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Mixture of the Riesz distribution with respect to the multivariate Poisson

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Running title: *Mixture of the Riesz distribution*

Abstract The aim of this paper is to study the mixture of the Riesz distribution on symmetric matrices with respect to the multivariate Poisson distribution. We show, in particular, that this distribution is related to the modified Bessel function of the first kind. We also study the generated natural exponential family. We determine the domain of the means and the variance function of this family.

Keywords: Mixed distribution, Riesz distribution, Bessel function, natural exponential family, variance function.

1 Introduction

Let μ_λ be a probability distribution on a finite dimensional linear space E depending on a parameter λ which belongs to a subset Λ of \mathbb{R}^r . Suppose that

$$\mu_\lambda = f(x, \lambda)\sigma(dx),$$

where σ is some reference measure, and that for each x in E , the map $\lambda \mapsto f(x, \lambda)$ defined on Λ is measurable. For a probability distribution $\nu(d\lambda)$ on the set Λ , define

$$h(x) = \int_{\Lambda} f(x, \lambda)\nu(d\lambda).$$

Then the probability measure

$$\mu_\nu(dx) = h(x)\sigma(dx)$$

is called the mixture of the distribution μ_λ with respect to ν . (See Feller [4], Vol. II, page 53 or Johnson *et al.* [8], page 360). Usually, μ_λ is said the mixed distribution and ν the mixing distribution (see Karlis and Meligkotsidou [9]).

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A special case of interest is when μ is not concentrated on an affine hyperplane of E , μ_λ is the λ -power of convolution of μ , and Λ is the so called Yørgensen set of μ . Specifically, let

$$L_\mu(\theta) = \int_E \exp(\langle \theta, x \rangle) \mu(dx) \quad (1.1)$$

denote the Laplace transform of μ , where $\langle \cdot, \cdot \rangle$ is the duality crocket, and suppose that the set

$$\Theta(\mu) = \text{interior}\{\theta \in E^*; L_\mu(\theta) < +\infty\} \quad (1.2)$$

is nonempty. Then the set

$$\Lambda = \{\lambda > 0; \exists \mu_\lambda \text{ such that } L_{\mu_\lambda}(\theta) = (L_\mu(\theta))^\lambda \text{ for all } \theta \in \Theta(\mu)\} \quad (1.3)$$

is called the Yørgensen set of μ and the measure μ_λ is its λ -power of convolution. This parameter λ appears in the most common models, it is in particular, the intensity in a Poisson model, the variance in a Gaussian model, and the shape parameter in a gamma model. For λ and λ' in Λ , we have that

$$\mu_\lambda * \mu_{\lambda'} = \mu_{\lambda+\lambda'}.$$

The set Λ contains always the set \mathbb{N}^* of positive integers and it is equal to $]0, +\infty[$ if and only if μ is infinitely divisible (see Seshadri [13], page 155).

In fact, if X_1, \dots, X_N are iid random variables with distribution μ , then the distribution of $X_1 + \dots + X_N$ is $\mu_N = \mu^{*N}$, the N -power of convolution of μ . Accordingly, for any distribution μ and any positive integer N , one may consider the distribution μ_N as defined in (1.3). When μ is discrete, i.e., with countable support, then the mixture of μ with respect of a distribution ν on the parameter N is known as a compound distribution. The most famous compound distribution is the one corresponding to the case where ν is Poisson (see Feller [4], Vol. I, page 286 or Vol. II, page 451 or Aalen [1], or Yørgensen [14], page 140). In fact, the real Poisson distribution appears in numerous works either as a mixed distribution (see Bhattacharya and Holla [2] or Johnson *et al.* [8], page 366) or as a mixing distribution (see Perline [12]). In the present work, we will be interested in a very special case in which the mixed distribution is defined on the cone of (r, r) positive definite symmetric matrices Ω with a parameter which belongs to a subset of \mathbb{R}^r . More precisely, the mixed distribution will be the absolutely continuous Riesz model introduced in Hassairi and Lajmi [6]

$$\left\{ R(s, \sigma), s \in \prod_{i=1}^r \left] \frac{i-1}{2}, +\infty \right[\right\}.$$

This model contains the Wishart model, since $R(s, \sigma)$ reduces to a Wishart distribution when $s_1 = s_2 = \dots = s_r$, and it has a convolution property which is analogous to the one satisfied by the ordinary powers of convolution. In fact, if s and s' are in $\prod_{i=1}^r \left] \frac{i-1}{2}, +\infty \right[$, then

$$R(s, \sigma) * R(s', \sigma) = R(s + s', \sigma).$$

The mixing distribution will be the multivariate Poisson distribution on \mathbb{N}^r . For simplicity, we will be interested in the case where σ is equal to the identity matrix of size r denoted

I_r . We first show that the mixture distribution is expressed in terms of the modified Bessel function. We then determine the domain of the means of the generated natural exponential family, and we calculate its variance function. This provides a rich class of natural exponential families.

2 The Riesz exponential dispersion model

A dispersion model is a class of natural exponential families where each family is a power of convolution of the other. In this section, we will first recall some general facts concerning the exponential dispersion models in an Euclidean space. Then we introduce the Riesz dispersion model on symmetric matrices.

2.1 Exponential dispersion model

Let E be an Euclidean space with finite dimension n , and let $\langle \cdot, \cdot \rangle$ denote the scalar product in E . If μ is a positive measure on E , we denote by $\mathcal{M}(E)$ the set of measures μ such that $\Theta(\mu)$ given in (1.2) is not empty and μ is not concentrated on an affine hyperplane of E . If μ is in $\mathcal{M}(E)$, we denote

$$k_\mu(\theta) = \log L_\mu(\theta), \text{ for all } \theta \text{ in } \Theta(\mu)$$

the cumulant function of μ , where L_μ is the Laplace transform of μ defined in (1.1).

To each μ in $\mathcal{M}(E)$ and θ in $\Theta(\mu)$, we associate the probability distribution on E

$$P(\theta, \mu)(dx) = \exp(\langle \theta, x \rangle - k_\mu(\theta)) \mu(dx).$$

The set

$$F = F(\mu) = \{P(\theta, \mu); \theta \in \Theta(\mu)\}$$

is called the natural exponential family (NEF) generated by μ . We also say that μ is a basis of F .

The function k_μ is strictly convex and real analytic. Its first derivative k'_μ defines a diffeomorphism between $\Theta(\mu)$ and its image M_F . Since $k'_\mu(\theta) = \int_E x P(\theta, \mu)(dx)$, M_F is called the domain of the means of F . The inverse function of k'_μ is denoted by ψ_μ and setting $P(m, F) = P(\psi_\mu(m), \mu)$ the probability of F with mean m , we have

$$F = \{P(m, F); m \in M_F\},$$

which is the parametrization of F by the mean.

Now the covariance operator of $P(m, F)$ is denoted by $V_F(m)$ and the map

$$M_F \longrightarrow L_s(E); m \longmapsto V_F(m) = k''_\mu(\psi_\mu(m))$$

is called the variance function of the NEF F . It is easy proved that for all $m \in M_F$,

$$V_F(m) = (\psi'_\mu(m))^{-1},$$

and an important feature of V_F is that it characterizes F in the following sense: If F and F' are two NEFs such that $V_F(m)$ and $V_{F'}(m)$ coincide on a nonempty open set of

$M_F \cap M_{F'}$, then $F = F'$. In particular, knowledge of the variance function gives knowledge of the natural exponential family.

Let μ be an element of $\mathcal{M}(E)$ and let Λ be its Yørgensen set defined by (1.3). Then the set

$$\{P(\theta, \lambda) = \exp(\langle \theta, x \rangle - k_{\mu_\lambda}(\theta)) \mu; \theta \in \theta(\mu), \lambda \in \Lambda\}$$

is called the dispersion model generated by μ . For more details, we refer to Letac [11].

2.2 Riesz natural exponential families

Let E be the Euclidean space of (r, r) real symmetric matrices equipped with the scalar product $\langle x, y \rangle = \text{tr}(xy)$, and the inner product $x.y = \frac{1}{2}(xy + yx)$, where xy is the ordinary product of two matrices.

We denote by e_1, e_2, \dots, e_r the canonical basis of \mathbb{R}^r ; $e_i = (0, \dots, 0, 1, 0 \dots 0)$, (1 in the i^{th} place), and we set $c_i = \text{diag}(e_i)$ for all $1 \leq i \leq r$.

For $x \in E$, we consider the endomorphism $L(x)$ of E defined by

$$L(x) : y \longmapsto x.y$$

and we set

$$P(x) = 2(L(x))^2 - L(x^2).$$

We denote by Ω the cone of (r, r) real symmetric positive definite matrices.

For $x = (x_{ij})_{1 \leq i, j \leq r}$ in E and $1 \leq k \leq r$, we define the sub-matrices

$$P_k(x) = (x_{ij})_{1 \leq i, j \leq k} \text{ and } P_k^*(x) = (x_{ij})_{r-k+1 \leq i, j \leq r}.$$

For convenience, $P_k(x)$ and $P_k^*(x)$ are also considered as elements of the space E , where the other entries are equal to zero and we set $P_0^*(x) = 0$.

Let $\Delta_k(x)$ and $\Delta_k^*(x)$ denote the determinant of $P_k(x)$ and the determinant of $P_k^*(x)$ respectively. Then the generalized power of x in Ω is defined, for $s = (s_1, s_2, \dots, s_r) \in \mathbb{R}^r$, by

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \dots \Delta_r(x)^{s_r}. \quad (2.4)$$

Note that if for all $i \in \{1, \dots, r\}$, $s_i = p \in \mathbb{R}$, then $\Delta_s(x) = (\det x)^p$. We also define

$$\Delta_s^*(x) = (\Delta_1^*(x))^{s_1 - s_2} (\Delta_2^*(x))^{s_2 - s_3} \dots (\Delta_{r-1}^*(x))^{s_{r-1} - s_r} (\Delta_r^*(x))^{s_r}. \quad (2.5)$$

It is shown (see Hassairi and Lajmi [6]) that for all $x \in \Omega$ and all $s \in \mathbb{R}^r$, we have

$$\Delta_s(x^{-1}) = \Delta_{-s^*}^*(x), \quad (2.6)$$

where $s^* = (s_r, s_{r-1}, \dots, s_1)$.

We denote by \mathcal{T}_l^+ the set of lower triangular matrices with positive diagonal elements, and for $u \in \mathcal{T}_l^+$, we define on E the automorphism

$$u(y) = uyu^*,$$

where u^* denotes the transpose matrix of u .

It is well known that for all $x \in \Omega$, there exists a unique $u \in \mathcal{T}_l^+$ such that

$$x = u(I_r),$$

where I_r is the identity matrix of order r , it is the Cholesky decomposition of x .

We also have (see Hassairi and Lajmi [6]) that for all $1 \leq i \leq r$,

$$\left(P_i^* \left((u(I_r))^{-1} \right) \right)^{-1} = u \left(\sum_{k=r-i+1}^r c_k \right), \quad (2.7)$$

and for all $s = (s_1, s_2, \dots, s_r) \in \mathbb{R}^r$,

$$\Delta_s(u(I_r)) = \Delta_{-s^*}^*(u^{*-1}(I_r)). \quad (2.8)$$

Recall also that for $x \in \Omega$ and $u \in \mathcal{T}_l^+$, we have

$$\Delta_i(u(x)) = \Delta_i(u(I_r))\Delta_i(x) = u_1^2 \dots u_i^2 \Delta_i(x), \quad (2.9)$$

where for all $i \in \{1, \dots, r\}$,

$$u_i = \langle u, c_i \rangle. \quad (2.10)$$

(See Faraut and Koranyi [3], page 114).

Now let Ξ be the set of elements $s = (s_1, s_2, \dots, s_r) \in \mathbb{R}^r$ defined as follows:

Consider the function ξ defined from \mathbb{R}^+ into $\{0; 1\}$, by

$$\xi : a \longmapsto \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0. \end{cases}$$

For all $(u_1, u_2, \dots, u_r) \in \mathbb{R}_+^r$, we define

$$\begin{cases} s_1 = u_1 \\ s_k = u_k + \frac{\xi(u_1) + \dots + \xi(u_{k-1})}{2}, \quad \forall k \in \{2, \dots, r\}. \end{cases}$$

A result due to Gindikin [5] and proved in Faraut and Korányi [3], page 124, says that there exists a positive measure R_s such that for all $\theta \in -\Omega$,

$$L_{R_s}(\theta) = \int_E e^{\langle \theta, x \rangle} R_s(dx) = \Delta_s(-\theta^{-1})$$

if and only if s is in Ξ . This measure is called the Riesz measure with parameter s .

When $s = (s_1, s_2, \dots, s_r) \in \Xi \setminus \prod_{i=1}^r \left] \frac{i-1}{2}, +\infty \right[$, the measure R_s is concentrated on the

boundary $\partial\Omega$ of Ω and when $s = (s_1, s_2, \dots, s_r)$ is such that for all i , $s_i > \frac{i-1}{2}$, the measure R_s is absolutely continuous with respect to the Lebesgue measure and is given by

$$R_s(dx) = \frac{\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)} \mathbf{1}_\Omega(x)(dx)$$

with $n = \frac{r(r+1)}{2}$ the dimension of E and

$$\Gamma_{\Omega}(s) = (2\pi)^{\frac{r(r-1)}{4}} \prod_{i=1}^r \Gamma\left(s_i - \frac{i-1}{2}\right). \quad (2.11)$$

It is shown in Hassairi and Lajmi [6] that the measure R_s generates a natural exponential family if and only if $s_1 \neq 0$. In this case

$$F = F(R_s) = \left\{ R(s, \sigma) = \frac{e^{-\langle \sigma, x \rangle}}{\Delta_s(\sigma^{-1})} R_s, \sigma \in \Omega, s \in \Xi, s_1 \neq 0 \right\}.$$

The distribution $R(s, \sigma)$ is called the Riesz distribution with parameters s and σ . It is shown in Hassairi and Lajmi [6] that the Laplace transform of $R(s, \sigma)$ is defined for θ in $\sigma - \Omega$, by

$$L_{R(s, \sigma)}(\theta) = \frac{\Delta_s((\sigma - \theta)^{-1})}{\Delta_s(\sigma^{-1})}. \quad (2.12)$$

This implies that if σ is an element of Ω and if s and s' are in Ξ , then we have

$$R(s, \sigma) * R(s', \sigma) = R(s + s', \sigma).$$

Let σ be an element of Ω . If s satisfies the conditions $s_i > \frac{i-1}{2}$, for $1 \leq i \leq r$, then the Riesz distribution is given by

$$R(s, \sigma)(dx) = \frac{e^{-\langle \sigma, x \rangle} \Delta_{s - \frac{n}{r}}(x)}{\Gamma_{\Omega}(s) \Delta_s(\sigma^{-1})} \mathbf{1}_{\Omega}(x)(dx).$$

When $s_1 = s_2 = \dots = s_r = p > 0$, $R(s, \sigma)$ reduces to the Wishart distribution with parameters $p > \frac{r-1}{2}$ and $\sigma \in \Omega$,

$$W(p, \sigma)(dx) = \frac{1}{\Gamma_{\Omega}(p) \det(\sigma^{-p})} e^{-\langle \sigma, x \rangle} \det(x)^{p - \frac{n}{r}} \mathbf{1}_{\Omega}(x)(dx),$$

with Laplace transform equal for all $\theta \in \sigma - \Omega$, to

$$L_{W(p, \sigma)}(\theta) = \det(I_r - \sigma^{-1}\theta)^{-p}.$$

3 The mixture of the Riesz distribution with respect to the multivariate Poisson

Consider the Poisson distribution on \mathbb{N}^r with parameter $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in (\mathbb{R}^+)^r$

$$\nu(dx) = e^{-\sum_{i=1}^r \lambda_i} \sum_{q \in \mathbb{N}^r} \frac{\lambda^q}{q!} \delta_q(dx), \quad (3.13)$$

where $q! = q_1! q_2! \dots q_r!$ and $\lambda^q = \lambda_1^{q_1} \lambda_2^{q_2} \dots \lambda_r^{q_r}$. Then for all $\theta \in \mathbb{R}^r$,

$$L_{\nu}(\theta) = \prod_{i=1}^r e^{\lambda_i(e^{\theta_i} - 1)}. \quad (3.14)$$

The variance function of the natural exponential family generated by ν evaluated for $m \in]0, +\infty[$ is equal to

$$V(m) = \text{diag}(m).$$

Note that this variance function is homogeneous of order 1 and consequently the natural exponential family $F(\nu)$ belongs to the Tweedie scale of \mathbb{R}^r (see Hassairi and Louati [7]).

Next, we will show that the mixture of the distribution $R(s, I_r)$ with respect to ν is related to the modified Bessel function of the first kind and of order 1.

Consider the second-order linear differential equation

$$z^2 y''(z) + zy'(z) + (z^2 - \alpha^2)y(z) = 0, \quad (3.15)$$

where z is a complex variable and α is a real parameter. One of the solution of (3.15) is the function $J(\alpha, z)$ known as the Bessel function of the first kind of order α , and defined for z in \mathbb{C} by the series

$$J(\alpha, z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\alpha}}{k! \Gamma(k + \alpha + 1)}. \quad (3.16)$$

In many applications, one frequently encounters two functions $I(\alpha, z)$ and $K(\alpha, z)$ called respectively the modified Bessel functions of the first and second kinds of order α , defined on the complex plane cut along the negative real axis by

$$I(\alpha, z) = i^{-\alpha} J(\alpha, iz), \quad (3.17)$$

and

$$K(\alpha, z) = \frac{\Pi I(-\alpha, x) - I(\alpha, x)}{2 \sin(\alpha \Pi)}, \quad \alpha \neq 0, \pm 1, \pm 2, \dots$$

It is an immediate fact that $I(\alpha, z)$ and $K(\alpha, z)$ are linearly independent solutions of the differential equation:

$$z^2 y''(z) + zy'(z) - (z^2 + \alpha^2)y(z) = 0. \quad (3.18)$$

For more details about the Bessel functions, we refer to Lebedev [10], page 108 and to Feller [4], Vol. II, page 58.

We also need to establish the following result related to these functions.

Proposition 3.1 *Let $x > 0$ and let $b > 0$, then*

$$\sum_{k \in \mathbb{N}} \frac{x^k}{k! \Gamma(k + b)} = x^{\frac{1-b}{2}} I(b - 1, 2\sqrt{x}).$$

Proof

Consider the function

$$g_b(x) = \sum_{k \in \mathbb{N}} \frac{x^k}{k! \Gamma(k + b)}. \quad (3.19)$$

Deriving g_b gives

$$g'_b(x) = \sum_{k \in \mathbb{N}} \frac{x^k}{k! \Gamma(k + 1 + b)} = \sum_{k \in \mathbb{N}} \frac{x^k}{k!(k + b) \Gamma(k + b)}.$$

This implies that

$$\left(x^b g'_b(x)\right)' = \sum_{k \in \mathbb{N}} \frac{x^{k+b-1}}{k! \Gamma(k+b)} = x^{b-1} g_b(x).$$

Consequently, $g_b(x)$ is a solution of the differential equation

$$x^b y''(x) + bx^{b-1} y'(x) - x^{b-1} y(x) = 0.$$

Or equivalently, $g_b(x)$ is solution of the equation

$$xy''(x) + by'(x) = y(x). \quad (3.20)$$

Using now the fact that $I(\alpha, z)$ and $K(\alpha, z)$ are two linearly independent solutions of (3.18), we deduce that $x^{\frac{1-b}{2}} I(b-1, 2\sqrt{x})$ and $x^{\frac{1-b}{2}} K(b-1, 2\sqrt{x})$ are two linearly independent solutions of (3.20). It follows that the general solution of (3.20) is of the form

$$c_1 x^{\frac{1-b}{2}} I(b-1, 2\sqrt{x}) + c_2 x^{\frac{1-b}{2}} K(b-1, 2\sqrt{x}),$$

where c_1 and c_2 are two constants.

According to (3.19), (3.16) and (3.17), we deduce that

$$g_b(1) = \sum_{k \in \mathbb{N}} \frac{1}{k! \Gamma(k+b)} = I(b-1, 2).$$

Therefore $c_1 = 1$ and $c_2 = 0$.

Consequently, for all $x > 0$,

$$g_b(x) = \sum_{k \in \mathbb{N}} \frac{x^k}{k! \Gamma(k+b)} = x^{\frac{1-b}{2}} I(b-1, 2\sqrt{x}). \quad (3.21)$$

□

3.1 The probability density of the mixture of the Riesz distribution with respect to the multivariate Poisson

Let $\rho = \left(0, \frac{1}{2}, \dots, \frac{r-1}{2}\right)$, $k = (k_1, k_2, \dots, k_r)$ be in \mathbb{N}^r and let

$$\tilde{R}_k = R(k + \rho, I_r).$$

Suppose that $k = (k_1, k_2, \dots, k_r)$ has the multivariate Poisson distribution ν defined in (3.13). For simplicity, in what follows, we will denote by μ the mixture of \tilde{R}_k by ν . The following theorem gives the expression of μ in terms of the modified Bessel function of the first kind.

Theorem 3.2

$$\mu(dx) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \frac{\sqrt{\lambda_i} e^{-\lambda_i}}{\sqrt{\Delta_{i-1}(x)}} I\left(1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \mathbf{1}_{\Omega}(x)(dx),$$

where $\Delta_0(x) = 1$.

Proof

Let $a = (a_1, a_2, \dots, a_r)$ be an element of $]0, +\infty[^r$ and let

$$\tilde{R}_{k,a} = R(k + \rho + a, I_r).$$

Denote μ_a the mixture of $\tilde{R}_{k,a}$ with respect to ν . Then

$$\mu_a(dx) = h_a(x) \mathbf{1}_\Omega(x)(dx),$$

where

$$h_a(x) = e^{-\sum_{i=1}^r \lambda_i} \sum_{q \in \mathbb{N}^r} \frac{\lambda^q e^{-tr(x)} \Delta_{q+\rho+a-\frac{n}{r}}(x)}{q! \Gamma_\Omega(q + \rho + a)}, \quad (3.22)$$

and

$$\lim_{a \rightarrow 0} \mu_a(dx) = \mu(dx) = e^{-\sum_{i=1}^r \lambda_i} \sum_{q \in \mathbb{N}^r} \frac{\lambda^q e^{-tr(x)} \Delta_{q+\rho-\frac{n}{r}}(x)}{q! \Gamma_\Omega(q + \rho)} \mathbf{1}_\Omega(x)(dx). \quad (3.23)$$

According to (2.4) and (2.11), (3.22) may be written as

$$\begin{aligned} h_a(x) &= \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \sum_{q_i \in \mathbb{N}} \frac{\lambda_i^{q_i}}{q_i! \Gamma(q_i + a_i)} \left(\frac{\Delta_i(x)}{\Delta_{i-1}(x)} \right)^{q_i + a_i} \right). \\ &= \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \sum_{q_i \in \mathbb{N}} \frac{\lambda_i^{q_i}}{q_i! \Gamma(q_i + a_i)} \Delta_{e_i}(x)^{q_i + a_i} \right). \\ &= \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \Delta_{e_i}(x)^{a_i} \sum_{q_i \in \mathbb{N}} \frac{(\lambda_i \Delta_{e_i}(x))^{q_i}}{q_i! \Gamma(q_i + a_i)} \right). \end{aligned}$$

Therefore

$$h_a(x) = \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\frac{e^{-\lambda_i}}{\sqrt{\Delta_i(x)}} \Delta_{e_i}(x)^{a_i} g_{a_i}(\lambda_i \Delta_{e_i}(x)) \right), \quad (3.24)$$

where g_{a_i} is defined in (3.19). Inserting now (3.21) in (3.24), we obtain

$$h_a(x) = \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\frac{\lambda_i^{\frac{1-a_i}{2}} e^{-\lambda_i}}{\sqrt{\Delta_i(x)}} \Delta_{e_i}(x)^{\frac{1+a_i}{2}} I\left(a_i - 1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \right).$$

According to (3.23), we deduce that

$$\mu(dx) = \frac{e^{-tr(x)}}{(2\Pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^r \left(\frac{\sqrt{\lambda_i} e^{-\lambda_i}}{\sqrt{\Delta_{i-1}(x)}} I\left(-1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \right) \mathbf{1}_\Omega(x)(dx).$$

Invoking the fact that for all $x > 0$, we have

$$I(1, x) = I(-1, x),$$

(see Lebedev [10], page 110), the proof of the theorem is complete. \square

3.2 The variance function of $F(\mu)$

In this subsection, we study the natural exponential family F generated by μ . We first give the Laplace transform of μ , then we determine the domain of the means and the variance function of the family F .

In what follows, for $r \geq 1$, we denote $\kappa_r = \sum_{j=1}^r \frac{j}{2} e_j$ and we set $\kappa_0 = 0$.

Theorem 3.3 *For all $\theta \in I_r - \Omega$, we have*

$$L_\mu(\theta) = \Delta_{\kappa_{r-1}}^*(I_r - \theta) \exp \left(\sum_{i=1}^r \lambda_i \left(\Delta_{-e_{r-i+1}}^*(I_r - \theta) - 1 \right) \right). \quad (3.25)$$

Proof

Let X_k be a random variable such that $X_k \sim R(k + \rho, I_r)$. Then, according to (2.12), we have that for all $\theta \in I_r - \Omega$,

$$L_\mu(\theta) = E \left(e^{\langle \theta, X_k \rangle} \right) = E \left(E \left(e^{\langle \theta, X_k \rangle} \mid k \right) \right) = E \left(\Delta_{k+\rho}((I_r - \theta)^{-1}) \right).$$

Using (2.5) and (2.6), we can write

$$\begin{aligned} L_\mu(\theta) &= E \left(\Delta_{-(k+\rho)^*}^*(I_r - \theta) \right). \\ &= E \left((\Delta_1^*(I_r - \theta))^{k_{r-1}-k_r-\frac{1}{2}} \dots (\Delta_{r-1}^*(I_r - \theta))^{k_1-k_2-\frac{1}{2}} (\Delta_r^*(I_r - \theta))^{-k_1} \right). \\ &= \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} E \left(\prod_{i=1}^r \left(\frac{\Delta_{i-1}^*(I_r - \theta)}{\Delta_i^*(I_r - \theta)} \right)^{k_{r-i+1}} \right). \\ &= \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} E \left(\prod_{i=1}^r (\Delta_{-e_i}^*(I_r - \theta))^{k_{r-i+1}} \right). \end{aligned}$$

It follows that for all $\theta \in I_r - \Omega$,

$$L_\mu(\theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} E \left(\prod_{i=1}^r e^{k_{r-i+1} \log(\Delta_{-e_i}^*(I_r - \theta))} \right). \quad (3.26)$$

Setting $\alpha(\theta) = \left(\log(\Delta_{-e_r}^*(I_r - \theta)), \log(\Delta_{-e_{r-1}}^*(I_r - \theta)), \dots, \log(\Delta_{-e_1}^*(I_r - \theta)) \right)$, then (3.26) becomes

$$L_\mu(\theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} E \left(e^{\langle \alpha(\theta), k \rangle} \right).$$

As k has the multivariate Poisson distribution ν , then

$$L_\mu(\theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} L_\nu(\alpha(\theta)).$$

According to (3.14), we can write for all $\theta \in I_r - \Omega$,

$$L_\mu(\theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} \prod_{i=1}^r e^{\lambda_i \left(e^{\log(\Delta_{-e_{r-i+1}}^*(I_r - \theta)) - 1} \right)}.$$

Therefore

$$L_\mu(\theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}} \prod_{i=1}^r e^{\lambda_i \left((\Delta_{-e_{r-i+1}}^*(I_r - \theta))^{-1} \right)}. \quad (3.27)$$

On the other hand, using (2.5), we have

$$\Delta_{\kappa_{r-1}}^*(I_r - \theta) = \prod_{i=1}^{r-1} \left(\frac{1}{\Delta_i^*(I_r - \theta)} \right)^{\frac{1}{2}}.$$

Inserting this in (3.27), we get (3.25). \square

Theorem 3.4 *The domain of the means of the natural exponential family $F = F(\mu)$ generated by the mixture μ is Ω .*

Proof

From (3.25), we deduce that for all $\theta \in \Theta(\mu) = I_r - \Omega$,

$$k_\mu(\theta) = \sum_{i=1}^r \lambda_i \left((\Delta_{-e_{r-i+1}}^*(I_r - \theta))^{-1} + \log(\Delta_{\kappa_{r-1}}^*(I_r - \theta)) \right). \quad (3.28)$$

As for all $i \in \{1, \dots, r\}$, the map

$$\varphi_i : x \mapsto \log \Delta_i^*(x)$$

is differentiable on Ω and

$$\varphi_i'(x) = (P_i^*(x))^{-1}, \quad (3.29)$$

then, for all $i \in \{1, \dots, r\}$, we have

$$\left(\Delta_{-e_i}^*(x) \right)' = \left(\frac{\Delta_{i-1}^*(x)}{\Delta_i^*(x)} \right)' = \Delta_{-e_i}^*(x) \left((P_{i-1}^*(x))^{-1} - (P_i^*(x))^{-1} \right), \quad (3.30)$$

and for $r \geq 2$, we have

$$\left(\log(\Delta_{\kappa_{r-1}}^*(x)) \right)' = -\frac{1}{2} \sum_{i=1}^{r-1} (P_i^*(x))^{-1} \quad (3.31)$$

Differentiating (3.28) and taking into account (3.30) and (3.31), we get

$$k'_\mu(\theta) = \sum_{i=1}^r \left(\lambda_{r-i+1} \Delta_{-e_i}^*(I_r - \theta) - \lambda_{r-i} \Delta_{-e_{i+1}}^*(I_r - \theta) + \frac{1}{2} \right) (P_i^*(I_r - \theta))^{-1} - \frac{1}{2} (P_r^*(I_r - \theta))^{-1}, \quad (3.32)$$

where $\lambda_0 = 0$.

Let $\theta \in I_r - \Omega$, and let u be the unique element of \mathcal{T}_l^+ such that $I_r - \theta = u^{*-1}(I_r)$. Then, for all $i \in \{1, \dots, r\}$, we have

$$(P_i^*(I_r - \theta))^{-1} = \left(P_i^*(u^{*-1}(I_r)) \right)^{-1} = \left(P_i^* \left((u(I_r))^{-1} \right) \right)^{-1}.$$

According to (2.7), this implies that for all $i \in \{1, \dots, r\}$,

$$(P_i^*(I_r - \theta))^{-1} = u \left(\sum_{j=r-i+1}^r c_j \right). \quad (3.33)$$

On the other hand, using (2.8), we can write for all $i \in \{1, \dots, r\}$,

$$\Delta_{-e_i}^*(I_r - \theta) = \Delta_{-e_i}^* \left(u^{*-1}(I_r) \right) = \Delta_{e_i^*}(u(I_r)) = \Delta_{e_{r-i+1}}(u(I_r)) = \frac{\Delta_{r-i+1}(u(I_r))}{\Delta_{r-i}(u(I_r))}.$$

This with (2.9) imply that for all $i \in \{1, \dots, r\}$,

$$\Delta_{-e_i}^* = u_{r-i+1}^2, \quad (3.34)$$

where for all $i \in \{1, \dots, r\}$, u_i are defined in (2.10).

Using (3.33) and (3.34), we deduce from (3.32) that

$$k'_\mu(\theta) = \sum_{i=1}^r \left(\lambda_{r-i+1} u_{r-i+1}^2 - \lambda_{r-i} u_{r-i}^2 + \frac{1}{2} \right) u \left(\sum_{j=r-i+1}^r c_j \right) - \frac{1}{2} u \left(\sum_{i=1}^r c_i \right).$$

This after a standard calculation, gives

$$k'_\mu(\theta) = \sum_{i=1}^r \left(\lambda_i u_i^2 + \frac{i-1}{2} \right) u(c_i) = u \left(\sum_{i=1}^r a_i(\theta) c_i \right), \quad (3.35)$$

where

$$a_i(\theta) = \frac{i-1}{2} + \lambda_i u_i^2. \quad (3.36)$$

As the a_i are strictly positive, we deduce that

$$k'_\mu(\Theta(\mu)) = k'_\mu(I_r - \Omega) \subseteq \Omega. \quad (3.37)$$

Conversely, consider $y \in \Omega$, then using the Cholesky decomposition, there exists a unique $w \in \mathcal{T}_l^+$ such that

$$y = w(I_r) = w \left(\sum_{i=1}^r c_i \right) = w \left(P \left(\sum_{i=1}^r \frac{1}{\sqrt{a_i(\theta)}} c_i \right) \left(\sum_{i=1}^r a_i(\theta) c_i \right) \right).$$

Let θ such that $I_r - \theta = u^{*-1}(I_r) \in \Omega$, where $u = w \sum_{i=1}^r \frac{1}{\sqrt{a_i(\theta)}} c_i$. Then

$$y = u \left(\sum_{i=1}^r a_i(\theta) c_i \right).$$

This, using (3.35), gives

$$y = k'_\mu(\theta) \in k'_\mu(I_r - \Omega).$$

Therefore

$$\Omega \subseteq k'_\mu(I_r - \Omega) = k'_\mu(\Theta(\mu)).$$

This with (3.37) imply that the domain of the means of the NEF $F = F(\mu)$ is

$$M_F = k'_\mu(\Theta(\mu)) = \Omega. \quad \square$$

The following theorem gives the variance function of the natural exponential family $F = F(\mu)$.

Theorem 3.5 *For all $m \in \Omega$,*

$$\begin{aligned} V_F(m) = & -\frac{1}{2}P \left(\sum_{i=1}^r \frac{1}{b_i(m)} \left((P_{r-i+1}^*(m^{-1}))^{-1} - (P_{r-i}^*(m^{-1}))^{-1} \right) \right) \\ & + \sum_{i=1}^r \left(\frac{\lambda_{r-i+1} \Delta_{e_{r-i+1}}(m)}{b_{r-i+1}(m)} - \frac{\lambda_{r-i} \Delta_{e_{r-i}}(m)}{b_{r-i}(m)} + \frac{1}{2} \right) \\ & \times \left[P \left(\sum_{j=r-i+1}^r \frac{1}{b_j(m)} \left((P_{r-j+1}^*(m^{-1}))^{-1} - (P_{r-j}^*(m^{-1}))^{-1} \right) \right) \right] \\ & + \sum_{i=1}^r \frac{\lambda_{r-i+1} \Delta_{e_{r-i+1}}(m)}{(b_{r-i+1}(m))^3} \left[\left((P_i^*(m^{-1}))^{-1} - (P_{i-1}^*(m^{-1}))^{-1} \right) \right. \\ & \quad \left. \otimes \left((P_i^*(m^{-1}))^{-1} - (P_{i-1}^*(m^{-1}))^{-1} \right) \right], \end{aligned} \quad (3.38)$$

where for all $i \in \{1, \dots, r\}$, $b_i(m) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i \Delta_{e_i}(m)}$.

Usually, for the calculation of the variance function, we set $m = k'_\mu(\theta)$ and we determine its reciprocal $\theta = \psi_\mu(m)$. This is difficult to do in the present situation, however, we are able to determine $(P_i^*(I_r - \psi_\mu(m)))^{-1}$, $\Delta_{-e_i}^*(I_r - \psi_\mu(m))$, and $a_i(\psi_\mu(m))$ where a_i is defined in (3.36).

Theorem 3.6 *For all $i \in \{1, \dots, r\}$,*

$$\begin{aligned} i) \quad a_i(\psi_\mu(m)) &= \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i \Delta_{e_i}(m)}. \\ ii) \quad (P_i^*(I_r - \psi_\mu(m)))^{-1} &= \sum_{j=r-i+1}^r \frac{1}{a_j(m)} \left[(P_{r-j+1}^*(m^{-1}))^{-1} - (P_{r-j}^*(m^{-1}))^{-1} \right]. \\ iii) \quad \Delta_{-e_i}^*(I_r - \psi_\mu(m)) &= \frac{\Delta_{e_{r-i+1}}(m)}{a_{r-i+1}(m)}. \end{aligned} \quad (3.39)$$

Proof

i) As from Theorem 3.4, $m = k'_\mu(\theta)$ is in Ω , there exists a unique $v \in \mathcal{T}_l^+$ such that $m = v(I_r)$. According to (3.35), we have

$$v(I_r) = m = u \left(\sum_{i=1}^r a_i(\psi_\mu(m)) c_i \right) = u \left(P \left(\sum_{i=1}^r \sqrt{a_i(\psi_\mu(m))} c_i \right) \left(\sum_{i=1}^r c_i \right) \right).$$

Therefore

$$v(I_r) = u \left(P \left(\sum_{i=1}^r \sqrt{a_i(\psi_\mu(m))} c_i \right) (I_r) \right).$$

It follows that

$$v = u \sum_{i=1}^r \sqrt{a_i(\psi_\mu(m))} c_i.$$

Or equivalently,

$$u = v \sum_{i=1}^r \frac{1}{\sqrt{a_i(\psi_\mu(m))}} c_i. \quad (3.40)$$

On the other hand, using (2.9), (3.36) becomes

$$a_i(\psi_\mu(m)) = \frac{i-1}{2} + \lambda_i \Delta_{e_i}(u(I_r)).$$

Then using (3.40), we can write

$$\begin{aligned} a_i(\psi_\mu(m)) &= \frac{i-1}{2} + \lambda_i \Delta_{e_i} \left(\left(v \sum_{j=1}^r \frac{1}{\sqrt{a_j(\psi_\mu(m))}} c_j \right) \left(\sum_{j=1}^r \frac{1}{\sqrt{a_j(\psi_\mu(m))}} c_j v^* \right) \right). \\ &= \frac{i-1}{2} + \lambda_i \Delta_{e_i} \left(v \sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j v^* \right). \\ &= \frac{i-1}{2} + \lambda_i \Delta_{e_i} \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right). \end{aligned}$$

Therefore $a_i(\psi_\mu(m))$ satisfies the equation

$$a_i(\psi_\mu(m)) = \frac{i-1}{2} + \frac{\lambda_i v_i^2}{a_i(\psi_\mu(m))}, \quad (3.41)$$

where v_i is defined in (2.10).

As $a_i(\psi_\mu(m)) > 0$, we deduce that

$$a_i(\psi_\mu(m)) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i v_i^2}.$$

On the other hand, since $m = v(I_r)$, then using (2.9), we have that

$$v_i^2 = \Delta_{e_i}(m).$$

Consequently, for all $i \in \{1, \dots, r\}$,

$$a_i(\psi_\mu(m)) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i \Delta_{e_i}(m)}.$$

ii) With the notations used above, we can write for all $i \in \{1, \dots, r\}$,

$$\begin{aligned}
(P_i^*(I_r - \psi_\mu(m)))^{-1} &= (P_i^*(u^{*-1}(I_r)))^{-1} \\
&= u \left(\sum_{j=r-i+1}^r c_j \right) \\
&= \left(v \sum_{i=1}^r \frac{1}{\sqrt{a_i(\psi_\mu(m))}} c_i \right) \left(\sum_{j=r-i+1}^r c_j \right) \left(\sum_{i=1}^r \frac{1}{\sqrt{a_i(\psi_\mu(m))}} c_i v^* \right) \\
&= v \sum_{j=r-i+1}^r \frac{1}{a_j(\psi_\mu(m))} c_j v^* \\
&= v \left(\sum_{j=r-i+1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right)
\end{aligned}$$

Thus

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} = \sum_{j=r-i+1}^r \frac{1}{a_j(\psi_\mu(m))} v(c_j). \quad (3.42)$$

As $m = v(I_r)$, then for all $j \in \{1, \dots, r\}$, we have

$$v(c_j) = v \left(\sum_{i=j}^r c_i - \sum_{i=j+1}^r c_i \right) = (P_{r-j+1}^*(m^{-1}))^{-1} - (P_{r-j}^*(m^{-1}))^{-1}.$$

Inserting this in (3.42), we deduce that

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} = \sum_{j=r-i+1}^r \frac{1}{a_j(\psi_\mu(m))} \left[(P_{r-j+1}^*(m^{-1}))^{-1} - (P_{r-j}^*(m^{-1}))^{-1} \right]. \quad (3.43)$$

Consequently

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} - (P_{i-1}^*(I_r - \psi_\mu(m)))^{-1} = \frac{1}{a_{r-i+1}(\psi_\mu(m))} \left((P_i^*(m^{-1}))^{-1} - (P_{i-1}^*(m^{-1}))^{-1} \right). \quad (3.44)$$

iii) According to (2.5) we have

$$\Delta_i^*(I_r - \psi_\mu(m)) = \Delta_i^*(u^{*-1}(I_r)) = \Delta_{\vartheta_i}^*(u^{*-1}(I_r)),$$

where $\vartheta_i = \sum_{j=1}^i e_j$. Using (2.8) and (3.40), we deduce that

$$\begin{aligned}
\Delta_i^*(I_r - \psi_\mu(m)) &= \Delta_{-\vartheta_i}^*(u(I_r)) \\
&= \Delta_{-\vartheta_i}^*(uu^*) \\
&= \Delta_{-\vartheta_i}^* \left(\left(v \sum_{j=1}^r \frac{1}{\sqrt{a_j(\psi_\mu(m))}} c_j \right) \left(\sum_{j=1}^r \frac{1}{\sqrt{a_j(\psi_\mu(m))}} c_j v^* \right) \right).
\end{aligned}$$

$$= \Delta_{-\vartheta_i^*} \left(v \sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j v^* \right).$$

Therefore

$$\Delta_i^*(I_r - \psi_\mu(m)) = \Delta_{-\vartheta_i^*} \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right).$$

Thus, using (2.4), we obtain

$$\Delta_i^*(I_r - \psi_\mu(m)) = \frac{\Delta_{r-i} \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right)}{\Delta_r \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right)}.$$

It follows that for all $i \in \{1, \dots, r\}$,

$$\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \frac{\Delta_{i-1}^*(I_r - \psi_\mu(m))}{\Delta_i^*(I_r - \psi_\mu(m))} = \frac{\Delta_{r-i+1} \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right)}{\Delta_{r-i} \left(v \left(\sum_{j=1}^r \frac{1}{a_j(\psi_\mu(m))} c_j \right) \right)}.$$

Using (2.9), we deduce that for all $i \in \{1, \dots, r\}$,

$$\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \frac{v_{r-i+1}^2}{a_{r-i+1}(\psi_\mu(m))}.$$

Therefore

$$\begin{aligned} \Delta_{-e_i}^*(I_r - \psi_\mu(m)) &= \frac{\Delta_{r-i+1}(m)}{a_{r-i+1}(\psi_\mu(m)) \Delta_{r-i}(m)} \\ &= \frac{\Delta_{e_{r-i+1}}(m)}{a_{r-i+1}(\psi_\mu(m))}. \end{aligned} \quad \square$$

We are now in position to give the variance function of the natural exponential family F stated in Theorem 3.5.

Proof of Theorem 3.5

We have that for all $m \in M_F = \Omega$,

$$V_F(m) = k_\mu''(\psi_\mu(m)).$$

Differentiating (3.32) and using (3.30) and the fact that for all $i \in \{1, \dots, r\}$ and $x \in \Omega$,

$$\left((P_i^*(x))^{-1} \right)' = -P \left((P_i^*(x))^{-1} \right),$$

we get for all $\theta \in I_r - \Omega$,

$$\begin{aligned}
k''_{\mu}(\theta) &= -\frac{1}{2}P\left((P_r^*(I_r - \theta))^{-1}\right) \\
&\quad + \sum_{i=1}^r \left(\lambda_{r-i+1}\Delta_{-e_i}^*(I_r - \theta) - \lambda_{r-i}\Delta_{-e_{i+1}}^*(I_r - \theta) + \frac{1}{2} \right) \left(P\left((P_i^*(I_r - \theta))^{-1}\right) \right) \\
&\quad + \sum_{i=1}^r \lambda_{r-i+1}\Delta_{-e_i}^*(I_r - \theta) \left((P_i^*(I_r - \theta))^{-1} - (P_{i-1}^*(I_r - \theta))^{-1} \right) \otimes (P_i^*(I_r - \theta))^{-1} \\
&\quad - \sum_{i=1}^r \lambda_{r-i}\Delta_{-e_{i+1}}^*(I_r - \theta) \left((P_{i+1}^*(I_r - \theta))^{-1} - (P_i^*(I_r - \theta))^{-1} \right) \otimes (P_i^*(I_r - \theta))^{-1}.
\end{aligned}$$

It follows that for all $\theta \in I_r - \Omega$,

$$\begin{aligned}
k''_{\mu}(\theta) &= -\frac{1}{2}P\left((P_r^*(I_r - \theta))^{-1}\right) \\
&\quad + \sum_{i=1}^r \left(\lambda_{r-i+1}\Delta_{-e_i}^*(I_r - \theta) - \lambda_{r-i}\Delta_{-e_{i+1}}^*(I_r - \theta) + \frac{1}{2} \right) \left(P\left((P_i^*(I_r - \theta))^{-1}\right) \right) \\
&\quad + \sum_{i=1}^r \lambda_{r-i+1}\Delta_{-e_i}^*(I_r - \theta) \\
&\quad \times \left[\left((P_i^*(I_r - \theta))^{-1} - (P_{i-1}^*(I_r - \theta))^{-1} \right) \otimes \left((P_i^*(I_r - \theta))^{-1} - (P_{i-1}^*(I_r - \theta))^{-1} \right) \right].
\end{aligned}$$

We need only to replace θ by $\psi_{\mu}(m)$, then insert (3.39), (3.43) and (3.44) to get the expression of the variance function of $F = F(\mu)$ given in (3.38). \square

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