideal such that  $R/A$  is Hausdorff and non-complete in its ideal topology. By [7, Lemma V.6.1] there exists  $x \in \widehat{R} \setminus R$  such that  $A = c(x+R)$  (Clearly  $c(x+R) = B(x)$ , the breadth ideal of  $x$ ). Let  $U$  be a pure uniserial submodule of  $\widehat{R}/R$  containing  $x+R$  and let  $M$  be the inverse image of  $U$  by the natural map  $\widehat{R} \rightarrow \widehat{R}/R$ . From the pure-composition series of  $M$  with factors  $R$  and  $U$ , and a pure-composition series of  $\widehat{R}/M$  we get a pure-composition series

of  $\widehat{R}$ . Since each pure composition series has isomorphic uniserial factors by [7, Theorem XV.1.7], it follows that  $U \cong U_k$  for some  $k$ ,  $2 \leq k \leq n$ . So, by Lemma 1.6  $A^\sharp = U^\sharp = U_k^\sharp$ .

We apply Proposition 1.5 and deduce that  $R_{L_k}/L_{k+1}$  is almost maximal  $\forall k$ ,  $0 \leq k \leq m-1$  and  $R_{L_m}$  is maximal if  $L_m \neq 0$ .

To prove the last assertion we apply [5, Lemma 2] (The conclusion of this lemma holds if  $R_L/L'$  is almost maximal, where  $L$  and  $L'$  are prime ideals,  $L' \subset L$ ).  $\square$

The following completes the previous theorem.

**Proposition 1.8.** *Let  $R$  be a valuation domain such that  $d_R(0) < \infty$ , let  $(U_k)_{1 \leq k \leq n}$  be the family of uniserial factors of all pure-composition series of  $\widehat{R}$  and let  $(L_j)_{0 \leq j \leq m}$  be the family of prime ideals defined in Theorem 1.7. Then:*

- (1)  $\forall k$ ,  $1 \leq k \leq n$ ,  $U_k \cong R_{U_k^\sharp}$ ;
- (2)  $\widehat{R}$  has a pure-composition series

$$0 = F_0 \subset R = F_1 \subset \cdots \subset F_{m-1} \subset F_m = \widehat{R}$$

where  $F_{j+1}/F_j$  is a free  $R_{L_j}$ -module of finite rank,  $\forall j$ ,  $0 \leq j \leq m-1$ .

*Proof.* (1). Let  $A$  be an ideal such that  $\exists j$ ,  $0 \leq j \leq m$ ,  $A^\sharp = L_j$  and  $A \not\cong L_j$ . In the sequel we put  $L_{m+1} = 0$  if  $L_m \neq 0$ .

First, for each uniserial torsion-free module  $U$ , we will show that each family  $(x_r + rU)_{r \in R \setminus A}$  has a non-empty intersection if  $x_r \in x_t + tU$ ,  $\forall r, t \in R \setminus A$ ,  $r \in tR$ . As in the proof of Proposition 1.2 we may assume that  $L_{j+1} \subset A$ . Since  $R_{L_j}/L_{j+1}$  is almost maximal and  $A$  is an ideal of  $R_{L_j}$  the family  $(x_r + rU_{L_j})_{r \in R \setminus A}$  has a non-empty intersection. If  $r \in L_j \setminus A$ , we have  $r^{-1}A \subset L_j$ . So, if  $t \in L_j \setminus r^{-1}A$  then  $rt \notin A$  and  $rtU_{L_j} \subseteq rU$ . It follows that we can do as in the proof of Proposition 1.1 to show that the family  $(x_r + rU)_{r \in R \setminus A}$  has a non-empty intersection.

Let  $0 = G_0 \subset R = G_1 \subset \cdots \subset G_{n-1} \subset G_n = \widehat{R}$  be a pure-composition series of  $\widehat{R}$  whose factors are the  $U_k$ ,  $1 \leq k \leq n$ . By induction on  $k$  and by using the pure-exact sequence  $0 \rightarrow G_{k-1} \rightarrow G_k \rightarrow U_k \rightarrow 0$ , we get that each family  $(x_r + rG_k)_{r \in R \setminus A}$  for which  $x_r \in x_t + tG_k$ ,  $\forall r, t \in R \setminus A$ ,  $r \in tR$ , has a non-empty intersection.

Let  $k$ ,  $2 \leq k \leq n$ , be an integer, let  $0 \neq x \in G_k \setminus G_{k-1}$  and let  $A = c(x + G_{k-1})$ . Then  $A^\sharp = U_k^\sharp = L_j$  for some  $j$ ,  $1 \leq j \leq m$ . We shall prove that  $A \cong L_j$ . For each  $r \in R \setminus A$ ,  $x = g_r + ry_r$  for some  $g_r \in G_{k-1}$  and  $y_r \in G_k$ . Let  $r, t \in R \setminus A$  such that  $r \in tR$ . Then we get that  $g_r \in g_t + tG_k \cap G_{k-1} = g_t + tG_{k-1}$  since  $G_{k-1}$  is a pure submodule. If  $A \not\cong L_j$  the family  $(g_r + rG_{k-1})_{r \in R \setminus A}$  has a non-empty intersection. Let  $g \in g_r + rG_{k-1} \forall r \in R \setminus A$ . Then  $(x - g) \in rG_k$ ,  $\forall r \in R \setminus A$ . Since  $G_k$  is a pure-essential extension of a free module,  $G_k$  is a content module by [4, Proposition 23]. It follows that  $(x - g) \in AG_k$ . So  $x + G_{k-1} \in AU_k$ . But, since  $k \geq 2$ ,  $U_k = PU_k$  because  $\widehat{R}/P\widehat{R} \cong R/PR$ . So,  $x + G_{k-1} \notin AU_k$ . From this contradiction we get that  $A = sL_j$  for some  $0 \neq s \in R$ . If  $sL_j \neq L_j$  then  $x + G_{k-1} = sy + G_{k-1}$  for some  $y \in G_k$  because  $s \notin A$ . It follows that  $c(y + G_{k-1}) = L_j$ . We put  $y' = y + G_{k-1}$ . Then, for each  $z \in U_k \setminus Ry'$  there exists  $t \in R \setminus L_j$  such that  $y' = tz$ . We get that  $U_k = R_{L_j}y'$ .

(2). Let  $M = \widehat{R}/R$ . Then  $L_1 = M^\sharp$ . From above we get that  $M/L_1M \neq 0$ . By [4, Proposition 21]  $M$  contains a pure free  $R_{L_1}$ -submodule  $N$  such that  $N/L_1N \cong$

$M/L_1M$ . It follows that  $(M/N)^\sharp = L_2$ . We set  $F_2$  the inverse image of  $N$  by the natural map  $\widehat{R} \rightarrow M$ . We complete the proof by induction on  $j$ .  $\square$

## 2. TORSION-FREE MODULES OF FINITE RANK.

In this section we give some precisions on the structure of torsion-free  $R$ -modules of finite rank when  $R$  satisfies a condition weaker than  $d_R(0) < \infty$ . The following lemmas are needed.

**Lemma 2.1.** *Let  $R$  be a valuation ring (possibly with zerodivisors), let  $U$  be a uniserial module and let  $L$  be a prime ideal such that  $L \subset U^\sharp$ . Then  $U_L$  is a cyclic  $R_L$ -module.*

*Proof.* Let  $s \in U^\sharp \setminus L$  and let  $x \in U \setminus sU$ . Let  $y \in U \setminus Rx$ . There exists  $t \in P$  such that  $x = ty$ . Then  $t \notin Rs$ , whence  $t \notin L$ . It follows that  $U_L = R_Lx$ .  $\square$

**Lemma 2.2.** *Let  $R$  be a valuation ring (possibly with zerodivisors), let  $U$  and  $V$  be uniserial modules such that  $V^\sharp \subset U^\sharp$ . Assume that  $U_L$  is a faithful  $R_L$ -module, where  $L = V^\sharp$ . Then  $\text{Ext}_R^1(U, V) = 0$ .*

*Proof.* Let  $M$  be an extension of  $V$  by  $U$ . By lemma 2.1  $U_L$  is a free cyclic  $R_L$ -module. Since  $V$  is a module over  $R_L$ , it follows that  $V$  is a summand of  $M_L$ . We deduce that  $V$  is a summand of  $M$  too.  $\square$

**Lemma 2.3.** *Let  $R$  be a valuation domain for which there exists a prime ideal  $L \neq P$  such that  $R/L$  is almost maximal. Then  $\text{Ext}_R^1(U, V) = 0$  for each pair of ideals  $U$  and  $V$  such that  $L \subset U^\sharp \cap V^\sharp$ .*

*Proof.* Let  $M$  be an extension of  $V$  by  $U$ . It is easy to check that  $U/LU$  and  $V/LV$  are non-zero and non divisible  $R/L$ -modules. Since  $R/L$  is almost maximal  $M/LM \cong U/LU \oplus V/LV$  by [6, Proposition VI.5.4]. If  $L \neq 0$ , it follows that there exist two submodules  $H_1$  and  $H_2$  of  $M$ , containing  $LM$ , such that  $H_1/LM \cong U/LU$  and  $H_2/LM \cong V/LV$ . For  $i = 1, 2$  let  $x_i \in H_i \setminus LM$  and let  $A_i$  be the submodule of  $H_i$  such that  $A_i/Rx_i$  is the torsion submodule of  $H_i/Rx_i$ . Then  $A_i + LM/LM$  is a non-zero pure submodule of  $H_i/LM$  which is of rank one over  $R/L$ . It follows that  $H_i = A_i + LM$ . By Lemma 2.1  $M_L \cong V_L \oplus U_L$ . We deduce that  $LM \cong LM_L$  is a direct sum of uniserial modules. Since  $A_i \cap LM$  is a non-zero pure submodule of  $LM$  there exists a submodule  $C_i$  of  $LM$  such that  $LM = (A_i \cap LM) \oplus C_i$  by [7, Theorem XII.2.2]. It is easy to check that  $H_i = A_i \oplus C_i$ . From  $M = H_1 + H_2$  and  $LM = H_1 \cap H_2$  we deduce that the following sequence is pure exact:

$$0 \rightarrow LM \rightarrow H_1 \oplus H_2 \rightarrow M \rightarrow 0,$$

where the homomorphism from  $LM$  is given by  $x \mapsto (x, -x)$ ,  $x \in LM$ , and the one onto  $M$  by  $(x, y) \mapsto x + y$ ,  $x \in H_1$ ,  $y \in H_2$ . Since  $H_1 \oplus H_2$  is a direct sum of uniserial modules, so is  $M$  by [7, Theorem XII.2.2]. Consequently  $M \cong V \oplus U$ .  $\square$

**Proposition 2.4.** *Let  $R$  be a valuation domain. Let  $G$  be a torsion-free  $R$ -module of finite rank. Then  $G$  has a pure-composition series with uniserial factors  $(U_k)_{1 \leq k \leq n}$  such that  $U_k^\sharp \supseteq U_{k+1}^\sharp$ ,  $\forall k$ ,  $1 \leq k \leq n - 1$ .*

*Proof.*  $G$  has a pure-composition series

$$0 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$$

such that,  $\forall k, 1 \leq k \leq n, V_k = H_k/H_{k-1}$  is a uniserial torsion-free module. Suppose there exists  $k, 1 \leq k \leq (n-1)$  such that  $V_k^\# \subset V_{k+1}^\#$ . By Lemma 2.2,  $H_{k+1}/H_{k-1} \cong V_k \oplus V_{k+1}$ . So, if  $H'_k$  is the inverse image of  $V_{k+1}$  by the surjection  $H_{k+1} \rightarrow H_{k+1}/H_{k-1}$ , if  $U_k = H'_k/H_{k-1}$  and  $U_{k+1} = H_{k+1}/H'_k$  then  $U_k^\# \supset U_{k+1}^\#$ . So, in a finite number of similar steps, we get a pure composition series with the required property.  $\square$

**Proposition 2.5.** *Let  $R$  be a valuation domain. Assume that there exists a finite family of prime ideals  $P = L_0 \supset L_1 \supset \cdots \supset L_{m-1} \supset L_m = 0$  such that  $R_{L_k}/L_{k+1}$  is almost maximal  $\forall k, 0 \leq k \leq m-1$ . Let  $G$  be a torsion-free  $R$ -module of finite rank. Then  $G$  has a pure-composition series*

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{m-1} \subseteq G_m \subseteq G_{m+1} = G$$

where  $G_{j+1}/G_j$  is a finite direct sum of ideals of  $R_{L_j}, \forall j, 0 \leq j \leq m$ .

*Proof.* By Proposition 2.4  $G$  has a pure-composition series

$$0 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$$

such that,  $\forall k, 1 \leq k \leq n, U_k = H_k/H_{k-1}$  is a uniserial torsion-free module and  $U_k^\# \supseteq U_{k+1}^\#, \forall k, 1 \leq k \leq n-1$ . Now, for each  $j, 1 \leq j \leq m$ , let  $k_j$  be the greatest index such that  $L_j \subset U_{k_j}^\#$ . We put  $G_j = H_{k_j}$ . Then  $G_{j+1}/G_j$  is an  $R_{L_j}$ -module which is a direct sum of ideals by Lemma 2.3.  $\square$

### 3. VALUATION DOMAINS $R$ WITH $fr(R) < \infty$ .

First we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. So, except in Theorem 3.6, we assume that  $R$  is an almost maximal valuation ring. We put  $K = Q/R$ . For each  $R$ -module  $M$ ,  $d(M)$  is the divisible submodule of  $M$  which is the union of all divisible submodules and  $M$  is said to be **reduced** if  $d(M) = 0$ . We denote by  $\widehat{M}$  the **pure-injective hull** of  $M$  (see [7, chapter XIII]). If  $U$  is a uniserial module then  $\widehat{U} \cong \widehat{R} \otimes_R U$  because  $R$  is almost maximal. Let  $G$  be a torsion-free module of finite rank  $r$ . By Proposition 2.5  $G$  contains a submodule  $B$  which is a direct sum of ideals and such that  $G/B$  is a  $Q$ -vector space. We put **corank**  $G = \text{rank } G/B$ . Now, it is easy to prove the following.

**Proposition 3.1.** *Let  $G$  be a torsion-free  $R$ -module of rank  $r$  and corank  $c$ . Then:*

- (1)  $G$  contains a pure direct sum  $B$  of ideals, of rank  $r - c$ , such that  $G/B$  is a  $Q$ -vector space of dimension  $c$ .
- (2)  $G$  contains a pure direct sum  $B'$  of ideals, of rank  $r$  such that  $G/B'$  is isomorphic to a quotient of  $K^c$ .

An element of  $Q \otimes_R \text{Hom}_R(G, H)$  is called a **quasi-homomorphism** from  $G$  to  $H$ , where  $G$  and  $H$  are  $R$ -modules. Let  $\mathcal{C}_{\text{ab}}$  be the category having **weakly polyserial**  $R$ -modules (i.e modules with composition series whose factors are uniserial) as objects and quasi-homomorphisms as morphisms and let  $\mathcal{C}$  be full subcategory of  $\mathcal{C}_{\text{ab}}$  having torsion-free  $R$ -modules of finite rank as objects. Then  $\mathcal{C}_{\text{ab}}$  is abelian by [7, Lemma XII.1.1]. If  $G$  and  $H$  are torsion-free of finite rank, then the quasi-homomorphisms from  $G$  to  $H$  can be identified with the  $Q$ -linear maps  $\phi : Q \otimes_R G \rightarrow Q \otimes_R H$  such that  $r\phi(G) \subseteq H$  for some  $0 \neq r \in R$ . We say that  $G$  and

$H$  are **quasi-isomorphic** if they are isomorphic objects of  $\mathcal{C}$ . A torsion-free module of finite rank is said to be **strongly indecomposable** if it is an indecomposable object of  $\mathcal{C}$ .

**Lemma 3.2.** *Let  $\phi : Q \otimes_R G \rightarrow Q \otimes_R H$  be a  $Q$ -linear map. Then  $\phi$  is a quasi-homomorphism if and only if  $(\widehat{R} \otimes_R \phi)(d(\widehat{R} \otimes_R G)) \subset d(\widehat{R} \otimes_R H)$ .*

*Proof.* Assume that  $\phi$  is a quasi-homomorphism. There exists  $0 \neq r \in R$  such that  $r\phi(G) \subseteq H$ . It successively follows that  $r(\widehat{R} \otimes_R \phi)(\widehat{R} \otimes_R G) \subseteq \widehat{R} \otimes_R H$  and  $(\widehat{R} \otimes_R \phi)(d(\widehat{R} \otimes_R G)) \subseteq d(\widehat{R} \otimes_R H)$ .

Conversely, let  $B$  be a finite direct sum of ideals which satisfies that  $G/B$  is a  $Q$ -vector space. There exists a free submodule  $F$  of  $Q \otimes_R G$  such that  $B \subseteq F$ . So,  $\exists 0 \neq r \in R$  such that  $r\phi(B) \subseteq r\phi(F) \subseteq H$ . Since  $\widehat{R} \otimes_R G = (\widehat{R} \otimes_R B) \oplus (d(\widehat{R} \otimes_R G))$ , it follows that  $r(\widehat{R} \otimes_R \phi)(\widehat{R} \otimes_R G) \subseteq (\widehat{R} \otimes_R H)$ . We deduce that  $r\phi(G) \subseteq (\widehat{R} \otimes_R H) \cap (Q \otimes_R H) = H$ .  $\square$

**Proposition 3.3.** *Let  $G$  be a torsion-free  $R$ -module of rank  $r$  and corank  $c$ .*

- (1) *If  $G$  has no summand isomorphic to an ideal, then  $\text{End}(G)$  can be embedded in the ring of  $c \times c$  matrices over  $\widehat{Q}$ . In particular if  $c = 1$ ,  $\text{End}(G)$  is a commutative integral domain.*
- (2) *If  $G$  is reduced, then  $\text{End}(G)$  can be embedded in the ring of  $(r-c) \times (r-c)$  matrices over  $\widehat{R}$ . In particular if  $c = r - 1$ ,  $\text{End}(G)$  is a commutative integral domain.*

*Proof.* See the proof of [8, Theorem 3.1].  $\square$

In the sequel we assume that  $n = c_R(0) < \infty$ . So, there are  $n-1$  units  $\pi_2, \dots, \pi_n$  in  $\widehat{R} \setminus R$  such that  $1, \pi_2, \dots, \pi_n$  is a basis of  $\widehat{Q}$  over  $Q$ . By [7, Theorem XV.6.3] there exists an indecomposable torsion-free  $R$ -module  $E$  with rank  $n$  and corank 1. We can define  $E$  in the following way: if  $(e_k)_{2 \leq k \leq n}$  is the canonical basis of  $\widehat{R}^{n-1}$ , if  $e_1 = \sum_{k=2}^{k=n} \pi_k e_k$  and  $V$  is the  $Q$ -vector subspace of  $\widehat{Q}^{n-1}$  generated by  $(e_k)_{1 \leq k \leq n}$ , then  $E = V \cap \widehat{R}^{n-1}$ . Then a basis element for  $d(\widehat{R} \otimes E)$  can be written  $u_1 + \pi_2 u_2 + \dots + \pi_n u_n$ , where  $u_1, \dots, u_n \in E$ . Since  $E$  is indecomposable it follows that  $u_1, \dots, u_n$  is a basis for  $Q \otimes E \cong V$ .

**Theorem 3.4.** *Let  $G$  be a torsion-free  $R$ -module of rank  $r$  and corank  $c$ . Then the following assertions hold:*

- (1) *The reduced quotient of  $G$  is isomorphic to a pure submodule of  $\widehat{B}$  where  $B$  is a direct sum of  $(r-c)$  ideals.*
- (2)  *$G$  is the direct sum of ideals of  $R$  with a quasi-homomorphic image of  $E^c$ .*

*Proof.* (1) can be shown as the implication (1)  $\Rightarrow$  (2) of [8, Theorem 4.1] and (2) as the implication (1)  $\Rightarrow$  (3) of [8, Theorem 4.1].  $\square$

**Corollary 3.5.** *Let  $G$  be a torsion-free  $R$ -module of rank  $r$  and corank  $c$ . Then:*

- (1) *If  $G$  has no summand isomorphic to an ideal, then  $r \leq nc$ .*
- (2) *If  $G$  is reduced, then  $nc \leq (n-1)r$ .*

*Proof.* This corollary is a consequence of Theorem 3.4 and can be shown as [8, Corollary 4.2].  $\square$

**Theorem 3.6.** *Let  $R$  be a valuation domain such that  $d_R(0) = 2$ . Then  $fr(R) = 2$ . Moreover  $fr^o(R) = 1$  if  $c_R(0) = 2$  and  $fr^o(R) = 2$  if  $c_R(0) = 1$ .*

*Proof.* First suppose that  $c_R(0) = 2$ . So,  $R$  is almost maximal and  $fr^o(R) = 1$ . Let  $G$  be an indecomposable torsion-free module with rank  $r$  and corank  $c$  which is not isomorphic to  $Q$  and to an ideal. Then  $G$  is reduced and has no summand isomorphic to an ideal of  $R$ . From Corollary 3.5 we deduce that  $r = 2c$ . By Theorem 3.4  $G$  is isomorphic to a pure submodule of  $\widehat{B}$  where  $B$  is a direct sum of  $c$  ideals. Since rank  $\widehat{B} = 2c$  it follows that  $G \cong \widehat{B}$ . So,  $c = 1$  and  $G \cong \widehat{A}$  for a non-zero ideal  $A$ .

If  $c_R(0) = 1$  let  $L$  be the non-zero prime ideal such that  $c_R(L) = d_{R/L}(0) = 2$ . Then  $fr(R/L) = 2$ . Since  $R_L$  is maximal it follows that  $fr(R) = fr^o(R) = 2$  by [12, Lemma 9 and Lemma 4].  $\square$

**Lemma 3.7.** *Every proper subobject of  $E$  in  $\mathcal{C}$  is a direct sum of ideals.*

*Proof.* Let  $G$  be a proper object of  $E$  in  $\mathcal{C}$  and let  $H$  be the pure submodule of  $E$  such that  $H/G$  is the torsion submodule of  $E/G$ . Since  $E$  is indecomposable,  $E$  has no summand isomorphic to a direct sum of ideals. So, corank  $E/H = 1$  and corank  $H = 0$ . As corank  $H \geq$  corank  $G$  we get that  $G$  is a direct sum of ideals.  $\square$

**Proposition 3.8.**  *$E$  is an indecomposable projective object of  $\mathcal{C}$ .*

*Proof.* Let  $\phi : H \rightarrow E$  be a quasi-epimorphism where  $H$  is a torsion-free module of finite rank. Suppose that  $H = F \oplus G$  where  $F$  is a direct sum of ideals. By Lemma 3.7,  $\phi(G)$  is quasi-isomorphic to  $E$ . So, we may assume that  $H$  has no summand isomorphic to an ideal. By Theorem 3.4 there is a quasi-epimorphism  $\psi : E^c \rightarrow H$  where  $c = \text{corank } H$ . It is sufficient to see that  $\phi \circ \psi$  is a split epimorphism in  $\mathcal{C}$ . But by Proposition 3.3(1),  $Q \otimes \text{End}(E)$  is a subfield of  $\widehat{Q}$ , so every quasi-homomorphism  $E \rightarrow E$  is either a quasi-isomorphism or trivial and the splitting follows immediately.  $\square$

In the sequel,  $Q \otimes_R \text{Hom}_R(R \oplus E, M)$  is denoted by  $\ddot{M}$  for each  $R$ -module  $M$  and the ring  $Q \otimes_R \text{End}_R(R \oplus E)$  by  $\Lambda$ .

**Theorem 3.9.** *The foncteur  $Q \otimes_R \text{Hom}_R(R \oplus E, \_)$  is an exact fully faithful functor from  $\mathcal{C}$  into  $\text{mod-}\Lambda$ , the category of finitely generated right  $\Lambda$ -modules.*

*Proof.* By Theorem 3.4 and Proposition 3.8,  $R \oplus E$  is a progenerator of  $\mathcal{C}$ . For each finite rank torsion-free  $R$ -module  $H$ , the natural map  $Q \otimes_R \text{Hom}_R(R \oplus E, H) \rightarrow \text{Hom}_\Lambda(\ddot{R} \oplus \ddot{E}, \ddot{H})$  is an isomorphism because  $\Lambda = \ddot{R} \oplus \ddot{E}$ . Thus  $Q \otimes_R \text{Hom}_R(F, H) \rightarrow \text{Hom}_\Lambda(\ddot{F}, \ddot{H})$  is an isomorphism if  $F$  is a summand of a finite direct sum of modules isomorphic to  $R \oplus E$ . Let  $G$  be a finite rank torsion-free  $R$ -module. We may assume that  $G$  has no summand isomorphic to an ideal of  $R$ . By Proposition 3.8 and Lemma 3.7, there is an exact sequence  $0 \rightarrow R^{nc-r} \rightarrow E^c \rightarrow G \rightarrow 0$  in  $\mathcal{C}$ . Since both functors are left exact, we get that  $Q \otimes_R \text{Hom}_R(G, H) \cong \text{Hom}_\Lambda(\ddot{G}, \ddot{H})$ .  $\square$

**Lemma 3.10.** *If  $M$  is a right  $\Lambda$ -module and  $M \subseteq \ddot{G}$  for some finite rank torsion-free  $R$ -module  $G$ , then  $M \cong \ddot{H}$  for some torsion-free  $R$ -module  $H$ .*

*Proof.* See the proof of [8, Lemma 5.2].  $\square$

**Proposition 3.11.** *The ring  $\Lambda$  is a hereditary Artinian  $Q$ -algebra such that  $(\text{rad } \Lambda)^2 = 0$ . There are two simple right  $\Lambda$ -modules,  $\tilde{R}$  which is projective and  $\tilde{K}$  which is injective.*

*Proof.* See the proof of [8, Proposition 5.3]. □

**Proposition 3.12.**  *$\tilde{Q}$  is an injective hull for  $\tilde{R}$  and  $\tilde{E}$  is a projective cover for  $\tilde{K}$ .*

*Proof.* See the proof of [8, Proposition 5.4]. □

**Theorem 3.13.** *The image of  $\mathcal{C}$  under the functor  $Q \otimes_R \text{Hom}_R(R \oplus E, \_)$  is the full subcategory of  $\text{mod-}\Lambda$  consisting of modules with no summand isomorphic to  $\tilde{K}$ .*

*Proof.* See the proof of [8, Theorem 5.5]. □

Let  $M$  be a finitely generated (i.e., finite length) right  $\Lambda$ -module. We define **rank**  $M$  to be the number of factors in a composition series for  $M$  isomorphic to  $\tilde{R}$  and **corank**  $M$  to be the number of composition factors isomorphic to  $\tilde{K}$ .

**Proposition 3.14.** *The foncteur  $Q \otimes_R \text{Hom}_R(R \oplus E, \_)$  preserves rank and corank.*

*Proof.* See the proof of [8, Proposition 5.6]. □

We now consider the functors  $\mathbf{D} = \text{Hom}_R(\_, Q)$  and  $\mathbf{Tr} = \text{Ext}_\Lambda(\_, \Lambda)$  which take right  $\Lambda$ -modules to left  $\Lambda$ -modules and conversely. It is well known that  $\mathbf{D}$  is an exact contravariant length preserving functor taking projectives to injectives and conversely, and that  $\mathbf{D}^2$  is the identity for finitely generated  $\Lambda$ -modules. Since  $\Lambda$  is hereditary,  $\mathbf{Tr}$  is right exact and  $\mathbf{Tr}^2 M \cong M$  if  $M$  has no projective summand,  $\mathbf{Tr} M = 0$  if  $M$  is projective. We consider the Coxeter functors  $C^+ = \mathbf{DTr}$  and  $C^- = \mathbf{TrD}$ . Thus  $C^+ : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda$  is left exact and  $C^- : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda$  is right exact. If  $M$  has no projective (respectively injective) summand, it is easy to check that  $M$  is indecomposable if and only if  $C^+ M$  (respectively  $C^- M$ ) is indecomposable.

**Proposition 3.15.** *Let  $M$  be a right  $\Lambda$ -module with rank  $r$  and corank  $c$ .*

- (1) *If  $M$  has no projective summand, then  $\text{corank } C^+ M = (n-1)c - r$  and  $\text{rank } C^+ M = nc - r$ .*
- (2) *If  $M$  has no injective summand, then  $\text{rank } C^- M = (n-1)r - nc$  and  $\text{corank } C^- M = r - c$ .*

*Proof.* See the proof of [8, Proposition 5.7]. □

**Theorem 3.16.** *The following assertions hold:*

- (1) *If  $n = 3$ , then, up to quasi-homomorphism, the strongly indecomposable torsion-free  $R$ -modules are  $R$ ,  $Q$ ,  $E$ ,  $\tilde{R}$  and an  $R$ -module with rank 2 and corank 1 (corresponding to  $C^+ \tilde{Q} = C^- \tilde{R}$ ).*
- (2) *If  $n \geq 4$ , there are strongly indecomposable torsion-free  $R$ -modules with arbitrarily large rank.*

*Proof.* We show (1) and (2) as Lady in the proof of [8, Theorem 5.11] by using Proposition 3.15 and [10, Theorem 2]. □

## 4. SOME OTHER RESULTS AND OPEN QUESTIONS.

Let  $G$  be a finite rank torsion-free module over an almost maximal valuation domain  $R$ . A **splitting field** for  $G$  is a subfield  $Q'$  of  $\widehat{Q}$  containing  $Q$  such that  $(Q' \cap \widehat{R}) \otimes_R G$  is a completely decomposable  $(Q' \cap \widehat{R})$ -module (i.e. a direct sum of rank one modules). If  $Q'$  is a splitting field for  $G$ ,  $G$  is called  **$Q'$ -decomposable**. By [9, Theorem 7], each finite rank torsion-free module  $G$  has a unique minimal splitting field  $Q'$  and  $[Q' : Q] < \infty$ . So, Lady's results on splitting fields of torsion-free modules of finite rank over rank one discrete valuation domains can be extended to almost maximal valuation domains by replacing  $\widehat{Q}$  by  $Q'$  in the previous section and by taking  $\mathcal{C}$  to be the category whose objects are the finite rank torsion-free  $Q'$ -decomposable modules.

Now,  $R$  is a valuation domain which is not necessarily almost maximal. We say that an  $R$ -module  $G$  is **strongly flat** if it is an extension of a free module by a divisible torsion-free module (see [2]). By [6, Lemma V.1.1 and Proposition V.1.2]  $\widetilde{Q}$  is a splitting field for each finite rank strongly flat module. So, each finite rank strongly flat module  $G$  has a unique minimal splitting field  $Q' \subseteq \widetilde{Q}$  and  $[Q' : Q] < \infty$ . We also can extended Lady's results. In particular:

**Theorem 4.1.** *Let  $R$  be a valuation domain. We consider the following conditions:*

- (1)  $\exists l \geq 1$  such that each finite rank strongly flat module is a direct sum of modules of rank at most  $l$ ;
- (2)  $c_R(0) \leq 3$ .

Then (1)  $\Rightarrow$  (2). If  $c_R(0) \leq 2$  then (1) holds and  $l = c_R(0)$ .

If  $c_R(0) = 3$ , it is possible that the proof of [1, Theorem 2.6] should be generalized.

*Proof.* (1)  $\Rightarrow$  (2). We show that  $c_R(0) \leq \infty$  in the same way that the condition (a) of [12, Theorem 3] is proven. Then, as above, we use Lady's methods to get  $c_R(0) \leq 3$ .

If  $c_R(0) = 2$  we do as in the proof of Theorem 3.6. □

**Some open questions:**

- (1) Does henselian valuation domains with finite total defect, which are not strongly discrete, exist?
- (2) For a valuation domain  $R$ , does the condition  $d_R(0) = 3$  imply  $fr(R) = 3$  or  $< \infty$ ? It is possible that the proof of [1, Theorem 2.6] should be generalized.
- (3) For a valuation domain  $R$  with finite total defect, does the condition  $c_R(L_i) = p$ ,  $\forall i$ ,  $1 \leq i \leq m$ , where  $p = 2$  or  $3$  and  $(L_i)_{1 \leq i \leq m}$  is the family of prime ideals defined in Theorem 1.7, imply  $fr(R) < \infty$  for some  $m > 1$ ? (If  $m > 1$ , by [12, Theorem 10],  $R$  is Henselian and  $p$  is the characteristic of its residue field.)

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