



**HAL**  
open science

# Asymptotic behaviour of reversible chemical reaction-diffusion equations

Ivan Gentil, Boguslaw Zegarlinski

► **To cite this version:**

Ivan Gentil, Boguslaw Zegarlinski. Asymptotic behaviour of reversible chemical reaction-diffusion equations. 2009. hal-00351445v1

**HAL Id: hal-00351445**

**<https://hal.science/hal-00351445v1>**

Preprint submitted on 9 Jan 2009 (v1), last revised 11 Apr 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Asymptotic behaviour of a general reversible chemical reaction-diffusion equation

Ivan Gentil and Boguslaw Zegarlinski

January 9, 2009

## Abstract

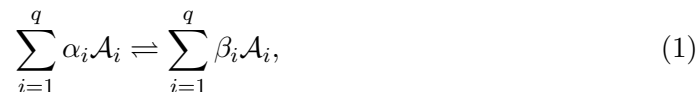
In this work, we prove the existence and the exponential decay to equilibrium of a general reversible chemical reaction-diffusion equation with same but general diffusion. Moreover, we prove the optimal asymptotic behaviour in the "two-by-two" case.

**Key words:** Reaction-diffusion equation, Spectral gap inequality, Poincaré inequality, Exponential decay.

**AMS subject classification:** 35B40, 35A05, 35F25

## 1 Introduction

In this paper we study the asymptotic behaviour of the following reversible chemical reaction-diffusion



where  $q$  is a positive integer and  $\alpha_i, \beta_i \in \mathbb{N}$ , with  $\mathbb{N}$  denoting the set of nonnegative integers. We assume that for any  $1 \leq i \leq q$ ,  $\alpha_i - \beta_i \neq 0$  which correspond to the case of a reaction without catalyzer.

First we will introduce the differential equations that describe the evolution of the species  $\mathcal{A}_i$  for  $1 \leq i \leq q$  in the diagram (1). For some  $0 \leq i \leq q$  we will denote by  $a_i$  the concentration of the specie  $\mathcal{A}_i$ . We assume existence of two non-negative real-valued rate functions

$$\mathcal{K}_{\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightarrow \sum_{i=1}^q \beta_i \mathcal{A}_i}, \quad \mathcal{K}_{\sum_{i=1}^q \beta_i \mathcal{A}_i \rightarrow \sum_{i=1}^q \alpha_i \mathcal{A}_i},$$

which describe the evolution of the concentration  $a_i$  in two reactions. In the first one we lose  $\alpha_i$  molecules of the specie  $\mathcal{A}_i$  and in the second there is a gain of  $\beta_i$  molecule of the same specie  $\mathcal{A}_i$ . We get the opposite for the reverse reaction. Thus we can write

$$\frac{d}{dt} a_i = (\beta_i - \alpha_i) \mathcal{K}_{\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightarrow \sum_{i=1}^q \beta_i \mathcal{A}_i}(a) - (\beta_i - \alpha_i) \mathcal{K}_{\sum_{i=1}^q \beta_i \mathcal{A}_i \rightarrow \sum_{i=1}^q \alpha_i \mathcal{A}_i}(a), \quad (2)$$

where  $a = (a_1, \dots, a_q)$ . We assume the kinetics to be of *mass action type*, which means that

$$\mathcal{K}_{\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightarrow \sum_{i=1}^q \beta_i \mathcal{A}_i}(a_i) = l \prod_{j=1}^q a_j^{\alpha_j},$$

where  $l$  is a positive constant called *the rate constant* of the reaction. The product form represents the probability which is assumed to be independent, that  $\alpha_i$  molecules  $\mathcal{A}_i$  met the other  $\alpha_j$  molecules  $\mathcal{A}_j$  for the reaction. Let us denote by  $k > 0$  the rate constant for the reverse reaction. Of course  $l$  and  $k$  could be very different.

We will presume that the pot of the reaction is not stirred. Then concentrations of species depend on the position in the pot and we have to add a diffusion term which depends also on the species. We obtain then the problem of the following general form

$$\partial_t a_i = L_i a_i + (\beta_i - \alpha_i) \left( l \prod_{j=1}^q a_j^{\alpha_j} - k \prod_{j=1}^q a_j^{\beta_j} \right),$$

where for all  $i$ ,  $L_i$  is a diffusion operator.

Throughout the entire paper we will assume that all species diffuse with the same speed, i.e. for all  $i \in \{1, \dots, q\}$ ,  $L_i = L$ , where  $L$  is a diffusion generator. This is a restrictive case but it we will treat, using semigroup tools, the case of a general diffusion in the domain.

In more detail we let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 1$ ) to be an open set and assume that the boundary  $\partial\Omega$  of  $\Omega$ , is  $C^\infty$  - smooth. Let a diffusion operator  $L$  be given by

$$Lf(x) = \sum_{i,j=1}^n b_{i,j}(x) \partial_{i,j}^2 f(x) + \sum_{i=1}^n b_i(x) \partial_i f(x),$$

for any smooth functions  $f$ , with  $b_{i,j}$  and  $b_i$  in  $C^\infty(\Omega)$  and the matrix  $(b_{i,j}(x))_{i,j}$  symmetric and positive for all  $x \in \Omega$ . The choice of coefficients  $b_{i,j}$  and  $b_i$  may strongly depend on the domain  $\Omega$ .

If we denote by  $a_i(t, x)$  ( $t \geq 0$ ,  $x \in \mathbb{R}^n$ ) the concentrations of the species  $\mathcal{A}_i$  at time  $t$  in the position  $x$ , then the following reaction-diffusion system is satisfied:

$$\left\{ \begin{array}{l} \forall t > 0, \forall x \in \Omega, \quad \partial_t a_i(t, x) = La_i + (\beta_i - \alpha_i) \left( l \prod_{j=1}^q a_j^{\alpha_j} - k \prod_{j=1}^q a_j^{\beta_j} \right), \\ \forall x \in \Omega, \quad a_i(0, x) = a_i^0(x) \\ \forall x \in \partial\Omega, \forall t > 0, \quad \frac{\partial a_i}{\partial \nu}(t, x) = 0. \end{array} \right. \quad (3)$$

where  $k, l > 0$  are rate constants of the reverse reaction. Initial conditions satisfy for all  $1 \leq i \leq q$ ,  $a_i^0 \geq 0$  and  $\int a_i^0 d\mu > 0$ .

The last equation in (3) represents the Neumann boundary conditions which is natural in the context of chemical reaction-diffusion. Let us denote by  $\mathcal{D}(\Omega)$  the set of smooth functions  $f$  on  $\Omega$  satisfying the Neumann boundary conditions, such that for all  $x \in \partial\Omega$ ,  $\frac{\partial f}{\partial \nu}(t, x) = 0$ .

In this paper we assume that there exists a probability measure  $\mu$  such that  $L$  is invariant with respect to  $\mu$ , i.e. for all functions  $f \in \mathcal{D}(\Omega)$ ,

$$\int Lf d\mu = 0.$$

Let  $(\mathbf{P}_t)_{t \geq 0}$  be the semi-group associated to  $L$  which is defined for all functions  $f \in \mathcal{D}(\Omega)$ ,

$$\left\{ \begin{array}{l} \forall t > 0, \forall x \in \Omega, \quad \frac{\partial}{\partial t} \mathbf{P}_t(f)(x) = L\mathbf{P}_t(f)(x) \\ \forall x \in \Omega, \quad \mathbf{P}_0(f)(x) = f(x) \\ \forall x \in \partial\Omega, \forall t > 0, \quad \frac{\partial \mathbf{P}_t(f)}{\partial \nu}(x) = 0. \end{array} \right.$$

We will assume furthermore that the operator  $L$  has a spectral gap  $C_{SG}$  in  $\mathcal{D}(\Omega)$  between zero eigenvalue and the rest of its spectrum. Then for all smooth function  $f \in \mathcal{D}(\Omega)$ ,

$$\mathbf{Var}_\mu(f) := \int \left( f - \int f d\mu \right)^2 d\mu \leq -C_{SG} \int f L f d\mu, \quad (4)$$

which is equivalent to, for all functions  $f \in \mathcal{D}(\Omega)$  and  $t \geq 0$ :

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq e^{-\frac{1}{C_{SG}}t} \mathbf{Var}_\mu(f). \quad (5)$$

See for example chapter 2 of [ABC<sup>+</sup>00] of a review on semigroups tools.

First we study the problem of a steady state of the differential system (3). Let  $v_i(t, x) = \lambda_i a_i(t, x)$  for some constant  $\lambda_i > 0$ , assuming that  $a_i$  is solution of (3) then  $v_i$  satisfies

$$\partial_t v_i = L v_i + \lambda_i (\beta_i - \alpha_i) \left( l \prod_{i=1}^q \left( \frac{v_i}{\lambda_i} \right)^{\alpha_i} - k \prod_{i=1}^q \left( \frac{v_i}{\lambda_i} \right)^{\beta_i} \right).$$

Let us chose constants  $\lambda_i > 0$  such that  $k \prod_{i=1}^q (\lambda_i)^{\alpha_i} = l \prod_{i=1}^q (\lambda_i)^{\beta_i}$ , or equivalently  $\prod_{i=1}^q (\lambda_i)^{\alpha_i - \beta_i} = k/l$ . We then obtain that  $v_i$  is solution of

$$\begin{cases} \forall t > 0, \forall x \in \Omega, & \partial_t v_i(t, x) = L v_i(t, x) + k_i (\beta_i - \alpha_i) G(v_1(t, x), \dots, v_q(t, x)), \\ \forall x \in \Omega, & v_i(0, x) = v_i^0(x) \\ \forall x \in \partial\Omega, \forall t > 0, & \frac{\partial v_i}{\partial \nu}(t, x) = 0 \end{cases} \quad (6)$$

where  $G(v_1, \dots, v_q) = \prod_{j=1}^q v_j^{\alpha_j} - \prod_{j=1}^q v_j^{\beta_j}$  and

$$k_i = \frac{\lambda_i l}{\prod_{i=1}^q (\lambda_i)^{\alpha_i}}. \quad (7)$$

Let us define the set  $\mathcal{S} = \left\{ (z_i)_{1 \leq i \leq q}, \text{ s.t. } \sum_{i=1}^q z_i k_i (\beta_i - \alpha_i) = 0 \right\}$ , where  $k_i$  is defined in (7). Then for all  $(z_i)_{1 \leq i \leq q} \in \mathcal{S}$  one gets

$$\partial_t \sum_{i=1}^q z_i v_i = L \sum_{i=1}^q z_i v_i,$$

which gives

$$\sum_{i=1}^q z_i v_i(t, x) = \mathbf{P}_t \left( \sum_{i=1}^q z_i v_i^0 \right) (x).$$

In particular, due to the fact that  $\mu$  is an invariant measure,  $\sum_{i=1}^q z_i \int v_i(t, x) d\mu(x) = \sum_{i=1}^q z_i \int v_i^0 d\mu$ .

The goal of this article is to understand the asymptotic behaviour of the reaction-diffusion problem in the spirit of [DF06, DF08] but for a general diffusion.

We will first define a *Steady State* of equation (6).

**Definition 1.1** *A steady state of equation (6) with non-negative initial conditions  $(v_i^0)_{1 \leq i \leq q}$  is a vector  $(s_i)_{1 \leq i \leq q} \in (\mathbb{R}^+)^q$  such that for all  $(z_i)_{1 \leq i \leq q} \in \mathcal{S}$ :*

$$\sum_{i=1}^q z_i s_i = \sum_{i=1}^q z_i \int v_i^0 d\mu \quad \text{and} \quad \prod_{i=1}^q s_i^{\alpha_i} = \prod_{i=1}^q s_i^{\beta_i}.$$

**Lemma 1.2** *Let  $(v_i)_{1 \leq i \leq q}$  satisfies equation (6) with initial conditions satisfying  $v_i^0 \geq 0$  and  $\int v_i^0 d\mu > 0$ . Then there exists a unique steady state  $(s_i)_{1 \leq i \leq q}$  of (6) such that for all  $i \in \{1, \dots, q\}$ ,  $s_i > 0$ .*

**Proof**

◁ The steady state  $(s_i)_{1 \leq i \leq q}$  has to satisfy:

$$\forall (z_i)_{1 \leq i \leq q} \in \mathcal{S}, \quad \sum_{i=1}^q z_i s_i = \sum_{i=1}^q z_i \int v_i^0 d\mu := M_z.$$

Let  $A = \{(s_i)_{1 \leq i \leq q}, \sum_{i=1}^q z_i s_i = M_z\}$ . Due to the fact that  $\mathcal{S}$  is a subspace of  $\mathbb{R}^n$  of dimension  $q - 1$  we get that  $A$  is a manifold of dimension 1. Then one gets

$$A = \left\{ \left( \int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) \right)_{1 \leq i \leq q}, t \in \mathbb{R} \right\}.$$

We have to find  $t \in \mathbb{R}$  such that for all  $1 \leq i \leq q$ ,  $\int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) > 0$  and

$$\prod_{i=1}^q \left( \int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) \right)^{\beta_i - \alpha_i} = 1.$$

A simple computation gives that the function

$$\varphi(t) = \prod_{i=1}^q \left( \int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) \right)^{\beta_i - \alpha_i}$$

is defined on the set  $[a, b)$  with

$$\begin{cases} a = \inf \left\{ t, \text{ s.t. } \forall i, \int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) \geq 0 \right\} \\ b = \sup \left\{ t, \text{ s.t. } \forall i, \int v_i^0 d\mu + t k_i (\beta_i - \alpha_i) \geq 0 \right\}. \end{cases}$$

For all  $t \in [a, b)$  one gets

$$\frac{\varphi'(t)}{\varphi(t)} = \sum_{i=1}^q \frac{k_i (\beta_i - \alpha_i)^2}{\int v_i^0 d\mu + t k_i (\beta_i - \alpha_i)},$$

then  $\varphi$  is increasing and satisfies  $\varphi(a) = 0$  and  $\varphi(b) = +\infty$ . Thus there exists a unique  $t \in (a, b)$  such that  $\varphi(t) = 1$ , which ends the proof. ▷

The goal of this paper is to prove, using the Poincaré or spectral gap inequality, that the solution of reaction-diffusion equation (6) converges to the unique steady state associated to the initial condition.

Due to the fact that all species are moving according to the same diffusion, one can reduce the problem as follow: for all  $1 \leq i, j \leq q$ , one gets

$$\partial_t \left( \frac{v_i}{k_i (\beta_i - \alpha_i)} - \frac{v_j}{k_j (\beta_j - \alpha_j)} \right) = L \left( \frac{v_i}{k_i (\beta_i - \alpha_i)} - \frac{v_j}{k_j (\beta_j - \alpha_j)} \right)$$

and thus

$$\begin{aligned} v_i &= \frac{k_i (\beta_i - \alpha_i)}{k_j (\beta_j - \alpha_j)} v_j + \mathbf{P}_t \left( v_i^0 - \frac{k_i (\beta_i - \alpha_i)}{k_j (\beta_j - \alpha_j)} v_j^0 \right) \\ &:= \frac{k_i (\beta_i - \alpha_i)}{k_j (\beta_j - \alpha_j)} v_j + C_{i,j}(t, x). \end{aligned} \tag{8}$$

Let us fix  $1 \leq i \leq q$ . Then the study of (6) is equivalent to the study of the following PDE with boundary conditions

$$\begin{aligned} \partial_t v_i &= Lv_i + k_i(\beta_i - \alpha_i) \times \\ &\left( \prod_{j=1}^q \left( \frac{k_j(\beta_j - \alpha_j)}{k_i(\beta_i - \alpha_i)} v_j + C_{j,i}(t, x) \right)^{\alpha_j} - \prod_{j=1}^q \left( \frac{k_j(\beta_j - \alpha_j)}{k_i(\beta_i - \alpha_i)} v_j + C_{j,i}(t, x) \right)^{\beta_j} \right) \quad (9) \\ &:= Lv_i + k_i(\beta_i - \alpha_i) F(t, x, v_i). \end{aligned}$$

We will consider different cases. The first one is without diffusion that means that concentration of different species do not depend on  $x \in \Omega$ . We will prove that there exists a solution in that case and it converges to the steady state with an exponential and optimal rate of convergence.

Then we study the following “two-by-two” case

$$\mathcal{A} + \mathcal{B} \rightleftharpoons \mathcal{C} + \mathcal{D},$$

treated in [DF08] by entropy methods. This case is interesting because it gives a tool to understand the general cases.

We finish in the last section with the general case

$$\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightleftharpoons \sum_{i=1}^q \beta_i \mathcal{A}_i.$$

We prove that there exists a non-negative solution and, under the assumption that  $\alpha_i \beta_i = 0$ , we get that solution converges with an exponential rate to the steady state.

## 2 Case without diffusion

Assume that concentrations of species do not depend on  $x \in \Omega$ . It is the case when the pot used for the chemical reaction is stirred constantly so that its contents remain spatially homogeneous. This case is important because we can solve it explicitly and it gives tools to study the general case in the later sections.

The chemical reaction, without diffusion is given by the following system, for all  $i \in \{1, \dots, q\}$ ,

$$\frac{d}{dt} v_i = k_i(\beta_i - \alpha_i) \left( \prod_{j=1}^q v_j^{\alpha_j} - \prod_{j=1}^q v_j^{\beta_j} \right), \quad (10)$$

where  $k_i$  is defined in (7), with initial conditions  $v_i(0) > 0$  for  $1 \leq i \leq q$ . Equivalently, using the same method as for (9), for some  $1 \leq i \leq q$  fixed,

$$\begin{aligned} \frac{d}{dt} v_i &= k_i(\beta_i - \alpha_i) \left( \prod_{j=1}^q \left( \frac{k_j(\beta_j - \alpha_j)}{k_i(\beta_i - \alpha_i)} v_j + C_{j,i} \right)^{\alpha_j} - \prod_{j=1}^q \left( \frac{k_j(\beta_j - \alpha_j)}{k_i(\beta_i - \alpha_i)} v_j + C_{j,i} \right)^{\beta_j} \right) \\ &:= k_i(\beta_i - \alpha_i) F(v_i), \end{aligned}$$

where  $C_{j,i} = k_j(\beta_j - \alpha_j) \left( \frac{v_j(0)}{k_j(\beta_j - \alpha_j)} - \frac{v_i(0)}{k_i(\beta_i - \alpha_i)} \right)$ .

**Theorem 2.1** *Let the initial conditions  $(v_j(0))_{1 \leq j \leq q}$  be positive. Then equation (10) has a unique solution defined on  $[0, +\infty)$ , which satisfies for all  $1 \leq i \leq q$ ,*

$$|v_i(t) - s_i| \leq e^K |v_i(0) - s_i| \exp(-Ct),$$

where  $K$  is a constant depending on initial conditions, the steady state  $(s_i)_{1 \leq i \leq q}$  is defined in Lemma 1.2 and

$$C = \prod_{i=1}^q s_i^{\alpha_i} \sum_{i=1}^q \frac{k_i(\beta_i - \alpha_i)^2}{s_i}. \quad (11)$$

Moreover the constant  $C$  is the optimal rate of convergence.

**Proof**

◁ Let  $j_0$  be such that  $\beta_{j_0} - \alpha_{j_0} > 0$  and for all  $i$  s.t.  $\beta_i - \alpha_i > 0$ , one has  $\frac{v_{j_0}(0)}{k_{j_0}(\beta_{j_0} - \alpha_{j_0})} \leq \frac{v_i(0)}{k_i(\beta_i - \alpha_i)}$ . If the set  $\{i; \beta_i - \alpha_i > 0\}$  is empty one can use the negative part. Assume for simplicity that  $j_0 = 1$ . Then the reaction equation becomes

$$\frac{d}{dt}v_1 = k_1(\beta_1 - \alpha_1) \times \left( \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1 + C_{i,1} \right)^{\alpha_i} - \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1 + C_{i,1} \right)^{\beta_i} \right). \quad (12)$$

By the definition of  $j_0$  one have,  $C_{i,1} \geq 0$  if  $\beta_i - \alpha_i > 0$  and  $C_{i,1} \leq 0$  if  $\beta_i - \alpha_i < 0$ .

Getting a positive solution  $(v_i)_{1 \leq i \leq q}$  of (10) is equivalent to getting a solution  $v_1$  of (12) which satisfies the following inequality

$$\forall t \in [0, +\infty), \quad 0 < v_1(t) < M,$$

where  $M = v_1(0) - k_1(\beta_1 - \alpha_1) \max \left\{ \frac{v_i(0)}{k_i(\beta_i - \alpha_i)}; \beta_i - \alpha_i < 0 \right\}$ . By convention, if the set  $\{i; \beta_i - \alpha_i < 0\}$  is empty, then we have  $\max \left\{ \frac{v_i(0)}{k_i(\beta_i - \alpha_i)}; \beta_i - \alpha_i < 0 \right\} = -\infty$ .

Let us denote by

$$F(X) = k_1(\beta_1 - \alpha_1) \times \left( \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} X + C_{i,1} \right)^{\alpha_i} - \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} X + C_{i,1} \right)^{\beta_i} \right). \quad (13)$$

Lemma 1.2 proves that the polynome  $F$  has only one solution  $s_1$  in the set  $(0, M)$ . Let  $Q$  be a factor of  $F$ , i.e. we have a factorization  $F(X) = (X - s_1)Q(X)$ . Then a simple computation yields

$$F'(s_1) = - \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} s_1 + C_{i,1} \right)^{\alpha_i} \sum_{i=1}^q \frac{k_i(\beta_i - \alpha_i)^2}{\frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} s_1 + C_{i,1}} = - \prod_{i=1}^q s_i^{\alpha_i} \sum_{i=1}^q \frac{k_i(\beta_i - \alpha_i)^2}{s_i} := -C < 0,$$

which proves that  $F'(s_1) = Q(s_1) < 0$  and then  $Q(X) < 0$  for all  $X \in (0, M)$ .

Function  $F$  is locally a Lipschitz function, thus by Cauchy-Lipschitz's Theorem, there exists a unique maximal solution starting at  $v_1(0)$  of the equation

$$\forall t \in [0, T), \quad \frac{dv_1(t)}{dt} = F(v_1(t)) = (v_1(t) - s_1)Q(v_1(t)),$$

for some  $T > 0$ .

One has  $Q(X) < 0$  for all  $X \in (0, M)$ , it implies that if  $v_1(0) > s_1$  then  $v_1$  is non-increasing and moreover  $v_1(t) \geq s_1$  for all  $t \in [0, T[$ , if  $v_1(0) < s_1$  then  $v_1$  is non-decreasing and moreover  $v_1(t) \leq s_1$  for all  $t \in [0, T[$  and if  $v_1(0) = s_1$  then  $v_1(t) = s_1$  for all  $t \in [0, T[$ . Then one gets that  $T = +\infty$  and for all  $t \geq 0$ ,  $v_1(t) \in (0, M)$ .

Using the identity

$$\frac{1}{F(X)} = \frac{1/Q(s_1)}{X - s_1} + \frac{R(X)}{Q(X)},$$

with  $R(X) = (Q(s_1) - Q(X))/(Q(s_1)(x - s_1))$  we get for all  $t \geq 0$

$$|v_1(t) - s_1| = |v_1(0) - s_1| \exp \left( Q(s_1)t + \int_{v_1(0)}^{v_1(t)} \frac{Q(a) - Q(s_1)}{(a - s_1)Q(a)} da \right). \quad (14)$$

This identity proves that  $v_1$  goes to  $s_1$  and moreover if

$$K = \int_{v_1(0)}^{s_1} \frac{Q(a) - Q(s_1)}{(a - s_1)Q(a)} da,$$

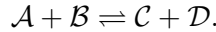
then for all  $t \geq 0$ ,

$$|v_1(t) - s_1| \leq e^K |v_1(0) - s_1| \exp(-Ct).$$

Links between  $v_i$  and  $v_1$  gives the last inequalities for any  $v_i$ . Moreover, equality (14) proves that the constant  $C$  is the optimal rate.  $\triangleright$

### 3 The “two-by-two” case

The goal of this section is to investigate the asymptotic behaviour of a particular chemical reaction



We will assume that all species are moving according to the same generator and the speed of the two reactions are the same, for instance equal to 1.

Note that this case was globally treated in dimension 1, but for the general problem, in [DF08].

Let  $\Omega \subset \mathbb{R}^n$  be a domain, not necessary bounded. Let us consider an infinitesimal generator  $L$ , with a domain  $\mathcal{D}_L$ , associated to a semigroup  $(\mathbf{P}_t)_{t \geq 0}$ . As in the introduction, assume that there exists an invariant probability measure  $\mu$  associated to the semigroup.

Let us denote by  $a, b, c$  and  $d$  concentrations of  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  in the domain  $\Omega$ . In this case the functions  $a, b, c$  and  $d$  are solutions of the following system on  $\Omega$ ,

$$\begin{cases} \partial_t a = La - ab + cd \\ \partial_t b = Lb - ab + cd \\ \partial_t c = Lc + ab - cd \\ \partial_t d = Ld + ab - cd \end{cases} \quad (15)$$

with non-negative initial conditions in  $\mathcal{D}_L$ ,  $a_0, b_0, c_0$  and  $d_0$  such that  $\int a_0 d\mu > 0$  and the same for  $b_0, c_0$  and  $d_0$ . The boundaries conditions are contained in the domain  $\mathcal{D}_L$ .

If  $a, b, c, d$  are solutions, then  $a + c = \mathbf{P}_t(a_0 + c_0)$ ,  $a + d = \mathbf{P}_t(a_0 + d_0)$  and  $a - b = \mathbf{P}_t(a_0 - b_0)$  which gives that the function  $a$  is solution of the linear equation

$$\partial_t a = La - aD_t + C_t,$$



where  $C_t = \mathbf{P}_t(a_0 + c_0)\mathbf{P}_t(a_0 + d_0)$  and  $D_t = \mathbf{P}_t(a_0 + b_0 + c_0 + d_0)$ .

We will assume in this section that there exists a smooth and positive solution of the problem. Results about existence are given in section 4.

We also obtain the estimation useful for the asymptotic behaviour.

**Lemma 3.1** *Solution  $a, b, c$  and  $d$  of (15) satisfy for all  $t \geq 0$  and  $x \in \Omega$ ,*

$$\begin{cases} 0 \leq a(t, x) \leq \min \{ \mathbf{P}_t(a_0 + c_0), \mathbf{P}_t(a_0 + d_0) \} \\ 0 \leq b(t, x) \leq \min \{ \mathbf{P}_t(b_0 + c_0), \mathbf{P}_t(b_0 + d_0) \} \\ 0 \leq c(t, x) \leq \min \{ \mathbf{P}_t(c_0 + a_0), \mathbf{P}_t(c_0 + b_0) \} \\ 0 \leq d(t, x) \leq \min \{ \mathbf{P}_t(d_0 + a_0), \mathbf{P}_t(d_0 + b_0) \} \end{cases} \quad (16)$$

**Theorem 3.2** *Assume that the semigroup  $(\mathbf{P}_t)_{t \geq 0}$  satisfies a spectral gap inequality (5) with respect to the invariant probability measure  $\mu$ . Let  $a_0, b_0, c_0$  and  $d_0$  be non-negative initial conditions satisfying  $a_0, b_0, c_0, d_0 \in L^4(d\mu)$ .*

We set  $M_{a+b+c+d} = \int (a_0 + b_0 + c_0 + d_0) d\mu$ ,  $M_4 = \left( \int (a_0 + b_0 + c_0 + d_0)^4 d\mu \right)^{\frac{1}{2}}$  and  $C_{SG}$  denotes the constant in the spectral gap inequality (5).

Let  $s_a$  is the steady state associated to the initial condition  $a_0$ , if  $M_{a+b+c+d} \neq \frac{1}{8C_{SG}}$ , then for all  $t \geq 0$ ,

$$\sqrt{\int (a - s_a)^2 d\mu} \leq \left( \sqrt{\int (a_0 - s_a)^2 d\mu} + \left| \frac{5M_4}{M_{a+b+c+d} - \frac{1}{8C_{SG}}} \right| \right) \times \exp \left( - \min \left\{ M_{a+b+c+d}, \frac{1}{8C_{SG}} \right\} t \right), \quad (17)$$

and if  $M_{a+b+c+d} = \frac{1}{8C_{SG}}$  then for all  $t \geq 0$ ,

$$\sqrt{\int (a - s_a)^2 d\mu} \leq \left( \sqrt{\int (a_0 - s_a)^2 d\mu} + 5M_4 t \right) \exp(-M_{a+b+c+d} t).$$

The same inequality holds for  $b, c$  and  $d$  associated to  $s_b, s_c$  and  $s_d$ .

If the initial conditions satisfy  $M_{a+b+c+d} < \frac{1}{8C_{SG}}$ , then the rate of the convergence is optimal.

Let us start with a general estimate.

**Lemma 3.3** *Assume that the semigroup  $(\mathbf{P}_t)_{t \geq 0}$  satisfies a spectral gap inequality (5) with respect to the invariant probability measure  $\mu$ , then for all functions  $f \in L^4(\mu)$  and all  $t \geq 0$ ,*

$$\int \left( \mathbf{P}_t f - \int f d\mu \right)^4 d\mu \leq 4e^{-\frac{1}{2C_{SG}} t} \int f^4 d\mu. \quad (18)$$

**Proof**

$\triangleleft$  Set  $\tilde{f} = f - \int f d\mu$ , then using semigroup properties and the Cauchy-Schwartz inequality applied to  $(\mathbf{P}_t)_{t \geq 0}$ , one gets

$$\int \left( \mathbf{P}_t f - \int f d\mu \right)^4 d\mu = \int \left( \mathbf{P}_t \tilde{f} \right)^4 d\mu = \int \left( \mathbf{P}_{\frac{t}{2}} \mathbf{P}_{\frac{t}{2}} \tilde{f} \right)^4 d\mu \leq \int \left( \mathbf{P}_{\frac{t}{2}} \left( \mathbf{P}_{\frac{t}{2}} \tilde{f} \right)^2 \right)^2 d\mu.$$

If we set  $F = \mathbf{P}_{\frac{t}{2}}(\tilde{f})$ , then one has

$$\int (\mathbf{P}_t \tilde{f})^4 d\mu \leq 2 \int \left( \mathbf{P}_{\frac{t}{2}}(F^2) - \int F^2 d\mu \right)^2 d\mu + 2 \left( \int \left( \mathbf{P}_{\frac{t}{2}}(f) - \int f d\mu \right)^2 d\mu \right)^2.$$

We apply twice inequality (5) to  $F$  and to  $f$  to obtain

$$\int (\mathbf{P}_t \tilde{f})^4 d\mu \leq 2e^{-\frac{t}{2c_{SG}}} \mathbf{Var}_\mu(F^2) + 2e^{-\frac{t}{c_{SG}}} (\mathbf{Var}_\mu(f^2))^2,$$

which implies (18).  $\triangleright$

**Proof of Theorem 3.2**

$\triangleleft$  In this case, the steady state is the following limit,

$$s_a = \frac{\int (a_0 + c_0) d\mu \int (a_0 + d_0) d\mu}{\int (a_0 + b_0 + d_0 + c_0) d\mu} = \lim_{t \rightarrow +\infty} \frac{C_t}{D_t}, \quad (19)$$

the limit can be seen in  $L^4(d\mu)$ . Let us denote by  $M_{a+c} = \int (a_0 + c_0) d\mu$  and define similarly  $M_{a+d}$  and  $M_{a+b+c+d}$ .

Using (19) one obtains

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int (a - s_a)^2 d\mu &= - \int |\nabla a|^2 d\mu - M_{a+b+c+d} \int (a - s_a)^2 d\mu \\ &+ \int a(a - s_a)(M_{a+b+c+d} - D_t) d\mu + \int (a - s_a)(C_t - M_{a+b}M_{a+d}) d\mu. \end{aligned} \quad (20)$$

Let us consider the last term:

$$\begin{aligned} \int (a - s_a)(C_t - M_{a+c}M_{a+d}) d\mu &= \int (a - s_a)M_{a+d}(\mathbf{P}_t(a_0 + c_0) - M_{a+c}) d\mu \\ &+ \int (a - s_a)\mathbf{P}_t(a_0 + b_0)(\mathbf{P}_t(a_0 + d_0) - M_{a+d}) d\mu. \end{aligned}$$

Setting  $\varphi(t) = \sqrt{\int (a - s_a)^2 d\mu}$ , Cauchy-Schwarz inequality yields

$$\begin{aligned} \int (a - s_a)(C_t - M_{a+b}M_{a+d}) d\mu &\leq \varphi(t) \left( M_{a+d} \sqrt{\int (\mathbf{P}_t(a_0 + c_0) - M_{a+c})^2 d\mu} \right) \\ &+ \varphi(t) \left( \left( \int \mathbf{P}_t(a_0 + c_0)^4 d\mu \right)^{1/4} \left( \int (\mathbf{P}_t(a_0 + d_0) - M_{a+d})^4 d\mu \right)^{1/4} \right). \end{aligned}$$

First spectral gap inequality gives

$$\int (\mathbf{P}_t(a_0 + c_0) - M_{a+c})^2 d\mu \leq e^{-\frac{1}{c_{SG}}t} \mathbf{Var}_\mu(a_0 + c_0) \leq e^{-\frac{1}{c_{SG}}t} \int (a_0 + c_0)^2 d\mu.$$

Since the semigroup  $(\mathbf{P}_t)_{t \geq 0}$  is contractive :  $\frac{d}{dt} \int \mathbf{P}_t(a_0 + b_0)^4 d\mu \leq 0$ , one obtains

$$\left( \int \mathbf{P}_t(a_0 + c_0)^4 d\mu \right)^{1/4} \leq \left( \int (a_0 + c_0)^4 d\mu \right)^{1/4}.$$

To finish, Lemma 3.3 gives

$$\left( \int (\mathbf{P}_t(a_0 + d_0) - M_{a+d})^4 d\mu \right)^{1/4} \leq \sqrt{2} e^{-\frac{t}{8c_{SG}}} \left( \int (a_0 + d_0)^4 d\mu \right)^{1/4},$$

which implies for the last term of (20):

$$\int (a - s_a)(D_t - M_{a+b}M_{a+d})d\mu \leq 3\varphi(t) \exp\left(-\frac{t}{8C_{SG}}\right) \left(\int (a_0 + b_0 + c_0 + d_0)^4 d\mu\right)^{\frac{1}{2}}$$

For the other term one gets

$$\int a(a - s_a)(M_{a+b+c+d} - C_t)d\mu \leq \varphi(t) \left(\int a^4 d\mu\right)^{1/4} \left(\int (M_{a+b+c+d} - D_t)^4 d\mu\right)^{1/4}.$$

Using (16), we get

$$\left(\int a^4 d\mu\right)^{1/4} \leq \min \left\{ \left(\int (a_0 + c_0)^4 d\mu\right)^{1/4}, \left(\int (a_0 + d_0)^4 d\mu\right)^{1/4} \right\},$$

and (18) gives

$$\int a(a - s_a)(M_{a+b+c+d} - C_t)d\mu \leq 2\varphi(t) \left(\int (a_0 + b_0 + c_0 + d_0)^4 d\mu\right)^{\frac{1}{2}} \exp\left(-\frac{t}{8C_{SG}}\right).$$

Then we obtain

$$\varphi'(t) \leq -M_{a+b+c+d}\varphi(t) + 5M_4 \exp\left(-\frac{t}{8C_{SG}}\right),$$

where  $M_4 = \left(\int (a_0 + b_0 + c_0 + d_0)^4 d\mu\right)^{\frac{1}{2}}$ . Integration of the last differential inequality yields, if  $M_{a+b+c+d} \neq \frac{1}{8C_{SG}}$ , then

$$\begin{aligned} \sqrt{\int (a - s_a)^2 d\mu} &\leq \left( \sqrt{\int (a_0 - s_a)^2 d\mu} + \left| \frac{5M_4}{M_{a+b+c+d} - \frac{1}{8C_{SG}}} \right| \right) \times \\ &\quad \exp\left(-\min\left\{M_{a+b+c+d}, \frac{1}{8C_{SG}}\right\}t\right), \end{aligned}$$

and if  $M_{a+b+c+d} = \frac{1}{8C_{SG}}$  then

$$\sqrt{\int (a - s_a)^2 d\mu} \leq \left( \sqrt{\int (a_0 - s_a)^2 d\mu} + 5M_4 t \right) \exp(-M_{a+b+c+d}t),$$

which finished the proof of (17).

If  $M_{a+b+c+d} < \frac{1}{8C_{SG}}$ , then the rate becomes  $e^{-M_{a+b+c+d}t}$ , one can check that  $M_{a+b+c+d}$  is equal to the constant  $C$  defined in (11) which is optimal.  $\triangleright$

**Remark 3.4** *In the case of equation of diffusion, the optimal rate of convergence, in  $L^2(d\mu)$ , is given by the spectral gap constant and is independent of the initial condition. In reaction-diffusion equations we can see in Theorem 2.1 and 3.2, that the optimal rate of convergence strongly depends on the initial conditions.*

## 4 Study of the general case

Let us consider now the general case for  $q \geq 1$ ,

$$\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightleftharpoons \sum_{i=1}^q \beta_i \mathcal{A}_i.$$

Assume now that  $\Omega \in \mathbb{R}^n$  is bounded domain.

Let consider non-negative initial conditions  $v_i^0 \geq 0$  on  $\Omega$ , for all  $1 \leq i \leq q$ . A weak solution of (6) on the time interval  $I$  is  $q$  measurable functions  $(v_i)_{1 \leq i \leq q}$  such that for all  $t \in I$ ,  $v_i(\cdot, t) \in L_1(\Omega)$ ,  $G(v_1(\cdot, t), \dots, v_q(\cdot, t)) \in L_1(\Omega)$ ,  $\int_0^t \|G(v_1(\cdot, s), \dots, v_q(\cdot, s))\|_{L_1} ds < +\infty$  and moreover for all  $1 \leq i \leq q$ ,  $x \in \mathbb{R}^n$  and  $t > 0$ ,

$$v_i(t, x) = \mathbf{P}_t(v_i^0) + k_i(\beta_i - \alpha_i) \int_0^t \mathbf{P}_{t-s}(G(v_1(\cdot, s), \dots, v_q(\cdot, s))) ds, \quad (21)$$

which satisfies also for all  $x \in \Omega$ ,  $v_i(0, x) = v_i^0(x)$ .

**Theorem 4.1** *Assume that there exist  $1 \leq i_0, j_0 \leq q$  such that  $\beta_{i_0} - \alpha_{i_0} > 0$  and  $\beta_{j_0} - \alpha_{j_0} < 0$ .*

*Then, for any non-negative bounded and measurable initial condition  $(v_i^0)_{1 \leq i \leq q}$ , there exists a unique non-negative weak solution of the system (6).*

**Proof**

◁ One of the main problem is to prove that the solution is non-negative. For that we will use Theorem 4 of Rothe [Rot84]. As for equation (8), for all  $1 \leq i \leq q$  we note by  $C_{i,1}(t, x) = \mathbf{P}_t\left(v_i^0 - \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1^0\right)(x)$  and consider the following PDE

$$\partial_t v_1 = Lv_1 + \bar{F}(x, t, v_1), \quad (22)$$

where

$$\bar{F}(x, t, v_1) = k_1(\beta_1 - \alpha_1) \left( \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1 + C_{i,1}(t, x) \right)_+^{\alpha_i} - \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1 + C_{i,1}(t, x) \right)_+^{\beta_i} \right),$$

and for  $x \in \mathbb{R}$ ,  $(x)_+ = \max\{x, 0\}$ .

Function  $\bar{F}$  has to satisfies conditions (F0), (F1) and (F3) of [Rot84].

Since, for all  $1 \leq i \leq q$ ,  $C_{i,1}$  is measurable, so for all  $v_1 \in \mathbb{R}$  the function  $\bar{F}(\cdot, \cdot, v_1) : (x, t) \in \Omega \times [0, +\infty[ \rightarrow \bar{F}(x, t, v_1)$  is measurable and thus (F0) holds.

Let  $\gamma = \max\{\sum_{i=1}^q \alpha_i, \sum_{i=1}^q \beta_i\}$ . If the initial conditions are bounded, then for each  $1 \leq i \leq q$ ,  $C_{i,1}$  is bounded uniformly on  $t \geq 0$ , which implies that there exists  $K \geq 0$  such that

$$|\bar{F}(x, t, v)| \leq K(1 + |v|)^\gamma,$$

for all  $u, v, x \in \mathbb{R}$  and  $t \geq 0$ . The same method gives also that there exists  $K' \geq 0$  such that for all  $v, x \in \mathbb{R}$  and  $t \geq 0$ ,

$$|\bar{F}(x, t, v) - \bar{F}(x, t, u)| \leq K'(1 + |v| + |u|)^{\gamma-1} |u - v|.$$

Thus (F1) and (F3) hold.

Let us check now that the PDE satisfies also the last condition of Theorem 4 of [Rot84], this last condition proves that the solution is defined in  $\mathbb{R}^+$ .

Let  $\bar{v}_1^0$  be a non-negative bounded initial condition and assume that  $\bar{v}_1$  is a weak solution of (22) (with the same definition as for equation (6), replacing  $G$  by  $F$  in (21)).

Let for all  $1 \leq i \leq q$ ,

$$\bar{v}_i(t, x) = \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} \bar{v}_1(t, x) + C_{i,1}(t, x). \quad (23)$$

Then  $(\bar{v}_i)_{1 \leq i \leq q}$  is a solution of

$$\partial_t \bar{v}_i = L\bar{v}_i + k_i(\beta_i - \alpha_i)\bar{G}(\bar{v}_1, \dots, \bar{v}_q), \quad (24)$$

where

$$\bar{G}(\bar{v}_1, \dots, \bar{v}_q) = \prod_{i=1}^q (\bar{v}_i)_+^{\alpha_i} - \prod_{i=1}^q (\bar{v}_i)_+^{\beta_i}.$$

We will prove now that for all  $1 \leq i \leq q$ ,  $\bar{v}_i$  is non-negative. Let us multiply (24) by  $-(\bar{v}_i)_- := \min\{\bar{v}_i, 0\}$ . After integration, we obtain

$$\frac{d}{dt} \int \frac{1}{2} ((\bar{v}_i)_-)^2 d\mu = - \int |\nabla(\bar{v}_i)_-|^2 d\mu - k_i(\beta_i - \alpha_i) \int (\bar{v}_i)_- \bar{G}(\bar{v}_1, \dots, \bar{v}_q) d\mu, \quad (25)$$

where  $\int |\nabla(\bar{v}_i)_-|^2 d\mu = \int \nabla \bar{v}_i \cdot \nabla(\bar{v}_i)_- d\mu$ .

On the set  $\{\bar{v}_i \leq 0\}$ , we have

$$k_i(\beta_i - \alpha_i)\bar{G}(\bar{v}_1, \dots, \bar{v}_q) = k_i(\beta_i - \alpha_i)\bar{G}(\bar{v}_1, \dots, 0, \dots, \bar{v}_q),$$

where we put 0 at the position  $i$ . Since for all  $j$ ,  $(\bar{v}_j)_+ \geq 0$  then it is not difficult to see that in all cases  $k_i(\beta_i - \alpha_i)\bar{G}(\bar{v}_1, \dots, 0, \dots, \bar{v}_q) \geq 0$ .

All of this gives that

$$\frac{d}{dt} \int \frac{1}{2} ((\bar{v}_i)_-)^2 d\mu \leq 0,$$

and since at time  $t = 0$ ,  $\int ((\bar{v}_i^0)_-)^2 d\mu = 0$  then for all  $t \geq 0$ ,  $\bar{v}_i(t) \geq 0$  almost everywhere.

Assume that  $\beta_1 - \alpha_1 > 0$ . Then, since the solutions are non-negative, we get the following global estimate of the solution,

$$0 \leq \bar{v}_1 = \frac{k_1(\beta_1 - \alpha_1)}{k_{j_0}(\beta_{j_0} - \alpha_{j_0})} \bar{v}_{j_0} + \mathbf{P}_t \left( \bar{v}_1^0 - \frac{k_1(\beta_1 - \alpha_1)}{k_{j_0}(\beta_{j_0} - \alpha_{j_0})} \bar{v}_{j_0}^0 \right) \leq \|\bar{v}_1^0\|_\infty + \left| \frac{k_1(\beta_1 - \alpha_1)}{k_{j_0}(\beta_{j_0} - \alpha_{j_0})} \right| \|\bar{v}_{j_0}^0\|_\infty.$$

If  $\beta_1 - \alpha_1 > 0$  does not hold, we use  $i_0$  instead  $j_0$  to get the same result with  $j_0$ .

Then condition (197) of [Rot84] is satisfied and Theorem 4 can be applied (with  $q_1 = q_2 = r_1 = r_2 = \infty$ ). It gives a global weak solution  $\bar{v}_1$  of (22) on  $[0, +\infty)$ . As before in (23), we are able to build the functions  $\bar{v}_i$  to get a weak solution of (24).

The same method as above in (25) proves that for all  $0 \leq i \leq q$ ,  $\bar{v}_i \geq 0$ . This gives that  $\bar{G}(\bar{v}_1, \dots, \bar{v}_q) = G(\bar{v}_1, \dots, \bar{v}_q)$  and then  $(\bar{v}_i)_{1 \leq i \leq q}$  is also a non-negative weak solution of (6) which finished the proof of the existence.

Let us prove now the uniqueness of the solution. If  $(v_i)_{1 \leq i \leq q}$  is a weak non-negative solution of (6), then  $v_1$  is a weak solution of (9) with  $j = 1$ . Corollary of Theorem 1 of [Rot84] insures that  $v_1$  is the unique weak solution of (9).  $\triangleright$

This corollary is a direct consequence of (8).

**Corollary 4.2** *Assume that there exist  $1 \leq i_0, j_0 \leq q$  such that  $\beta_{i_0} - \alpha_{i_0} > 0$  and  $\beta_{j_0} - \alpha_{j_0} < 0$ .*

*Let  $(v_i)_{1 \leq i \leq q}$  be a solution of (6). Then for all  $1 \leq i \leq q$  such that  $\beta_i - \alpha_i > 0$  one gets for all  $t \geq 0$  and  $x \in \Omega$ :*

$$0 \leq v_i(t, x) \leq k_i(\beta_i - \alpha_i) \min \left\{ \mathbf{P}_t \left( \frac{v_i^0}{k_i(\beta_i - \alpha_i)} - \frac{v_j^0}{k_j(\beta_j - \alpha_j)} \right) (x), \beta_j - \alpha_j < 0 \right\}$$

and for all  $1 \leq i \leq q$  such that  $\beta_i - \alpha_i < 0$ :

$$0 \leq v_i(t, x) \leq k_i |\beta_i - \alpha_i| \min \left\{ \mathbf{P}_t \left( \frac{v_i^0}{k_j(\beta_j - \alpha_j)} - \frac{v_i^0}{k_i(\beta_i - \alpha_i)} \right) (x), \beta_j - \alpha_j > 0 \right\}.$$

In particular when initial conditions are bounded, solutions of (3) are also bounded with an explicit upper bound.

**Theorem 4.3** Assume that the semigroup  $(\mathbf{P}_t)_{t \geq 0}$  satisfies a spectral gap inequality (5) with respect to the invariant probability measure  $\mu$ . Assume also that for all  $1 \leq i \leq q$ ,  $\alpha_i \beta_i = 0$ .

Let  $(v_i^0)_{1 \leq i \leq q}$  be a non-negative bounded initial condition. We assume furthermore that for all  $1 \leq i \leq q$ ,  $\int v_i^0 d\mu > 0$ .

Let  $(s_i)_{1 \leq i \leq q}$  be the steady state given by Lemma 1.2. Then for all  $1 \leq i \leq q$ , one gets

$$\sqrt{\int (v_i - s_i)^2 d\mu} \leq K \exp(-\min\{a, M\}t), \quad (26)$$

where  $a > 0$  depends on  $\alpha_i$ ,  $\beta_i$  and  $C_{SG}$  instead  $M, K > 0$  depend on the initial conditions.

**Proof**

◁ The idea is almost the same as in Theorem 3.2 except that we do not obtain the optimal rate.

Assume that  $\beta_1 - \alpha_1 > 0$ , the opposite case could be treated in the similar way. Equation (9) applied for  $j = 1$  reads

$$\partial_t v_1 = Lv_1 + F(t, x, v_1),$$

where

$$F(t, x, y) = k_1(\beta_1 - \alpha_1) \times \left( \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} y + C_{i,1}(t, x) \right)^{\alpha_i} - \prod_{i=1}^q \left( \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} y + C_{i,1}(t, x) \right)^{\beta_i} \right)$$

and functions  $C_{i,1}(t, x)$  are defined in (8). Ergodicity property of the semi-group  $(\mathbf{P}_t)_{t \geq 0}$  implies that for almost all  $x \in \Omega$ ,

$$\lim_{t \rightarrow \infty} C_{i,1}(t, x) = \int \left( v_i^0 - \frac{k_i(\beta_i - \alpha_i)}{k_1(\beta_1 - \alpha_1)} v_1^0 \right) d\mu := C_{i,1}^\infty.$$

Denote by  $F_\infty(y)$  the limit of  $F(t, x, y)$  when  $t$  goes to infinity. Note that  $F_\infty$  does not depend on  $x \in \Omega$ . Then, one gets

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int (a_1 - s_1)^2 d\mu &\leq k_1(\beta_1 - \alpha_1) \int (a_1 - s_1)(F(t, \cdot, a_1) - F_\infty(a_1)) d\mu \\ &\quad + k_1(\beta_1 - \alpha_1) \int (a_1 - s_1) F_\infty(a_1) d\mu. \end{aligned} \quad (27)$$

Note that  $F_\infty$  is equal to the polynome  $F$  defined in (13) where  $v_i(0)$  is replaced by the mean value of initial conditions  $\int v_i^0 d\mu$ . Let us set

$$M_1(t, x) = k_1(\beta_1 - \alpha_1) \min \left\{ \mathbf{P}_t \left( \frac{v_1^0}{k_1(\beta_1 - \alpha_1)} - \frac{v_j^0}{k_j(\beta_j - \alpha_j)} \right) (x), \beta_j - \alpha_j < 0 \right\},$$

and for its limit as  $t$  goes to  $\infty$

$$M_1^\infty = k_1(\beta_1 - \alpha_1) \min \left\{ \int \left( \frac{v_1^0}{k_1(\beta_1 - \alpha_1)} - \frac{v_j^0}{k_j(\beta_j - \alpha_j)} \right) d\mu, \beta_j - \alpha_j < 0 \right\}.$$

As it was shown in the proof in Theorem 2.1,  $F_\infty(X) = (X - s_1)Q(X)$  with  $Q(X) < 0$  for all  $X \in (0, M_1^\infty)$ . Now using the fact that for all  $1 \leq i \leq q$ ,  $\alpha_i \beta_i = 0$  one gets also that  $Q(0) < 0$  and  $Q(M_1^\infty) < 0$ . Then by continuity of  $Q$  there exist  $\epsilon, \eta > 0$  such that  $Q(X) \leq -\epsilon$  for all  $X \in [0, M_1^\infty + \eta]$ .

For the second term in (27), we get

$$\begin{aligned} k_1(\beta_1 - \alpha_1) \int (v_1 - s_1)F_\infty(v_1)d\mu &\leq -k_1(\beta_1 - \alpha_1)\epsilon \int_{\{v_1 \leq M_1^\infty + \eta\}} (v_1 - s_1)^2 d\mu \\ &\quad + k_1(\beta_1 - \alpha_1) \int_{\{v_1 > M_1^\infty + \eta\}} (v_1 - s_1)F_\infty(v_1)d\mu, \end{aligned}$$

and then for some constant  $K$  depending on initial conditions

$$\begin{aligned} k_1(\beta_1 - \alpha_1) \int (v_1 - s_1)F_\infty(v_1)d\mu &\leq \\ &\quad - k_1(\beta_1 - \alpha_1)\epsilon \int (v_1 - s_1)^2 d\mu + K\mu\{v_1 > M_1^\infty + \eta\}. \end{aligned}$$

Corollary 4.2 implies that  $M_1(t, \cdot) \geq v_1$  and then Markov inequality gives

$$\begin{aligned} k_1(\beta_1 - \alpha_1) \int (v_1 - s_1)F_\infty(v_1)d\mu &\leq \\ &\quad - k_1(\beta_1 - \alpha_1)\epsilon \int (v_1 - s_1)^2 d\mu + K\mathbf{Var}_\mu(M_1(t, \cdot)). \end{aligned} \quad (28)$$

Since for  $q$  measurable functions  $g_i \in L_2(\mu)$  one has

$$\mathbf{Var}_\mu(\min \{g_i, 1 \leq i \leq q\}) \leq \frac{1}{2} \sum_{i=1}^q \mathbf{Var}_\mu(g_i),$$

so the last term of (28) gives

$$k_1(\beta_1 - \alpha_1) \int (v_1 - s_1)F_\infty(v_1)d\mu \leq -k_1(\beta_1 - \alpha_1)\epsilon \int (v_1 - s_1)^2 d\mu + K'e^{-\frac{1}{\sigma^2 S}t},$$

where  $K'$  is an another constant depending on initial conditions.

Let us note  $F(t, x, y) = \sum_{i=1}^\gamma K_{t,i,x}y^i$  and  $F_\infty(y) = \sum_{i=1}^\gamma K_{\infty,i}y^i$  where we note  $\gamma = \max \{\sum_{i=1}^q \beta_i, \sum_{i=1}^q \alpha_i\}$ . The first term of (27) gives

$$\int (v_1 - s_1)(F(t, \cdot, v_1) - F_\infty(v_1))d\mu \leq \sqrt{\int (v_1 - s_1)^2 d\mu} \sum_{i=1}^\gamma \sqrt{\int v_1^{2i}(K_{t,i,x} - K_{\infty,i})^2 d\mu}.$$

Let us consider one of them, with  $1 \leq i \leq q$ , one has

$$\int v_1^{2i}(K_{t,i,x} - K_{\infty,i})^2 d\mu \leq \sqrt{\int v_1^{4i} d\mu} \sqrt{\int (K_{t,i,x} - K_{\infty,i})^4 d\mu}.$$

There exist some sets  $\Gamma_{i,j}$  and  $\Delta_i$  and constants  $\mu_{i,j,k} \in \mathbb{R}$ ,  $\gamma_{i,j,k} \in \mathbb{N}$  such that

$$K_{t,i,x} = \sum_{j \in \Delta_i} \prod_{k \in \Gamma_{i,j}} \mu_{i,j,k}(C_{i,1}(t, x))^{\gamma_{i,j,k}}.$$

Then for some constant  $K_i > 0$ ,  $p_{i,j,k} \geq 2$  and  $q_{i,j,k} > 0$ ,

$$\int (K_{t,i,x} - K_{\infty,i})^4 d\mu \leq K_i \sum_{j \in \Delta_i} \prod_{k \in \Gamma_{i,j}} \left( \int ((C_{i,1}(t, \cdot))^{\gamma_{i,j,k}} - (C_{i,1}^\infty)^{\gamma_{i,j,k}})^{p_{i,j,k}} d\mu \right)^{q_{i,j,k}}.$$

Since the initial conditions are bounded and  $p_{i,j,k} \geq 2$ , one gets for some another constant  $K$

$$\int ((C_{i,1}(t, \cdot))^{\gamma_{i,j,k}} - (C_{i,1}^\infty)^{\gamma_{i,j,k}})^{p_{i,j,k}} d\mu \leq K \int (C_{i,1}(t, \cdot) - C_{i,1}^\infty)^2 d\mu,$$

and then spectral gap inequality gives for some  $K'$ ,

$$\int ((C_{i,1}(t, \cdot))^{\gamma_{i,j,k}} - (C_{i,1}^\infty)^{\gamma_{i,j,k}})^{p_{i,j,k}} d\mu \leq K' e^{-\frac{1}{C_{SG}} t}.$$

Then we have proved that there exists  $\gamma_i > 0$  and  $R_i > 0$  depending on initial conditions and  $C_{SG}$  such that

$$\int (K_{t,i,x} - K_{\infty,i})^4 d\mu \leq R_i e^{-\gamma_i t}.$$

All of these estimates give for some  $\alpha > 0$  depending on  $\alpha_i$ ,  $\beta_i$  and  $C_{SG}$  and  $R > 0$  depending on initial conditions the following bound

$$k_1(\beta_1 - \alpha_1) \int (a_1 - s_1)(F_t(a_1) - F_\infty(a_1)) d\mu \leq \sqrt{\int (a_1 - s_1)^2 d\mu} R e^{-\alpha t}.$$

If  $\varphi(t) = \sqrt{\int (a_1 - s_1)^2 d\mu}$ , the equation (27) becomes

$$\varphi'(t) \leq -k_1(\beta_1 - \alpha_1)\epsilon\varphi(t) + R e^{-\alpha t} + K' e^{-\frac{1}{C_{SG}} t},$$

which finishes the proof.  $\triangleright$

**Remark 4.4** *One can generalize the last theorem in the following way without assuming that for all  $i$ ,  $\alpha_i \beta_i = 0$ . Let consider  $i^+$  and  $j^-$  such that*

$$\sup_{j, \text{ s.t. } \beta_j - \alpha_j > 0} \left\{ -\frac{\int v_j^0 d\mu}{\beta_j - \alpha_j} \right\} = -\frac{\int v_{i^+}^0 d\mu}{\beta_{i^+} - \alpha_{i^+}}$$

and

$$\sup_{j, \text{ s.t. } \beta_j - \alpha_j < 0} \left\{ -\frac{\int v_j^0 d\mu}{\beta_j - \alpha_j} \right\} = -\frac{\int v_{j^-}^0 d\mu}{\beta_{j^-} - \alpha_{j^-}}.$$

*Assume only that  $\alpha_{i^+} \beta_{i^+} = \alpha_{j^-} \beta_{j^-} = 0$ . Then the computation for the species  $i^+$  and  $j^-$  are the same as in the proof of Theorem 4.3.*

## References

- [ABC<sup>+</sup>00] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.



- [DF06] L. Desvillettes and K. Fellner. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations. *J. Math. Anal. Appl.*, 319(1):157–176, 2006.
- [DF08] L. Desvillettes and K. Fellner. Entropy methods for reaction-diffusion equations: Slowly growing A-priori bounds. *Rev. Mat. Iberoamericana*, 24(2):407–431, 2008.
- [Rot84] F. Rothe. *Global solutions of reaction-diffusion systems*, volume 1072 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

Email: gentil@ceremade.dauphine.fr

Email: b.zegarlinski@imperial.ac.uk