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# Fuzzy Region Competition: A Convex Two-Phase Segmentation Framework

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**Abstract.** This paper introduces a new framework for two-phase image segmentation, namely the Fuzzy Region Competition. A generic formulation is developed that extends in a convex way several existing supervised or unsupervised models. The method provides globally minimizing solutions and can be solved with efficient convex optimization tools. We also show by some classical examples that the results are in practice almost insensitive to initial conditions. Motivated by medical applications, in particular angiography, we finally derive a fast algorithm for segmenting images into two non-overlapping smooth regions. Compared to existing methods, this last model has the unique advantage of featuring closed-form solutions for the approximation functions in each region based on normalized convolutions. Results are shown in 2D and 3D.

## 1 Introduction

Several successful variational approaches to region-based image segmentation are based on the minimization of a functional that includes both boundary and region integrals [5, 8, 12, 17]. When a two-phase partition of an image  $I$  is considered, a general form of the optimization problem reads :

$$(\mathcal{P}_0) \min_{\substack{\Sigma \subset \Omega \\ \alpha \in A}} \left\{ F_0(\Sigma, \alpha) = \int_{\Gamma} g(r(s))ds + \int_{\Sigma} r_1^{\alpha_1}(x)dx + \int_{\Omega \setminus \Sigma} r_2^{\alpha_2}(x)dx \right\}$$

where  $\Omega \subset \mathbb{R}^n$  is the image domain,  $\Sigma \subset \Omega$  the foreground region and  $\Gamma = \partial\Sigma$  its boundary. The error functions  $r_i^{\alpha_i} : \Omega \rightarrow \mathbb{R}$  encode the underlying model of each region in terms of intensity properties. They may depend on a set of *region parameters*  $\alpha = (\alpha_1, \alpha_2)$ , such as a couple of scalars [5], vectors [12, 17] or functions [13, 14] (see Table 1).  $g$  is a positive boundary potential, usually chosen to be a decreasing function of the image gradient. If the optimal  $\alpha$  is known *a priori*,  $(\mathcal{P}_0)$  is a *supervised* segmentation problem, constrained by the geodesic length of the boundary [12]. Otherwise, the segmentation is *unsupervised* and  $\alpha$  has to be included in the optimization process. A natural approach is then to perform successive minimization steps alternatively on  $\Sigma$  (the *partition* variable) and on the components of  $\alpha$  (the *region parameters*).

**Table 1.** Possible error functions  $r_i$  and region parameters

<i>Active Contours Without Edges</i> [5]	$r_i = \lambda (I - c_i)^2$	$A = \mathbb{R}^2$
<i>Region Competition</i> [17]	$r_i = -\lambda \log P_i(I \alpha_i)$	$A = (\mathbb{R}^k)^2$
<i>Geodesic Active Regions</i> [12]		
<i>Mumford-Shah</i> [10, 13, 14]	$r_i = \lambda (I - s_i)^2 + \mu  \nabla s_i ^2$	$A = (\mathcal{C}^1)^2$

Since the *Region Competition*<sup>1</sup> algorithm [17], a popular way of solving  $(\mathcal{P}_0)$  is to rewrite  $F_0$  as a functional that only contains boundary integrals, using *Green's* theorem. Gradient-descent evolution schemes can then be built upon optimality conditions for  $\Gamma$  (*Euler-Lagrange* equations). The boundary evolution is most often carried out numerically in the *Level Sets* framework [11] in which  $\Gamma$  is represented implicitly by a real-valued function  $\phi$  such that  $\Gamma = \phi^{-1}(0)$ . This boundary-oriented approach has intrinsic convergence limitations, since neither the involved functional nor the optimization space - the set of curves - are convex. In practice, this produces a dependency on initial conditions and prevents the use of powerful tools from convex optimization theory.

A different perspective on  $(\mathcal{P}_0)$  is to rewrite  $F_0$  as a functional that only contains integrals over the whole domain  $\Omega$ . To that end, *Level Sets* have also proved useful since the *Active Contours Without Edges* and related models [16, 5, 14]. The key idea is to replace in  $F_0$  the set  $\Sigma$  by the Heaviside function  $H(\phi)$  and derive *Euler-Lagrange* equations directly for  $\phi$ , without explicit use of the boundary. This approach inherits some aforementioned drawbacks. It is ill-posed<sup>2</sup> in  $\phi$  and the optimization space - the set of characteristic functions - is still clearly non-convex. This recently motivated very promising works [4, 2] that consider alternative convex methods based on total variation regularization for the global minimization with respect to  $\phi$ .

Along the same line, our present contribution is three-fold. In section 2, we first propose a generic formulation that has a number of interesting properties, including convexity with respect to the *partition* variable. This leads to a new framework, namely the *Fuzzy Region Competition*, that can be applied to any problem under the form  $(\mathcal{P}_0)$ . We illustrate the *supervised* case by using a statistical region term borrowed from the original *Region Competition* [17]. In section 3, we also apply our convex framework to the classical error function of the cartoon *Mumford-Shah* functional [5, 10], assuming constant regions. In section 4, motivated by medical applications such as angiography, we finally propose, develop and illustrate a new localized extension of this model that includes an intrinsic notion of scale, namely the *Smooth Region Competition*. Built on windowing and convolutions, this last segmentation method is very efficient and gives qualitative results that are similar to existing piecewise-smooth models based on *Level Sets*. Examples are shown in 2D and 3D.

<sup>1</sup> The name illustrates the *competition* between two repulsive forces applied at the boundary, depending on the *competition function*  $r = r_1 - r_2$ .

<sup>2</sup> An infinite number of solutions  $\phi$  are valid representations of a given optimal  $\Sigma$ .

## 2 Two-phase Fuzzy Region Competition

The crux of our general formulation is that  $(\mathcal{P}_0)$  can be solved by considering a closely related problem that is at least convex in its *partition* variable. Indeed,  $(\mathcal{P}_0)$  is not convex since the set of sub-domains  $\Sigma \subset \Omega$  is not convex. Nonetheless, it can be expressed as an optimization problem in the set of characteristic functions<sup>3</sup> (still non-convex) so that it also reads

$$\min_{\chi, \alpha} \left\{ \tilde{F}_0(\chi, \alpha) = \int_{\Omega} g(x) |\nabla \chi(x)| dx + \int_{\Omega} \chi(x) r_1^{\alpha_1}(x) dx + \int_{\Omega} (1 - \chi(x)) r_2^{\alpha_2}(x) dx \right\}$$

Under this form, we propose to extend  $(\mathcal{P}_0)$  into a problem that is convex in its *partition* variable, replacing the characteristic function  $\chi$  by a fuzzy membership function  $u$  belonging to a convex set. A suitable choice for this set is the space of functions of bounded variations taking their values in  $[0, 1]$ , noted hereafter  $BV_{[0,1]}(\Omega)$ . This extension leads to the new *Fuzzy Region Competition* problem:

$$(\mathcal{P}) \quad \min_{u \in BV_{[0,1]}, \alpha} \left\{ F(u, \alpha) = \int_{\Omega} g |\nabla u| + \underbrace{\int_{\Omega} u(x) r_1^{\alpha_1}(x) dx}_{P(x \in \Sigma)} + \underbrace{\int_{\Omega} (1 - u(x)) r_2^{\alpha_2}(x) dx}_{P(x \in \Omega \setminus \Sigma)} \right\}$$

A physical interpretation of the variable  $u$  is that for any  $x \in \Omega$ ,  $u(x)$  is the probability that  $x$  belongs to  $\Sigma$  (the foreground). Being convex in  $u$ , problem  $(\mathcal{P})$  has only solutions globally minimizing  $F$  and can be solved (for fixed  $\alpha$ ) with efficient algorithms. Furthermore, the interest of this formulation with respect to the segmentation problem  $(\mathcal{P}_0)$  resides in the following claim :

**Proposition 1.** *Fixing  $\alpha$ , if  $u^*$  is a global minimizer of  $F$  in  $BV_{[0,1]}(\Omega)$  then for almost every  $t \in [0, 1]$ , the characteristic function<sup>4</sup>  $x \mapsto \chi_{u^*}(x, t)$  of the set  $\Sigma_t = \{x \in \Omega, u^*(x) > t\}$  is also a global minimizer of  $F$ . In addition,  $\Sigma_t$  is a global minimizer of  $F_0$ .*

*Proof:* For any functions  $u$  in  $BV_{[0,1]}(\Omega)$  and  $r$  in  $L^1(\Omega)$ , we have:

$$\text{Coarea formula [7]:} \quad \int_{\Omega} g |\nabla u| = \int_0^1 \int_{\Omega} g(x) |\nabla \chi_u(x, t)| dx dt \quad (1)$$

$$\int_{\Omega} u r = \int_{\Omega} \left( \int_0^{u(x)} dt \right) r(x) dx = \int_{\Omega} \left( \int_0^1 \chi_u(x, t) dt \right) r(x) dx = \int_0^1 \int_{\Omega} \chi_u(x, t) r(x) dx dt \quad (2)$$

For sake of simplicity, we shall omit the fixed variable  $\alpha$  in the following. Applying (1) and (2) to the minimizer  $u^*$ , we obtain  $F(u^*) = \int_0^1 F(\chi_{u^*}(\cdot, t)) dt$ , which is equivalent to  $\int_0^1 \{F(\chi_{u^*}(\cdot, t)) - F(u^*)\} dt = 0$ . Since  $\forall t, F(u^*) \leq F(\chi_{u^*}(\cdot, t))$ ,  $F(\chi_{u^*}(\cdot, t)) = F(u^*)$  for a.e.  $t \in [0, 1]$ . This means that the function  $x \mapsto \chi_{u^*}(x, t)$  is also a minimizer of  $F(\cdot, \alpha)$  for almost every  $t \in [0, 1]$ . In addition, if  $\Sigma \subset \Omega$  is such that  $F_0(\Sigma) < F(u^*)$  then its characteristic function<sup>3</sup> also satisfies  $F_0(\Sigma) = F(\chi^{\Sigma}) < F(u^*)$ , which is a contradiction. Since for a.e  $t \in [0, 1]$   $F_0(\Sigma_t) = F(\chi_{u^*}(\cdot, t)) = F(u^*)$ , we finally have  $\forall \Sigma \subset \Omega, F_0(\Sigma_t) \leq F_0(\Sigma)$ .  $\square$

<sup>3</sup> The characteristic function of a set  $\Sigma \subset \Omega$  is the function  $\chi^{\Sigma}(x) = 1$  if  $x \in \Sigma$ , 0 otherwise. The perimeter of its boundary is given by  $Per(\partial \Sigma) = \int_{\Omega} |\nabla \chi|$ .

<sup>4</sup>  $\chi_u$  is the function defined in  $\Omega \times [0, 1]$  by  $\chi_u(x, t) = 1$  if  $u(x) > t$ , 0 otherwise.

In other words, we proved that the set of solutions of  $(\mathcal{P})$  is stable under thresholding and it is clear that every solution of  $\mathcal{P}_0$  corresponds to one of these thresholded functions. We can now develop an efficient optimization strategy capable of solving any two-phase segmentation problem that can be expressed through the general formulation  $(\mathcal{P}_0)$ , as the classical models listed in Table 1. Indeed, being convex in  $u$ , problem  $(\mathcal{P})$  can be solved through powerful tools from convex optimization theory [6]. In the general *unsupervised* setting,  $F$  can be minimized iteratively by alternating the following two steps:

- (A) Considering  $u$  fixed, optimize and update the region parameters  $\alpha$ ,  
 (B) Considering  $\alpha$  fixed, minimize w.r.t the *partition* variable and update  $u$ .  
 When a steady state  $u^*$  is found, simply threshold it.

Step (A) depends on the specific choice of error functions (see sections 3 and 4). Step (B), however, can be realized by applying a generic minimization scheme for the variable  $u$ : Fixing  $\alpha$ , minimizing  $F$  with respect to  $u$  in  $BV_{[0,1]}$  is equivalent to minimizing  $\int_{\Omega} g |\nabla u| + \int_{\Omega} ur$  in  $BV$  under the constraint  $0 \leq u \leq 1$ , where  $r = r_1^{\alpha_1} - r_2^{\alpha_2}$  is the *competition function*. Based on [4], this *constrained* problem has the same set of minimizers than the *unconstrained* problem  $(\tilde{\mathcal{P}})$  of minimizing  $\tilde{F}(u) = \int_{\Omega} g |\nabla u| + \int_{\Omega} ur + \beta \nu(u)$ , with the exact penalty term  $\nu(\xi) = \max(0, |2\xi - 1| - 1)$  and  $\beta > \frac{1}{2} \|r\|_{\infty}$ . Even though  $(\tilde{\mathcal{P}})$  can be numerically solved using a gradient-descent scheme based on the *Euler-Lagrange* equation, no advantage would be taken of the convexity of  $\tilde{F}$ . We thus choose to follow [1] and exploit the fast duality projection algorithm of *Chambolle* [3]. To that end, we add an auxiliary variable  $v$  and consider the following weak approximation:

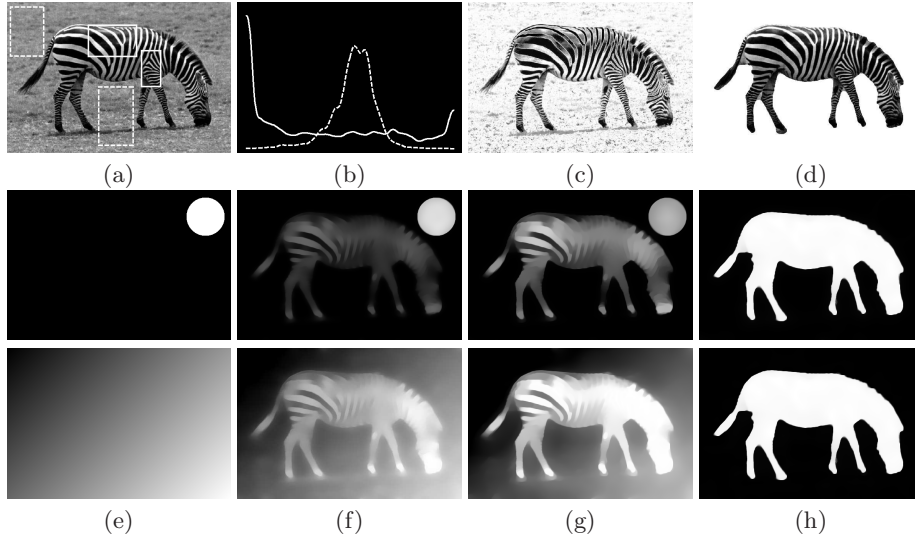
$$(\tilde{\mathcal{P}}_{\theta}) \quad \min_{(u,v) \in BV} \left\{ \tilde{F}_{\theta}(u,v) = \int_{\Omega} g |\nabla u| + \frac{1}{2\theta} \int_{\Omega} |u-v|^2 + \int_{\Omega} rv + \beta \nu(v) \right\},$$

where  $\theta$  is chosen to be small enough so that the two components of any minimizing couple  $(u^*, v^*)$  are almost identical w.r.t. the  $L^2$  norm. Note that this approximation is still componentwise convex in  $u$  and  $v$ . Moreover,  $u$  being fixed, it is easy to check that the optimal  $v$  is directly given by  $v = \max(\min(u - \theta r, 1), 0)$ . Now,  $v$  being fixed, the projection algorithm of [3] can be applied for the minimization of  $\tilde{F}_{\theta}$  with respect to  $u$ . Hence,  $u = v - \theta \operatorname{div}(\mathbf{p})$  where<sup>5</sup> the vector field  $\mathbf{p} \in C_c^1(\Omega, \mathbb{R}^n)$  is the solution of  $\nabla(\theta \operatorname{div}(\mathbf{p}) - v) - |\nabla(\theta \operatorname{div}(\mathbf{p}) - v)| \mathbf{p} = 0$ . This can be solved using a fixed point algorithm, iterating on  $n \geq 0$

$$\text{taking } \tau > 0, \quad \mathbf{p}^0 = \mathbf{0}, \quad \mathbf{p}^{n+1} = \frac{\mathbf{p}^n + \tau \nabla(\operatorname{div}(\mathbf{p}^n) - v/\theta)}{1 + \tau |\nabla(\operatorname{div}(\mathbf{p}^n) - v/\theta)|/g}.$$

where  $\tau$  is chosen to ensure stability (in the isotropic 2D case,  $\tau \leq \frac{1}{8}$  [3]). In practice, this algorithm is numerically very stable. Note that contrary to gradient-descent schemes, it does not rely on the explicit computation of the curvature of  $u$ . We also observed that decoupling  $u$  and  $v$  has a relaxation effect on the regularization and a very positive impact on the overall convergence.

<sup>5</sup>  $C_c^1(\Omega, \mathbb{R}^n)$  is the set of compactly-supported  $C^1$  functions.



**Fig. 1.** *Fuzzy Region Competition*, supervised segmentation of the zebra image: (a) The original image and patches used to learn the background (dashed) and foreground (solid) histograms given in (b). (c) The corresponding competition function  $r = \log(P_2/P_1)$ . (d) Final segmentation. Two different (e) initializations, (f)-(g) intermediary states (step 10-20) and (h) final partition function  $u$  (step 300).

In figure 1, we show an immediate application of the above generic scheme in a *supervised* segmentation experiment. The *competition function*  $r = r_1 - r_2$  has been built *a priori*, following the statistical model given in [12] or [17]. Probability distributions  $P_i(I)$  have been estimated from both background and foreground region patches, such that  $r = \lambda \log(P_2/P_1)$ , where  $\lambda$  is a regularization parameter. In practice, we have observed that this supervised method gives always the same binary partition function, regardless of initial conditions. However, there exist 'pathological' cases of  $r$  for which several minimizers of  $F$ , not necessarily binary, could solve the non strictly convex problem  $(\mathcal{P})$ . Those cases do not contradict proposition 1, neither do they easily show up in practice, but it may be interesting to characterize them in a more in-depth study. In the following sections, we apply the *Fuzzy Region Competition* to some *unsupervised* cases, by first considering the classical two-phase constant model (section 3), then extending it to a new smooth model (section 4).

### 3 Constant Region Competition

In [4], *Chan et al.* propose to solve the minimization involved in the *Active Contours Without Edges* [5] by considering an auxiliary convex problem. They show that if  $c_1$  and  $c_2$  are fixed and  $u^*$  is a solution of

$$\min_{u \in BV_{[0,1]}(\Omega)} \left\{ F_{[4]}(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} u ((I - c_1)^2 - (I - c_2)^2) \right\},$$

then the set  $\Sigma_t = \{x \in \Omega, u^*(x) > t\}$  is for a.e.  $t \in [0, 1]$  a minimizer of the following functional, considered in [5]:

$$F_{[5]}(\Sigma) = \int_{\partial\Sigma} ds + \lambda \int_{\Sigma} (I - c_1)^2 + \lambda \int_{\Omega \setminus \Sigma} (I - c_2)^2$$

This result inspired the general formulation of section 2, and also guides its first application. Solving the *Active Contours Without Edges* model using our framework boils down to choosing  $r_i^{c_i} = \lambda (I - c_i)^2$ ,  $\alpha = (c_1, c_2)$  and  $g = 1$  in  $(\mathcal{P})^6$ , such that the problem becomes

$$(\mathcal{P}_c) \quad \min_{\substack{u \in BV[0,1] \\ (c_1, c_2) \in \mathbb{R}^2}} \left\{ F_c(u, c_1, c_2) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} u(I - c_1)^2 + \lambda \int_{\Omega} (1 - u)(I - c_2)^2 \right\}$$

where  $\lambda$  is a parameter balancing the region error terms and the Total Variation regularization. Applied to  $(\mathcal{P}_c)$ , Proposition 1 gives the result already obtained in [4]. Nonetheless, although closely related by  $F_c(u, c_1, c_2) = F_{[4]}(u, c_1, c_2) + \int_{\Omega} (I - c_2)^2$ , the two involved functionals lead to a slightly different perspective.  $F_{[4]}$  is not to be considered as a minimization on the triplet  $(u, c_1, c_2)$  but as a convex alternative to obtain a minimizer of the original problem [5], when  $c_1$  and  $c_2$  are fixed. Indeed, no optimization of those constants is involved in [4]. In practice, this implies the choice of an arbitrary level set of  $u$  (e.g. 0.5) in step (A) of the alternate minimization scheme. This may at first seem a minor issue since any  $\Sigma_t$  should be a minimizer at convergence. However, full convergence of  $u$  is rarely obtained in practice at each step (B), in particular in the first iterations, and the levels of  $u$  are not equivalent. Hence the arbitrary choice of a level  $t$  for the computation of  $c_1$  and  $c_2$  may introduce a bias in the optimization process. On the contrary, in the proposed approach, the derivatives of  $F_c$  with respect to scalars  $c_1$  and  $c_2$  give directly new optimality conditions that naturally involve all levels of the fuzzy membership function<sup>7</sup>  $u$ :

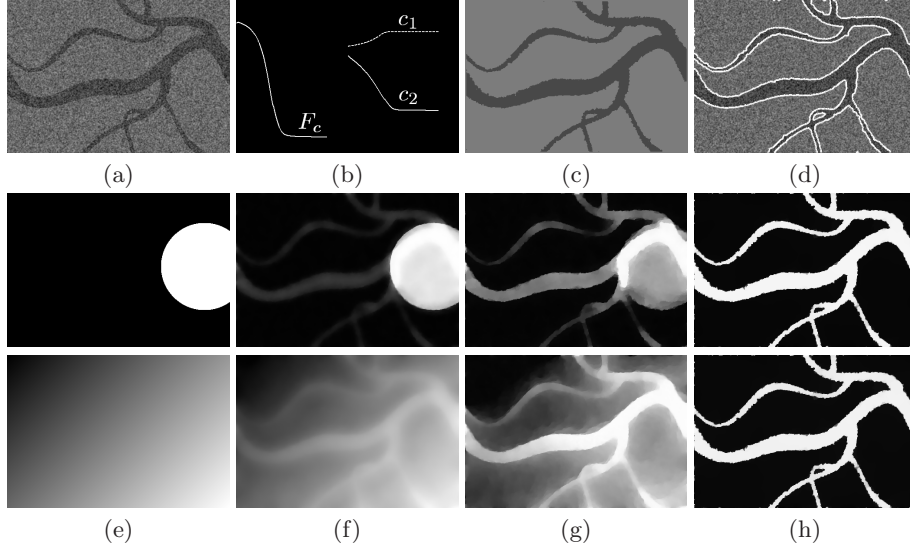
$$c_1^* = \frac{\int_{\Omega} u I}{\int_{\Omega} u}, \quad c_2^* = \frac{\int_{\Omega} (1 - u) I}{\int_{\Omega} (1 - u)} \quad (3)$$

Our *Constant Region Competition* algorithm follows the generic scheme presented in section 2: (A)  $u$  being fixed, compute the weighted averages  $c_1$  and  $c_2$  using equation (3). Then, (B)  $c_1$  and  $c_2$  being fixed, minimize w.r.t.  $u$  using the weak formulation  $(\tilde{\mathcal{P}}_{\theta})$  and perform a few iterations of *Chambolle's* projection algorithm. At convergence, the final segmentation is obtained by thresholding  $u$  at any level in  $[0, 1]$ , as justified by Proposition 1. In figure 2, we give an example of this simple *unsupervised* segmentation algorithm on a synthetic image. As in the *supervised* case, at least above a reasonable value of  $\lambda$ , we observed that the steady state is again always a binary function that is independent of the initialization. This method can thus be a useful tool for segmenting two-phase images corrupted by a gaussian noise of constant variance (the underlying assumption).

<sup>6</sup> For sake of simplicity,  $g$  will be kept constant in the remainder (euclidian length).

<sup>7</sup> If  $u$  is 0 almost everywhere (or 1 almost everywhere),  $c_1^* = c_2^* = \int_{\Omega} I / |\Omega|$ .





**Fig. 2.** *Constant Region Competition* : (a) A noisy synthetic angiography image where only two intensity means are present. (b) Convergence plots for  $F_c$ ,  $c_1$  and  $c_2$ . (c) Final approximation  $u c_1 + (1 - u) c_2$ . (d) Final segmentation. Two different (e) initial conditions, (f)-(g) intermediary steps and (h) final partition functions  $u$ .

Unfortunately, this is hardly realistic for medical images, in particular in angiography, since vessels usually exhibit a slowly varying contrast across the image (figure 3.a). In addition, extending the model in order to handle more than two phases would not necessarily be of any help if the observed variations are smooth. In the next section, we derive a new segmentation model by adding in the region error a localizing kernel, which explicitly provides an intrinsic notion of scale to our two-phase separation problem.

## 4 Smooth Region Competition

Even though two-phase images that could be accurately approximated by a single constant in each region are rarely encountered in practice, there are certainly some applications for which the hypothesis may still hold locally, i.e. at a certain *scale*. To that end, our purpose is to formulate a local extension of the previous model that would account for slowly varying intensities and still keep the advantages of the convex general framework ( $\mathcal{P}$ ). A possible way to achieve this localization is to introduce a symmetrical window function  $\omega : \Omega \mapsto \mathbb{R}^+$ , such that  $\omega(x) \rightarrow 0$  when  $|x| \rightarrow +\infty$ , and use it to design a continuous sum of local *Constant Region Competitions*. In the previous section, the global contribution of the foreground to the total error was  $E = \int_{\Omega} u(I - c_1)^2$ . Now windowing this error with  $\omega$  in a neighborhood of  $y \in \Omega$  gives :

$$e(y) = \int_{x \in \Omega} u(x)(I(x) - s_1(y))^2 \omega(x - y) dx$$



where the previously constant approximation  $c_1$  is now an integrable function  $s_1(y)$  that is allowed to vary in space. Hence, the new contribution of the foreground to the total error is  $\int_{y \in \Omega} e(y) dy$ . Adding a similar background contribution and switching the order of integrations finally exhibits our new region error terms, which leads to the *Smooth Region Competition* problem:

$$(\mathcal{P}_s) \quad \min_{\substack{u \in BV_{[0,1]} \\ (s_1, s_2) \in L^1}} \left\{ \begin{aligned} F_s(u, s_1, s_2) = & \int_{\Omega} |\nabla u| + \lambda \int_{x \in \Omega} u(x) \int_{y \in \Omega} \omega(x-y) (I(x) - s_1(y))^2 dy dx \\ & + \lambda \int_{x \in \Omega} (1-u(x)) \int_{y \in \Omega} \omega(x-y) (I(x) - s_2(y))^2 dy dx \end{aligned} \right.$$

where  $\lambda$  is a parameter balancing the error terms and the Total Variation regularization.  $(\mathcal{P}_s)$  corresponds to the specific choices  $\alpha = (s_1, s_2)$ ,  $A = L^1(\Omega)^2$  and  $r_i^{s_i} : x \rightarrow \lambda \int_{y \in \Omega} \omega(x-y) (I(x) - s_i(y))^2 dy$  in the general problem  $(\mathcal{P})$ , such that the generic scheme given in section 2 and Proposition 1 are still valid. Therefore, we only need to specify how to minimize with respect to  $s_1$  and  $s_2$ .

**Proposition 2.** *Fixing the partition variable  $u$ , the couple of functions  $(s_1^*, s_2^*)$  that minimize  $F_s(u, s_1, s_2)$  satisfy for almost every  $y \in \Omega$  the following equations:*

$$\begin{aligned} s_1^*(y) \int_{\Omega} u(x) \omega(x-y) dx - \int_{\Omega} u(x) I(x) \omega(x-y) dx &= 0 \\ s_2^*(y) \int_{\Omega} (1-u(x)) \omega(x-y) dx - \int_{\Omega} (1-u(x)) I(x) \omega(x-y) dx &= 0 \end{aligned}$$

Consequently, as soon as there is a measurable set  $M_0$  (resp.  $M_1$ ) where  $\forall x \in M_0$  (resp.  $M_1$ ),  $u(x) \neq 0$  (resp.  $u(x) \neq 1$ ) and if  $\omega$  is a positive function, optimal closed-form solutions are given by the following normalized convolutions<sup>8</sup>:

$$s_1^* = \frac{w * (uI)}{w * u} \quad \text{and} \quad s_2^* = \frac{w * ((1-u)I)}{w * (1-u)} \quad (4)$$

*Proof:* With  $f_{\eta}(t) = F_s(u, s_1 + t\eta, s_2)$ , the functional derivative of  $F_s$  with respect to  $s_1$  in the direction  $\eta$  (a compactly supported  $\mathcal{C}^\infty$  function) is given by

$$\begin{aligned} \left. \frac{df_{\eta}}{dt} \right|_{t=0} &= -2\lambda \int_{x \in \Omega} u(x) \int_{y \in \Omega} (I(x) - s_1(y)) \eta(y) \omega(x-y) dy dx \\ &= -2\lambda \int_{y \in \Omega} \eta(y) \int_{x \in \Omega} (I(x) - s_1(y)) u(x) \omega(x-y) dx dy \end{aligned}$$

A necessary condition for  $s_1^*$  to be a minimizer of  $F_s$  is :  $\forall \eta, df_{\eta}/dt|_{t=0} = 0$ . Thus  $s_1^*$  must satisfy for almost every  $y \in \Omega$ :  $\int_{x \in \Omega} (I(x) - s_1^*(y)) u(x) \omega(x-y) dx = 0$ . Similar derivations with respect to  $s_2$  complete the proof.  $\square$

The smooth case can be interpreted by analogy to the constant case of section 3, where the weighted averages  $c_1$  and  $c_2$  have been replaced by the normalized convolutions  $s_1$  and  $s_2$ . They should be considered as smooth approximations

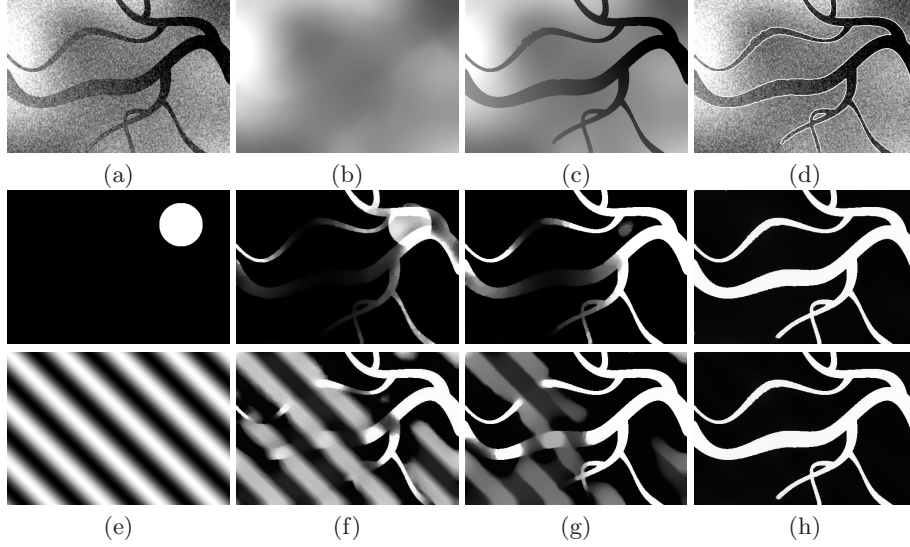
<sup>8</sup> The convolution between  $f$  and  $g$  is function  $f * g : x \rightarrow \int_{\Omega} f(y) g(x-y) dy$ .

of the image within each fuzzy region. Indeed, even though no regularity constraint is explicit in  $F_s$ , the resulting functions are as regular as the window  $\omega$ , provided that the latter is positive. We remind that the theory of normalized convolution, introduced in [9], is a simple and useful extension of convolution that accounts for uncertain or missing image samples. Here, normalized convolutions naturally appear to create smooth approximations for each phase, taking into account the fuzzy membership functions  $u$  and  $(1 - u)$  as certainty measures.

Our *Smooth Region Competition* algorithm follows again the generic scheme presented in section 2: (A)  $u$  being fixed, compute the normalized convolutions  $s_1$  and  $s_2$  using equation (4). Then, (B)  $s_1$  and  $s_2$  being fixed, compute the competition function  $r = r_1^{s_1} - r_2^{s_2}$  (this also involves convolutions<sup>9</sup>) and perform a few iterations of the dual projection algorithm. When a steady state is found, the final segmentation is obtained by thresholding  $u$  at any level in  $[0, 1]$ . The function  $S = us_1 + (1 - u)s_2$  gives then a piecewise-smooth approximation of the original image (see figure 3.c). Indeed, a binary  $u$  prevents the smoothing of  $s_1$  and  $s_2$  across the boundary. In this respect, our approach relates to segmentation methods assuming a piecewise-smooth underlying model, such as those based on the *Mumford-Shah* functional [10]. However, it also features an essential practical advantage. Indeed, for instance in [13, 14, 2], the approximation functions  $s_i$  are defined as solutions of second-order differential equations, only defined inside each phase<sup>10</sup>. Although theoretically sound, this has two drawbacks in practice. First, diffusion equations with conditions on the boundary of an arbitrary-shaped domain (the current contour/surface) must be solved at each step (A) of the alternate minimization scheme. This is tedious since the domain is moving between each iteration. Second, both approximation functions are only defined in their respective domain, but need to be extended in order to compute the competition function, at least in a local vicinity. The construction of the extension is arbitrary and needs the exact location of the boundary separating the phases at all times. For 3D volumes in particular, those steps involve quite heavy computations. In contrast, step (A) of the *Smooth Region Competition* algorithm is much lighter since smooth closed-form solutions for  $s_1$  and  $s_2$ , defined everywhere in  $\Omega$ , are directly constructed from  $u$ .

A very suitable choice for the window function is the normalized isotropic Gaussian kernel  $\omega(x) = (2\pi\sigma^2)^{-n/2} \exp(-|x|^2/2\sigma^2)$ . The standard deviation  $\sigma$  explicitly provides the model with an intrinsic notion of scale, related to the intensity variations that are expected to arise in each region. Note that when  $\sigma \rightarrow \infty$ , we recover the constant model. Furthermore, being positive, non-compactly supported and  $\mathcal{C}^\infty$ , the Gaussian window guarantees the regularity of  $s_i$  functions everywhere in the domain  $\Omega$ . As for the implementation, very efficient recursive filters that approximate the convolution and simulate the non-compact support with a computation cost that is independent on  $\sigma$  are also available [15]. In fig-

<sup>9</sup> The competition function here reads  $r(x) = [\omega * (s_1^2 - s_2^2)](x) - 2I(x)[\omega * (s_1 - s_2)](x)$ .  
<sup>10</sup>  $\int_{\Sigma} |\nabla s_1|^2 + \int_{\Omega \setminus \Sigma} |\nabla s_2|^2$  is typically added in the functional.



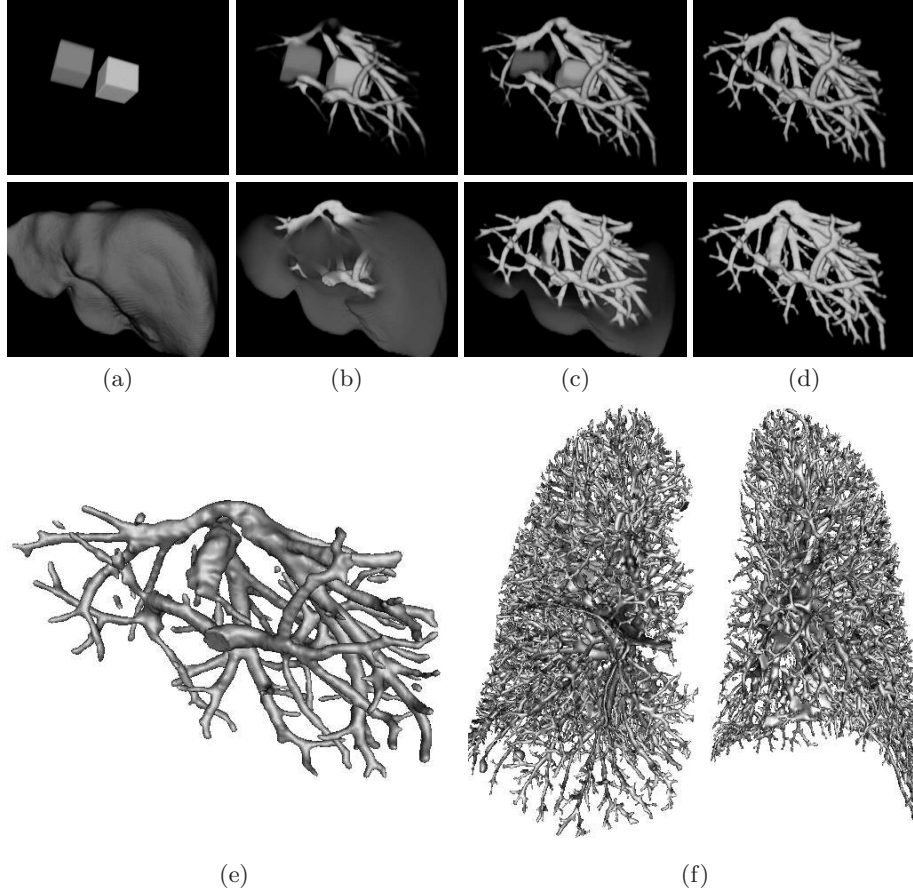
**Fig. 3.** *Smooth Region Competition*: (a) A noisy synthetic image of a vessel of smoothly-varying intensity on a smooth background. (b) Final  $s_2$ . (c) Final smooth approximation  $us_1 + (1 - u)s_2$ . (d) Segmentation. Two different (e) initial conditions, (f)-(g) intermediary steps and (h) final partition functions  $u$ . The circle radius in (e) is  $2\sigma$ .

ures 3 and 4, we show the application of this algorithm on a 2D synthetic image and on real 3D volumes.

## 5 Conclusion

We introduced a new framework for two-phase image segmentation, namely the *Fuzzy Region Competition*. It generalizes some existing *supervised* and *unsupervised* region-based models, if only two phases are considered, and can be used in all applications where a distinction foreground / background is meaningful, in 2D or 3D. The functional formulation we proposed is convex in its bounded variation *partition* variable  $u$ . The convexity guarantees to obtain in the *supervised* case a globally optimal solution. Although this property does not hold in the *unsupervised* setting, for which the *region parameters* have also to be optimized, it still produces very stable algorithms that turn out to be weakly sensitive to initial conditions. Furthermore, those algorithms can be very fast, in particular by using convex optimization tools and recent developments in total variation regularization. We believe that the use of a fuzzy membership function in region-based segmentation models leads to easy implementations that are more efficient and stable than their existing curve evolution *Level Sets* counterparts. This motivates the need for further work, in particular by including some a priori shape knowledge within the model in order to increase robustness.

Based on this framework, a new consistent model for the partition of an image into two smooth components was also derived. It is extensively built around



**Fig. 4.** 3D Applications : We show here typical results of the *Smooth Region Competition* applied to 3D vessel segmentation. In both experiments, a segmentation mask of the involved organs (liver, lungs) was available, such that the two-phase hypothesis (vessels + tissue) was approximately valid. In (a),(b),(c),(d), we show in volume rendering two evolutions of the partition function segmenting a liver vessel structure from a  $294 \times 215 \times 101$  CT angiography. Computation time is about 20 sec. on a standard PC platform (depending on the initial conditions). (e) shows the resulting mesh in surface rendering after the final thresholding. (f) is the result of the same algorithm applied to a pulmonary vessel structure, illustrating the topological complexity that the model can handle.

windowing principles, convolutions, and features closed-form approximations in each region. Thus, it can be faster than some of the existing variational methods that give qualitatively similar results. It also provides an intrinsic notion of scale, depending on the intensity variations expected to arise in the objects to be segmented. However, from a segmentation perspective, scale also intuitively relates to other geometrical quantities such as the smoothness of the boundary or the allowed size of foreground objects. Both are presently dependent on the

chosen balance between the error terms and the TV regularization. This would probably need also to be better understood and studied in the future.

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