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Some estimates and maximum principles for weakly coupled systems of elliptic PDE

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1 Introduction

In this paper we discuss maximum principles and Harnack type estimates for systems of linear elliptic PDE's of second order

$$\left\{\begin{array}{rcl} L_1u_1 + c_{11}(x)u_1 + c_{12}(x)u_2 + \ldots + c_{1n}(x)u_n & = & f_1(x) \\ L_2u_2 + c_{21}(x)u_1 + c_{22}(x)u_2 + \ldots + c_{2n}(x)u_n & = & f_2(x) \\ & \ldots & \\ L_nu_n + c_{n1}(x)u_1 + c_{n2}(x)u_2 + \ldots + c_{nn}(x)u_n & = & f_n(x) \end{array}\right. \tag{1}$$

given in a bounded domain $\Omega \subset \mathbb{R}^N$; $n, N \geq 1$. Here $L_1, \ldots, L_n$ are supposed to be uniformly elliptic operators in general non-divergence form

$$L_k = \sum_{i,j=1}^{N} a_{ij}^k(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^k(x)\frac{\partial}{\partial x_i}. \tag{2}$$

with $\lambda|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$, $\xi \in \mathbb{R}^N$, for some $0 < \lambda \leq \Lambda$.

Studying such systems is an object of ever increasing interest in recent years. The most important reason is that whenever one wants to study a *nonlinear* system of elliptic PDE's (such systems are abundant in all areas of applications) a first step often is gaining some knowledge on its *linearized* system, which is in the form (1). Further, many higher-order equations - like $(-\Delta)^m u = f(x), \Delta^2 u + \beta\Delta u = f(x)$, are particular cases of (1). Some problems in probability theory, namely in the study of infinitesimal generators of diffusion processes with jumps also lead to system (1).

So it is very natural to ask whether known results for linear elliptic PDE's extend to systems like (1). Here we shall be interested in the possibility of obtaining a generalized maximum principle (often referred to as Alexandrov-Bakelman-Pucci, ABP inequality) and a Harnack inequality for (1). In the scalar case these estimates play a fundamental role in the existence and regularity theory - see for instance GT, Chapters 8 and 9.

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Unfortunately, and as is well known, such estimates do not hold for all systems of type (1). One needs in general the additional assumption that the system has a (quasi-)monotonicity property, also called cooperativeness (this term comes from biology, where models in population dynamics for species which cooperate with each other lead to quasimonotone systems). We recall that system (1) is cooperative provided for all indices $i, j \in \{1, \dots, n\}$,

$$(H_0) \qquad i \neq j \quad \text{implies} \quad c_{ij} \geq 0 \ \text{ a.e. in } \ \Omega.$$

This condition is rather restrictive, but many important systems do satisfy it. For instance the higher order equations that we quoted above are equivalent to cooperative systems, the problems in probability and their applications to mathematical finance lead to cooperative systems. Further, whenever one has a nonlinear system like, say, $-\Delta u_1 = g_1(u_1, u_2)$, $-\Delta u_2 = g_2(u_1, u_2)$, its linearization is cooperative if g_i is non-decreasing in u_j, $i \neq j$. A simple example is provided by the widely studied Lane-Emden system $-\Delta u = |v|^p v, -\Delta v = |u|^q u$. Another example is the system $-\Delta u_i = -\lambda_i u_i + (c_{i1} u_1^2 + c_{i2} u_2^2) u_i$, $i = 1, 2$, which represents the stationary states of coupled Schrodinger systems, modeling some phenomena in nonlinear optics and low temperatures physics (these systems are an object of large interest recently).

Some time ago in [2] J. Busca and the author proved an ABP inequality and a Harnack inequality for cooperative systems of type (1). These results apply to rather more general systems than (1) (namely, to systems of fully nonlinear equations of Hamilton-Jacobi-Bellman-Isaac type) but their proofs are rather lengthy, involved and, in particular, rely on the difficult theory of L^N-viscosity solutions of fully nonlinear PDE, developed in the last twenty years. We have often been asked whether simpler proofs could be found, at least for linear systems.

This is the first goal of the present work – to give elementary and shorter proofs of the results in [2] in the linear case, which use only the standard theory of scalar linear PDE, as developed for instance in Chapter 9 of [8]. These new proofs, apart from being of course interesting in their own right, permit to wrap up within the classical framework the theory of solvability of several types of nonlinear systems, recently developed in [6], [7], [14] (these papers used in an essential way Theorems 1-3 below).

Further, the proofs we give here permit to us to improve the results in [2] by allowing the system to have unbounded coefficients with (optimal) Lebesgue integrability. Namely, we suppose that

$$(H_p) \qquad c_{ij}, f_i \in L^N(\Omega), \quad b_i^k \in L^p(\Omega), \quad i, j, k \in \{1, \dots, n\}.$$

Let μ be an upper bound for the L^p-norms of b_i^k, and ν be an upper bound for the L^N-norms of c_{ij}. We are going to prove the ABP inequality under

(H_N) and the Harnack inequality under (H_p), for some $p > N$ - these are the hypotheses under which these results are known for scalar equations.

We shall also discuss some explicit conditions for a system to satisfy the maximum principle.

Setting $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$, we write (1) in the form

$$LU + \mathcal{C}U = F,$$

with $L = \mathrm{diag}(L_1, \dots, L_n)$, $U = (u_1, \dots, u_n)^T$, $F = (f_1, \dots, f_n)^T$. For any vector $U \in \mathbb{R}^n$ we set $\overline{U} = \max\{u_1, \dots, u_n\}$, $\underline{U} = \min\{u_1, \dots, u_n\}$, $U^\pm = (u_1^\pm, \dots, u_n^\pm)$, $U = U^+ + U^-$. All through the paper we consider strong solutions of (1), that is, functions $u_i \in W^{2,N}_{\mathrm{loc}}(\Omega)$ which satisfy (1) a.e. in Ω. All (in)equalities between vectors are understood to hold component-wise.

We are going to use the following hypothesis.

(H_Ψ) there exists a function $\Psi = (\psi_1, \dots, \psi_n) \in W^{2,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, for some $p > N$, such that $L\Psi + \mathcal{C}\Psi \le 0$ in Ω and $\Psi \ge (1, \dots, 1)$ in $\overline{\Omega}$.

Theorem 1 (ABP estimate) *If $(H_0), (H_N), (H_\Psi)$ hold and U is such that $LU + \mathcal{C}U \ge -F$ in Ω, then*

$$\sup_\Omega \overline{U} \le \max_{\overline{\Omega}} \overline{\Psi} \left(\sup_{\partial\Omega} (\overline{U})^+ + C \|\overline{F}^+\|_{L^N(\Omega)} \right). \tag{3}$$

The constant C depends on $n, N, \lambda, \Lambda, \mu, \nu, \|\Psi\|_{C^1(\Omega)}$, and $|\Omega|$.

Remark. Note (3) with $F = 0$ gives a maximum principle for $L + \mathcal{C}$.

We turn to the Harnack inequality for non-negative solutions of (1). We shall limit ourselves here to the case of a fully coupled system - that is, a system which cannot be divided into two subsystems one of which does not depend on the other (extensions to more general systems are then not difficult to get, see Sections 8 and 9 of [2]).

Definition 1.1 *A matrix $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$, which satisfies (H_0), is called irreducible in Ω, and the system $LU + \mathcal{C}U = F$ is called fully coupled in Ω, provided for any non-empty sets $I, J \subset \{1, \dots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$, there exist $i_0 \in I$ and $j_0 \in J$ for which*

$$\mathrm{meas}\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\} > 0. \tag{4}$$

For simplicity, when (4) holds we write $c_{i_0 j_0} \not\equiv 0$ in Ω. Hence we can fix $\rho > 0$ such that the sets $\{x \in B_R \mid c_{i_0 j_0}(x) \ge \rho\}$ have positive measures. Let $\omega > 0$ be a lower bound for these measures.

Theorem 2 (Harnack inequality) *Suppose $(H_0), (H_p)$ are satisfied, for some $p > N$, and let $U \geq 0$ be a solution of (1) in Ω. Let $B_{2R} \subset \Omega$ be a ball with radius $2R$. Assume (1) is fully coupled. Then*

$$\sup_{B_R} \overline{U} \leq C \left(\inf_{B_R} \underline{U} + R\|F\|_{L^N(B_{2R})} \right), \tag{5}$$

where C depends on $n, N, \lambda, \Lambda, \mu R, \nu R^2, \rho, \omega$.

A large discussion on the importance of these estimates, extensions, counterexamples and applications can be found in [2] (we refer in particular to Sections 1, 3, 8, 10-15 of that paper). Here we only recall the following fundamental consequence of Theorems 1 and 2.

Theorem 3 (i) *Suppose (H_0) holds and $a_{ij}^k \in C(\overline{\Omega})$, $b_i^k, c_{ij} \in L^\infty(\Omega)$, for all $i, j, k = 1, \ldots, n$. Set $\nu = \max_{i,j}\{\|b_i\|_{L^\infty(\Omega)}, \|c_{ij}\|_{L^\infty(\Omega)}\}$. The following are equivalent :*

(a) Condition (H_Ψ) holds.

(b) the operator $L + \mathcal{C}$ satisfies the maximum principle in Ω, that is, if $LU + \mathcal{C}U \leq 0$ in Ω and $U \geq 0$ on $\partial\Omega$, then $U \geq 0$ in Ω.

(c) for any $F \in L^N(\Omega)$ and any solution U of $LU + \mathcal{C}U \geq -F$ there holds

$$\sup_{\Omega} \overline{U} \leq C \left(\sup_{\partial\Omega} (\overline{U})^+ + \|\overline{F}^+\|_{L^N(\Omega)} \right),$$

where C depends only on $n, N, \lambda, \Lambda, \nu$ and $|\Omega|$.

(ii) *Under any of (a), (b), (c) in (i), if Ω satisfies an uniform exterior cone condition, then for any $F \in L^p(\Omega), p \geq N$, there exists a unique solution $U \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of $LU + \mathcal{C}U = F$ in Ω, $U = 0$ on $\partial\Omega$. We have $\|U\|_{W^{2,p}(\Omega)} \leq C\|F\|_{L^p(\Omega)}$, C depends on $n, N, \lambda, \Lambda, \mu, \nu$, Ω.*

The not difficult (once we have ABP and Harnack inequalities) proof of Theorem 3 is given in Sections 13 and 14 of [2] (see also the remarks in Section 3.1 of [14], in particular Theorem 7 there). The proof is based on results of existence and properties of a principal eigenvalue of a matrix operator. The fact that (a) implies (b) was proved in [5], the implication $(b) \Rightarrow (a)$ was proved in [13] for systems with regular coefficients, and $(a) \Rightarrow (ii)$ was established in [15], in the fully coupled case.

The most important statements in Theorem 3 are the facts that the maximum principle implies a *quantitative estimate* of how it fails for systems

having a right-hand side with the wrong sign, *an a priori bound* for the solutions of the Dirichlet problem, and the unique *solvability* of this problem.

We are going to close this introduction with a review of the available explicit criteria for a quasimonotone system to satisfy the maximum principle. First, (H_Ψ) of course holds if $\mathcal{C}(x)V \leq 0$ a.e. in Ω, for some constant positive vector V. Second, the maximum principle is equivalent to the positivity of the principal eigenvalue of the system (or eigenvalues, if the system is not fully coupled). The results in [1] and their extensions in [2] give lower bounds for the eigenvalue in terms of the coefficients of L_i and the domain, which can be used to verify the condition of positivity of the eigenvalue. For instance, Proposition 14.1 in [2] shows that the maximum principle holds for domains with sufficiently small measure. Further, it was shown in [2] that the maximum principle is verified if either the matrix $(\sup_\Omega c_{ij})_{i,j}$ is semi-negative definite or the operators $L_1, \ldots, L_n$ coincide and $\mathcal{C}(x)$ is semi-negative definite a.e. in Ω (and an example was given showing that this last hypothesis is not enough if the operators are different). Finally, it was shown in [4] that the maximum principle holds provided the operators coincide, can be written in divergence form, the matrix $\mathcal{C}$ is constant and verifies $\mathcal{C} < \lambda_1(L_1)I$ (in the sense that $\mathcal{C} - \lambda_1(L_1)I$ is negative definite). By combining our arguments here with a reasoning from [14], we can extend this to an arbitrary operator and a nonconstant matrix satisfying $(\sup_\Omega c_{ij})_{i,j} < \lambda(L_1)I$.

Proposition 1.1 *Under the hypotheses of Theorem 3, set* $\overline{\mathcal{C}} := (\sup_\Omega c_{ij})_{i,j}$. *Suppose also* $L_1 \equiv \ldots \equiv L_n$ *and let* $\lambda_1 = \lambda_1(L_1)$ *be the principal eigenvalue of the linear operator* L_1 *(as defined in [1]). If* $\overline{\mathcal{C}} < \lambda_1 I$ *then the operator* $L + \mathcal{C}$ *satisfies the maximum principle in* Ω*, as in Theorem 3 (i).*

2 Proof of Theorem 1

We recall the following fundamental generalized maximum principle, due to Alexandrov and Bakelman, obtained independently by Pucci. It is Theorem 9.1 in [8].

Theorem 4 (ABP inequality) *Let* $c \leq 0$. *For any* $f \in L^N(\Omega)$ *and any solution* u *of* $L_i u + c(x)u \geq -f$ *there holds*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C_{ABP}\|f^+\|_{L^N(\Omega)},$$

where $C_{ABP} = C|\Omega|^{1/N}$ *and* C *depends only on* $n, N, \lambda, \Lambda, \mu$ *and* $|\Omega|$.

Remark. In classical form the constant C_{ABP} depends on the diameter of Ω. The fact that it can be replaced by the volume of Ω (and in fact an even more precise quantity, describing the "thickness" of Ω) is proved in [1], [3], see also [17] for more recent results.

As a simple computation on page 571 of [2] shows, it is sufficient to prove the result under the following condition:

$$\sum_{j=1}^{n} c_{ij}(x) \leq 0 \quad \text{a.e. in} \quad \Omega, \quad \text{for every } i \in \{1, \dots, n\}. \tag{6}$$

Note this is another way of saying (H_Ψ) is verified with $\Psi \equiv (1, \dots, 1)$. So in the following we assume that (6) is satisfied.

We take v_i to be the solution of the problem

$$\begin{cases} L_i v_i + c_{ii} v_i &= c_{ii} \quad \text{in} \quad \Omega \\ v_i &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Since $c_{ii} \leq -\sum_{i \neq j} c_{ij} \leq 0$ (by (6) and (H_0)) this problem has an unique solution, such that $v_i \geq 0$ in Ω, by the maximum principle for scalar equations.

Note if $c_{ii} \equiv 0$ then by (6) and (H_0) we have $c_{ij} \equiv 0$ for all j, so the i-th inequality in the system is scalar, and Theorem 4 applies to it. Hence we can suppose $c_{ii} \not\equiv 0$ for all i.

Lemma 2.1 *There exists a number $\delta > 0$, depending on $N, \lambda, \Lambda, \mu, \nu, |\Omega|$, such that*

$$0 \leq v_i \leq 1 - \delta \qquad \textit{in } \Omega, \quad i = 1, \dots, n.$$

Proof. We note that the function $z_i = 1 - v_i$ satisfies $L_i z_i + c_{ii} z_i = 0$ in Ω, $z_i = 1$ on $\partial\Omega$, so $z_i \geq 0$ in Ω, by the maximum principle. By the strong maximum principle (Theorem 3.5 in [8]) $z_i > 0$ in Ω.

We set $\overline{z}_i = g(z_i) := z_i^{-\alpha}$, where $\alpha > 0$ is a positive number to be chosen later. Since g is a smooth convex function, it is simple to check that

$$L_i \overline{z}_i \geq g'(z_i) L_i z_i$$

(this is the classical Kato inequality). Hence we get

$$L_i \overline{z}_i \geq \alpha c_{ii} \overline{z}_i \qquad \text{in } \Omega,$$

and $\overline{z}_i = 1$ on $\partial\Omega$. So by Theorem 4 we have

$$\sup_\Omega \overline{z}_i \leq 1 + \alpha \nu C_{ABP} \sup_\Omega \overline{z}_i.$$

Choosing $\alpha = (2\nu C_{ABP})^{-1}$ we obtain $\overline{z}_i \leq 2$ in Ω, that is,

$$v_i = 1 - z_i \leq 1 - 2^{-1/\alpha} \quad \text{in } \Omega,$$

which proves the lemma. □

Next, take w_i to be the solution of

$$\begin{cases} L_i w_i &= -f_i^+ \quad \text{in} \quad \Omega \\ w_i &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By Theorem 4 we have

$$\sup_\Omega w_i \leq C_{ABP} \| \max_{1\leq i\leq n} f_i^+ \|_{L^N(\Omega)}, \tag{7}$$

and $w \geq 0$ in Ω, by the maximum principle.

Proof of Theorem 1. Replacing u_i by $\widetilde{u}_i = u_i - \sup_{\partial\Omega}(\overline{U})^+$ we see that we can suppose $u_i \leq 0$ on $\partial\Omega$ for each i (note $\widetilde{u}_i$ satisfies the same inequality as u_i, because of (6)). Let $M = \sup_\Omega \overline{U} = \sup_{i,\Omega} u_i$ be the quantity we want to estimate. We have, by (6) and (H_0),

$$\begin{aligned} L_i u_i + c_{ii} u_i &\geq -f_i - \sum_{i\neq j} c_{ij}(x) u_j \\ &\geq -f_i - M \sum_{i\neq j} c_{ij}(x) \\ &\geq -f_i + M c_{ii}(x) \quad \text{in } \Omega. \end{aligned}$$

Consider the function $h_i = w_i + M v_i$. It satisfies $h_i = 0 \geq u_i$ on $\partial\Omega$ and

$$L_i h_i + c_{ii} h_i = -f_i^+ + c_{ii} w_i + M c_{ii} \leq -f_i + M c_{ii},$$

since $c_{ii} \leq 0$, $w_i \geq 0$. So by the maximum principle $u_i \leq h_i$ in Ω, for all i. This implies, by Lemma 2.1 and (7), that

$$M \leq C_{ABP} \| \max_{1\leq i\leq n} f_i^+ \|_{L^N(\Omega)} + M(1-\delta),$$

from which Theorem 1 follows. □

3 Proof of Theorem 2

As usual, the proof of the Harnack inequality is divided into two half-Harnack inequalities (the so-called local maximum principle for subsolutions and weak Harnack inequality for supersolutions), each of which is important in itself.

In what follows C will denote a positive constant which may change from line to line, and which depends only on the appropriate quantities.

Proposition 3.1 (local maximum principle) *Suppose* $(H_0), (H_p)$ *hold, for some* $p > N$*, and let* U *be a solution of* $LU + \mathcal{C}U \geq -F$ *in* Ω*. Let* $B_{2R} \subset \Omega$ *be a ball with radius* $2R$*. Then for each* $p > 0$ *we have*

$$\sup_{B_R} \overline{U} \leq C \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} |\overline{U}^+|^p + R\|f\|_{L^N(B_{2R})} \right), \tag{8}$$

where C *depends on* $p, N, \lambda, \Lambda, \mu R, \nu R^2$*.*

Proposition 3.2 (weak Harnack inequality) *Suppose* $(H_0), (H_p)$ *hold, for some* $p > N$*, and let* $U \geq 0$ *be a solution of* $LU + \mathcal{C}U \geq -F$ *in* Ω*. Let* $B_{2R} \subset \Omega$ *be a ball with radius* $2R$*. Assume (1) is fully coupled. Then there exists a number* $p = p(n, N, \lambda, \Lambda, \mu R, \nu R^2)$ *such that*

$$\frac{1}{|B_{2R}|} \int_{B_{2R}} |\overline{U}^+|^p \leq C \left(\inf_{B_R} \underline{U} + R\|f\|_{L^N(B_{2R})} \right), \tag{9}$$

where C *depends on* $n, N, \lambda, \Lambda, \mu R, \nu R^2, \rho, \omega$*.*

For the scalar case these theorems can be found in [8], for operators with bounded coefficients, and in [16], [10], [9], for operators with only Lebesgue integrable coefficients. Putting them together gives the full Harnack inequality.

3.1 Proof of Proposition 3.1

So we have a solution of

$$L_i u_i \geq -f_i - \sum_j c_{ij} u_j \geq -f_i^+ - c_{ii} u_i - \sum_{i \neq j} c_{ij} u_j^+$$

and we want to show (8). The idea of the proof is to find a regular approximation for $\overline{U}^+$ and a linear operator to which Theorem 9.20 of [8] applies.

We introduce the real function

$$h_\varepsilon(t) = \begin{cases} (t^4 + \varepsilon^4)^{1/4} - \varepsilon & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases}$$

It is very simple to check that $h_\varepsilon \in C^2(\mathbb{R})$, h_ε is *convex*, and has the following properties

$$0 \leq h'_\varepsilon(t) < 1, \quad h'_\varepsilon(t) \to 1 \quad \text{for all } t > 0, \tag{10}$$

$$\|t h'_\varepsilon(t) - h_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \varepsilon, \qquad \|t^+ - h_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \varepsilon. \tag{11}$$

Again, since h_ε is convex, the Kato inequality gives $L(h_\varepsilon(u)) \geq h'_\varepsilon(u)Lu$, for each linear elliptic operator of our type and for each $u \in W^{2,N}$. So we obtain

$$L_i(h_\varepsilon(u_i)) \geq h'_\varepsilon(u_i)(-f_i^+ - c_{ii}u_i - \sum_{i\neq j} c_{ij}u_j^+) \geq -f_{i,\varepsilon} - c_{ii}h_\varepsilon(u_i) - \sum_{i\neq j} c_{ij}h_\varepsilon(u_j),$$

where we have used (10), (11), and have noted $f_{i,\varepsilon} = f_i^+ + \varepsilon \sum_j c_{ij}$. In the sequel we shall in general denote with $f_{i,\varepsilon}$ any function which converges to f_i^+ in $L^N(B_{2R})$ as $\varepsilon \to 0$.

We have proved that the nonnegative vector $\widetilde{U} = (h_\varepsilon(u_1), \ldots, h_\varepsilon(u_n))$, $\widetilde{u}_i = h_\varepsilon(u_i)$, satisfies the same system as U, with f_i replaced by $f_{i,\varepsilon}$. We now set

$$w = w_\varepsilon = \frac{1}{2}\left(h_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2) + h_\varepsilon(\widetilde{u}_2 - \widetilde{u}_1) + \widetilde{u}_1 + \widetilde{u}_2\right).$$

We easily see that $w \geq 0$ is in $W^{2,N}(B_{2R})$ and

$$\|w - \max\{u_1^+, u_2^+\}\|_{L^\infty(B_{2R})} \leq \varepsilon + \|w - \max\{\widetilde{u}_1, \widetilde{u}_2\}\|_{L^\infty(B_{2R})} \leq 2\varepsilon. \quad (12)$$

We now fix a second order operator L whose coefficients admit the same bounds as those of L_1, L_2, and such that $Lw \geq L_1 w$ and $Lw \geq L_2 w$ in Ω (such an operator is easy to construct, see for instance Lemma 4.1 (b) in [2]). Note the coefficients of L depend on w but the respective bounds on these coefficients are still $\lambda, \Lambda, \mu, \nu$.

By applying the Kato inequality again we obtain

$$\begin{aligned} Lw \geq L_i w \quad \geq \quad & \tfrac{1}{2}\left(h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)(L_i\widetilde{u}_1 - L_i\widetilde{u}_2)\right. \\ & \left. + h'_\varepsilon(\widetilde{u}_2 - \widetilde{u}_1)(L_i\widetilde{u}_2 - L_i\widetilde{u}_1) + L_i\widetilde{u}_1 + L_i\widetilde{u}_2\right) \end{aligned}$$

a.e. in B_{2R}, $i = 1, 2$. Note that, since $\widetilde{u}_1, \widetilde{u}_2$ are in $W^{2,N}(B_{2R})$, we have $L_i\widetilde{u}_1 = L_i\widetilde{u}_2$ almost everywhere on the set $\{\widetilde{u}_1 = \widetilde{u}_2\}$. Hence a.e. on the set $\{\widetilde{u}_1 \geq \widetilde{u}_2\}$ we have, by the definition of h_ε and (10),

$$\begin{aligned} Lw \quad & \geq \quad \frac{1}{2}\left(1 + h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)\right) L_1\widetilde{u}_1 + \frac{1}{2}\left(1 - h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)\right) L_1\widetilde{u}_2 \\ & \geq \quad \frac{1}{2}\left(1 + h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)\right) L_1\widetilde{u}_1 + \frac{1}{2}\left(1 - h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)\right) L_1\widetilde{u}_1|_{\{\widetilde{u}_1 = \widetilde{u}_2\}} + \omega_\varepsilon \\ & \geq \quad L_1\widetilde{u}_1 + \omega_\varepsilon \\ & \geq \quad -f_{1,\varepsilon}^+ - c_{11}\widetilde{u}_1 - c_{12}\widetilde{u}_2 + \omega_\varepsilon \\ & \geq \quad -f_{1,\varepsilon}^+ - \max\{c_{11}^+, c_{12}\}\max\{\widetilde{u}_1, \widetilde{u}_2\} + \omega_\varepsilon \\ & \geq \quad -f_{1,\varepsilon}^+ - \max\{c_{11}^+, c_{12}\}(w + \varepsilon) + \omega_\varepsilon \end{aligned}$$

(recall (12)), where we have denoted

$$\omega_\varepsilon = \begin{cases} \frac{1}{2}\left(1 - h'_\varepsilon(\widetilde{u}_1 - \widetilde{u}_2)\right) L_1 \widetilde{u}_2 & \text{on } \{\widetilde{u}_1 > \widetilde{u}_2\} \\ 0 & \text{elsewhere .} \end{cases}$$

By (10) and the Lebesgue dominated convergence theorem we have $\omega_\varepsilon \to 0$ in $L^N(B_{2R})$ as $\varepsilon \to 0$.

We repeat the same argument, replacing L_1 by L_2 and $\{\widetilde{u}_1 \geq \widetilde{u}_2\}$ by $\{\widetilde{u}_1 \leq \widetilde{u}_2\}$, and we finally obtain

$$Lw \geq -f_\varepsilon - c(x)w$$

a.e. in B_{2R}, where $c = \max\{c_{11}^+, c_{12}, c_{21}, c_{22}^+\}$ and $f_\varepsilon = \max\{f_{1,\varepsilon}, f_{2,\varepsilon}\} + \varepsilon c - \omega_\varepsilon$, so $f_\varepsilon \to f$ in B_{2R}. Now Theorem 9.20 in [8] (or its extension to operators with unbounded coefficients in [16]) applies to this inequality and gives

$$\sup_{B_R} w \leq C \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} w^p + R \|f_\varepsilon\|_{L^N(B_{2R})} \right),$$

where C depends on $p, N, \lambda, \Lambda, \mu R, \nu R^2$. So if $n = 2$ we let ε tend to zero and finish the proof, using (12).

If $n = 3$ we repeat the above argument, replacing $\widetilde{u}_1, \widetilde{u}_2$ by $w, \widetilde{u}_3$ and L_1, L_2 by L, L_3. So doing the same procedure $n - 1$ times we obtain Proposition 3.1. □

3.2 Proof of Proposition 3.2

Now we have a *nonnegative* solution of $L_i u_i + \sum_j c_{ij} u_j \leq f_i$ and we want to show (9). We are going to use the argument of [2], which simplifies greatly in the linear case.

Up to a change of coordinates we can suppose that $R = 1$. By (H_0) we have $L_i u_i + c_{ii} u_i \leq f_i$ for each i, so applying the scalar weak Harnack inequality (Theorem 9.22 in [8], see also [16], [10] for the case of unbounded coefficients[2]) we obtain

$$\frac{1}{|B_2|} \int_{B_2} |\overline{U}^+|^p \leq C \left(\max_{1 \leq i \leq n} \inf_{B_2} u_i + \|f\|_{L^N(B_2)} \right).$$

Hence it only remains to show that

$$\max_{1 \leq i \leq n} \inf_{B_2} u_i \leq C(\min_{1 \leq i \leq n} \inf_{B_1} u_i + \|f\|_{L^N(B_2)}). \tag{13}$$

[2]Recently, Koike and Swiech obtained the Harnack inequality in an even more general context, see [9]

Recall that the system is supposed to be fully coupled. It is easy to see that this implies that there exists a permutation $\{j_1, \dots, j_n\}$ of $\{1, \dots, n\}$ such that $c_{kj_k} \not\equiv 0$ in B_1.

We are going to show that for each $k = 1, \dots, n$, we have

$$\inf_{B_{(1+(k-1)/n)}} u_k \geq \inf_{B_{(1+k/n)}} u_{j_k} - C\|f\|_{L^N(B_2)},$$

from which the desired inequality (13) easily follows. Say $k = 1$. Note if $\inf_{B_{(1+1/n)}} u_{j_1} = 0$ there is nothing to prove, so we can suppose this quantity is positive.

Let w be the solution of the Dirichlet problem

$$-L_1 w + c_{11}^- w = (\inf_{B_{(1+1/n)}} u_{j_1}) c_{1j_1}(x) \qquad \text{in } B_{(1+1/n)},$$

and $w = 0$ on $\partial B_{(1+1/n)}$. The function w is a solution of a problem whose right-hand side is nonnegative everywhere (so w is positive), and larger than the positive constant $\rho(\inf_{B_{(1+1/n)}} u_{j_1})$ on a set of measure $\omega > 0$. By applying a (deep) theorem by Krylov – Theorem 12 on page 129 in [12], or Theorem 9.2 of [1], an easier to read proof of this result is given in the Appendix of [11] – we get

$$\inf_{B_1} w \geq C^{-1} \rho(\inf_{B_{(1+1/n)}} u_{j_1}).$$

On the other hand $L_1 u_1 + c_{11} u_1 + c_{1j_1} u_{j_1} \leq f_1$ implies

$$L_1(w - u_1) - c_{11}^-(w - u_1) \geq -(\inf_{B_{(1+1/n)}} u_{j_1}) c_{1j_1}(x) - f_1 + c_{1j_1}(x) u_{j_1} \geq -f_1$$

in $B_{(1+1/n)}$. Since $w - u_1 \leq 0$ on $\partial B_{(1+1/n)}$ the ABP inequality (Theorem 4) gives

$$\sup_{B_{(1+1/n)}} (w - u_1) \leq C\|f_1\|_{L^N(B_{1+1/n})},$$

which implies $\inf_{B_1} u_1 \geq \inf_{B_1} w - C\|f_1\|_{L^N(B_{(1+1/n)})}$, and we conclude. $\square$

3.3 Proof of Proposition 1.1

First, the operator $L + \overline{\mathcal{C}}$ satisfies the maximum principle in Ω. This is proved exactly by the same argument as the one on page 123 of [14] - we only have to take for the function ψ there the vector $(\phi_1, \dots, \phi_1)$, where ϕ_1 is the first eigenfunction of L_1 in a slightly larger domain $\Omega' \Supset \Omega$, such that we still have $\lambda_1(L_1, \Omega')I > \overline{\mathcal{C}}$. Let us briefly recall this reasoning. We replace u_i by $v_i = \phi_1 u_i$, and see that V satisfies $\widetilde{L}V + (\mathcal{C} - \lambda(L_1, \Omega')I)V \geq 0$, for some modified operator $\widetilde{L}$, of the same type as L. Then we evaluate the i-th inequation in

this system at a point of positive maximum of v_i (removing from the system the equations with indices j for which v_j is nonpositive in Ω), and see that the vector W_0 of positive maxima of u_i satisfies $(\mathcal{C} - \lambda(L_1, \Omega')I)W_0 \geq 0$. Multiplying this inequality by W_0 we obtain $W_0 = 0$.

Let U be a vector function such that $LU + \mathcal{C}U \geq 0$ in Ω and $U \leq 0$ on $\partial\Omega$. We set $h_\varepsilon(U) = (h_\varepsilon(u_1), \ldots, h_\varepsilon(u_n)) \geq 0$, where h_ε is the function from the proof of Proposition 3.1. Then by (10), (11) and the Kato inequality we have

$$Lh_\varepsilon(U) + \overline{\mathcal{C}}h_\varepsilon(U) \geq h'_\varepsilon(U)LU + \mathcal{C}h_\varepsilon(U) \geq \mathcal{C}(-h'_\varepsilon(U)U + h_\varepsilon(U)) \geq -\varepsilon\|\mathcal{C}\|,$$

and $h_\varepsilon(U) = 0$ on $\partial\Omega$. Since $L + \overline{\mathcal{C}}$ satisfies the maximum principle, by Theorem 3 it also satisfies the ABP inequality, which implies

$$\sup h_\varepsilon(U) \leq C\varepsilon.$$

Letting $\varepsilon \to 0$ gives $U^+ = 0$ in Ω, which is what we wanted to prove.

Remark. Note the above quoted argument in [14] was carried out under the hypothesis $\overline{\mathcal{C}} \prec \lambda_1(L_1)I$, defined by

$$\overline{\mathcal{C}} \prec \lambda_1(L_1)I \Longleftrightarrow \forall\, V \in \mathbb{R}^n \ : \ \begin{cases} \overline{\mathcal{C}}V & \geq & \lambda_1(L_1)V \\ V & \geq & 0 \end{cases} \quad \text{implies} \quad V = 0.$$

In general this is a weaker hypothesis than $\overline{\mathcal{C}} < \lambda_1(L_1)I$ but in this particular situation these two turn out to be equivalent.

Lemma 3.1 *If C is a constant cooperative matrix, then $C \prec \lambda I$ is equivalent to $C < \lambda I$.*

Note this lemma strongly depends on the cooperativity of C and on the direction of the inequalities - for instance, it is not true that $\lambda I \prec C$ implies $\lambda I < C$ – take for example $C = \begin{pmatrix} 0 & 2\lambda \\ 2\lambda & 0 \end{pmatrix}$. That is why the other results in [14] cannot be stated using the relation "$<$" between matrices.

Proof of Lemma 3.1. It is obvious that $C < \lambda I$ implies $C \prec \lambda I$. So let us show the contrary.

By our hypotheses the matrix $B = \lambda I - C$ satisfies the assumptions of Lemma 2.3 in [4], so by this lemma the positive definiteness of B follows from the implication

$$BX > 0 \Rightarrow X > 0, \qquad \text{for all } \ X \in \mathbb{R}^n.$$

Suppose this is wrong, that is, there exists $Y \in \mathbb{R}^n$ such that $BY < 0$ and $Y \not< 0$.

If $Y_i = 0$ for some i then the i-th equation in $BY < 0$ and $b_{ij} \leq 0$ for $i \neq j$ imply that $y_j > 0$ for some j. This and $Y \not< 0$ imply that Y^+ is not the zero vector.

We are going to show that $BY^+ \leq 0$ - this is then a contradiction with $0 \prec B$. Let us suppose $n = 2$ for simplicity (the argument is exactly the same for any n). Say $y_1 > 0$. If $y_2 \geq 0$ we are done. If $y_2 \leq 0$ we have $b_{12}y_2 \geq 0$ (by the cooperativity of B) so the first line of $BY < 0$ gives $b_{11}y_1 < 0$. This inequality together with $b_{12}y_1 \leq 0$ (again by the cooperativity) gives exactly $BY^+ \leq 0$. $\square$

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