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ALMOST CLEAN RINGS AND ARITHMETICAL RINGS

FRANÇOIS COUCHOT

ABSTRACT. It is shown that a commutative Bézout ring R with compact minimal prime spectrum is an elementary divisor ring if and only if so is R/L for each minimal prime ideal L . This result is obtained by using the quotient space $\text{pSpec } R$ of the prime spectrum of the ring R modulo the equivalence generated by the inclusion. When every prime ideal contains only one minimal prime, for instance if R is arithmetical, $\text{pSpec } R$ is Hausdorff and there is a bijection between this quotient space and the minimal prime spectrum $\text{Min } R$, which is a homeomorphism if and only if $\text{Min } R$ is compact. If x is a closed point of $\text{pSpec } R$, there is a pure ideal $A(x)$ such that $x = V(A(x))$. If R is almost clean, i.e. each element is the sum of a regular element with an idempotent, it is shown that $\text{pSpec } R$ is totally disconnected and, $\forall x \in \text{pSpec } R$, $R/A(x)$ is almost clean; the converse holds if every principal ideal is finitely presented. Some questions posed by Facchini and Faith at the second International Fez Conference on Commutative Ring Theory in 1995, are also investigated. If R is a commutative ring for which the ring $Q(R/A)$ of quotients of R/A is an IF-ring for each proper ideal A , it is proved that R_P is a strongly discrete valuation ring for each maximal ideal P and R/A is semicoherent for each proper ideal A .

1. INTRODUCTION

In this paper we consider the following two questions:

Question 1.1. *Is every Bézout domain an elementary divisor ring?*

Question 1.2. *More generally, is every Bézout semihereditary ring an elementary divisor ring?*

The first question was posed by M. Henriksen in [17] in 1955, and the second by M.D. Larsen, W.J Lewis and T.S. Shores in [20] in 1974.

In Section 3 we prove that these two questions are equivalent but they are still unsolved.

To show this equivalence, we use the quotient space $\text{pSpec } R$ of $\text{Spec } R$ modulo the equivalence generated by the inclusion, where R is a commutative ring. When R is a *Gelfand ring*, i.e. each prime ideal is contained in only one maximal, $\text{pSpec } R$ is Hausdorff and homeomorphic to $\text{Max } R$ (Proposition 2.1). On the other hand, if each prime ideal contains a unique minimal prime, then $\text{pSpec } R$ is Hausdorff and there is a continuous bijection from $\text{Min } R$ into $\text{pSpec } R$ which is a homeomorphism if and only if $\text{Min } R$ is compact. There is also a continuous surjection $\tau_R : \text{pSpec } R \rightarrow \text{Spec } B(R)$, where $B(R)$ is the Boolean ring associated to R , and

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τ_R is a homeomorphism if and only if $\text{pSpec } R$ is totally disconnected. In this case, it is possible to get some interesting algebraic results by using Lemma 2.9.

A ring R is said to be *clean* (respectively *almost clean* (see [24])) if each element of R is the sum of an idempotent with a unit (respectively a regular element). In Section 4 we show that the total disconnectedness of $\text{pSpec } R$ is necessary if the ring R is almost clean. Recall that a ring R is clean if and only if R is Gelfand and $\text{Max } R$ totally disconnected: see [25, Theorem 1.7], [22, Corollary 2.7] or [9, Theorem I.1]. Almost clean rings were introduced by McGovern in [24] and studied by several authors: Ahn and Anderson [1], Burgess and Raphael [4] and [3], Varadarajan [30]. If Q is the quotient ring of R and if each prime ideal of R contains a unique minimal prime, we show that $\text{pSpec } R$ and $\text{pSpec } Q$ are homeomorphic and $B(R) = B(Q)$; moreover, if R is arithmetical, then R is almost clean if Q is clean, and the converse holds if Q is coherent.

In Section 5 we give partial answers to some questions posed by Facchini and Faith at the second International Fez Conference on Commutative Ring Theory in 1995 [12]. If R is fractionally IF, it is shown that R/A is semicoherent for each ideal A and R_P is a strongly discrete valuation ring for each maximal ideal P . We give an example of a finitely fractionally self FP-injective ring which is not arithmetical; recall that Facchini and Faith proved that each fractionally self FP-injective ring is arithmetical. It is also proven that any ring which is either clean, coherent and arithmetical or semihereditary is finitely fractionally IF. However, there exist examples of clean coherent arithmetical rings with a non-compact minimal prime spectrum; recall that the author proved that $\text{Min } R/A$ is compact for any ideal A of a fractionally self FP-injective ring R .

All rings in this paper are associative and commutative with unity, and all modules are unital. We denote respectively $\text{Spec } R$, $\text{Max } R$ and $\text{Min } R$, the space of prime ideals, maximal ideals and minimal prime ideals of R , with the Zariski topology. If A a subset of R , then we denote $V(A) = \{P \in \text{Spec } R \mid A \subseteq P\}$ and $D(A) = \text{Spec } R \setminus V(A)$.

2. A QUOTIENT SPACE OF THE PRIME SPECTRUM OF A RING

If R is a ring, we consider on $\text{Spec } R$ the equivalence relation \mathcal{R} defined by $L\mathcal{R}L'$ if there exists a finite sequence of prime ideals $(L_k)_{1 \leq k \leq n}$ such that $L = L_1$, $L' = L_n$ and $\forall k, 1 \leq k \leq (n-1)$, either $L_k \subseteq L_{k+1}$ or $L_k \supseteq L_{k+1}$. We denote by $\text{pSpec } R$ the quotient space of $\text{Spec } R$ modulo \mathcal{R} and by $\lambda_R : \text{Spec } R \rightarrow \text{pSpec } R$ the natural map. The quasi-compactness of $\text{Spec } R$ implies the one of $\text{pSpec } R$, but generally $\text{pSpec } R$ is not T_1 : see [21, Propositions 6.2 and 6.3]. However:

Proposition 2.1. *The following conditions are equivalent for a ring R :*

- (1) *the restriction of λ_R to $\text{Max } R$ is a homeomorphism;*
- (2) *the restriction of λ_R to $\text{Max } R$ is injective;*
- (3) *R is Gelfand.*

In this case $\text{pSpec } R$ is Hausdorff.

Proof. It is obvious that (i) \Rightarrow (ii).

(ii) \Rightarrow (iii). If a prime ideal is contained in two maximal ideals P_1 and P_2 we get that $\lambda_R(P_1) = \lambda_R(P_2)$.

(iii) \Rightarrow (i). If L is a prime ideal we denote by $\mu(L)$ the unique maximal ideal containing L . It is easy to verify that $\mu(L) = \mu(L')$ if $L\mathcal{R}L'$. So, μ induces a map

$\bar{\mu} : \text{pSpec } R \rightarrow \text{Max } R$. We easily show that $\bar{\mu}^{-1} = \lambda_R|_{\text{Max } R}$. By [10, Theorem 1.2] μ is continuous and $\text{Max } R$ is Hausdorff. Hence $\bar{\mu}$ is a homeomorphism and $\text{pSpec } R$ is Hausdorff. \square

Proposition 2.2 ([9, Proposition IV.1]). *Let R be a ring such that each prime ideal contains only one minimal prime. Then $\text{pSpec } R$ is Hausdorff and $\lambda_R|_{\text{Min } R}$ is bijective. Moreover $\lambda_R|_{\text{Min } R}$ is a homeomorphism if and only if $\text{Min } R$ is compact.*

Proposition 2.3. *Let $\varphi : R \rightarrow T$ be a ring homomorphism. Then φ induces a continuous map ${}^b\varphi : \text{pSpec } T \rightarrow \text{pSpec } R$ such that $\lambda_R \circ {}^a\varphi = {}^b\varphi \circ \lambda_T$, where ${}^a\varphi : \text{Spec } T \rightarrow \text{Spec } R$ is the continuous map induced by φ .*

Proof. If L and L' are prime ideals of T such that $L \subseteq L'$ then ${}^a\varphi(L) \subseteq {}^a\varphi(L')$. Hence, if $x \in \text{pSpec } T$, we can put ${}^b\varphi(x) = \lambda_R({}^a\varphi(L))$ where $L \in x$. Since λ_R and ${}^a\varphi$ are continuous, so is ${}^b\varphi$. \square

An exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is *pure* if it remains exact when tensoring it with any R -module. Then, we say that F is a *pure* submodule of E . By [13, Proposition 8.6] F is a pure submodule of E if every finite system of equations

$$\sum_{i=1}^{i=n} r_{j,i}x_i = y_j \in F, \quad (1 \leq j \leq p),$$

with coefficients $r_{j,i} \in R$ and unknowns x_1, \dots, x_n , has a solution in F whenever it is solvable in E . The following proposition is well known.

Proposition 2.4. *Let A be an ideal of a ring R . The following conditions are equivalent:*

- (1) A is a pure ideal of R ;
- (2) for each finite family $(a_i)_{1 \leq i \leq n}$ of elements of A there exists $t \in A$ such that $a_i = a_it$, $\forall i$, $1 \leq i \leq n$;
- (3) for all $a \in A$ there exists $b \in A$ such that $a = ab$;
- (4) R/A is a flat R -module.

Moreover, if A is finitely generated, then A is pure if and only if it is generated by an idempotent.

Proof. (ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (iv). Let B be an ideal of R . We must prove that $A \cap B = AB$. If $a \in A \cap B$, there exists $t \in A$ such that $a = at$. Hence $a \in AB$.

(iv) \Rightarrow (iii). If $a \in A$, then $Ra = A \cap Ra = Aa$ by (iv).

(i) \Rightarrow (iii). If $a \in A$, 1 is solution of the equation $ax = a$. So, this equation has a solution in A .

(iii) \Rightarrow (ii). Let a_1, \dots, a_n be elements of A . We proceed by induction on n . There exists $t \in A$ such that $a_n = ta_n$. By induction hypothesis there exists $s \in A$ such that $a_i - ta_i = s(a_i - ta_i)$, $\forall i$, $1 \leq i \leq (n-1)$. Now, it is easy to check that $(s + t - st)a_i = a_i$, $\forall i$, $1 \leq i \leq n$.

(ii) \Rightarrow (i). We consider the following system of equations:

$$\sum_{i=1}^{i=n} r_{j,i}x_i = a_j \in A, \quad 1 \leq j \leq p.$$

Assume that (c_1, \dots, c_n) is a solution of this system in R . There exists $s \in A$ such that $a_j = sa_j$, $\forall j$, $1 \leq j \leq p$. So, (sc_1, \dots, sc_n) is a solution of this system in A . \square

We set 0_P the kernel of the natural map $R \rightarrow R_P$ where $P \in \text{Spec } R$.

Lemma 2.5. *Let R be a ring and let C a closed subset of $\text{Spec } R$. Then C is the inverse image of a closed subset of $\text{pSpec } R$ by λ_R if and only if $C = V(A)$ where A is a pure ideal. Moreover, in this case, $A = \bigcap_{P \in C} 0_P$.*

Proof. Let A be a pure ideal, and let P and L be prime ideals such that $A \subseteq P$ and $L \subseteq P$. Since A is pure, for each $a \in A$ there exists $b \in A$ such that $a = ab$. Then $(1 - b)a = 0$ and $(1 - b) \notin P$, whence $(1 - b) \notin L$ and $a \in L$. So, $L \in V(A)$ and $V(A)$ is the inverse image of a closed subset of $\text{pSpec } R$ by λ_R .

Let $C = V(B)$ where $B = \bigcap_{L \in C} L$. Suppose that C is the inverse image of a closed subset of $\text{pSpec } R$ by λ_R . We put $A = \bigcap_{P \in C} 0_P$. Let $b \in B$ and $P \in C$. Then C contains each minimal prime ideal contained in P . So, the image of b , by the natural map $R \rightarrow R_P$, belongs to the nilradical of R_P . It follows that there exist $0 \neq n_P \in \mathbb{N}$ and $s_P \in R \setminus P$ such that $s_P b^{n_P} = 0$. Hence, $\forall L \in D(s_P) \cap C$, $b^{n_P} \in 0_L$. A finite family $(D(s_{P_j}))_{1 \leq j \leq m}$ covers C . Let $n = \max\{n_{P_1}, \dots, n_{P_m}\}$. Then $b^n \in 0_L$, $\forall L \in C$, whence $b^n \in A$. We deduce that $C = V(A)$. Now, we have $A_P = 0$ if $P \in V(A)$ and $A_P = R_P$ if $P \in D(A)$. Hence A is a pure ideal. \square

Corollary 2.6. *For any ring R the following assertions hold:*

- (1) *a subset U of $\text{pSpec } R$ is open and closed if and only if there exists an idempotent $e \in R$ such that $\lambda_R^{\leftarrow}(U) = D(e)$;*
- (2) *R is indecomposable if and only if $\text{pSpec } R$ is connected.*

Proof. A subset U of $\text{pSpec } R$ is open and closed if and only if it is so $\lambda_R^{\leftarrow}(U)$ and it is well known that a subset U' of $\text{Spec } R$ is open and closed if and only if $U' = D(e)$ for some idempotent $e \in R$. The second assertion is an immediate consequence of the first. \square

If $\{x\}$ is closed in $\text{pSpec } R$ we denote by $A(x)$ the pure ideal of R for which $x = V(A(x))$. A topological space is called *totally disconnected* if each of its connected components contains only one point. Every Hausdorff topological space X with a base of clopen neighbourhoods is totally disconnected and the converse holds if X is compact (see [16, Theorem 16.17]).

Proposition 2.7. *Let R be a ring. Then the following conditions are equivalent:*

- (1) *$\text{pSpec } R$ is totally disconnected;*
- (2) *for each $x \in \text{pSpec } R$, $\{x\}$ is closed and $A(x)$ is generated by idempotents.*

Proof. (ii) \Rightarrow (i). Let $x, y \in \text{pSpec } R$, $x \neq y$. Then $V(A(x)) \cap V(A(y)) = \emptyset$. So, $A(x) + A(y) = R$, whence $\exists a \in A(x)$ such that $(1 - a) \in A(y)$. There exists an idempotent $e \in A(x)$ such that $a = ae$. So, $(1 - e)(1 - a) = (1 - e) \in A(y)$. We easily deduce that $x \subseteq D(1 - e)$ and $y \subseteq D(e)$.

(i) \Rightarrow (ii). Let $x \in \text{pSpec } R$ and $a \in A(x)$. There exists $b \in A(x)$ such that $a = ab$. So $(1 - b)a = 0$. Clearly $x \subseteq D(1 - b)$. Since $\text{pSpec } R$ is Hausdorff and $\text{Spec } R$ is quasi-compact, $\lambda_R^{\rightarrow}(V(1 - b))$ is closed. Therefore $U = \text{pSpec } R \setminus \lambda_R^{\rightarrow}(V(1 - b))$ is open and contains x . The condition $\text{pSpec } R$ is totally disconnected implies that there exists an idempotent e such that $x \subseteq D(e) \subseteq \lambda_R^{\leftarrow}(U) \subseteq D(1 - b)$. It follows

that $e \in R(1 - b)$. So $ea = 0$ and consequently $a = a(1 - e)$. From $x \subseteq D(e)$ and $e(1 - e) = 0$ we deduce that $(1 - e) \in A(x)$ by Lemma 2.5. \square

For any ring R , $B(R)$ is the set of idempotents of R . For any $e, e' \in B(R)$ we put $e \oplus e' = e + e' - ee'$ and $e \odot e' = ee'$. With these operations $B(R)$ is a Boolean ring. The space $\text{Spec } B(R)$ is denoted by $X(R)$. Then $X(R)$ is Hausdorff, compact and totally disconnected. If $x \in X(R)$ the *stalk* of R at x is the quotient of R by the ideal generated by the idempotents contained in x .

Proposition 2.8. *Let R be a ring. The following assertions hold:*

- (1) *there exists a surjective continuous map $\tau_R : \text{pSpec } R \rightarrow X(R)$;*
- (2) *$\text{pSpec } R$ is totally disconnected if and only if τ_R is a homeomorphism. In this case, for each $x \in \text{pSpec } R$, $R/A(x)$ is the stalk of R at $\tau_R(x)$.*

Proof. (i). If L and L' are prime ideals of R , $L \subseteq L'$, then $L \cap B(R) = L' \cap B(R)$ since each prime ideal of $B(R)$ is maximal. So, τ_R is well defined. It is easy to check that for any idempotent $e \in R$, $\tau_R^{\leftarrow}(D(e)) = \lambda_R^{\rightarrow}(D(e))$. Hence τ_R is continuous. For each $x \in X(R)$, if L is a maximal ideal containing all elements of x , then $x = \tau_R(\lambda_R(L))$, whence τ_R is surjective.

(ii). It is obvious that $\text{pSpec } R$ is totally disconnected if τ_R is a homeomorphism. Conversely, since $\text{pSpec } R$ is compact and $X(R)$ is Hausdorff it is enough to show that τ_R is injective. Let $x, x' \in \text{pSpec } R$, $x \neq x'$. There exists an idempotent e such that $x \in \lambda_R^{\rightarrow}(D(e))$ and $x' \in \lambda_R^{\rightarrow}(D(1 - e))$. It follows that $e \notin \tau_R(x)$ and $e \in \tau_R(x')$. Hence $\tau_R(x) \neq \tau_R(x')$. The last assertion is a consequence of Proposition 2.7 and Lemma 2.5. \square

The following lemma will be useful to show some important results of this paper.

Lemma 2.9. *Let R be a ring such that $\text{pSpec } R$ is totally disconnected. Then any R -algebra S (which is not necessarily commutative) satisfies the following condition: let f_1, \dots, f_k be polynomials over S in noncommuting variables $x_1, \dots, x_m, y_1, \dots, y_n$. Let $a_1, \dots, a_m \in S$. Assume that, $\forall x \in \text{pSpec } R$ there exist $b_1, \dots, b_n \in S$ such that:*

$$f_i(a_1, \dots, a_m, b_1, \dots, b_n) \in A(x)S, \forall i, 1 \leq i \leq k.$$

Then there exist $d_1, \dots, d_n \in S$ such that:

$$f_i(a_1, \dots, a_m, d_1, \dots, d_n) = 0, \forall i, 1 \leq i \leq k.$$

Proof. Let $x \in \text{pSpec } R$ and let $b_{x,1}, \dots, b_{x,n} \in S$ such that

$$f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) \in A(x)S, \forall i, 1 \leq i \leq k.$$

Then there exists a finitely generated ideal $A \subseteq A(x)$ such that

$$f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) \in AS, \forall i, 1 \leq i \leq k.$$

There exists an idempotent e_x such that $A \subseteq R(1 - e_x) \subseteq A(x)$. Hence

$$e_x f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) = 0, \forall i, 1 \leq i \leq k.$$

A finite family $(\lambda_R^{\rightarrow}(D(e_{x_j})))_{1 \leq j \leq p}$ covers $\text{pSpec } R$. We may assume that $(e_{x_j})_{1 \leq j \leq p}$ is a family of orthogonal idempotents. We put $d_\ell = e_{x_1} b_{x_1, \ell} + \dots + e_{x_p} b_{x_p, \ell}$, $\forall \ell, 1 \leq \ell \leq n$. Then $f_i(a_1, \dots, a_m, d_1, \dots, d_n) = 0$, $\forall i, 1 \leq i \leq k$. \square

We denote by $\text{gen } M$ the minimal number of generators of a finitely generated R -module M . The following proposition is an example of an algebraic result that can be proven by using Lemma 2.9. Recall that the *trivial extension* $R \ltimes M$ of R by M is defined by: $R \ltimes M = \left\{ \begin{pmatrix} r & x \\ 0 & r \end{pmatrix} \mid r \in R, x \in M \right\}$. It is convenient to identify $R \ltimes M$ with the R -module $R \oplus M$ endowed with the following multiplication: $(r, x)(s, y) = (rs, ry + sx)$, where $r, s \in R$ and $x, y \in M$.

Proposition 2.10. *Let R be a ring such that $\text{pSpec } R$ is totally disconnected. Let M be a finitely generated R -module and F a finitely presented R -module. Then:*

- (1) *if, $\forall x \in \text{pSpec } R$, $M/A(x)M$ is a homomorphic image of $F/A(x)F$, then M is a homomorphic image of F ;*
- (2) *$\text{gen } M = \sup\{\text{gen } (M/A(x)M) \mid x \in \text{pSpec } R\}$;*
- (3) *if M is finitely presented and, $\forall x \in \text{pSpec } R$, $M/A(x)M \cong F/A(x)F$, then $M \cong F$.*

Proof. (i). Let $\{m_1, \dots, m_p\}$ be a spanning set of M . Let $\{f_1, \dots, f_n\}$ be a spanning set of F with the following relations: $\forall \ell, 1 \leq \ell \leq n', \sum_{i=1}^{i=n} c_{\ell,i} f_i = 0$. We put $S = R \ltimes M$ the trivial extension of R by M . We consider the following system \mathcal{E}_1 of polynomial equations in variables $X_{j,i}, Y_i, Z_{i,j}, 1 \leq j \leq p, 1 \leq i \leq n$:

$$\sum_{i=1}^{i=n} X_{j,i} Y_i = (0, m_j), \forall j, 1 \leq j \leq p; \quad Y_i = \sum_{j=1}^{j=p} Z_{i,j} (0, m_j), \forall i, 1 \leq i \leq n;$$

$$\sum_{i=1}^{i=n} (c_{\ell,i}, 0) Y_i = 0, \forall \ell, 1 \leq \ell \leq n'.$$

Thus \mathcal{E}_1 has a solution modulo $A(x)S$ for each $x \in \text{pSpec } R$. By Lemma 2.9 \mathcal{E}_1 has a solution $x_{j,i}, y_i, z_{i,j}, 1 \leq j \leq p, 1 \leq i \leq n$ in S . It is easy to check that $y_i = (0, m'_i), \forall i, 1 \leq i \leq n$, and if $x_{j,i} = (r_{j,i}, x'_{j,i}), \forall j, i, 1 \leq j \leq p, 1 \leq i \leq n$, then $m_j = \sum_{i=1}^{i=n} r_{j,i} m'_i, \forall j, 1 \leq j \leq p$. We have also: $\forall \ell, 1 \leq \ell \leq n', \sum_{i=1}^{i=n} c_{\ell,i} m'_i = 0$. Hence we get an epimorphism $\phi: F \rightarrow M$ defined by $\phi(f_i) = m'_i, \forall i, 1 \leq i \leq n$.

(ii) is an easy consequence of (i).

(iii). Let the notations be as in (i). We assume that m_1, \dots, m_p verify the following relations: $\forall k, 1 \leq k \leq p', \sum_{j=1}^{j=p} d_{k,j} m_j = 0$.

Observe that $F \cong M$ if M has a spanning set $\{m'_1, \dots, m'_n\}$ with the relations: $\forall \ell, 1 \leq \ell \leq n', \sum_{i=1}^{i=n} c_{\ell,i} m'_i = 0$. In this case there exist $r_{j,i} \in R$ such that $m_j = \sum_{i=1}^{i=n} r_{j,i} m'_i, \forall j, 1 \leq j \leq p$. Thus

$$\sum_{j=1}^{j=p} d_{k,j} m_j = \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} d_{k,j} r_{j,i} \right) m'_i = 0, \forall k, 1 \leq k \leq p'.$$

It follows that there exist $w_{k,\ell} \in R$ such that:

$$\sum_{j=1}^{j=p} d_{k,j} m_j = \sum_{\ell=1}^{\ell=n'} w_{k,\ell} \left(\sum_{i=1}^{i=n} c_{\ell,i} m'_i \right) = \sum_{i=1}^{i=n} \left(\sum_{\ell=1}^{\ell=n'} w_{k,\ell} c_{\ell,i} \right) m'_i = 0, \forall k, 1 \leq k \leq p'.$$

We deduce that

$$(1) \quad \sum_{j=1}^{j=p} d_{k,j} r_{j,i} = \sum_{\ell=1}^{\ell=n'} w_{k,\ell} c_{\ell,i}, \forall k, 1 \leq k \leq p', \forall i, 1 \leq i \leq n.$$

Conversely, if there exists an epimorphism $\phi : F \rightarrow M$ defined by $\phi(f_i) = m'_i$, then ϕ is bijective if each relation $(rel) \sum_{i=1}^{i=n} a_i m'_i = 0$ is a linear combination of the relations $\sum_{i=1}^{i=n} c_{\ell,i} m'_i = 0$. Since m'_i is a linear combination of m_1, \dots, m_p , from the relation (rel) we get a relation $(rel1)$ which is a linear combination of the relations $\sum_{j=i}^{j=p} d_{k,j} m_j = 0$. By using the equalities 1, we get that (rel) is a linear combination of the relations $\sum_{i=1}^{i=n} c_{\ell,i} m'_i = 0$.

Let \mathcal{E}_2 be the system of polynomial equations in variables $X_{j,i}, W_{k,\ell}, 1 \leq j \leq p, 1 \leq i \leq n, 1 \leq k \leq p', 1 \leq \ell \leq n'$,

$$\sum_{j=1}^{j=p} (d_{k,j}, 0) X_{j,i} = \sum_{\ell=1}^{\ell=n'} W_{k,\ell} (c_{\ell,i}, 0), \quad \forall k, 1 \leq k \leq p', \quad \forall i, 1 \leq i \leq n.$$

We put $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. As in (i) and by using the above observation we show that \mathcal{E} has a solution. We define $\phi : F \rightarrow M$ as in (i), and by using the fact that \mathcal{E}_2 has a solution, we prove that ϕ is injective by using the above observation. \square

3. HERMITE RINGS AND ELEMENTARY DIVISOR RINGS

An R -module is called *uniserial* if the set of its submodules is totally ordered by inclusion. A ring R is a *valuation ring* if it is a uniserial R -module. We say that R is *arithmetical* if R_L is a valuation ring for each maximal ideal L . A ring is a *Bézout ring* if every finitely generated ideal is principal. A ring R is *Hermite* if R satisfies the following property : for every $(a, b) \in R^2$, there exist d, a', b' in R such that $a = da', b = db'$ and $Ra' + Rb' = R$. We say that R is an *elementary divisor ring* if for every matrix A , with entries in R , there exist a diagonal matrix D and invertible matrices P and Q , with entries in R , such that $PAQ = D$. Then we have the following implications:

elementary divisor ring \Rightarrow Hermite ring \Rightarrow Bézout ring \Rightarrow arithmetical ring;
but these implications are not reversible: see [14] or [5].

Theorem 3.1. *Let R be a ring such that $\text{pSpec } R$ is totally disconnected. Assume that $R/A(x)$ is Bézout for each $x \in \text{pSpec } R$. Then R is Hermite.*

Proof. By Proposition 2.10 R is Bézout. It follows that each prime ideal contains a unique minimal prime, whence $R/A(x)$ has a unique minimal prime ideal. By [17, Theorem 2], a ring with a unique minimal ideal is Hermite if and only if it is Bézout. So, $R/A(x)$ is Hermite. Now, let $a, b \in R$. Consider the following polynomial equations: $a = XZ, b = YZ$ and $1 = SX + TY$. For each $x \in \text{pSpec } R$, these equations have a solution modulo $A(x)$. By Lemma 2.9 they have a solution in R . \square

Recall that a ring R is *(semi)hereditary* if each (finitely generated) ideal is projective. If F is a submodule of a module E and x an element of E , then the ideal $\{r \in R \mid rx \in F\}$ is denoted by $(F : x)$.

The following was already proved, see [8, Theorem III.3] and [20, Theorem 2.4].

Corollary 3.2. *Let R be a Bézout ring. Then the following assertions hold:*

- (1) R is Hermite if $\text{Min } R$ is compact;
- (2) R is Hermite if it is semihereditary.

Proof. (i). Since each prime ideal contains a unique minimal prime, $\lambda_R|_{\text{Min } R}$ is bijective. Moreover, by [9, Proposition IV.1] $\text{pSpec } R$ is Hausdorff. It follows that $\lambda_R|_{\text{Min } R}$ is a homeomorphism because $\text{Min } R$ is compact. We can apply the previous theorem because $\text{Min } R$ is always totally disconnected ([18, Corollary 2.4]):

$$\forall a \in R, D(a) \cap \text{Min } R = V((N : a)) \cap \text{Min } R \text{ where } N \text{ is the nilradical of } R.$$

(ii) is an immediate consequence of (i) because $\text{Min } R$ is compact if R is semihereditary by [26, Proposition 10]. \square

The following example shows that $\text{pSpec } R$ is not generally totally disconnected, even if R is arithmetical.

Example 3.3. Consider [31, Example 6.2 (due to Jensen)] defined in the following way: let \mathcal{I} be a family of pairwise disjoint intervals of the real line with rational endpoints, such that between any two intervals of \mathcal{I} there is at least another interval of \mathcal{I} ; let R be the ring of continuous maps $\mathbb{R} \rightarrow \mathbb{R}$ which are rational constant by interval except on finitely many intervals of \mathcal{I} on which it is given by a rational polynomial. It is easy to check that R is a reduced indecomposable ring. It is also Bézout (left as an exercise!). Let $[a, b] \in \mathcal{I}$ et $f \in R$ defined by $f(x) = (x - a)(b - x)$ if $a \leq x \leq b$ and $f(x) = 0$ elsewhere. Then $(0 : f)$ is not finitely generated, whence R is not semihereditary. So, $\text{pSpec } R$ is an infinite set and a compact connected topological space.

Theorem 3.4. Let R be a ring such that $\text{pSpec } R$ is totally disconnected. Assume that $R/A(x)$ is Bézout for each $x \in \text{pSpec } R$. Then R is an elementary divisor ring if and only if so is R/L , for each minimal prime ideal L .

Proof. Only "if" requires a proof. By Theorem 3.1 R is Hermite. Let $x \in \text{pSpec } R$ and $R' = R/A(x)$. Then R' has a unique minimal prime ideal. Let L be the minimal prime ideal of R such that $L/A(x)$ is the minimal prime of R' . Thus $L/A(x)$ is contained in the Jacobson radical $\mathcal{J}(R')$ of R' . So, $R'/\mathcal{J}(R')$ is an elementary divisor ring since it is a homomorphic image of R/L . By [17, Theorem 3] a Hermite ring S is an elementary divisor ring if and only if so is $S/\mathcal{J}(S)$. Hence $R/A(x)$ is an elementary divisor ring. Let $a, b, c \in R$ such that $Ra + Rb + Rc = R$. We consider the polynomial equation in variables X, Y, S, T : $aSX + bTX + cTY = 1$. By [15, Theorem 6], this equation has a solution modulo $A(x)$, $\forall x \in \text{pSpec } R$. So, by Lemma 2.9 there is a solution in R . We conclude by [15, Theorem 6]. \square

With a similar proof as in Corollary 3.2, we get Corollary 3.5. The second condition shows that the two questions 1.1 and 1.2 have the same answer.

Corollary 3.5. The following assertions hold:

- (1) Let R be a Bézout ring with compact minimal prime spectrum. Then R is an elementary divisor ring if and only if so is R/L , for each minimal prime ideal L .
- (2) Let R be a semihereditary ring. Then R is an elementary divisor ring if and only if so is R/L , for each minimal prime ideal L .
- (3) Let R be a hereditary ring. Then R is an elementary divisor ring if and only if R/L is Bézout for each minimal prime ideal L .

The third assertion can be also deduced from [28, Corollary].

4. ALMOST CLEAN RINGS

In [24, Proposition 15], McGovern proved that each element of a ring R is the product of an idempotent with a regular element if and only if R is a *PP-ring*, i.e. each principal ideal is projective, and he showed that each PP-ring is almost clean ([24, Proposition 16]). The aim of this section is to study almost clean rings.

In the sequel, if R is a ring, $\mathfrak{R}(R)$ is its set of regular elements of R and

$$\Phi_R = \{L \in \text{Spec } R \mid L \cap \mathfrak{R}(R) = \emptyset\}.$$

By [2, Corollaire 2 p.92] each zero-divisor is contained in an element of Φ_R . So, the following proposition is obvious.

Proposition 4.1. *Let R be a ring. The following conditions are equivalent:*

- (1) *For each $a \in R$, either a or $(a - 1)$ is regular.*
- (2) *R is almost clean and indecomposable.*
- (3) *$\forall L, L' \in \Phi_R, L + L' \neq R$.*

Corollary 4.2. *Let R be an arithmetical ring, Q its ring of fractions and N its nilradical. Then:*

- (1) *R is almost clean and indecomposable if and only if Q is a valuation ring;*
- (2) *R/A is almost clean and indecomposable for each ideal $A \subseteq N$ if and only if N is prime and uniserial.*

Proof. (i). Assume that R is almost clean and indecomposable. Then if $L, L' \in \Phi_R$ then there exists a maximal ideal P such that $L + L' \subseteq P$. Since R_P is a valuation ring, either $L \subseteq L'$ or $L' \subseteq L$. By [2, Corollaire p.129] Φ_R is homeomorphic to $\text{Spec } Q$. It follows that Q is local. Conversely, Φ_R contains a unique maximal element.

(ii). First, assume that R/A is almost clean and indecomposable for each ideal $A \subseteq N$. Since Q is a valuation ring then N is prime. By way of contradiction suppose $\exists a, b \in N$ such that neither divides the other. We may assume that $Ra \cap Rb = 0$. Let A and B be maximal submodules of Ra and Rb respectively. We may replace R by $R/(A + B)$ and assume that Ra and Rb are simple modules. Let L and P be their respective annihilators. Since R_L is a valuation ring and $R_L a \neq 0$, we have $R_L b = 0$. So, $L \neq P$. It follows that $\exists c \in L$ such that $(1 - c) \in P$. Neither c nor $(1 - c)$ is regular. This contradicts that R is almost clean.

Conversely, suppose that N is prime and uniserial. Then, if A is an ideal contained in N , N/A is also uniserial. So, the ring of fractions of R/A is a valuation ring: see [29, p.218, between the definition of a torch ring and Theorem B]. \square

Following Vámos [29], we say that R is a *torch ring* if the following conditions are satisfied :

- (1) R is an arithmetical ring with at least two maximal ideals;
- (2) R has a unique minimal prime ideal N which is a nonzero uniserial module.

We follow T.S. Shores and R. Wiegand [27], by defining a *canonical form* for an R -module E to be a decomposition $E \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$, where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \neq R$, and by calling a ring R a *CF-ring* if every direct sum of finitely many cyclic modules has a canonical form.

Corollary 4.3. *Each CF-ring is almost clean.*

Proof. By [27, Theorem 3.12] every CF-ring is arithmetical and a finite product of indecomposable CF-rings. If R is indecomposable then R is either a domain, or a local ring, or a torch ring. By Corollary 4.2 R is almost clean. \square

By Proposition 2.8 there is some similarity between [3, Theorem 2.4] and the following theorem.

Theorem 4.4. *Let R be a ring. Consider the following conditions:*

- (1) R is almost clean;
- (2) $\text{pSpec } R$ is totally disconnected and $\forall r \in R, \forall x \in \text{pSpec } R, \exists s_x \in \mathfrak{A}(R)$ such that either $r \equiv s_x$ modulo $A(x)$ or $(r - 1) \equiv s_x$ modulo $A(x)$;
- (3) $\text{pSpec } R$ is totally disconnected and for each $x \in \text{pSpec } R, R/A(x)$ is almost clean.

Then (i) \Leftrightarrow (ii) \Rightarrow (iii) and the three conditions are equivalent if every principal ideal of R is finitely presented.

Proof. (i) \Rightarrow (ii). Let x and y be two distinct points of $\text{pSpec } R$. Let P and P' be two minimal prime ideals of R such that $P \in x$ and $P' \in y$. There is no maximal ideal containing P and P' . So, $P + P' = R$ and there exist $a \in P$ and $a' \in P'$ such that $a + a' = 1$. We have $a = s + e$ where s is regular and e idempotent. Since $s \notin P$ we get that $e \notin P$. It follows that for each $L \in x, (1 - e) \in L$ and $e \notin L$. So, $x \subseteq D(e)$. We have $a' = -s + (1 - e)$. In the same way we get $y \subseteq D(1 - e)$. Therefore x and y belong to disjoint clopen neighbourhoods. Hence $\text{pSpec } R$ is totally disconnected. Now, let $r \in R$ and $z \in \text{pSpec } R$. We have $r = s + e$ where s is regular and e idempotent. If $z \subseteq V(e)$ then $e \in A(z)$ and $r \equiv s$ modulo $A(z)$; and, if $z \subseteq V(1 - e)$ then $(1 - e) \in A(z)$ and $(r - 1) \equiv s$ modulo $A(z)$.

(ii) \Rightarrow (i). Let $a \in R$. Let Q be the ring of fractions of R and let $S = R \times Q$ be the trivial extension of R by Q . We consider the following polynomial equations in S : $E^2 = E, E + X = (a, 0)$ and $XY = (0, 1)$. Let $x \in \text{pSpec } R$. If $a \equiv s_x$ modulo $A(x)$ where s_x is a regular element of R , then $E = (0, 0), X = (s_x, 0), Y = (0, 1/s_x)$ is a solution of these polynomial equations modulo $A(x)S$; if $(a - 1) \equiv s_x$ modulo $A(x)$, we take $E = (1, 0)$. So, by Lemma 2.9, these equations have a solution in S : $E = (e, q), X = (s, u), Y = (t, v)$. From $E^2 = E$ we deduce that $e^2 = e$ and $(2e - 1)q = 0$. So, $q = 0$ since $(2e - 1)$ is a unit. From $E + X = (a, 0)$ we deduce that $u = 0$, and from $XY = (0, 1)$ we deduce that $sv = 1$. Hence s is a regular element of R and $a = e + s$. We conclude that R is almost clean.

(ii) \Rightarrow (iii). Clearly, if $r \in R$ then, $\forall x \in \text{pSpec } R$ either r or $(r - 1)$ is regular modulo $A(x)$. So, $R/A(x)$ is almost clean $\forall x \in \text{pSpec } R$.

(iii) \Rightarrow (ii). We assume that each principal ideal is finitely presented. Let $x \in \text{pSpec } R$. It remains to show that any regular element a modulo $A(x)$ is congruent to a regular element of R modulo $A(x)$. Since $(0 : a)$ is finitely generated and $A(x)$ is generated by idempotents, there exists an idempotent $e \in A(x)$ such that $(0 : a) \subseteq Re$. Now, it is easy to check that $a(1 - e) + e$ is a regular element. \square

The following examples show that the conditions (i) and (iii) are not generally equivalent.

Example 4.5. [3, Examples 2.2 and 2.9.(i)] *are non-almost clean arithmetical rings (the second is reduced) with almost clean stalks. These are defined in the following way: let D be a principal ideal domain and $\forall n \in \mathbb{N}$, let R_n be a quotient of D by*

a non-zero proper ideal I_n ; let R be the set of elements $r = (r_n)_{n \in \mathbb{N}}$ of $\prod_{n \in \mathbb{N}} R_n$ which satisfy $\exists m_r \in \mathbb{N}$ and $\exists d_r \in D$ such that $\forall n \geq m_r$, $r_n = d_r + I_n$. We put $e_m = (\delta_{m,n})_{n \in \mathbb{N}}$, $\forall m \in \mathbb{N}$. It is easy to check that the points of $\text{pSpec } R$ are: $x_\infty = V(\bigoplus_{n \in \mathbb{N}} R_n)$ and $\forall n \in \mathbb{N}$, $x_n = V(1 - e_n)$. Since $x_\infty \subseteq D(1 - e_m)$ and $x_m \subseteq D(e_m)$, $\forall m \in \mathbb{N}$, $\text{pSpec } R$ is totally disconnected. So, (by using Proposition 2.8) these examples satisfy condition (iii) of Theorem 4.4.

The following example shows that the condition "each principal ideal is finitely presented" is not necessary if R is almost clean.

Example 4.6. We consider [31, Example 1.3b]. Let $R = \mathbb{Z} \oplus S$ where $S = (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$. The multiplication is defined by $(m, x)(n, y) = (mn, nx + my + xy)$, where $m, n \in \mathbb{Z}$ and $x, y \in S$. It is known that R is a reduced arithmetical ring which is not semihereditary. Let $p \in \mathbb{Z}$ and $s \in S$. Then $(2p - 1, s) = (2p - 1, 0) + (0, s)$ and $(2p, s) = (2p - 1, 0) + (1, s)$. It is easy to check that $(2p - 1, 0)$ is regular and, $(0, s)$ and $(1, s)$ are idempotents. Hence R is almost clean. But $(0 : (2, 0)) = 0 \times S$ which is not finitely generated.

Proposition 4.7. Let R be a ring and Q its ring of fractions. Assume that each prime ideal of R contains only one minimal prime. Then:

- (1) $\text{pSpec } Q$ and $\text{pSpec } R$ are homeomorphic;
- (2) each idempotent of Q belongs to R ;
- (3) if Q is clean then R is almost clean.

Proof. (i). If $\varphi : R \rightarrow Q$ is the natural map then $\text{Min } R \subseteq \Phi_R = \text{Im } {}^a\varphi$. It follows that each prime ideal of Q contains only one minimal prime and ${}^b\varphi$ is bijective. Moreover, since $\text{pSpec } Q$ and $\text{pSpec } R$ are compact, ${}^b\varphi$ is a homeomorphism.

(ii). Let e an idempotent of Q . Then $\lambda_{\vec{Q}}(D(e))$ is a clopen subset of $\text{pSpec } Q$, whence its image by ${}^b\varphi$ is a clopen subset of $\text{pSpec } R$ and consequently it is of the form $\lambda_{\vec{R}}(D(e'))$ where e' is an idempotent of R . But the inverse image of $D(e') \subseteq \text{Spec } R$ by ${}^a\varphi$ is $D(e') \subseteq \text{Spec } Q$. So, $e = e' \in R$.

(iii). Assume that Q is clean. Let $r \in R$. Then $r = q + e$ where q is a unit of Q and e an idempotent. Since $e \in R$, q is a regular element of R . \square

Corollary 4.8. Let R be an almost clean arithmetical ring and Q its ring of fractions. If Q is coherent then Q is clean. In this case Q is an elementary divisor ring.

Proof. Recall that an arithmetical ring is coherent if and only if each principal ideal is finitely presented because the intersection of any two finitely generated ideals is finitely generated by [27, Corollary 1.11]. By Proposition 4.7 we may assume that $\text{pSpec } Q = \text{pSpec } R$. This space is totally disconnected by Theorem 4.4. Let $x \in \text{pSpec } Q$ and let $A(x)$ be the pure ideal of R such that $x = V(A(x))$. Then $x = V(QA(x))$. If s is a regular element of R then $s + A(x)$ is a regular element of $R/A(x)$. So, $R/A(x)$ and $Q/QA(x)$ have the same ring of fractions which is a valuation ring by Corollary 4.2. Hence $Q/QA(x)$ is almost clean, $\forall x \in \text{pSpec } Q$. By Theorem 4.4 Q is almost clean too. We conclude that Q is clean since each regular element is a unit.

The last assertion is a consequence of [9, Theorem I.1 and Corollary II.2]. \square

We don't know if the assumption " Q is coherent" can be omitted. The following example shows that the conclusion of the previous corollary doesn't hold if R is not arithmetical, even if R has a unique minimal prime ideal.

Example 4.9. Let K be a field, $D = K[x, y]_{(x, y)}$ where x, y are indeterminates, $E = D/Dx \oplus D/Dy$ and R the trivial extension of D by E . Since R contains a unique minimal prime ideal, Q is indecomposable. We put $r = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $s = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$. Clearly r and s are zerodivisors but $r + s$ is regular. It follows that $\frac{r}{r+s}$ and $\frac{s}{r+s}$ are two zerodivisors of Q whose sum is 1. So, Q is not almost clean.

5. FRACTIONALLY IF-RINGS

Let \mathcal{P} be a ring property. We say that a ring R is (finitely) fractionally \mathcal{P} if the classical ring of quotients $Q(R/A)$ of R/A satisfies \mathcal{P} for each (finitely generated) ideal A . In [12] Facchini and Faith studied fractionally self FP-injective rings. They proved that these rings are arithmetical ([12, Theorem 1]) and they gave some examples ([12, Theorem 6]). In the first part of this section we investigate fractionally IF-rings and partially answer a question posed by Facchini and Faith in [12, question Q1 p.301].

Some preliminary results are needed. As in [23] a ring R is said to be *semicoherent* if $\text{Hom}_R(E, F)$ is a submodule of a flat R -module for any pair of injective R -modules E, F . An R -module E is *FP-injective* if $\text{Ext}_R^1(F, E) = 0$ for any finitely presented R -module F , and R is *self FP-injective* if R is FP-injective as R -module. We recall that a module E is FP-injective if and only if it is a pure submodule of every overmodule. If each injective R -module is flat we say that R is an *IF-ring*. By [6, Theorem 2], R is an IF-ring if and only if it is coherent and self FP-injective.

Proposition 5.1. *Let R be a self FP-injective ring. Then R is coherent if and only if it is semicoherent.*

Proof. If R is coherent then $\text{Hom}_R(E, F)$ is flat for any pair of injective modules E, F by [13, Theorem XIII.6.4(b)]; so, R is semicoherent. Conversely, let E be the injective hull of R . Since R is a pure submodule of E , then, for each injective R -module F , the following sequence is exact:

$$0 \rightarrow \text{Hom}_R(F \otimes_R E/R, F) \rightarrow \text{Hom}_R(F \otimes_R E, F) \rightarrow \text{Hom}_R(F \otimes_R R, F) \rightarrow 0.$$

By using the natural isomorphisms $\text{Hom}_R(F \otimes_R B, F) \cong \text{Hom}_R(F, \text{Hom}_R(B, F))$ and $F \cong \text{Hom}_R(R, F)$ we get the following exact sequence:

$$0 \rightarrow \text{Hom}_R(F, \text{Hom}_R(E/R, F)) \rightarrow \text{Hom}_R(F, \text{Hom}_R(E, F)) \rightarrow \text{Hom}_R(F, F) \rightarrow 0.$$

So, the identity map on F is the image of an element of $\text{Hom}_R(F, \text{Hom}_R(E, F))$. Consequently the following sequence splits:

$$0 \rightarrow \text{Hom}_R(E/R, F) \rightarrow \text{Hom}_R(E, F) \rightarrow F \rightarrow 0.$$

It follows that F is a direct summand of a flat module. So, R is an IF-ring. \square

Corollary 5.2. *Let R be a ring. Assume that its ring of quotients Q is self FP-injective. Then R is semicoherent if and only if Q is coherent.*

Proof. If R is semicoherent, then so is Q by [23, Proposition 1.2]. From Proposition 5.1 we deduce that Q is coherent. Conversely, let E and F be injective R -modules. It is easy to check that the multiplication by a regular element of R in

$\text{Hom}_R(E, F)$ is injective. So, $\text{Hom}_R(E, F)$ is a submodule of the injective hull of $Q \otimes_R \text{Hom}_R(E, F)$ which is flat over Q and R because Q is an IF-ring. \square

Corollary 5.3. *Let R be a valuation ring and A an ideal. Then R/A is semicoherent if and only if A is either prime or the inverse image of a proper principal ideal of $R_{A^\#}$ by the natural map $R \rightarrow R_{A^\#}$, where $A^\# = \{r \in R \mid rA \subset A\}$.*

Proof. Assume that A is not prime and let $A' = AR_{A^\#}$. Then $R_{A^\#}/A'$ is the ring of quotients of R/A . So, by [7, Théorème 2.8], $R_{A^\#}/A'$ is self FP-injective because each non-unit is a zero-divisor. By [8, Corollary II.14] it is coherent if and only if A' is principal. So, we conclude by Corollary 5.2. \square

A valuation ring R is called *strongly discrete* if there is no non-zero idempotent prime ideal.

Corollary 5.4. *Let R be a valuation ring. Then R/A is semicoherent for each ideal A if and only if R is strongly discrete.*

Proof. Assume that R is strongly discrete. Each ideal A is of the form $A = aL$, where L is a prime ideal and $a \in R$. Clearly $L = A^\#$. Then, $L^2 \neq L$ implies that AR_L is principal over R_L . Since A is the inverse image of AR_L by the natural map $R \rightarrow R_L$, R/A is semicoherent by Corollary 5.3.

Conversely, let L be non-zero prime ideal, let $A = aLR_L$, where $0 \neq a \in R_L$ and let A' be the inverse image of A by the natural map $R \rightarrow R_L$. Clearly $L = (A')^\#$. Since R/A' is semicoherent, A is principal over R_L by Corollary 5.3. It follows that L is principal over R_L . So, $L \neq L^2$. \square

Now, we can prove one of the main results of this section.

Theorem 5.5. *Let R be a fractionally IF-ring. Then, R/A is semicoherent for each ideal A and R_P is a strongly discrete valuation ring for each maximal ideal P .*

Proof. By [12, Theorem 1] R is arithmetical because it is fractionally self FP-injective. Let P be a maximal ideal and let A be an ideal of R_P . If B is the kernel of the following composition of natural maps $R \rightarrow R_P \rightarrow R_P/A$, then $Q(R_P/A) = Q(R/B)$ is an IF-ring. We conclude by Corollaries 5.3 and 5.4. \square

It is obvious that each von Neumann regular ring is a fractionally IF-ring. Moreover:

Proposition 5.6. *Let R be an arithmetical ring which is locally strongly discrete. Then R is a fractionally IF-ring in the following cases:*

- (1) R is fractionally semilocal;
- (2) R is semilocal;
- (3) R is a Prüfer domain of finite character, i.e. each non-zero element is contained in but a finite number of maximal ideals.

Proof. (i). We may assume that $R = Q(R)$. By [12, Lemma 7], for each maximal ideal P , $R_P = Q(R_P)$. It follows that R_P is self FP-injective. Moreover it is coherent by Corollary 5.4 and Proposition 5.1. Since $\prod_{P \in \text{Max } R} R_P$ is a faithfully flat R -module and an IF-ring, we deduce that R is IF too.

(ii) follows from (i) by [12, Lemma 5].

(iii) follows from (ii) since R/A is semilocal for each non-zero ideal A . \square

Question 5.7. *What are the locally strongly discrete Prüfer domains which are fractionally IF?*

The following example shows that an arithmetical ring which is locally Artinian is not necessarily fractionally IF.

Example 5.8. *Let K be a field, $V = K[X]/(X^2)$ where X is an indeterminate and let x be the image of X in V . For each $p \in \mathbb{N}$ we put $R_{2p} = V$ and $R_{2p+1} = V/xV \cong K$. Let $S = \prod_{n \in \mathbb{N}} R_n$, $J = \bigoplus_{n \in \mathbb{N}} R_n$ and let R be the unitary V -subalgebra of S generated by J . For each $n \in \mathbb{N}$ we set $\mathbf{e}_n = (\delta_{n,p})_{p \in \mathbb{N}}$ and we denote by $\mathbf{1}$ the identity element of R . Let P be a maximal ideal of R :*

- either $J \subseteq P$; in this case $P = P_\infty = J + xR$ and $R_{P_\infty} = R/J \cong V$;
- or $\exists n \in \mathbb{N}$ such that $\mathbf{e}_n \notin P$; in this case $P = P_n = R(\mathbf{1} - \mathbf{e}_n) + Rxe_n$, $R_{P_n} = R/R(\mathbf{1} - \mathbf{e}_n) \cong V$ if n is even and $R_{P_n} \cong K$ if n is odd.

Then, for each maximal ideal P , R_P is an artinian valuation ring. Now it is easy to check that $(0 : x\mathbf{1}) = Rx\mathbf{1} + \bigoplus_{n \in \mathbb{N}} Re_{2n+1}$. So, R is not coherent.

The following proposition is a short answer to another question posed by Facchini and Faith in [12, question Q3 p.301].

Proposition 5.9. *There exists a non-arithmetical zero-Krull-dimensional ring R which is finitely fractionally self FP-injective.*

Proof. Let V be the artinian valuation ring of Example 5.8, $S = V^{\mathbb{N}}$ and $J = V^{(\mathbb{N})}$. Let $y = (y_n)_{n \in \mathbb{N}}$, $z = (z_n)_{n \in \mathbb{N}} \in S$ such that, $\forall p \in \mathbb{N}$, $y_{2p} = z_{2p+1} = x$ and $y_{2p+1} = z_{2p} = 0$, and let R be the unitary V -subalgebra of S generated by y, z and J . The idempotents $(\mathbf{e}_n)_{n \in \mathbb{N}}$ are defined as in Example 5.8. Let $P \in \text{Max } R$:

- either $J \subseteq P$; in this case $P = P_\infty = J + yR + zR$ and $R_{P_\infty} = R/J \cong K[Y, Z]/(Y, Z)^2$;
- or $\exists n \in \mathbb{N}$ such that $\mathbf{e}_n \notin P$; in this case $P = P_n = R(\mathbf{1} - \mathbf{e}_n) + Rxe_n$, $R_{P_n} = R/R(\mathbf{1} - \mathbf{e}_n) \cong V$.

Clearly, R_{P_∞} is not a valuation ring. So, R is not arithmetical. First, we show that R is a pure submodule of S . It is sufficient to prove that R_P is a pure submodule of S_P for each maximal ideal P . It is obvious that $R_{P_n} \cong S_{P_n} \cong \mathbf{e}_n S \cong V$. It remains to be shown that R_{P_∞} is a pure submodule of $S_{P_\infty} \cong S/J$. We consider the following equations:

$$\forall i, 1 \leq i \leq p, \sum_{1 \leq j \leq m} r_{i,j} x_j \equiv s_i \text{ modulo } J,$$

where $r_{i,j}, s_i \in R$, $\forall i, 1 \leq i \leq p$, $\forall j, 1 \leq j \leq m$. When these equations have a solution in S , we must prove they have a solution in R too. This can be done by using the basis $\{\mathbf{1}, y, z, \mathbf{e}_n, x\mathbf{e}_n \mid n \in \mathbb{N}\}$ of R over K . Consequently R is pure in S . Now, let A be a finitely generated ideal of R . Then R/A is a pure submodule of S/SA . We have $S/SA \cong \prod_{n \in \mathbb{N}} (R/A)_{P_n}$. For each $n \in \mathbb{N}$, $(R/A)_{P_n}$ is self injective, whence it is an injective (R/A) -module. We deduce that S/SA is injective over R/A . Hence R/A is self FP-injective. \square

Finally, for the second question posed by Facchini and Faith in [12, question Q2 p.301], we shall prove Theorem 5.11. The following lemma is needed.

Lemma 5.10. *Let R be a clean ring such that $R = Q(R)$ and $(0 : a)$ is finitely generated for each $a \in R$. Then, for each maximal ideal P , $R_P = Q(R_P)$.*

Proof. Let P be a maximal ideal of R . By way of contradiction, suppose that R_P contains a regular element which is not a unit. So, $\exists a \in P$ such that $(0_P : a) = 0_P$ (since R is clean, $R_P = R/0_P$ by [9, Proposition III.1]). It follows that $(0 : a) \subseteq 0_P$. Since $(0 : a)$ is finitely generated and 0_P is generated by idempotents, there exists an idempotent $e \in 0_P$ such that $(0 : a) \subseteq Re$. Now, it is easy to check that $a(1-e) + e$ is a regular element contained in P . This contradicts that $R = Q(R)$. \square

Theorem 5.11. *The following assertions hold:*

- (1) *let R be an almost clean coherent arithmetical ring. Assume that $R/A(x)$ is either torch or local or a domain $\forall x \in \text{pSpec } R$. Then R is finitely fractionally IF;*
- (2) *each clean coherent arithmetical ring is finitely fractionally IF;*
- (3) *each semihereditary ring is finitely fractionally IF;*
- (4) *let R be a zero-Krull-dimensional ring or a one-Krull-dimensional domain. Then R is finitely fractionally IF if and only if R is coherent and arithmetical.*

Proof. (i). Let A be a finitely generated ideal of an almost clean coherent arithmetical ring R and let N be the nilradical of R . Since each prime ideal contains only one minimal prime, by [9, Lemme IV.2] $D((N : A))$ is the inverse image by λ_R of an open subset U of $\text{pSpec } R$. For each $x \in U$ there exists an idempotent $e_x \in (N : A)$ such that $x \subseteq D(e_x) \subseteq D((N : A))$. Hence $(N : A) = \sum_{x \in U} (Re_x + N)$. Since R is coherent $(0 : A)$ is finitely generated. It follows that there exists an idempotent $e \in (N : A)$ such that $(0 : A) \subseteq (Re + N)$. We have $R/A \cong (R(1-e)/A(1-e)) \times (Re/Ae)$. Since $(0 :_{R(1-e)} A(1-e)) \subseteq N(1-e)$, $(R(1-e)/A(1-e))$ is IF by [8, Proposition II.15]. On the other hand, Re is an almost coherent arithmetical ring and Ae is a finitely generated ideal contained in Ne . Let $T = (Re/Ae)$ and let $\phi : R \rightarrow T$ be the natural epimorphism. Then $\text{pSpec } T$ is totally disconnected because it is homeomorphic to $\text{pSpec } (Re)$. Let $x \in \text{pSpec } T$. Then $T/A(x)$ is the quotient of $R/A({}^b\phi(x))$ modulo an ideal contained in the minimal prime of $R/A({}^b\phi(x))$. By Corollary 4.2 $T/A(x)$ is almost clean. We deduce that T is coherent and almost clean by Theorem 4.4. By Corollary 4.8 $T' = Q(T)$ is clean. By Lemma 5 $T'_P = Q(T'_P)$ for each maximal ideal P of T' . We deduce that T'_P is IF because it is a valuation ring. So, since T' is locally IF, it is IF too. Hence $Q(R/A)$ is IF.

(ii) and (iii). If R is either clean, coherent and arithmetical or semihereditary, then R satisfies the conditions of (i). Hence R is finitely fractionally IF.

(iv). Observe that $Q(R/A) = R/A$ for each non-zero proper ideal A .

First assume that R is arithmetical and coherent. We deduce that R is finitely fractionally IF from (ii) and (iii).

Conversely, let P be a maximal ideal of R and A a finitely generated ideal of R_P . There exists a finitely generated ideal B of R such that $A = B_P$. So, $R_P/A \cong (R/B)_P$. Since R/B is IF, so is R_P/A by [7, Proposition 1.2]. Hence we may assume that R is local and we must prove that R is a valuation ring. If not, there exist $a, b \in R$ such that $a \notin Rb$ and $b \notin Ra$. The coherence of $R/(ab)$ implies that $Ra \cap Rb$ is finitely generated. It follows that $R/(Ra \cap Rb)$ is IF. We

may assume that $Ra \cap Rb = 0$. By [19, Corollary 2.5] $A = (0 : (0 : A))$ for each finitely generated ideal A . We deduce that $0 = Ra \cap Rb = (0 : (0 : a) + (0 : b))$. Then $(0 : a) + (0 : b)$ is a faithful finitely generated proper ideal. By [19, Corollary 2.5] this is not possible. Hence R is a valuation ring. \square

If R is fractionally self FP-injective, then, by [8, Theorem III.1] $\text{Min } R/A$ is compact for each proper ideal A . The following example shows that this is not true if R is finitely fractionally IF, even if R is a coherent clean arithmetical ring.

Example 5.12. *Let D be a valuation domain. Assume that its maximal ideal P' is the only non-zero prime and it is not finitely generated. Let $0 \neq d \in P'$. We put $V = D/dD$, $P = P'/dD$ and $R = V^{\mathbb{N}}$. It is easy to check that R is clean, Bézout and coherent. So, by (ii) of Theorem 5.11 R is finitely fractionally IF. Since P is not finitely generated, $\forall n \in \mathbb{N}$, $\exists b_n \in P$ such that $b_n^n \neq 0$. We set $b = (b_n)_{n \in \mathbb{N}}$. Let N be the nilradical of R . If there exists $c = (c_n)_{n \in \mathbb{N}} \in R$ such that $(b - bcb) \in N$, then $\exists m \in \mathbb{N}$ such that $b^m(1 - cb)^m = 0$. If $n \in \mathbb{N}$, $n \geq m$ we get that $b_n^n(1 - c_n b_n)^n = 0$. Clearly there is a contradiction. So, R/N is not Von Neumann regular. Now, let $c \in R$ such that $(N : c) = N$. We shall prove that c is a unit. By way of contradiction, suppose $\exists k \in \mathbb{N}$ such that $c_k \in P$. We put $e_k = (\delta_{k,n})_{n \in \mathbb{N}}$. Then $ce_k = c_k e_k \in N$. It follows that $e_k \in N$, which is absurd. So, $\forall k \in \mathbb{N}$, $c_k \notin P$. Therefore c is a unit and R/N is equal to its quotient ring. By [11, Theorem 5] a reduced arithmetic ring S is semihereditary if and only if $Q(S)$ is von Neumann regular. By [26, Proposition 10] a reduced arithmetic ring S is semihereditary if and only if $\text{Min } S$ is compact. Consequently $\text{Min } R/N$ is not compact. Hence $\text{Min } R$ is not compact too. (If P' is finitely generated by p and $R = \prod_{n \in \mathbb{N}} D/p^{n+1}D$, then R is clean, arithmetical, coherent and finitely fractionally IF, but $\text{Min } R$ is not compact. We do the same proof by taking $b_n = p + p^{n+1}D$, $\forall n \in \mathbb{N}$.)*

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