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ON THEORIES OF RANDOM VARIABLES

ITAÏ BEN YAACOV

ABSTRACT. Nous étudions des théories d'espaces de variables aléatoires : en un premier temps, nous considérons les variables aléatoires à valeurs dans l'intervalle $[0, 1]$, puis à valeur dans des structures métriques quelconques, généralisant la procédure d'aléatoirisation de structures classiques due à Keisler. Nous démontrons des résultats de préservation et de non-préservation de propriétés modèle-théoriques par cette construction :

- (i) L'aléatoirisée d'une structure ou théorie stable est stable.
- (ii) L'aléatoirisée d'une structure ou théorie simple instable n'est pas simple.

Nous démontrons également que dans la structure aléatoirisée, tout type est un type de Lascar.

We study theories of spaces of random variables: first, we consider random variables with values in the interval $[0, 1]$, then with values in an arbitrary metric structure, generalising Keisler's randomisation of classical structures. We prove preservation and non-preservation results for model theoretic properties under this construction:

- (i) The randomisation of a stable structure is stable.
- (ii) The randomisation of a simple unstable structure is not simple.

We also prove that in the randomised structure, every type is a Lascar type.

INTRODUCTION

Mathematical structures arising in the theory of probabilities are among the most natural examples for metric structures which admit a model theoretic treatment, albeit not in the strict setting of classical first order logic. Examples include the treatment of adapted spaces by Keisler & Fajardo [FK02], in which no logic of any kind appears explicitly (even though many model theoretic notions, such as types, do appear). Another example, which is the main topic of the present paper, is Keisler's randomisation construction [Kei99], in which one considers spaces of random variables whose values lie in some given structures. The randomisation construction was originally set up in the formalism of classical first order logic, viewing probability algebras as classical structures. We consider that this formalism was not entirely adequate for the purpose, and that it restricted considerably what could be done or proved. To the best of our knowledge,

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the first model theoretic treatment of a probabilistic structure in which notions such as stability and model theoretic independence were considered was carried out by the author in [Ben06], in which probability spaces were considered via their probability algebras, in the formalism of compact abstract theories. While this latter formalism was adequate, in the sense that it did allow one to show that probability algebras are stable and that the model theoretic independence coincides with the probabilistic one, it was quite cumbersome, and soon to be obsolete.

Continuous first order logic is a relatively new formalism, at least in its present form, developed by Alexander Usvyatsov and the author [BU] as a formalism for a model theoretic treatment of (classes of) complete metric structures. For example, we observe there that the class of probability algebras is elementary, its theory admitting a simple set of axioms, and that the theory of atomless probability algebras admits quantifier elimination, thus simplifying considerably many of the technical considerations contained in [Ben06]. Viewing probability algebras as metric structures in this fashion, rather than as classical structures, allowed Keisler and the author [BK] to present randomisation as metric structures. With the metric formalism one can prove several preservation results, e.g., if a structure is stable then so is its randomisation, which are false in the old formalism of [Kei99]. In addition, types in the metric randomisation correspond to Borel probability measures on the space of types of the original theory, also referred to nowadays as *Keisler measures*, and which turned out to be particularly useful for the study of dependent theories, e.g., in [HPP08]. A preservation result for dependence (i.e.e, if a theory is dependent then so is its randomisation) was proved by the author in [Bena].

The present paper comes to fill in and complete the state of knowledge regarding the randomisation in some points where we find it wanting. One point regards the formalism. As we pointed above, the correct formalism for the randomised structures is that of metric structures and continuous logic, even when the original structures are classical. It is therefore natural to extend the construction of [BK], which only randomises classical structures and theories, to randomisation of metric structures and of continuous theories. This extension is quite straightforward, with one technical complication that probability algebras need to be replaced with spaces of $[0, 1]$ -valued random variables, whose theory, while again relatively straightforward, has not yet been looked at. (It is worthwhile to point out that the description of random types as Borel probability measures on the original type spaces holds just as well for randomisation of metric structures, allowing the preservation of dependence to be proved in [Bena] for the randomisation of metric structures even before the latter was formally defined.) Another point is that the treatment of randomisation in [BK] is closely based on [Kei99], many times referring to it for proofs. In the present paper we seek to develop the theory of randomisation directly in the metric setting, rather than as a modification of a development in another formalism. We even gain a little doing that. For example, the classical formalism used in [Kei99] makes it necessary to name the randomisation of each \mathcal{L} -formula in \mathcal{L}^R (the randomisation language), so T^R (the randomisation theory) must contain axioms which

deal explicitly with the relations between the randomised formulae, an approach also adopted in [BK]. Here we observe that it is enough to put in \mathcal{L}^R the randomisations of the relation symbols of \mathcal{L} , leading to much more compact language and set of axioms. A third and relatively minor point is that our proof of preservation of stability in [BK] had a corollary that types over arbitrary sets are stationary, or equivalently, are Lascar types. In the present paper we prove that in an arbitrary randomisation (stable or not), a type over any set is a Lascar type.

Let us also point out, even though this is not treated in the present paper, that the work of Fajardo & Keisler [FK02] on adapted spaces can also be translated to a true metric model theoretic setting. To a probability space $(\Omega, \mathcal{F}, \mu)$ equipped with a filtration $(\mathcal{F}_i)_{i \in I}$ we can associate a structure consisting of the probability algebra $\bar{\mathcal{F}}$ of Ω , equipped with unary predicates measuring the distances to the sub-algebras in the filtration: $P_i(A) = \inf\{\mu(A \Delta B) : B \in \mathcal{F}_i\}$. If each sub-algebra \mathcal{F}_i in the filtration is in addition atomless over the previous ones, we obtain a structure in which types, in the sense of continuous logic, coincide with the abstract types defined by Fajardo and Keisler. Such a structure is a model of the theory of “beautiful tuples” of probability algebras (defined for classical structures by Bouscaren and Poizat [BP88]). It is stable, and its notion of independence can be characterised similarly to the characterisation of independence in beautiful and lovely pairs given by Pillay, Vassiliev and the author [BPV03]. If we replace probability algebras with spaces of $[0, 1]$ -valued random variables, whose basic model theoretic properties are studied in Section 2 below, we can replace the distance predicates P_i with function symbols for conditional expectation $E_i(X) = \mathbb{E}[X | \mathcal{F}_i]$. In this functional language the theory of atomless adapted spaces admits quantifier elimination.

The present paper is organised as follows. In Section 1 we consider formal deductions in propositional continuous logic, after Rose, Rosser and Church. These are used to give axioms for the theory of spaces $[0, 1]$ -valued random variables in Section 2. Model theoretic properties of this theory are deduced from those of the theory of probability algebras, with which it is bi-interpretable. In Section 3 we define and study the randomisations of metric structures, namely spaces of random variables whose values lie in metric structures. We give axioms for the theory of these random structures, prove quantifier elimination in the appropriate language, characterise types and so on. We also prove a version of Loś’s Theorem for randomisations, in which the ultra-filter, which is a finitely additive 0/1 measure, is replaced with an arbitrary finitely additive probability measure. In Section 4 we prove several preservation and non-preservation results, e.g., that the randomisation of a stable theory (or structure) is again stable, but the randomisation of a simple unstable structure or theory is not simple. In Section 5 we prove that in random structures, types over sets are Lascar types, so in the stable case they are stationary.

1. ON RESULTS OF ROSE, ROSSER AND CHANG

In the late 1950s Rose and Rosser [RR58] proved the completeness of a proof system for Łukasiewicz's many valued propositional logic, subsequently improved and simplified by Chang [Cha58b, Cha58a, Cha59]. This logic is very close to propositional continuous logic. Syntactically, the notation is quite different, partially stemming from the fact we identify *True* with 0, rather than with 1. Also, the connective $\frac{1}{2}$ does not exist in Łukasiewicz's logic. Semantically, we only allow the standard unit interval $[0, 1]$ as a set of truth values, while some fuzzy logicians allow non-standard extensions thereof (namely, they allow infinitesimal truth values). We should therefore be careful in how we use their results.

In these references, Propositional Łukasiewicz Logic is presented using Polish (prefix) notation, without parentheses. A formula is either an atomic proposition, $C\varphi\psi$ or $N\varphi$, where φ and ψ are simpler formulae. We shall prefer to use the notation of continuous logic, replacing $C\varphi\psi$ with $\psi \dot{\div} \varphi$ and $N\varphi$ with $\neg\varphi$.

Definition 1.1. Let $\mathcal{S}_0 = \{P_i : i \in I\}$ be a set distinct symbols, which we view as atomic proposition. Let \mathcal{S} be freely generated from \mathcal{S}_0 with the formal binary operation $\dot{\div}$ and unary operation \neg . Then \mathcal{S} is a *Łukasiewicz logic*.

Definition 1.2. Let \mathcal{S} be a Łukasiewicz logic.

- (i) For every mapping $v_0: \mathcal{S}_0 \rightarrow [0, 1]$, let $v: \mathcal{S} \rightarrow [0, 1]$ be the unique mapping extending v_0 such that $v(\varphi \dot{\div} \psi) = v(\varphi) \dot{\div} v(\psi)$ and $v(\neg\varphi) = 1 - v(\varphi)$. We call v the *truth assignment* defined by v_0 , and $v(\varphi)$ is the *truth value* of φ .
- (ii) If $v(\varphi) = 0$ we say that v is a *model* of φ , in symbols $v \models \varphi$. If $\Sigma \subseteq \mathcal{S}$, then $v \models \Sigma$ if $v \models \varphi$ for all $\varphi \in \Sigma$. We say that φ (or Σ) is *satisfiable* if it has a model.
- (iii) Let $\Sigma \subseteq \mathcal{S}$ and $\varphi \in \mathcal{S}$. We say that Σ *entails* φ , or that φ is a *logical consequence* of Σ , if every model of Σ is a model of φ . This is denoted $\Sigma \models \varphi$.

Let \mathcal{S} be a Łukasiewicz logic generated by $\{P_i : i \in I\}$, and $\varphi \in \mathcal{S}$. Then the truth assignments to \mathcal{S} are in bijection with $[0, 1]^I$, and every formula $\varphi \in \mathcal{S}$ can be identified with a continuous function $\hat{\varphi}: [0, 1]^I \rightarrow [0, 1]$ by $\hat{\varphi}(v) = v(\varphi)$.

Let us start with a few observations concerning the semantics of the Łukasiewicz logic.

Notation 1.3. (i) We define $\psi \dot{\div} n\varphi$ by induction on n :

$$\psi \dot{\div} 0\varphi = \psi, \quad \psi \dot{\div} (n+1)\varphi = (\psi \dot{\div} n\varphi) \dot{\div} \varphi.$$

We shall follow the convention that $\dot{\div}$ binds from left to right, so $\varphi \dot{\div} n\psi \dot{\div} m\chi$ should be read as $(\varphi \dot{\div} n\psi) \dot{\div} m\chi$.

- (ii) We use 1 as abbreviation for $\neg(\varphi_0 \dot{\div} \varphi_0)$, where φ_0 is any formula.

Lemma 1.4. *Let \mathcal{S} be a Łukasiewicz logic, and assume that $\Sigma \subseteq \mathcal{S}$ has no model. Then there are $n, m < \omega$ and $\varphi_i \in \Sigma$ for $i < m$ such that $\models 1 \dot{\div} n\varphi_0 \dot{\div} \dots \dot{\div} n\varphi_{m-1}$.*

Proof. For every $n, m < \omega$ and $\bar{\varphi} \in \Sigma^m$, let $\psi_{n,\bar{\varphi}} = 1 \dot{\div} n\varphi_0 \dot{\div} \dots \dot{\div} n\varphi_{m-1}$, and assume that $\not\vdash \psi_{n,\bar{\varphi}}$ for all n and $\bar{\varphi}$. In particular, for all n, m and $\bar{\varphi} \in \Sigma^m$ there is v such that $v(\psi_{2n,\bar{\varphi}}) > 0$, whereby $v(\psi_{n,\bar{\varphi}}) > \frac{1}{2}$. Call this $v_{n,\bar{\varphi}}$. Note that if $n \leq n'$ and $\bar{\varphi} \subseteq \bar{\varphi}'$ then $v_{n',\bar{\varphi}'}(\psi_{n,\bar{\varphi}}) > \frac{1}{2}$ as well. As $[0, 1]^I$ is compact, we obtain an accumulation point $v \in [0, 1]^I$ such that $v(\psi_{n,\bar{\varphi}}) \geq \frac{1}{2}$ for all $n, m < \omega$ and $\bar{\varphi} \in \Sigma^m$. It follows that $v(\varphi) = 0$ for all $\varphi \in \Sigma$. ■_{1.4}

The previous result is fairly robust in the sense that it remains valid if we extend the logic by arbitrary continuous connectives (i.e., allow any continuous function $f : [0, 1]^n \rightarrow [0, 1]$ as an n -ary connective).

The following is more delicate and depends on our connectives being not merely continuous but in fact piecewise linear. In addition, it requires the set of premises to be finite.

Lemma 1.5. *Let φ be a Łukasiewicz formula, say in the propositional variables $\{P_i : i < n\}$, and let $\hat{\varphi} : [0, 1]^n \rightarrow [0, 1]$ be the corresponding truth function. Then $\hat{\varphi}$ is piecewise affine, that is to say that the space $[0, 1]^n$ can be presented as a finite union compact convex sets on each of which $\hat{\varphi}$ is affine.*

Proof. By induction on the structure of φ , using the fact that $x \mapsto \neg x$ and $(x, y) \mapsto x \dot{\div} y$ are piecewise affine in this manner. ■_{1.5}

Proposition 1.6. *Let \mathcal{S} be a Łukasiewicz logic, $\varphi, \psi_i \in \mathcal{S}$ for $i < n$, $\Sigma = \{\psi_i : i < n\}$. Then $\Sigma \vDash \varphi$ if and only if there exists m such that $\vDash \varphi \dot{\div} m\psi_0 \dot{\div} \dots \dot{\div} m\psi_{n-1}$.*

Proof. Right to left is immediate (and holds of course in much more generality). For left to right we may replace Σ with $\{\bigvee_{i < k} \psi_i\}$, so we may assume that $\Sigma = \{\psi\}$ contains a single formula. Let $\{P_i : i < k\}$ be the propositional variables appearing in φ, Σ . Thus our assumption is that for every $v \in [0, 1]^k$: $v(\psi) = 0 \implies \varphi(v) = 0$.

For $m < \omega$ let $\chi_m = \varphi \dot{\div} m\psi$ and assume for a contradiction that $\vDash \chi_m$ fails for all m . In other words, for every m there is $v_m \in [0, 1]^k$ such that $v_m(\chi_m) > 0$. This implies that $\hat{\psi}(v_m) < 1/m$ and $\hat{\varphi}(v_m) > 0$. By our assumption $\hat{\psi}(v_m) > 0$ as well.

By Lemma 1.5, there is a compact convex set $C \subseteq [0, 1]^k$ and an unbounded set $J \subseteq \omega$ such that $\hat{\varphi}$ and $\hat{\psi}$ are both affine on C and $v_m \in C$ for all $m \in J$. Possibly passing to a smaller set J we may assume that $v = \lim_{m \in J} v_m$ exists in which case it belongs to C as well. By continuity and our assumption $\hat{\psi}(v) = \hat{\varphi}(0) = 0$.

Let $J_0 \subseteq J$ be maximal such that $\{v_m : m \in J_0\}$ is affinely independent over v , so $|J_0| \leq k$. Let

$$b = \min\{\hat{\psi}(v_m) : m \in J_0\} \qquad c = \max\{\hat{\varphi}(v_m) : m \in J_0\}.$$

Then $b > 0$ so we may choose $m_0 \in J$ such that $m_0 > c/b$. We may write $v_{m_0} = v + \sum_{m \in J_0} d_m v_m$. Then:

$$\begin{aligned} \hat{\varphi}(v_{m_0}) &= \hat{\varphi}(v) + \sum_{m \in J_0} d_m \hat{\varphi}(v_m) \\ &\leq \sum_{m \in J_0} c d_m \leq \sum_{m \in J_0} m_0 b d_m \leq m_0 \left(\hat{\psi}(v) \sum_{m \in J_0} d_m \hat{\psi}(v_m) \right) \\ &= m_0 \hat{\psi}(v_{m_0}). \end{aligned}$$

It follows that $v_{m_0}(\chi_{m_0}) = 0$ after all. This contradiction concludes the proof. $\blacksquare_{1.6}$

Remark 1.7. If we allow non-affine connectives Proposition 1.6 may fail. Indeed, $P^2 \vDash P$ even though for no m do we have $\vDash P \div mP^2$.

Remark 1.8. Logical implication in Łukasiewicz logic can be infinitary by nature. Indeed, let $\varphi_n = 1 \div 2(1 \div nP)$. Then $\varphi_n = 0$ if and only if $1 \div nP \geq \frac{1}{2}$, i.e., if and only if $P \leq \frac{1}{2n}$. Letting $\Sigma = \{\varphi_n : n < \omega\}$ we have $\Sigma \vDash P$ even though there is no finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash P$. This means that finiteness assumption in Proposition 1.6 cannot be discarded.

Let us now consider formal deductions in Łukasiewicz logic. As mentioned in the introduction, $\varphi \wedge \psi$ is abbreviation for $\varphi \div (\varphi \div \psi)$ (which would be $A\psi\varphi$ in the notation of [RR58]). Thus, the four axiom schemes which, according to [RR58, Cha58b], form a complete deduction system, are:

- (A1) $(\varphi \div \psi) \div \varphi$
- (A2) $((\rho \div \varphi) \div (\rho \div \psi)) \div (\psi \div \varphi)$
- (A3) $(\varphi \wedge \psi) \div (\psi \wedge \varphi)$
- (A4) $(\varphi \div \psi) \div (\neg\psi \div \neg\varphi)$

While Modus Ponens becomes:

$$(MP) \quad \frac{\varphi, \psi \div \varphi}{\varphi}$$

Formal deductions and the relation $\Sigma \vdash \varphi$ (φ is *deducible* from Σ) are defined as usual. Soundness of this deduction system (i.e., $\Sigma \vdash \varphi \implies \Sigma \vDash \varphi$) is easy to verify. A subset $\Sigma \subseteq \mathcal{S}$ is *contradictory* if $\Sigma \vdash \varphi$ for all $\varphi \in \mathcal{S}$. Otherwise it is *consistent*. The completeness result we referred to can be now stated as:

Fact 1.9 ([RR58, Cha59]). *Let \mathcal{S} be a Łukasiewicz logic, and $\varphi \in \mathcal{S}$. Then $\vDash \varphi$ if and only if $\vdash \varphi$.*

Using previous results this extends to finite sets of premises:

Proposition 1.10. *Let \mathcal{S} be a Łukasiewicz logic, $\varphi \in \mathcal{S}$ and $\Sigma \subseteq \mathcal{S}$ finite. Then $\Sigma \vDash \varphi$ if and only if $\Sigma \vdash \varphi$.*

Proof. Right to left is by soundness (and does not require Σ to be finite). For left to right, let $\Sigma = \{\psi_i : i < n\}$. Assuming $\Sigma \vDash \varphi$, by Proposition 1.6 there exists m such that $\vDash \varphi \dot{\div} m\psi_0 \dot{\div} \dots \dot{\div} \psi_{n-1}$. By Fact 1.9: $\vdash \varphi \dot{\div} m\psi_0 \dot{\div} \dots \dot{\div} m\psi_{n-1}$. It follows that $\Sigma \vdash \varphi$. $\blacksquare_{1.10}$

On the other hand, by Remark 1.8 and finiteness of formal deduction, Proposition 1.10 may fail for infinite sets of premises. We can still prove a weaker statement:

Proposition 1.11. *Let \mathcal{S} be a Lukasiewicz logic, and let $\Sigma \subseteq \mathcal{S}$. Then Σ is consistent if and only if it has a model.*

Proof. One direction is by soundness. For the other, assume that Σ has no model. Then by Lemma 1.4 there are n and $\varphi_i \in \Sigma$ such that letting $\psi = 1 \dot{\div} n\varphi_0 \dot{\div} \dots \dot{\div} n\varphi_{m-1}$ we have $\vDash \psi$. By Fact 1.9 we have $\vdash \psi$, and by Modus Ponens $\Sigma \vdash 1$. By Fact 1.9 we also have $\vdash \varphi \dot{\div} 1$ for every formula φ , so $\Sigma \vdash \varphi$ and Σ is contradictory. $\blacksquare_{1.11}$

Unfortunately, this result is still not precisely what we need, and we shall require the following modifications:

- (i) We wish to allow non-free logics, i.e., logics which are not necessarily freely generated from a set of atomic propositions. In particular, such logics need not be well-founded (i.e., we may have an infinite sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ such that each φ_{i+1} is a ‘‘proper sub-formula’’ of φ_i).
- (ii) The set of connectives $\{\neg, \dot{\div}\}$ is not full in the sense of [BU]. We would therefore like to introduce an additional unary connective, denoted $\frac{1}{2}$, which consists of multiplying the truth value by one half.

Definition 1.12. A *continuous propositional logic* is a (non-empty) structure $(\mathcal{S}, \neg, \frac{1}{2}, \dot{\div})$, where $\dot{\div}$ is a binary function symbol and $\neg, \frac{1}{2}$ are unary function symbols.

Given a continuous propositional logic \mathcal{S} , a *truth assignment* to \mathcal{S} is a mapping $v : \mathcal{S} \rightarrow [0, 1]$ such that for all $\varphi, \psi \in \mathcal{S}$:

- (i) $v(\neg\varphi) = 1 - v(\varphi)$.
- (ii) $v(\frac{1}{2}\varphi) = v(\varphi)/2$.
- (iii) $v(\varphi \dot{\div} \psi) = v(\varphi) \dot{\div} v(\psi)$.

Models and logical entailment are defined in the same manner as above.

We say that \mathcal{S} is *free (over \mathcal{S}_0)* if there exists a subset $\mathcal{S}_0 \subseteq \mathcal{S}$ such that \mathcal{S} is freely generated from \mathcal{S}_0 by the connectives $\{\neg, \frac{1}{2}, \dot{\div}\}$. In that case every mapping $v_0 : \mathcal{S}_0 \rightarrow [0, 1]$ extends to a unique truth assignment.

The new connective $\frac{1}{2}$ requires two more axioms:

$$(A5) \quad \frac{1}{2}\varphi \dot{\div} (\varphi \dot{\div} \frac{1}{2}\varphi)$$

$$(A6) \quad (\varphi \dot{\div} \frac{1}{2}\varphi) \dot{\div} \frac{1}{2}\varphi$$

Formal deductions in the sense of continuous propositional logic are defined as usual, allowing A1-6 as logical axiom schemes.

Theorem 1.13. *Let \mathcal{S} be a free continuous logic, $\Sigma \subseteq \mathcal{S}$ finite and $\varphi \in \mathcal{S}$. Then $\Sigma \models \varphi$ if and only if $\Sigma \vdash \varphi$.*

Proof. Throughout this proof we only consider free continuous propositional logics. We only prove left to right.

We shall construct a sequence of pairs Σ_n, φ_n by induction on n , where Σ_n is finite and $\Sigma_n \models \varphi_n$. We start with $\Sigma_0 = \Sigma$, $\varphi_0 = \varphi$. Assume Σ_n, φ_n are known. If a formula of the form $\frac{1}{2}\psi_n$ appears as a sub-formula in $\Sigma_n \cup \{\varphi_n\}$, add a new atomic proposition Q_n , replace every instance of $\frac{1}{2}\psi_n$ with Q_n , and add to Σ_n the formulae $\psi_n \div 2Q_n$ and $Q_n \div (\psi_n \div Q_n)$. We obtain Σ_{n+1} and φ_{n+1} such that Σ_{n+1} is still finite and $\Sigma_{n+1} \models \varphi_{n+1}$.

After finitely many steps of this kind we obtain Σ_n, φ_n in which the connective $\frac{1}{2}$ does not appear. Thus Σ_n, φ_n consist of Łukasiewicz formulae, and we can apply Proposition 1.10 to conclude that $\Sigma_n \vdash \varphi_n$ in the deduction system consisting of A1-4. Now undo the translation above, replacing each atomic proposition Q_i with $\frac{1}{2}\psi_i$. Then the new premises we added become instances of A5-6, so $\Sigma \vdash \varphi$ (using all of A1-6). ■_{1.13}

Remark 1.14. The assumptions that the logic is free and Σ is finite are both required. Indeed the argument of Remark 1.8 works just as well in any free continuous propositional logic \mathcal{S} , so we can find an infinite set $\Sigma \subseteq \mathcal{S}$ and $\varphi \in \mathcal{S}$ such that $\Sigma \models \varphi$ and yet $\Sigma \not\vdash \varphi$. For $\psi, \chi \in \mathcal{S}$, say that $\psi \sim_\Sigma \chi$ if $\Sigma \vdash \psi \div \chi$ and $\Sigma \vdash \chi \div \psi$. By Theorem 1.13 we can easily show that this is an equivalence and congruence relation. Let $\mathcal{S}_\Sigma = \mathcal{S}/\sim_\Sigma$ and let $[\varphi]$ denote the equivalence class of φ . Then $\models_{\mathcal{S}_\Sigma} [\varphi]$ and yet $\not\vdash_{\mathcal{S}_\Sigma} [\varphi]$.

Lemma 1.15. *For every $\varphi, \psi \in \mathcal{S}$, $\Sigma \subseteq \mathcal{S}$ and $n < \omega$:*

- (i) $\vdash \varphi \div \varphi$.
- (ii) $\vdash (\varphi \div \psi) \div (1 \div n(\psi \div \varphi))$.
- (iii) *If $\Sigma, \varphi \div \psi$ is contradictory then $\Sigma \vdash \psi \div \varphi$.*

Proof. (i) Work in a Łukasiewicz logic as in Fact 1.9. Clearly $\models P \div P$ for every atomic proposition P , whereby $\vdash P \div P$ by the said Fact, and by substitution we get a deduction for $\varphi \div \varphi$.

(ii) Same argument.

(iii) If $\Sigma, \varphi \div \psi$ is contradictory then it has no model. By the proof of Proposition 1.11 there is $n < \omega$ such that $\Sigma \vdash 1 \div n(\varphi \div \psi)$. Therefore $\Sigma \vdash \psi \div \varphi$. ■_{1.15}

Theorem 1.16. *Let \mathcal{S} be a (non-free) continuous propositional logic, and let $\Sigma \subseteq \mathcal{S}$. Then Σ is consistent if and only if it is satisfiable.*

Proof. Let \mathcal{S}^f be the Łukasiewicz logic freely generated by $\{P_\varphi : \varphi \in \mathcal{S}\}$, and let:

$$\begin{aligned} \Sigma_0^f &= \{P_{\neg\varphi} \div \neg P_\varphi, \neg P_\varphi \div P_{\neg\varphi} : \varphi \in \mathcal{S}\} \\ &\cup \{P_{\varphi \div \psi} \div (P_\varphi \div P_\psi), (P_\varphi \div P_\psi) \div P_{\varphi \div \psi} : \varphi, \psi \in \mathcal{S}\} \\ &\cup \{P_{\frac{1}{2}\varphi} \div P_{\varphi \div \frac{1}{2}\varphi}, P_{\varphi \div \frac{1}{2}\varphi} \div P_{\frac{1}{2}\varphi} : \varphi \in \mathcal{S}\} \\ \Sigma^f &= \{P_\varphi : \varphi \in \Sigma\} \cup \Sigma_0^f. \end{aligned}$$

Assume that Σ^f has a model v^f . Define $v: \mathcal{S} \rightarrow [0, 1]$ by $v(\varphi) = v^f(P_\varphi)$. As $v^f \models \Sigma_0^f$, v is a truth assignment in the sense of \mathcal{S} , and is clearly a model of Σ .

Thus, if Σ has no model, neither does Σ^f . By Proposition 1.11 Σ^f is contradictory. Thus, for every $\psi \in \mathcal{S}$ we have $\Sigma^f \vdash P_\psi$. Take any deduction sequence witnessing this, replacing every atomic proposition P_φ with φ . If a formula was obtained from previous ones using Modus Ponens, the same holds after this translation. Premises from Σ^f become translated to one of several cases:

- (i) Premises of the form P_φ for $\varphi \in \Sigma$ are replaced with $\varphi \in \Sigma$.
- (ii) Premises of the first two kinds from Σ_0^f are replaced with something of the form $\varphi \div \varphi$, which we know is deducible without premises.
- (iii) Premises of the last kind from Σ_0^f are translated to instances of the axioms schemes A5-6.

We conclude that $\Sigma \vdash \psi$ for all $\psi \in \mathcal{S}$, and Σ is contradictory. The other direction is by easy soundness. ■_{1.16}

As we said earlier Theorem 1.13 fails with infinite sets of premises or non-free logics. We can prove the full version if we are willing to weaken a little the conclusion that $\Sigma \vdash \varphi$. Let 2^{-n} be abbreviation for $\frac{1}{2} \cdots \frac{1}{2} 1$ (n times). Clearly, if v is any truth assignment then $v(2^{-n}) = 2^{-n}$.

Corollary 1.17. *Let \mathcal{S} be a continuous propositional logic, $\Sigma \subseteq \mathcal{S}$ and $\varphi \in \mathcal{S}$. Then $\Sigma \models \varphi$ if and only if $\Sigma \vdash \varphi \div 2^{-n}$ for all n .*

Proof. Right to left is clear.

Assume now that $\Sigma \models \varphi$. Then $\Sigma \cup \{2^{-n} \div \varphi\}$ is non-satisfiable, and therefore contradictory by Theorem 1.16. By Lemma 1.15: $\Sigma \vdash \varphi \div 2^{-n}$. ■_{1.17}

These completeness results are extended to the full continuous first order logic in [BP]. We conclude with a word regarding the semantics of continuous propositional logics.

Definition 1.18. A *homomorphism* of continuous propositional logics is a mapping which respects the connectives \neg , $\frac{1}{2}$ and \div .

Let \mathcal{S} be a continuous propositional logic. Its *Stone space* is defined to be the set $\tilde{\mathcal{S}} = \text{Hom}(\mathcal{S}, [0, 1])$, where $[0, 1]$ is equipped with the natural interpretation of the connectives. Equivalently, $\tilde{\mathcal{S}}$ is the collection of truth assignments to \mathcal{S} . We equip \mathcal{S} with the induced topology as a subset of $[0, 1]^{\mathcal{S}}$ (i.e., with the point-wise convergence topology).

For each $\varphi \in \mathcal{S}$ we define a function $\hat{\varphi}: \tilde{\mathcal{S}} \rightarrow [0, 1]$ by $\hat{\varphi}(v) = v(\varphi)$.

Proposition 1.19. *Let \mathcal{S} be a continuous propositional logic, $\tilde{\mathcal{S}}$ its Stone space, and let $\theta_{\mathcal{S}}$ denote the mapping $\varphi \mapsto \hat{\varphi}$.*

- (i) *The space $\tilde{\mathcal{S}}$ is compact and Hausdorff.*
- (ii) *$\theta_{\mathcal{S}} \in \text{Hom}(\mathcal{S}, C(\tilde{\mathcal{S}}, [0, 1]))$. In particular, each $\hat{\varphi}$ is continuous.*
- (iii) *For $\varphi, \psi \in \mathcal{S}$ we have $\theta_{\mathcal{S}}(\varphi) = \theta_{\mathcal{S}}(\psi)$ if and only if $\varphi \equiv \psi$.*

(iv) *The image of $\theta_{\mathcal{S}}$ is dense in the uniform convergence topology on $C(\tilde{\mathcal{S}}, [0, 1])$.*

Moreover, the properties characterise the pair $(\tilde{\mathcal{S}}, \theta_{\mathcal{S}})$ up to a unique homeomorphism.

Proof. That the image is dense is a direct application of a variant of the Stone-Weierstrass theorem proved in [BU, Proposition 1.4]. The other properties are immediate from the construction.

We are left with showing uniqueness. Indeed, assume that X is a compact Hausdorff space and $\theta: \mathcal{S} \rightarrow C(X, [0, 1])$ satisfies all the properties above. Define $\zeta: X \rightarrow \tilde{\mathcal{S}}$ by $\zeta(x)(\varphi) = \theta(\varphi)(x)$. Thus ζ is the unique mapping satisfying $\theta_{\mathcal{S}}(\varphi) \circ \zeta = \theta(\varphi)$, and we need to show that it is a homeomorphism. Continuity is immediate. The image of θ is dense in uniform convergence and therefore separates points, so ζ is injective. Since X is compact and Hausdorff ζ must be a topological embedding. In order to see that ζ is surjective it will be enough to show that its image is dense. So let $U \subseteq \tilde{\mathcal{S}}$ be a non empty open set, which must contain a non empty set of the form $\{v \in \tilde{\mathcal{S}}: f(v) > 0\}$ for some $f \in C(\tilde{\mathcal{S}}, [0, 1])$. For n big enough there is $v_0 \in \tilde{\mathcal{S}}$ such that $f(v_0) > 2^{-n+1}$. By density find $\varphi_0 \in \mathcal{S}$ such that $\|\hat{\varphi}_0 - f\|_{\infty} < 2^{-n}$. and let $\varphi = \varphi_0 - 2^{-n} \in \mathcal{S}$. Then $\{v \in \tilde{\mathcal{S}}: v(\varphi) > 0\} \subseteq U$ and $v_0(\varphi) \neq 0$. Since $\varphi \neq 0$ there is $x \in X$ such that $\zeta(x)(\varphi) = \theta(\varphi)(x) \neq 0$, i.e., $\zeta(x) \in U$. This concludes the proof. $\blacksquare_{1.19}$

2. THE THEORY OF $[0, 1]$ -VALUED RANDOM VARIABLES

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. In [Ben06] we considered such a space via its probability algebra $\tilde{\mathcal{F}}$, namely the Boolean algebra of events \mathcal{F} modulo null measure difference. Equivalently, the probability algebra $\tilde{\mathcal{F}}$ can be viewed as the space of $\{0, 1\}$ -valued random variables (up to equality a.e.). Here we shall consider a very similar object, namely the space of $[0, 1]$ -valued random variables. This space will be denoted $L^1((\Omega, \mathcal{F}, \mu), [0, 1])$, or simply $L^1(\mathcal{F}, [0, 1])$, where we consider that the measure μ is implicitly part of the structure of \mathcal{F} . We equip this space with the natural interpretation of the connectives \neg , $\frac{1}{2}$ and \div (e.g., $(X \div Y)(\omega) = X(\omega) \div Y(\omega)$), as well as with the L^1 distance $d(X, Y) = \mathbb{E}[|X - Y|]$, for which it is a complete metric space. It is thus naturally a structure in the random variable language

$$\mathcal{L}_{RV} = \{0, \neg, \frac{1}{2}, \div\}.$$

Throughout, we shall use 1 as an abbreviation for $\neg 0$ and $E(x)$ as an abbreviation for $d(x, 0)$. The intended interpretation of $E(x)$ is the expectation. Notice that by definition, if \mathcal{M} is any \mathcal{L}_{RV} -structure and $a \in M$ then $a = 0 \iff d(a, 0) = 0 \iff E(a) = 0$.

2.1. The theories RV and ARV . We shall use the results of Section 1 to give axioms for the theory of $[0, 1]$ -valued random variables equipped with the L^1 metric, in the language \mathcal{L}_{RV} given above.

The term algebra \mathcal{T}_{RV} of \mathcal{L}_{RV} is a free propositional continuous logic (freely generated by the variables of the language together with the symbol 0). Thus, if $\tau(\bar{x}) \in \mathcal{T}_{RV}$

then $\vdash_{\mathcal{T}_{RV}} \tau \iff \models_{\mathcal{T}_{RV}} \tau$ by Theorem 1.13. On the other hand, an \mathcal{L}_{RV} -structure \mathcal{M} is a non-free continuous propositional logic. Translating proofs from \mathcal{T}_{RV} to \mathcal{M} we have $\vdash_{\mathcal{T}_{RV}} \tau \implies \vdash_{\mathcal{M}} \tau(\bar{a})$ for all $\bar{a} \in \mathcal{M}$.

We define the theory RV to consist of the following axioms. In each axiom we quantify universally on all free variables. Keep in mind also that $x \wedge y$ is abbreviation for $x \div (x \div y)$.

- (RV1) $E(x) = E(x \div y) + E(y \wedge x)$
 (RV2) $E(1) = 1$
 (RV3) $d(x, y) = E(x \div y) + E(y \div x)$
 (RV4) $\tau = 0$ whenever $\models_{\mathcal{T}_{RV}} \tau$

ARV is defined by adding the following axiom:

$$(ARV) \quad \inf_y \left(E(y \wedge \neg y) \vee \left| E(y \wedge x) - \frac{E(x)}{2} \right| \right) = 0.$$

Lemma 2.1. *Let \mathcal{M} be a model of $RV1$. Then for every $a, b \in M$:*

$$E(a) - E(b) \leq E(a \div b) \leq E(a).$$

In particular \mathcal{M} respects Modus Ponens: if $b = 0$ and $a \div b = 0$ then $a = 0$.

Proof. Axiom RV1 implies first that $E(a) \geq E(a \div b)$. But then $E(b) \geq E(b \div (b \div a)) = E(b \wedge a)$ whereby $E(a) - E(b) \leq E(a) - E(b \wedge a) = E(a \div b)$. Modus Ponens follows. $\blacksquare_{2.1}$

Thus, modulo RV1, the axiom scheme RV4 is equivalent to the finite set:

- (RV4.1) $(x \div y) \div x = 0$
 (RV4.2) $((x \div z) \div (x \div y)) \div (y \div z) = 0$
 (RV4.3) $(x \wedge y) \div (y \wedge x) = 0$
 (RV4.4) $(x \div y) \div (\neg y \div \neg x) = 0$
 (RV4.5) $\frac{1}{2}x \div (x \div \frac{1}{2}x) = 0$
 (RV4.6) $(x \div \frac{1}{2}x) \div \frac{1}{2}x = 0$

Furthermore, modulo RV1, RV3 and RV4.1-4, axioms RV4.5-6 are further equivalent to:

$$(RV5) \quad \frac{1}{2}x = x \div \frac{1}{2}x$$

Indeed, left to right is by RV3. Axioms RV1 and RV4.1-4 imply that $x \div x = 0$, giving right to left.

The following is fairly obvious:

Fact 2.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\mathcal{M} = L^1(\mathcal{F}, [0, 1])$. Then $\mathcal{M} \models RV$, and if $(\Omega, \mathcal{F}, \mu)$ is atomless then $\mathcal{M} \models ARV$.*

We now aim to prove the converse of Fact 2.2.

Lemma 2.3. *Let $\mathcal{M} \models RV$, $a, b \in M$. Then:*

- (i) $d(a, a \dot{\div} b) = E(a \wedge b) \leq E(b)$. In particular, $a \dot{\div} 0 = a$.
- (ii) $a \dot{\div} a = 0$.
- (iii) $a \dot{\div} \frac{1}{2}a = \frac{1}{2}a$, $\frac{1}{2}a \dot{\div} a = 0$ and $E(\frac{1}{2}a) = \frac{1}{2}E(a)$.
- (iv) Define by induction $2^0 = 1$ (i.e., $2^0 = -0$) and $2^{-(n+1)} = \frac{1}{2}2^{-n}$. Then for all $n \in \mathbb{N}$: $E(2^{-n}) = 2^{-n}$.
- (v) $a = 0 \iff \vdash_{\mathcal{M}} a \iff \vDash_{\mathcal{M}} a$.
- (vi) $a = b \iff a \equiv_{\mathcal{M}} b$.

Proof. (i) From RV4 we have $(a \dot{\div} b) \dot{\div} a = 0$ and using RV3 we obtain $d(a, a \dot{\div} b) = E(a \wedge b)$. By RV1 $E(a \wedge b) \leq E(b)$. The rest follows.

- (ii) By RV4 we have $a \dot{\div} a = 0$.
- (iii) Axiom RV4 yields $\frac{1}{2}a \dot{\div} a = \frac{1}{2}a \dot{\div} (a \dot{\div} \frac{1}{2}a) = (a \dot{\div} \frac{1}{2}a) \dot{\div} \frac{1}{2}a = 0$, and by RV3 $\frac{1}{2}a = a \dot{\div} \frac{1}{2}a$. Thus

$$E(a) = E(a \dot{\div} \frac{1}{2}a) + E(\frac{1}{2}a \dot{\div} (\frac{1}{2}a \dot{\div} a)) = E(\frac{1}{2}a) + E(\frac{1}{2}a \dot{\div} 0) = 2E(\frac{1}{2}a).$$

- (iv) Immediate from the previous item.
- (v) Assume that $\vdash_{\mathcal{M}} a$. Then by RV1 (which implies Modus Ponens) and RV4.1-6 we have $a = 0$. Thus $a = 0 \iff \vdash_{\mathcal{M}} a$. The implication $\vdash_{\mathcal{M}} a \implies \vDash_{\mathcal{M}} a$ is by soundness. Finally assume that $\vDash_{\mathcal{M}} a$. Then for all n we have $\vdash_{\mathcal{M}} a \dot{\div} 2^{-n}$, whereby $a \dot{\div} 2^{-n} = 0$. Thus $E(a) = E(a \wedge 2^{-n}) \leq E(2^{-n}) = 2^{-n}$, for arbitrary n . It follows that $E(a) = 0$, i.e., that $a = 0$.
- (vi) Assume that $a \equiv_{\mathcal{M}} b$, i.e., that $\vDash_{\mathcal{M}} a \dot{\div} b$ and $\vDash_{\mathcal{M}} b \dot{\div} a$. Be the previous item $a \dot{\div} b = b \dot{\div} a = 0$ whereby $a = b$. ■_{2.3}

Let $\widetilde{\mathcal{M}}$ be the Stone space of \mathcal{M} , viewed as a continuous propositional logic, and let $\theta_{\mathcal{M}}: \mathcal{M} \rightarrow C(\widetilde{\mathcal{M}}, [0, 1])$ be as in Proposition 1.19. Recall the notation $\hat{a} = \theta_{\mathcal{M}}(a)$. By Lemma 2.3.(vi) and Proposition 1.19 $\theta_{\mathcal{M}}$ is injective.

The space $C(\widetilde{\mathcal{M}}, [0, 1])$ is naturally equipped with the supremum metric, denoted $\|f - g\|_{\infty}$. On the other hand, we aim is to show that $d^{\mathcal{M}}$ is an L^1 distance, i.e., that for an appropriate measure we have $d^{\mathcal{M}}(a, b) = \|\hat{a} - \hat{b}\|_1$, which need not be equal to $\|\hat{a} - \hat{b}\|_{\infty}$. Nonetheless, we can relate the two metrics as follows (we essentially say that L^{∞} -convergence of random variables implies L^1 -convergence).

Lemma 2.4. *Assume that $\{a_n\}_{n \in \mathbb{N}} \subseteq M$ is such that $\{\hat{a}_n\}_{n \in \mathbb{N}} \subseteq C(\widetilde{\mathcal{M}}, [0, 1])$ is a Cauchy sequence in the supremum metric. Then $\{a_n\}_{n \in \mathbb{N}}$ converges in \mathcal{M} and $\lim \hat{a}_n = \widehat{\lim a_n}$.*

Proof. By assumption, for every $k < \omega$ there is N_k such that for all $n, m < N_k$ $\|\hat{a}_n - \hat{a}_m\|_{\infty} \leq 2^{-k}$ for all $n, m < N_k$. Therefore $(\hat{a}_n \dot{\div} \hat{a}_m) \dot{\div} 2^{-k} = 0$, and since $\theta_{\mathcal{M}}$ is injective: $a_n \dot{\div} a_m \dot{\div} 2^{-k} = 0$. Thus $E(a_n \dot{\div} a_m) = E((a_n \dot{\div} a_m) \wedge 2^{-k}) \leq E(2^{-k}) = 2^{-k}$. Similarly $E(a_m \dot{\div} a_n) \leq 2^{-k}$, whereby $d(a_n, a_m) \leq 2^{-k+1}$. Since \mathcal{M} is a (complete) \mathcal{L} -structure, it contains a limit a .

Now fix $n \geq N_k$ and let $m \rightarrow \infty$. Then $a_m \rightarrow a$, and therefore $a_m \dot{\div} a_n \dot{\div} 2^{-k} \rightarrow a \dot{\div} a_n \dot{\div} 2^{-k}$. Thus $a \dot{\div} a_n \dot{\div} 2^{-k} = 0$, and by a similar argument $a_n \dot{\div} a \dot{\div} 2^{-k} = 0$. We have thus shown that $\hat{a}_n \rightarrow \hat{a}$ uniformly as desired. $\blacksquare_{2.4}$

Corollary 2.5. *The mapping $\theta_{\mathcal{M}}: \mathcal{M} \rightarrow C(\widetilde{\mathcal{M}}, [0, 1])$ is bijective.*

Proof. We already know it is injective, and by Proposition 1.19 its image is dense. By the previous lemma its image is complete, so it is onto. $\blacksquare_{2.5}$

We shall identify \mathcal{M} with $C(\widetilde{\mathcal{M}}, [0, 1])$.

Lemma 2.6. *For all $a, b \in \mathcal{M}$ and $r \in \mathbb{R}^+$:*

- (i) *If $a + b \in \mathcal{M}$ (i.e., $\|a + b\|_{\infty} \leq 1$) then $E(a + b) = E(a) + E(b)$.*
- (ii) *If $ra \in \mathcal{M}$ (i.e., $r\|a\|_{\infty} \leq 1$) then $E(ra) = rE(a)$.*

Proof. (i) Let $c = a + b$. Then $c \dot{\div} b = a$ and $b \dot{\div} c = 0$, whereby:

$$E(c) = E(c \dot{\div} b) + E(b \dot{\div} (b \dot{\div} c)) = E(a) + E(b \dot{\div} 0) = E(a) + E(b).$$

- (ii) For integer r this follows from the previous item, and the rational case follows. If $r_n \rightarrow r$ then $r_n a \rightarrow ra$ in \mathcal{M}^* and *a fortiori* in \mathcal{M} , so the general case follows by continuity of E . $\blacksquare_{2.6}$

Theorem 2.7. *Let $\mathcal{M} \models RV$, $\widetilde{\mathcal{M}}$ its Stone space and $\theta_{\mathcal{M}}: \mathcal{M} \rightarrow C(\widetilde{\mathcal{M}}, [0, 1])$ as in Proposition 1.19.*

- (i) *As a topological space, $\widetilde{\mathcal{M}}$ is compact and Hausdorff.*
- (ii) *The mapping $\theta_{\mathcal{M}}: \mathcal{M} \rightarrow C(\widetilde{\mathcal{M}}, [0, 1])$ is bijective and respects the operations \neg , $\frac{1}{2}$ and $\dot{\div}$ (i.e., it is an isomorphism of continuous propositional logics).*
- (iii) *There exists a regular Borel probability measure μ on $\widetilde{\mathcal{M}}$ such that the natural mapping $\rho_{\mu}: C(\widetilde{\mathcal{M}}, [0, 1]) \rightarrow L^1(\mu, [0, 1])$ is bijective as well, and the composition $\rho_{\mu} \circ \theta_{\mathcal{M}}: \mathcal{M} \rightarrow L^1(\mu, [0, 1])$ is an isomorphism of L_{RV} -structures.*

Moreover, these properties characterise $(\widetilde{\mathcal{M}}, \mu, \theta_{\mathcal{M}})$ up to a unique measure preserving homeomorphism.

Proof. The first two properties are already known. By Lemma 2.6 we can extend E by linearity from $C(\Omega, [0, 1])$ to $C(\Omega, \mathbb{R})$, yielding a positive linear functional. By the Riesz Representation Theorem [Rud66, Theorem 2.14] there exists a unique regular Borel measure μ on Ω such that $E(f) = \int f d\mu$. Since $E(1) = 1$, μ is a probability measure.

The mapping $\mathcal{M} \rightarrow L^1(\mu, [0, 1])$ is isometric and in particular injective. Its image is dense (continuous functions are always dense in the L^1 space of a regular Borel measure). Moreover, since \mathcal{M} is a complete metric space the image must be all of $L^1(\mu, [0, 1])$, whence the last item.

The uniqueness of $\widetilde{\mathcal{M}}$ as a topological space verifying the first two properties follows from Proposition 1.19 and Lemma 2.3.(vi). The Riesz Representation Theorem then yields the uniqueness of μ . $\blacksquare_{2.7}$

We may refer to $(\widetilde{\mathcal{M}}, \mu)$ (viewed as a topological space equipped with a Borel measure) as the *Stone space* of \mathcal{M} or say that \mathcal{M} is *based on* $(\widetilde{\mathcal{M}}, \mu)$.

Corollary 2.8. *Let \mathcal{M} be an \mathcal{L}_{RV} -structure. Then:*

- (i) *The structure \mathcal{M} is a model of RV if and only if it is isomorphic to some $L^1(\mathcal{F}, [0, 1])$.*
- (ii) *A structure of the form $L^1(\mathcal{F}, [0, 1])$ is a model of ARV if and only if $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space.*

Corollary 2.9. *Let $\mathcal{M} \models RV$ be based on $(\widetilde{\mathcal{M}}, \mu)$. Then every Borel function $\widetilde{\mathcal{M}} \rightarrow [0, 1]$ is equal almost everywhere to a unique continuous function.*

2.2. Interpreting random variables in events and vice versa. In the previous section we attached to every probability space $(\Omega, \mathcal{F}, \mu)$ the space $L^1(\mathcal{F}, [0, 1])$ of $[0, 1]$ -valued random variables and axiomatised the class of metric structures arising in this manner. While we cannot quite recover the original space Ω from $L^1(\mathcal{F}, [0, 1])$ we do consider that $L^1(\mathcal{F}, [0, 1])$ retains all the pertinent information

An alternative approach to coding a probability space in a metric structure goes through its probability algebra, namely the space of $\{0, 1\}$ -valued random variables. It can be constructed directly as the Boolean algebra quotient $\widetilde{\mathcal{F}} = \mathcal{F} / \mathcal{F}_0$ where \mathcal{F}_0 is the ideal of null measure sets. In addition to the Boolean algebra structure, it is equipped with the induced measure function $\mu: \widetilde{\mathcal{F}} \rightarrow [0, 1]$ and the metric $d(a, b) = \mu(a \Delta b)$ (in fact, the measure μ is superfluous and can be recovered as $\mu(x) = d(x, 0)$). The metric is always complete, so a probability algebra is a structure in the language $\mathcal{L}_{Pr} = \{0, 1, \cap, \cup, \cdot^c, \mu\}$.

Let us define the theory Pr to consist of the following axioms, quantified universally:

- (Bool) The theory of Boolean algebras: $(x \cap y)^c = x^c \cup y^c, \dots$
- (Pr1) $\mu(1) = 1$
- (Pr2) $\mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y)$
- (Pr3) $d(x, y) = \mu(x \Delta y)$.

The theory APr (atomless probability algebras) consists of PA_0 along with:

$$(APr) \quad \sup_x \inf_y \left| \mu(y \wedge x) - \frac{\mu(x)}{2} \right| = 0$$

Fact 2.10. *The class of probability algebras is elementary, axiomatised by Pr . The class of atomless probability algebras is elementary as well, axiomatised by APr .*

Moreover, the theory APr eliminates quantifiers (it is the model completion of Pr). It is \aleph_0 -categorical (there is a unique complete separable atomless probability algebra), and admits no compact model, whereby it is complete. It is \aleph_0 -stable and its notion of independence coincides with probabilistic independence. All types over sets (in the real sort) are stationary.

Proof. Most of this is shown in Ben Yaacov & Usvyatsov [BU, Example 4.3]. The fact regarding stability and independence were shown by the author in [Ben06] using the more technically involved (and now somewhat obsolete) setting of compact abstract theories. The arguments carry nonetheless to models of APr in continuous logic. $\blacksquare_{2.10}$

We would like to show that these two ways of coding a probability space in a metric structure are equivalent. Specifically we shall show that for any probability space $(\Omega, \mathcal{F}, \mu)$, the probability algebra $\tilde{\mathcal{F}}$ and the space $L^1(\mathcal{F}, [0, 1])$ of $[0, 1]$ -valued random variables are (uniformly) interpretable in one another.

Proposition 2.11. *Let \mathcal{M} be a model of RV , say $\mathcal{M} = L^1(\mathcal{F}, [0, 1])$. Then the \mathcal{L}_{Pr} -structure $\tilde{\mathcal{F}}$ is quantifier-free definable in \mathcal{M} in a manner which does not depend on Ω . More precisely:*

- (i) *We may identify an event $a \in \tilde{\mathcal{F}}$ with its characteristic function $\mathbb{1}_a \in M$. This identifies $\tilde{\mathcal{F}}$ with the subset $L^1(\mathcal{F}, \{0, 1\}) \subseteq M$ consisting of all $\{0, 1\}$ -valued random variables over $(\Omega, \mathcal{F}, \mu)$.*
- (ii) *Under the identification of the previous item, $\tilde{\mathcal{F}}$ is a quantifier-free definable subset of \mathcal{M} , that is, the predicate $d(x, \tilde{\mathcal{F}})$ is quantifier free definable in \mathcal{M} . Moreover, the Boolean algebra operations of $\tilde{\mathcal{F}}$ are definable by terms in \mathcal{M} , and the predicates of $\tilde{\mathcal{F}}$ (measure and distance) are quantifier free definable in \mathcal{M} .*

(See the first section of [Benb] for facts regarding definable sets in continuous logic.)

Proof. The first item is a standard fact. For the second item, let $g \in L^1(\mathcal{F}, [0, 1])$, and let $a = \{g \geq \frac{1}{2}\}$ (i.e., $a = \{\omega : g(\omega) \geq \frac{1}{2}\}$). Then:

$$d(g, \tilde{\mathcal{F}}) = d(g, \mathbb{1}_a) = E(g \wedge \neg g).$$

Given $a, b \in \tilde{\mathcal{F}}$ we have $\mathbb{1}_{a^c} = \neg \mathbb{1}_a$ and $\mathbb{1}_{a \setminus b} = \mathbb{1}_a \dot{-} \mathbb{1}_b$, from which the rest of the Boolean algebra structure can be recovered. In addition $d^{\tilde{\mathcal{F}}}(a, b) = d^{\mathcal{M}}(\mathbb{1}_a, \mathbb{1}_b)$ and $\mu(a) = E(\mathbb{1}_a)$. $\blacksquare_{2.11}$

Since $\tilde{\mathcal{F}}$ is (uniformly) definable we may quantify over it. Thus, modulo the theory RV , axiom ARV can be written more elegantly as:

$$(ARV') \quad \inf_{y \in \tilde{\mathcal{F}}} \left| E(y \wedge x) - \frac{E(x)}{2} \right| = 0.$$

The converse is a little more technical, since the interpretation of $L^1(\mathcal{F}, [0, 1])$ in the structure $\tilde{\mathcal{F}}$ will necessarily be in an imaginary sort. A similar interpretation of the space of $[0, \infty]$ -valued random variables in a *hyperimaginary* sort has already been discussed in [Ben06, Section 3]. The result we prove here is a little stronger and easier to work with. It is based on the notion of imaginary sorts for metric structures, which were introduced with the development of continuous logic in [BU].

Let $D = \{k/2^n : n \in \mathbb{N}, 0 < k < 2^n\}$ denote the set of all dyadic fractions in $]0, 1[$, $D' = D \cup \{0, 1\}$. For $r \in D$, let $n(r)$ be the least n such that $2^n r$ is an integer (so

$n(0) = 0$, and for $r \neq 0$, $n = n(r)$ is unique such that $2^n r$ is an odd integer). We shall now construct by induction on $n(r)$ a family of \mathcal{L}_{Pr} -terms $(\tau_r)_{r \in D'}$ in a sequence of distinct variables $(x_r)_{r \in D}$. We start with $\tau_0 = 1$ and $\tau_1 = 0$. If $n(r) = m > 0$ then $n(r \pm 2^{-m}) < m$ and we define:

$$\tau_r = (x_r \cup \tau_{r-2^{-m}}) \cap \tau_{r+2^{-m}}.$$

We may write such a term as $\tau_r(x_s)_{s \in D}$, where it is understood that only finitely many of $(x_s)_{s \in D}$ appear in τ_r . Let $\tilde{\mathcal{F}}$ be a probability algebra. Let $(a_s)_{s \in D'} \subseteq \tilde{\mathcal{F}}$ be any sequence of events, and let $b_r = \tau_r(a_s)_{s \in D}$. Then the sequence $(b_r)_{r \in D}$ is necessarily decreasing, and if the original sequence $(a_s)_{s \in D}$ is decreasing then the two sequences coincide: $b_r = a_r$ for all $r \in D$.

Let us also define:

$$\begin{aligned} \varphi_n(y, x_r)_{r \in D} &= \sum_{k < 2^n} 2^{-n} \mu(y \cap \tau_{k/2^n}) \\ \varphi(y, x_r)_{r \in D} &= \mathcal{F}\text{lim}_n \varphi_n. \end{aligned}$$

Each φ_n is an \mathcal{L}_{Pr} -formula (in particular, it only mentions finitely many variables); their forced limit $\varphi(y, x_r)_{r \in D}$ is a definable predicate in \mathcal{L}_{Pr} .

Proposition 2.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\mathcal{M} = L^1(\mathcal{F}, [0, 1])$. Let $\tilde{\mathcal{F}}_\varphi$ be the sort of canonical parameters for instances $\varphi(y, a_r)_{r \in D}$ over $\tilde{\mathcal{F}}$. For each random variable $f \in M$, let $f_r = \{f \leq r\}$ for $r \in D$ and let $\tilde{f} \in \tilde{\mathcal{F}}_\varphi$ be the canonical parameter of $\varphi(y, f_r)_{r \in D}$.*

- (i) *For every event $c \in \tilde{\mathcal{F}}$: $\varphi(c, \tilde{f}) = \varphi(c, f_r)_{r \in D} = \int_c f$.*
- (ii) *The mapping $f \mapsto \tilde{f}$ is a bijection between M and $\tilde{\mathcal{F}}_\varphi$.*
- (iii) *Identifying M with $\tilde{\mathcal{F}}$ in this manner, the \mathcal{L}_{RV} -structure on M is definable in $\tilde{\mathcal{F}}$ in a manner which does not depend on Ω .*

Moreover, if we compose this interpretation of $L^1(\mathcal{F}, [0, 1])$ in $\tilde{\mathcal{F}}$ with the definition of $\tilde{\mathcal{F}}$ in $L^1(\mathcal{F}, [0, 1])$ discussed in Proposition 2.11 above in either order, there is a definable bijection between the original structure and its interpreted copy in a manner which is uniform in Ω .

Proof. For the first item, the sequence $(f_r)_{r \in D}$ is decreasing so $\tau_r(f_s)_{s \in D} = f_r$. It follows that $|\varphi_n(c, f_r)_{r \in D} - \int_c f| < 2^{-n}$, and passing to the (forced) limit we obtain $\varphi(c, f_r)_{r \in D} = \int_c f$.

We now show the second item. To see that $f \mapsto \tilde{f}$ is injective assume that $\tilde{f} = \tilde{g}$. By the previous item this means that $\int_c f = \int_c g$ for every $c \in \tilde{\mathcal{F}}$, whereby $f = g$. To see it is surjective let $\varphi(x, a_r)_{r \in D}$ be any instance of φ . Define:

$$\begin{aligned} b_r &= \tau_r(a_s)_{s \in D} \in \tilde{\mathcal{F}} & r \in D, \\ f_n &= \sum_{k < 2^n} 2^{-n} \mathbb{1}_{b_{k/2^n}} \in M & n \in \mathbb{N}. \end{aligned}$$

One readily checks that $d(f_n, f_m) < 2^{-\min(n,m)}$, so the sequence f_n converges to a limit $g \in M$, $d(f_n, g) \leq 2^{-n}$. For every event $c \in \mathcal{F}$ we have $\varphi_n(c, a_r)_{r \in D} = \int_c f_n$. It follows that $|\varphi_n(c, a_r)_{r \in D} - \int_c g| \leq 2^{-n}$ and passing to the limit $\varphi(c, a_r)_{r \in D} = \int_c g$. In other words, $\varphi(x, a_r)_{r \in D} \equiv \varphi(x, g_r)_{r \in D}$, so \tilde{g} is a canonical parameter for $\varphi(x, a_r)_{r \in D}$.

Let us now prove the third item. In order to prove that $(\tilde{f}, \tilde{g}) \mapsto \widetilde{f \dot{\div} g}$ is definable it is enough to show that we can define the predicate $\varphi(x, \widetilde{f \dot{\div} g})$ uniformly from \tilde{f} and \tilde{g} . Indeed:

$$\begin{aligned} \varphi(x, \widetilde{f \dot{\div} g}) &= \int_x (f \dot{\div} g) = \sup_y \left[\int_{x \cap y} f \dot{\div} \int_{x \cap y} g \right] \\ &= \sup_y \left[\varphi(x \cap y, \tilde{f}) \dot{\div} \varphi(x \cap y, \tilde{g}) \right]. \end{aligned}$$

Similarly:

$$\int_x 0 = 0, \quad \int_x \neg f = \neg \int_x f, \quad \int_x \frac{1}{2} f = \frac{1}{2} \int_x f.$$

It follows that all the connectives which one can construct from these primitives are definable, and in particular $(x, y) \mapsto |x - y|$. Thus the distance $d(f, g) = \varphi(1, \widetilde{|f - g|})$ is definable.

We leave the moreover part to the reader. ■_{2.12}

The intrinsic distance on the imaginary sort \mathcal{F}_φ is by definition:

$$d_\varphi(f, g) = \sup_{b \in \mathcal{F}} \left| \int_b (f - g) \right| = \max(\|f \dot{\div} g\|_1, \|g \dot{\div} f\|_1).$$

The distance d_φ is easily verified to be uniformly equivalent to the L^1 metric on the space of $[0, 1]$ -valued random variables. This is a special case of the general fact that any two definable distance functions on a sort are uniformly equivalent. At the cost of additional technical complexity we could have arranged to recover $L^1(\mathcal{F}, [0, 1])$ on an imaginary sort in which the intrinsic distance is already the one coming from L^1 . Indeed, we could have defined a formula $\psi(y, z, x_r)_{r \in D}$ such that $\psi(b, c, f_r)_{r \in D} = \int_b f + \int_{c \setminus b} \neg f$, obtaining further down the road:

$$d_\psi(f, g) = \sup_{b, c \in \mathcal{F}} \left| \int_b (f - g) + \int_{c \setminus b} (g - f) \right| = \|f - g\|_1.$$

2.3. Additional properties of RV and ARV. Models of RV admits quantifier free definable continuous functional calculus on models of RV.

Lemma 2.13. *If $\theta: [0, 1]^\ell \rightarrow [0, 1]$ is a continuous function, then the function $\bar{f} \mapsto \theta \circ (\bar{f})$ is uniformly quantifier free definable in models of RV. By “quantifier free definable” we mean that for every definable predicate $P(\bar{y}, z)$, the definable predicate $P(\bar{y}, \theta \circ (\bar{x}))$ is*

definable with the same variable complexity. Specifically, $d(y, \theta \circ (\bar{x}))$ is quantifier free definable.

Proof. We can uniformly approximate θ by a sequence of terms $\tau_n(\bar{x})$ in $\neg, \frac{1}{2}, \div$. Assuming the sequence converges fast enough we get $P(\bar{y}, \theta \circ (\bar{x})) = \mathcal{F}\text{lim } P(\bar{y}, \tau_n \circ (\bar{x}))$. $\blacksquare_{2.13}$

For example, the predicates $E(x^p)$ or $E(|x - y|^p)$ are definable for every $p \in [1, \infty[$, and thus the L^p distance $\|x - y\|_p = E(|x - y|^p)^{1/p}$ is definable as well, all the definitions being quantifier free and uniform.

For $A \subseteq L^1(\mathcal{F}, [0, 1])$, let $\sigma(A) \subseteq \mathcal{F}$ denote the minimal σ -sub-algebra by which every member of A is measurable, i.e., such that $A \subseteq L^1(\sigma(A), [0, 1])$ (For this to be entirely well-defined we may require $\sigma(A)$ to contain the null measure ideal of \mathcal{F} .)

Lemma 2.14. *Let \mathcal{M} be a model of RV, say $\mathcal{M} = L^1(\mathcal{F}, [0, 1])$. Then for every σ -sub-algebra $\mathcal{F}_1 \subseteq \mathcal{F}$, the space $L^1(\mathcal{F}_1, [0, 1])$ is a sub-structure of \mathcal{M} . Conversely, every sub-structure $\mathcal{N} \subseteq \mathcal{M}$ arises in this manner as $L^1(\sigma(N), [0, 1])$.*

Proof. The first assertion is clear, so we prove the converse. It is also clear that $N \subseteq L^1(\sigma(N), [0, 1])$.

Let $f \in N$, and define $\dot{m}f = f \dot{+} \dots \dot{+} f$ (m times). Then $\dot{m}f \in N$, and as $m \rightarrow \infty$ we have $\dot{m}f \rightarrow \mathbb{1}_{\{f > 0\}}$ in L^1 , so $\mathbb{1}_{\{f > 0\}} \in N$. Since N is complete and closed under \neg and \div , it follows that $\mathbb{1}_A \in N$ for every $A \in \sigma(N)$. Considering finite sums of the form $(\frac{1}{2})^k \mathbb{1}_{A_0} \dot{+} \dots \dot{+} (\frac{1}{2})^k \mathbb{1}_{A_{n-1}}$ we see that every simple function in $L^1(\sigma(N), [0, 1])$ whose range consists solely of dyadic fractions belongs to N . Using the completeness of N one last time we may conclude that $L^1(\sigma(N), [0, 1]) \subseteq N$. $\blacksquare_{2.14}$

Lemma 2.15. *Let \mathcal{M} and \mathcal{N} be two models of RV, say $\mathcal{M} = L^1(\mathcal{F}, [0, 1])$, $\mathcal{N} = L^1(\mathcal{G}, [0, 1])$. Then two ℓ -tuples $\bar{f} \in M^\ell$ and $\bar{g} \in N^\ell$ have the same quantifier free type in \mathcal{L}_{RV} if and only if they have the same joint distribution as random variables.*

Proof. Assume that $\bar{f} \equiv^{qf} \bar{g}$. By the previous Lemma we have $E(\theta \circ (\bar{f})) = E(\theta \circ (\bar{g}))$ for every continuous function $\theta: [0, 1]^\ell \rightarrow [0, 1]$, which is enough in order to conclude that \bar{f} and \bar{g} have the same joint distribution. Conversely, assume that \bar{f} and \bar{g} have the same joint distribution. Then $E(\tau(\bar{f})) = E(\tau(\bar{g}))$ for every term $\tau(\bar{x})$. It follows that $\bar{f} \equiv^{qf} \bar{g}$. $\blacksquare_{2.15}$

Let $\bar{\mathcal{F}}_a$ denote the set of atoms in $\bar{\mathcal{F}}$, which we may enumerate as $\{A_i: i \in I\}$. Then I is necessarily countable and every $f \in L^1(\bar{\mathcal{F}}, [0, 1])$ can be written uniquely as $f_0 + \sum_{i \in I} \alpha_i \mathbb{1}_{A_i}$, where f_0 is over the atomless part and $\alpha_i \in [0, 1]$.

Lemma 2.16. *The set $\bar{\mathcal{F}}_a \cup \{0\}$ is uniformly definable in $\bar{\mathcal{F}}$. In $L^1(\bar{\mathcal{F}}, [0, 1])$, both the sets $\bar{\mathcal{F}}_a \cup \{0\}$ (i.e., $\{\mathbb{1}_A: A \in \bar{\mathcal{F}}_a\} \cup \{0\}$) and $\{\alpha \mathbb{1}_A: \alpha \in [0, 1], A \in \bar{\mathcal{F}}_a\}$ are uniformly definable.*

Proof. For the first assertion let $\varphi(x)$ be the \mathcal{L}_{Pr} -formula $\sup_y (\mu(x \cap y) \wedge \mu(x \setminus y))$. If A is an atom or zero then clearly $\varphi(A) = 0$. If A is an event which is not an atom then the

nearest atom to A is the biggest atom in A (or any of them if there are several of largest measure, or 0 if A contains no atoms). Let us construct a partition of A into two events A_1 and A_2 by assigning the atoms in A (if any) sequentially to A_1 or to A_2 , whichever has so far the lesser measure, and by splitting the atomless part of A equally between A_1 and A_2 . If $B \subseteq A$ is an atom of greatest measure (or zero if there are none) then $|\mu(A_1) - \mu(A_2)| \leq \mu(B)$ and:

$$\begin{aligned} \varphi(A) &\geq \mu(A_1) \wedge \mu(A_2) \geq \frac{1}{2}\mu(A) - \frac{1}{2}\mu(B) = \frac{1}{2}\mu(A \setminus B) \\ &= \frac{1}{2}d(A, \bar{\mathcal{F}}_a \cup \{0\}). \end{aligned}$$

Thus $\bar{\mathcal{F}}_a \cup \{0\}$ is definable.

For the second assertion, $\bar{\mathcal{F}}_a \cup \{0\}$ is relatively definable in $\bar{\mathcal{F}}$ which is in turn definable in $L^1(\mathcal{F}, [0, 1])$, so $\bar{\mathcal{F}}_a \cup \{0\}$ is definable in $L^1(\mathcal{F}, [0, 1])$. We may therefore quantify over $\bar{\mathcal{F}}_a \cup \{0\}$, and define:

$$\psi_n(x) = \inf_{A \in \bar{\mathcal{F}}_a \cup \{0\}} \bigwedge_{k \leq 2^n} d(x, \frac{k}{2^n} \mathbb{1}_A).$$

Then \mathcal{F} lim ψ_n defines the distance to the last set. ■_{2.16}

It follows that for each n , the set of events which can be written as the union of at most n atoms is definable, as is the set of all finite sums $\sum_{i < n} \alpha \mathbb{1}_{A_i}$ where each A_i is an atom (or zero). These definitions cannot be uniform in n , though. Indeed, an easy ultra-product argument shows that the set of all atomic events (i.e., which are unions of atoms) cannot be definable or even type-definable, and similarly for the set of all random variables whose support is atomic.

The atoms of a probability space always belong to the algebraic closure of the empty set (to the definable closure if no other atom has the same measure). They are therefore quite uninteresting from a model theoretic point of view, so we shall mostly consider atomless probability spaces.

Theorem 2.17. (i) *The theory ARV is complete and \aleph_0 -categorical.*

(ii) *The theory ARV eliminates quantifiers.*

(iii) *The universal part of ARV is RV, and ARV is the model completion of RV.*

(iv) *If $\mathcal{M} = L^1(\mathcal{F}, [0, 1]) \models \text{ARV}$ and $A \subseteq M$ then $\text{dcl}(A) = \text{acl}(A) = L^1(\sigma(A), [0, 1]) \subseteq M$.*

(v) *Two tuples f and \bar{g} have the same type over a set A (all in a model of ARV) if and only if they have the same joint conditional distribution over $\sigma(A)$.*

(vi) *The theory ARV is \aleph_0 -stable, independence coinciding with probabilistic independence, i.e.: $A \downarrow_B C \iff \sigma(A) \downarrow_{\sigma(B)} \sigma(C)$. Moreover, types over any sets in the home sort (i.e., not over imaginary elements) are stationary.*

Proof. Categoricity and completeness of ARV follow from the analogous properties for APR.

Assume that \bar{f} and \bar{g} are two ℓ -tuples in models of ARV , $\bar{f} =^{qf} \bar{g}$. By Lemma 2.15 they have the same joint distribution. For every $(r, i) \in D \times \ell$ define events $a_{f,i} = \{f_i \leq r\}$, $b_{r,i} = \{g_i \leq r\}$. The joint distribution implies that $(a_{r,i})_{(r,i) \in D \times \ell} \equiv^{qf} (b_{r,i})_{(r,i) \in D \times \ell}$, and since APr eliminates quantifiers: $(a_{r,i})_{(r,i) \in D \times \ell} \equiv (b_{r,i})_{(r,i) \in D \times \ell}$. Under the interpretation of Proposition 2.12 we have $f_i \in \text{dcl}((a_{r,i})_{r \in D})$, $g_i \in \text{dcl}((b_{r,i})_{r \in D})$, so $\bar{f} \equiv \bar{g}$. In other words, the quantifier-free type of a type determines its type, whence quantifier elimination.

The theory RV is universal and all its models embed in models of ARV , whereby $RV = ARV_{\forall}$. Since ARV eliminates quantifiers it is the model completion of its universal part.

Let now $\mathcal{M} = L^1(\mathcal{F}, [0, 1]) \models ARV$, and let $A \subseteq M$. By Lemma 2.14, $\langle A \rangle$ (the sub-structure generated by A in \mathcal{M}) is $L^1(\sigma(A), [0, 1])$. Identifying \mathcal{F} with its definable copy in \mathcal{M} we obtain $\langle A \rangle = \langle \sigma(A) \rangle = L^1(\sigma(A), [0, 1])$ and $\text{dcl}(A) = \text{dcl}(\sigma(A)) \supseteq L^1(\sigma(A), [0, 1])$. On the other hand, $\sigma(A)$ is a complete sub-algebra of $\mathcal{F} \models APr$ and therefore definably and even algebraically closed there. By our bi-interpretability result, $\sigma(A)$ is relatively algebraically closed in the definable copy of \mathcal{F} in \mathcal{M} . Therefore, if $g \in \text{acl}(A) = \text{acl}(\sigma(A))$ then $\sigma(g) \subseteq \sigma(A)$, i.e., $g \in L^1(\sigma(A), [0, 1])$.

Let us identify \mathcal{F} with its definable copy in \mathcal{M} , and let $\mathcal{A} = \sigma(A)$. By the previous item we have $\text{tp}(\bar{f}/A) \equiv \text{tp}(\bar{f}/\mathcal{A})$. When $\mathcal{A} = \{a_0, \dots, a_{m-1}\}$ is finite sub-algebra, it is easy to verify that the joint conditional distribution of \bar{f} over \mathcal{A} is the same as the joint distribution of the $(n+m)$ -tuple $\bar{f}, \mathbb{1}_{\bar{a}}$. The result for types over infinite algebras follows.

Stability and the characterisation of independence for ARV follow from the analogous properties for APr via bi-interpretability. ■_{2.17}

3. KEISLER RANDOMISATION

In this section we use earlier results to extend H. Jerome Keisler's notion of a *randomisation* of a classical structure, or of a classical theory, to continuous ones. For the original constructions we refer the reader to [Kei99, BK]. Throughout, we work with a fixed continuous signature \mathcal{L} . For the sake of simplicity we shall assume that \mathcal{L} is single-sorted, but everything we do can be carried out in a multi sorted setting.

3.1. Randomisation. We shall want to consider some notion of probability integration of functions on a space Ω , which is going to be additive, although not always σ -additive (i.e., not always verifying the Monotone Convergence Theorem and its consequences). Thus, even though we do deal with concrete functions, it will be far more convenient to keep track of an abstract integration functional alone, rather than of the standard integration theory data (measurable sets, measurable functions, the measure, and the notion of integration all these give rise to).

Definition 3.1. A *finitely additive probability integration space*, or simply an *integration space*, is a triplet (Ω, \mathcal{A}, E) where Ω is any set, $\mathcal{A} \subseteq [0, 1]^\Omega$ is non empty and closed

under the connectives \neg , $\frac{1}{2}$ and \div , and $E: \mathcal{A} \rightarrow [0, 1]$ satisfies $E(1) = 1$ and $E(X + Y) = E(X) + E(Y)$ whenever X, Y and $X + Y$ are all in \mathcal{A} .

In this case we also say that E is a *finitely additive probability integration functional*, or simply an *integration functional*, on \mathcal{A} .

Fact 3.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $\mathcal{A} \subseteq [0, 1]^\Omega$ consist of all \mathcal{F} -measurable functions and let $E: \mathcal{A} \rightarrow [0, 1]$ be integration $d\mu$. Then (Ω, \mathcal{A}, E) is an integration space.*

The same still holds even if \mathcal{F} is a mere Boolean algebra and μ is merely finitely additive.

Lemma 3.3. *Let (Ω, \mathcal{A}, E) be an integration space. Equip \mathcal{A} with the distance $d(X, Y) = E(|X - Y|)$. Then $E(X) = d(X, 0)$ for all $X \in \mathcal{A}$ and $(\mathcal{A}, 0, \neg, \frac{1}{2}, \div, d)$ is a pre-model of RV. In this situation we shall say that (\mathcal{A}, E) is a pre-model of RV, or that E renders \mathcal{A} a pre-model of RV.*

Proof. Indeed, $d(X, 0) = E(|X|) = E(X)$. Now RV1,2 follows from the hypothesis and the fact that $X = (X \div Y) + (X \wedge Y)$. RV3 holds by definition. It follows from the hypothesis that $E(0) = 2E(0) = 0$, whence RV4. ■_{3.3}

If E renders \mathcal{A} a pre-model of ARV then we say that (Ω, \mathcal{A}, E) is *atomless*.

Let Ω be an arbitrary set and let $\mathcal{M} = \mathcal{M}_\Omega = \{\mathcal{M}_\omega\}_{\omega \in \Omega}$ be a family of \mathcal{L} -structures. Its product $\prod \mathcal{M} = \prod_{\omega \in \Omega} \mathcal{M}_\omega$ consists of all functions $\mathbf{a}: \Omega \rightarrow \bigcup M_\omega$ which verify $\mathbf{a}(\omega) \in M_\omega$ for all $\omega \in \Omega$. Function symbols and terms of \mathcal{L} are interpreted naturally on $\prod \mathcal{M}$. For an \mathcal{L} -formula $\varphi(\bar{x})$ we define

$$\langle \varphi(\bar{\mathbf{a}}) \rangle \in \Omega^{[0,1]}, \quad \langle \varphi(\bar{\mathbf{a}}) \rangle: \omega \mapsto \varphi^{\mathcal{M}_\omega}(\bar{\mathbf{a}}(\omega)).$$

Definition 3.4. Let Ω be a set, $\mathcal{M}_\Omega = \{\mathcal{M}_\omega\}_{\omega \in \Omega}$ a family of \mathcal{L} -structures. Let also $\mathbf{M} \subseteq \prod \mathcal{M}$, $\mathcal{A} \subseteq [0, 1]^\Omega$ and $E: \mathcal{A} \rightarrow [0, 1]$. We say that $(\mathbf{M}, \mathcal{A}, E)$ is a *randomisation based on \mathcal{M}_Ω* if

- (i) The triplet (Ω, \mathcal{A}, E) is an integration space.
- (ii) The subset $\mathbf{M} \subseteq \prod \mathcal{M}$ is non empty, closed under function symbols, and $\langle P(\bar{\mathbf{a}}) \rangle \in \mathcal{A}$ for every n -ary predicate symbol $P \in \mathcal{L}$ and $\bar{\mathbf{a}} \in \mathbf{M}^n$.

We equip \mathbf{M} with the pseudo-metric $d(\mathbf{a}, \mathbf{b}) = E\langle d(\mathbf{a}, \mathbf{b}) \rangle$ and \mathcal{A} with the L^1 pseudo-metric $d(X, Y) = E(|X - Y|)$.

We may choose to consider E as part of the structure on \mathcal{A} , in which case the randomisation is denoted by the pair $(\mathbf{M}, \mathcal{A})$ alone.

If $(\Omega, \mathcal{F}, \mu)$ is a probability space, every $X \in \mathcal{A}$ is \mathcal{F} -measurable and $E[X] = \int X d\mu$ then we say that $(\mathbf{M}, \mathcal{A})$ is based on the *random family* $\mathcal{M}_{(\Omega, \mathcal{F}, \mu)}$ (and then we almost always omit E from the notation).

The *randomisation signature* \mathcal{L}^R is defined as follows:

- The sorts of \mathcal{L}^R include of the sorts of \mathcal{L} , referred to as *main sorts*, plus a new *auxiliary sort*.

- Every function symbol of \mathcal{L} is present in \mathcal{L}^R , between the corresponding main sorts. It is equipped with the same uniform continuity moduli as in \mathcal{L} .
- For every predicate symbol P of \mathcal{L} , \mathcal{L}^R contains a function symbol $\llbracket P \rrbracket$ from the corresponding main sorts into the auxiliary sort. It is equipped with the same uniform continuity moduli as P in \mathcal{L} .
- The auxiliary sort is equipped with the signature \mathcal{L}_{RV} .

A randomisation $(\mathbf{M}, \mathcal{A})$ admits a natural interpretation as an \mathcal{L}^R -pre-structure $(\mathcal{M}, \mathcal{A})$. The corresponding structure will be denoted $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$. We also say that the randomisation $(\mathbf{M}, \mathcal{A})$ is a *representation* of the structure $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$.

Example 3.5. A special case of a randomisation is when $\mathbf{M} = \prod \mathcal{M}$ (i.e., the set of all sections), $\mathcal{A} = [0, 1]^\Omega$, \mathcal{U} is an ultra-filter on Ω , and $E_{\mathcal{U}}: [0, 1]^\Omega \rightarrow [0, 1]$ is the integration functional corresponding to the 0/1 measure defined by \mathcal{U} , i.e., $E_{\mathcal{U}}(X) = \lim_{\omega \in \mathcal{U}} X(\omega)$. In this case $\widehat{\mathcal{A}} = [0, 1]$ and $\widehat{\mathcal{M}}$ can be identified with the ultra-product $\prod_{\mathcal{U}} \mathcal{M}$.

Definition 3.6. We say that a randomisation $(\mathbf{M}, \mathcal{A})$ is *full* if for every $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ and $X \in \mathcal{A}$, there is a function $\mathbf{c} \in \mathbf{M}$ satisfying:

$$\mathbf{c}(\omega) = \begin{cases} \mathbf{a}(\omega) & X(\omega) = 1, \\ \mathbf{b}(\omega) & X(\omega) = 0, \\ \text{anything} & \text{otherwise.} \end{cases}$$

We shall sometimes refer to it as $\langle X, \mathbf{a}, \mathbf{b} \rangle$, even though there is no uniqueness here.

We say that $(\mathbf{M}, \mathcal{A})$ is *atomless* if \mathcal{A} is a pre-model of ARV (i.e., if (Ω, \mathcal{A}, E) is atomless).

Example 3.7 (Randomisation of a single structure). Let \mathcal{M} be a structure, $(\Omega, \mathcal{F}, \mu)$ an atomless probability space. Let $\mathbf{M}_c \subseteq M^\Omega$ consist of all functions $\mathbf{a}: \Omega \rightarrow M$ which take at most countably many values in M , each on a measurable set. Define $\mathcal{A}_c \subseteq [0, 1]^\Omega$ similarly, equipping it with integration with respect to μ . Then $(\mathbf{M}_c, \mathcal{A}_c)$ is a full atomless randomisation.

Assume now that $(\Omega, \mathcal{F}, \mu)$ is merely a finitely additive probability space, namely that \mathcal{F} is a mere Boolean algebra and μ is finitely additive. Let $\mathbf{M}_f \subseteq M^\Omega$ and $\mathcal{A}_f \subseteq [0, 1]^\Omega$ consist of functions which take at most finitely many values, each on a measurable set. Again, $(\mathbf{M}_f, \mathcal{A}_f)$ is an atomless, full randomisation.

If $(\Omega, \mathcal{F}, \mu)$ is a true (i.e., σ -additive) probability space then both constructions are possible and $(\mathcal{M}_f, \mathcal{A}_f) \subseteq (\mathcal{M}_c, \mathcal{A}_c)$. It is not difficult to check that they have the same completion $(\widehat{\mathcal{M}}_f, \widehat{\mathcal{A}}_f) = (\widehat{\mathcal{M}}_c, \widehat{\mathcal{A}}_c)$. In particular, $\widehat{\mathcal{A}}_f = \widehat{\mathcal{A}}_c = L^1(\mathcal{F}, [0, 1])$.

Moreover, the resulting structure only depends on $\mathcal{A} = L^1(\mathcal{F}, [0, 1])$, and we denote it by $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ (or just $\mathcal{M}^{\mathcal{A}}$).

3.2. The randomisation theory. Our first task is to axiomatise the class of \mathcal{L}^R -structures which can be obtained from full atomless randomisations (and in particular

show that it is elementary). We shall use x, y, \dots to denote variables of \mathcal{L} , $\mathbf{x}, \mathbf{y}, \dots$ to denote the corresponding variables in the main sort of \mathcal{L}^R and U, V, \dots to denote variables in the auxiliary sort of \mathcal{L}^R . For simplicity of notation, an \mathcal{L}^R -structure $(\mathcal{M}, \mathcal{A})$ may be denoted by \mathcal{M} alone. In this case, the auxiliary sort will be denoted by $\mathcal{A}^{\mathcal{M}}$ and we may write somewhat informally $\mathcal{M} = (\mathcal{M}, \mathcal{A}^{\mathcal{M}})$. When $\mathcal{A}^{\mathcal{M}} \models RV$ we shall refer to the probability algebra of $\mathcal{A}^{\mathcal{M}}$ as $\mathcal{F}^{\mathcal{M}}$ (so $\mathcal{A}^{\mathcal{M}} = L^1(\mathcal{F}^{\mathcal{M}}, [0, 1])$).

The “base theory” for randomisation, which will be denoted by T_0^{Ra} , consists of the theory RV for the auxiliary sort along with the following additional axioms:

$$\begin{aligned}
(\text{R1}_f) \quad & (\delta_{f,i}(\varepsilon) \dot{\div} \llbracket d(\mathbf{x}, \mathbf{y}) \rrbracket) \wedge (\llbracket d(f(\bar{\mathbf{x}}', \mathbf{x}, \bar{\mathbf{y}}'), f(\bar{\mathbf{x}}', \mathbf{y}, \bar{\mathbf{y}}')) \rrbracket \dot{\div} \varepsilon) = 0 \\
(\text{R1}_P) \quad & (\delta_{P,i}(\varepsilon) \dot{\div} \llbracket d(\mathbf{x}, \mathbf{y}) \rrbracket) \wedge (\llbracket P(\bar{\mathbf{x}}', \mathbf{x}, \bar{\mathbf{y}}') \rrbracket \dot{\div} \llbracket P(\bar{\mathbf{x}}', \mathbf{y}, \bar{\mathbf{y}}') \rrbracket \dot{\div} \varepsilon) = 0 \\
(\text{R2}) \quad & d(\mathbf{x}, \mathbf{y}) = E\llbracket d(\mathbf{x}, \mathbf{y}) \rrbracket \\
(\text{R3}) \quad & \sup_{U \in \mathcal{F}} \inf_{\mathbf{z}} E \left[(\llbracket d(\mathbf{x}, \mathbf{z}) \rrbracket \wedge U) \vee (\llbracket d(\mathbf{y}, \mathbf{z}) \rrbracket \wedge \neg U) \right]
\end{aligned}$$

In axiom R1, $\delta_{s,i}$ denotes the uniform continuity modulus of the symbol s with respect to its i th argument, with $|\bar{\mathbf{x}}'| = i$ and $|\bar{\mathbf{y}}'| = n_s - i - 1$. In axiom R3, \mathcal{F} denotes the probability algebra of the auxiliary sort, over which, modulo RV , we may quantify.

The role of axiom R1 is to ensure that the values of $\llbracket P(\bar{\mathbf{a}}) \rrbracket(\omega)$, $f(\bar{\mathbf{a}})(\omega)$ only depends on $\bar{\mathbf{a}}(\omega)$ and respect the uniform continuity moduli prescribed by \mathcal{L} . Axiom R2 is straightforward, requiring the distance in the main sort of be the expectation of the random variable associated to \mathcal{L} -distance. Axiom R3 is a gluing property, corresponding to fullness of a randomisation. It can be informally stated as

$$(\text{R3}') \quad (\forall \mathbf{x}, \mathbf{y}) (\forall U \in \mathcal{F}) (\exists \mathbf{z}) \left(\llbracket d(\mathbf{x}, \mathbf{z}) \rrbracket \wedge U = \llbracket d(\mathbf{y}, \mathbf{z}) \rrbracket \wedge \neg U = 0 \right),$$

where the existential quantifier is understood to hold in the approximate sense. We prove in Lemma 3.10 below that it actually holds in the precise sense.

Lemma 3.8. *Let $(\mathbf{M}, \mathcal{A})$ be a randomisation. Then $(\mathcal{M}, \mathcal{A})$ is a pre-model of RV (in the auxiliary sort) and of R1,2. If $(\mathbf{M}, \mathcal{A})$ is full then $(\mathcal{M}, \mathcal{A})$ is a pre-model of T_0^{Ra} .*

Proof. All we have to show is that if $(\mathbf{M}, \mathcal{A})$ is full then $(\mathcal{M}, \mathcal{A})$ verifies R3, or equivalently, $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$ does. However, we chose to write R3 using a quantifier over a definable set, a construct which need not have the apparent semantics in a pre-structure such as $(\mathcal{M}, \mathcal{A})$, and we find ourselves forced to work with $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$. (Indeed, since \mathcal{A} is a mere pre-model of RV , the algebra of characteristic functions in \mathcal{A} may well be trivial.)

Let $\widehat{\mathcal{F}}$ denote the probability algebra of $\widehat{\mathcal{A}}$ and let $A \in \widehat{\mathcal{F}}$, $\mathbf{a}, \mathbf{b} \in \mathbf{M}$. Then there is $X \in \mathcal{A}$ whose image $\widehat{X} \in \widehat{\mathcal{A}}$ is very close to $\mathbb{1}_A$. Define:

$$\begin{aligned}
Y &= \dot{2}(X \dot{\div} 1/4) \in \mathcal{A}, \\
\mathbf{c} &= \langle Y, \mathbf{a}, \mathbf{b} \rangle \in \mathbf{M} && \text{(by fullness),} \\
W &= (\llbracket d(\mathbf{a}, \mathbf{c}) \rrbracket \wedge Y) \vee (\llbracket d(\mathbf{b}, \mathbf{c}) \rrbracket \wedge \neg Y) \in \mathcal{A}.
\end{aligned}$$

For every $\omega \in \Omega$ we have $Y(\omega) \in \{0, 1\} \implies W(\omega) = 0$. In other words, $W(\omega) \neq 0 \implies 0 < Y(\omega) < 1 \implies 1/4 < X(\omega) < 3/4$. Thus $W \leq (\dot{4}X) \wedge (\dot{4}\neg X)$, and:

$$\widehat{W} = (\llbracket d(\hat{\mathbf{a}}, \hat{\mathbf{c}}) \rrbracket \wedge \widehat{Y}) \vee (\llbracket d(\hat{\mathbf{b}}, \hat{\mathbf{c}}) \rrbracket \wedge \neg \widehat{Y}) \leq (\dot{4}\widehat{X}) \wedge (\dot{4}\neg \widehat{X}).$$

Choosing \widehat{X} close enough to $\mathbb{1}_A$ (let us write $\widehat{X} \sim \mathbb{1}_A$), we have:

$$\begin{aligned} \widehat{Y} &\sim \mathbb{1}_A, \\ \widehat{W} &\sim (\llbracket d(\hat{\mathbf{a}}, \hat{\mathbf{c}}) \rrbracket \wedge A) \vee (\llbracket d(\hat{\mathbf{b}}, \hat{\mathbf{c}}) \rrbracket \wedge \neg A), \\ (\dot{4}\widehat{X}) \wedge (\dot{4}\neg \widehat{X}) &\sim 0. \end{aligned}$$

Thus $(\llbracket d(\hat{\mathbf{a}}, \hat{\mathbf{c}}) \rrbracket \wedge A) \vee (\llbracket d(\hat{\mathbf{b}}, \hat{\mathbf{c}}) \rrbracket \wedge \neg A)$ can be arbitrarily close to 0 in $\widehat{\mathcal{A}}$.

We have shown that for all $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ and $A \in \widehat{\mathcal{A}}$:

$$\inf_{\mathbf{z}} E \left[(\llbracket d(\hat{\mathbf{a}}, \mathbf{z}) \rrbracket \wedge A) \vee (\llbracket d(\hat{\mathbf{b}}, \mathbf{z}) \rrbracket \wedge \neg A) \right] = 0.$$

Since the image of \mathbf{M} is dense in $\widehat{\mathbf{M}}$ this shows that $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$ is a model of R3. $\blacksquare_{3.8}$

In order to prove a converse we need to construct, for every model $\mathcal{M} \models T_0^{Ra}$, a corresponding randomisation.

Definition 3.9. Assume $(\mathcal{M}, \mathcal{A}) \models T_0^{Ra}$. Let $(\Omega, \mu) = (\Omega^{\mathcal{M}}, \mu^{\mathcal{M}})$ be the Stone space of \mathcal{A} as per Theorem 2.7. Then we say that $(\mathcal{M}, \mathcal{A})$ is *based* on (Ω, μ) .

We recall that Ω is a compact Hausdorff topological space, μ is a regular Borel probability measure and we may identify $\mathcal{A} = C(\Omega, [0, 1]) = L^1(\mu, [0, 1])$. Under this identification $\int_{\Omega} X d\mu = E(X)$ for all $X \in \mathcal{A}$.

For each $\omega \in \Omega$ we define an \mathcal{L} -pre-structure $\mathcal{M}_{0,\omega}$. Its underlying set is $M_{0,\omega} = \mathbf{M}$ and the interpretations of the symbols are inherited naturally from \mathcal{M} :

$$f^{\mathcal{M}_{0,\omega}} = f^{\mathcal{M}}: \mathbf{M}^n \rightarrow \mathbf{M}, \quad P^{\mathcal{M}_{0,\omega}}(\bar{\mathbf{a}}) = \llbracket P(\bar{\mathbf{a}}) \rrbracket(\omega) \in [0, 1].$$

Notice that axiom R1_d implies that $\llbracket d(\mathbf{x}, \mathbf{y}) \rrbracket \div \llbracket d(\mathbf{x}, \mathbf{z}) \rrbracket \leq \llbracket d(\mathbf{y}, \mathbf{z}) \rrbracket$ and axiom R2 implies $\llbracket d(\mathbf{x}, \mathbf{x}) \rrbracket = 0$. Symmetry of $\llbracket d(\mathbf{x}, \mathbf{y}) \rrbracket$ and the usual form of the triangle inequality follow, so $d^{\mathcal{M}_{0,\omega}}$ is a pseudo-metric for every ω . Other instances of axiom R1 imply that $\mathcal{M}_{0,\omega}$ respects uniform continuity moduli prescribed by \mathcal{L} . Thus $\mathcal{M}_{0,\omega}$ is indeed an \mathcal{L} -pre-structure. The structure associated to $\mathcal{M}_{0,\omega}$ will be denoted \mathcal{M}_{ω} . Let \mathcal{M} denote the family $\{\mathcal{M}_{\omega}\}_{\omega \in \Omega}$ and let a_{ω} denote the image of \mathbf{a} in \mathcal{M}_{ω} .

Assume that $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ are distinct. Then $E \llbracket d(\mathbf{a}, \mathbf{b}) \rrbracket > 0$, whereby $\llbracket d(\mathbf{a}, \mathbf{b}) \rrbracket(\omega) > 0$ for some $\omega \in \Omega$. Thus $a_{\omega} \neq b_{\omega}$ and the mappings $\omega \mapsto a_{\omega}, \omega \mapsto b_{\omega}$ are distinct. In other words, we may identify $\mathbf{a} \in \mathbf{M}$ with the mapping $\mathbf{a}: \omega \mapsto a_{\omega}$. Viewed in this manner we have $\mathbf{M} \subseteq \prod \mathcal{M}$. By construction, if $f \in \mathcal{L}$ is a function symbol then its coordinate-wise action on \mathbf{M} as a subset of $\prod \mathcal{M}$ coincides with $f^{\mathcal{M}}$. Similarly, if $P \in \mathcal{L}$ is a predicate symbol then $\langle P(\bar{\mathbf{a}}) \rangle = (\omega \mapsto P^{\mathcal{M}_{\omega}}(\bar{\mathbf{a}}(\omega))) = \llbracket P(\bar{\mathbf{a}}) \rrbracket \in \mathcal{A}$. We have thus

identified $(\mathbf{M}, \mathcal{A})$ with a randomisation base on (Ω, μ) . This randomisation is called the *canonical representation* of $(\mathcal{M}, \mathcal{A})$.

Lemma 3.10. *Let $\mathcal{M} \models T_0^{Ra}$, $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ and $A \in \mathcal{F}^{\mathcal{M}}$. Then there exists (a unique) $\mathbf{c} = \langle A, \mathbf{a}, \mathbf{b} \rangle \in \mathbf{M}$ which is equal to \mathbf{a} over A and to \mathbf{b} elsewhere:*

$$\llbracket d(\mathbf{a}, \mathbf{c}) \rrbracket \wedge A = \llbracket d(\mathbf{b}, \mathbf{c}) \rrbracket \wedge \neg A = 0.$$

Identifying $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ with its canonical representation based on Ω , A is identified with a (unique) clopen set $A \subseteq \Omega$ and we have:

$$\mathbf{c}(\omega) = \begin{cases} \mathbf{a}(\omega) & \omega \in A, \\ \mathbf{b}(\omega) & \omega \notin A. \end{cases}$$

Proof. By axiom R3, for every $\varepsilon > 0$, there is \mathbf{c}_ε such that:

$$E \left[\left(\llbracket d(\mathbf{a}, \mathbf{c}_\varepsilon) \rrbracket \wedge A \right) \vee \left(\llbracket d(\mathbf{b}, \mathbf{c}_\varepsilon) \rrbracket \wedge \neg A \right) \right] < \varepsilon.$$

Passing to the canonical representation it is easy to check that $d(\mathbf{c}_\varepsilon, \mathbf{c}_{\varepsilon'}) < \varepsilon + \varepsilon'$ for any $\varepsilon, \varepsilon' > 0$. Thus $(\mathbf{c}_\varepsilon)_{\varepsilon \rightarrow 0^+}$ is a Cauchy sequence whose limit $\mathbf{c} = \langle A, \mathbf{a}, \mathbf{b} \rangle$ is as desired. Uniqueness is clear. ■_{3.10}

Theorem 3.11. *An \mathcal{L}^R -structure is a model of T_0^{Ra} if and only if it has a full representation, i.e., if and only if it is isomorphic to a structure $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$ associated to a full randomisation $(\mathbf{M}, \mathcal{A})$.*

Moreover, let $(\mathcal{M}, \mathcal{A})$ be a model of T_0^{Ra} . Then the canonical representation of $(\mathcal{M}, \mathcal{A})$ is full, and as an \mathcal{L}^R -pre-structure it is isomorphic to $(\mathcal{M}, \mathcal{A})$. In particular, the \mathcal{L}^R -pre-structure associated to the canonical representation is already a structure.

Proof. One direction is Lemma 3.8, so it is enough to prove the moreover part. It is clear that the identity mapping is an isomorphism between the structure $(\mathcal{M}, \mathcal{A})$ and the pre-structure associated to the canonical representation, so all that is left to show is that the latter is full.

Let $\mathbf{a}, \mathbf{b} \in \mathbf{M}$, $X \in \mathcal{A}$. The set $\{X \leq \frac{1}{2}\} \subseteq \Omega$ is Borel and therefore equal outside a null measure set to some clopen set $U \subseteq \Omega$. We now have

$$\begin{aligned} X \dot{\div} \mathbb{1}_{\{X \geq 1/2\}} \leq \frac{1}{2} &\implies X \dot{\div} \mathbb{1}_U \leq \frac{1}{2}, \implies U \supseteq \{X = 1\}, \\ \mathbb{1}_{\{X \geq 1/2\}} \dot{\div} X \leq \frac{1}{2} &\implies \mathbb{1}_U \dot{\div} X \leq \frac{1}{2}, \implies U \cap \{X = 0\} = \emptyset. \end{aligned}$$

Thus $\mathbf{c} = \langle U, \mathbf{a}, \mathbf{b} \rangle$ will do as $\langle X, \mathbf{a}, \mathbf{b} \rangle$. ■_{3.11}

From now on we shall identify a model of T_0^{Ra} with its canonical representation whenever that is convenient and without further mention.

3.3. Quantifiers. It is a classical fact that $\mathcal{A} = L^1(\mathcal{F}, [0, 1])$ is a complete lattice. More precisely, let $\mathcal{A} \subseteq \mathcal{A}$ be any subset. We may assume that \mathcal{A} is closed under \wedge . Let $r = \inf\{E(X) : X \in \mathcal{A}\}$ and let $(X_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ satisfy $E(X_n) \rightarrow r$. By hypothesis $E(X_n \wedge X_m) \geq r$ whereby $d(X_n, X_m) \leq |E(X_n) - r| + |E(X_m) - r|$. The sequence $(X_n)_{n \in \mathbb{N}}$ is therefore Cauchy and its limit is $\inf \mathcal{A}$.

Let now $(\mathcal{M}, \mathcal{A})$ be a model of T_0^{Ra} ,

Definition 3.12. Let $(\mathcal{M}, \mathcal{A}) \models T_0^{Ra}$, $t : \mathbf{M}^n \rightarrow \mathcal{A}$ a function. We say that t is *local* if it is always true that:

$$t(\dots, \langle A, \mathbf{a}, \mathbf{b} \rangle, \dots) = t(\dots, \mathbf{a}, \dots) \wedge A + t(\dots, \mathbf{b}, \dots) \wedge \neg A.$$

For a function $t : \mathbf{M}^{n+1} \rightarrow \mathcal{A}$ we define $\inf_{\mathbf{y}} t(\bar{\mathbf{x}}, \mathbf{y}) : \mathbf{M}^n \rightarrow \mathcal{A}$ by

$$\inf_{\mathbf{y}} t(\bar{\mathbf{a}}, \mathbf{y}) = \inf\{t(\bar{\mathbf{a}}, \mathbf{b}) : \mathbf{b} \in \mathbf{M}\} \in \mathcal{A}.$$

Lemma 3.13. *Let $t(\bar{\mathbf{x}}, \mathbf{y})$ be a uniformly definable local function in models of T_0^{Ra} from the main sort into the auxiliary sort. Then the function $s(\bar{\mathbf{x}}) = \inf_{\mathbf{y}} t(\bar{\mathbf{x}}, \mathbf{y})$ is uniformly definable and local as well, and T_0^{Ra} implies that:*

$$\inf_{\mathbf{z}} d(\inf_{\mathbf{y}} t(\bar{\mathbf{x}}, \mathbf{y}), t(\bar{\mathbf{x}}, \mathbf{z})) = 0.$$

Moreover, for every $\bar{\mathbf{a}}$ in a model of T_0^{Ra} and $\varepsilon > 0$ there is \mathbf{b} such that:

$$t(\bar{\mathbf{a}}, \mathbf{b}) \leq \inf_{\mathbf{y}} t(\bar{\mathbf{a}}, \mathbf{y}) + \varepsilon$$

(Similarly for $\sup_{\mathbf{y}} t$.)

Proof. It follows directly from the definition that if t is local then so is $\inf_{\mathbf{y}} t$ (no definability is needed here).

We start by proving the moreover part. Let $(\mathcal{M}, \mathcal{A}) \models T_0^{Ra}$, $\bar{\mathbf{a}} \in \mathbf{M}^n$. Following the discussion of the completeness of the lattice structure on \mathcal{A} there is a sequence $\{\mathbf{c}_n\}_{n \in \mathbb{N}}$ such that $\inf_{\mathbf{y}} t(\bar{\mathbf{a}}, \mathbf{y}) = \inf_n t(\bar{\mathbf{a}}, \mathbf{c}_n)$. Let us define a sequence $\{\mathbf{b}_n\}$ by:

$$\mathbf{b}_0 = \mathbf{c}_0,$$

$$\mathbf{b}_{n+1} = \langle \{t(\bar{\mathbf{a}}, \mathbf{b}_n) - t(\bar{\mathbf{a}}, \mathbf{c}_{n+1}) > \varepsilon\}, \mathbf{c}_{n+1}, \mathbf{b}_n \rangle.$$

In other words, when passing from \mathbf{b}_n to \mathbf{b}_{n+1} we use \mathbf{c}_{n+1} only where this means a decrease of more than ε , and elsewhere keep \mathbf{b}_n .

Clearly $\sum_n \mu\{t(\bar{\mathbf{a}}, \mathbf{b}_n) - t(\bar{\mathbf{a}}, \mathbf{c}_{n+1}) > \varepsilon\} \leq 1/\varepsilon$. Since t is local, $d(\mathbf{b}_n, \mathbf{b}_{n+1}) \leq \mu\{t(\bar{\mathbf{a}}, \mathbf{b}_n) - t(\bar{\mathbf{a}}, \mathbf{c}_{n+1}) > \varepsilon\}$, so the sequence $\{\mathbf{b}_n\}$ converges to some \mathbf{b} . For every n we have $t(\bar{\mathbf{a}}, \mathbf{b}) \leq t(\bar{\mathbf{a}}, \mathbf{c}_n) + \varepsilon$, whence $t(\bar{\mathbf{a}}, \mathbf{b}) \leq \inf_{\mathbf{y}} t(\bar{\mathbf{a}}, \mathbf{y}) + \varepsilon$, as desired.

We can now prove the first assertion. Indeed, it follows from the moreover part that the graph of $\inf_{\mathbf{y}} t$ is uniformly definable as:

$$X = \inf_{\mathbf{y}} t(\bar{\mathbf{a}}, \mathbf{y}) \iff \begin{cases} \sup_{\mathbf{z}} E(X \div t(\bar{\mathbf{a}}, \mathbf{z})) = 0, \\ \inf_{\mathbf{z}} E(t(\bar{\mathbf{a}}, \mathbf{z}) \div X) = 0. \end{cases}$$

Once we know that $\inf_{\mathbf{y}} f$ is definable, the sentence in the second assertion is expressible, and holds true by the moreover part. ■_{3.13}

We now proceed to define by induction, for every \mathcal{L} -formula $\varphi(\bar{x})$, a T_0^{Ra} -definable local function $\llbracket \varphi(\bar{x}) \rrbracket$ to the auxiliary sort, in the following natural manner:

- *Atomic formulae:* $\llbracket P(\bar{\tau}) \rrbracket = \llbracket P \rrbracket \circ (\bar{\tau})$ is a term, the composition of the function symbol $\llbracket P \rrbracket$ with the \mathcal{L} -terms $\bar{\tau}$, which are also \mathcal{L}^R -terms. These are local by Theorem 3.11.
- *Connectives:* $\llbracket \varphi \div \psi \rrbracket = \llbracket \varphi \rrbracket \div \llbracket \psi \rrbracket$, and so on. Locality is clear.
- *Quantifiers:* $\llbracket \inf_{\mathbf{y}} \varphi(\bar{x}, y) \rrbracket = \inf_{\mathbf{y}} \llbracket \varphi(\bar{x}, \mathbf{y}) \rrbracket$, $\llbracket \sup_{\mathbf{y}} \varphi(\bar{x}, y) \rrbracket = \sup_{\mathbf{y}} \llbracket \varphi(\bar{x}, \mathbf{y}) \rrbracket$. Locality follows from Lemma 3.13.

Our somewhat minimalist approach differs from that of Keisler, who introduces a function symbol $\llbracket \varphi(\bar{x}) \rrbracket$ for every \mathcal{L} -formula φ (see [Kei99, BK]). Keisler's *Boolean Axioms* and *Fullness Axiom* are valid in our setting by definition of $\llbracket \varphi \rrbracket$ (using Lemma 3.13 for fullness). Keisler's *Distance Axiom* for the main sort is our R2. While not entirely equivalent, Keisler's *Event Axiom* corresponds to our axiom R3. (More precisely, Keisler's Event Axiom is equivalent to R3 plus $\sup_{\mathbf{x}, \mathbf{y}} d(\mathbf{x}, \mathbf{y}) = 1$. We do not find it necessary or desirable to assume the latter.) Other axioms related to the auxiliary sort, with the exception of atomlessness, are coded in *RV* (with one big difference, namely that here the auxiliary sort is the full $L^1(\mathcal{F}, [0, 1])$ whereas in [Kei99, BK] it only consists of \mathcal{F}). We shall add atomlessness later on, when it is needed for Theorem 3.32. We are left with the *Validity Axioms* which we also claim follow from T_0^{Ra} .

Theorem 3.14. *Let $(\mathcal{M}, \mathcal{A})$ be a model of T_0^{Ra} which we identify as usual with its canonical representation, based on (Ω, μ) . Then for every formula $\varphi(\bar{x})$ and tuple $\bar{\mathbf{a}}$ of the appropriate length we have $\langle \varphi(\bar{\mathbf{a}}) \rangle = \llbracket \varphi(\bar{\mathbf{a}}) \rrbracket$ as functions on Ω (and not merely up to a null measure set).*

Proof. We prove by induction on φ . If φ is atomic this is known by construction and the induction step for connectives is immediate. We are left with the case of a formula $\inf_x \varphi(x, \bar{y})$. First of all, by construction, we have:

$$\begin{aligned} \langle \inf_x \varphi(x, \bar{\mathbf{a}}) \rangle &= \inf^s \{ \langle \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rangle : \mathbf{b} \in \mathbf{M} \}, \\ \llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket &= \inf_{\mathbf{x}} \llbracket \varphi(\mathbf{x}, \bar{\mathbf{a}}) \rrbracket = \inf^{L^1} \{ \llbracket \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rrbracket : \mathbf{b} \in \mathbf{M} \}. \end{aligned}$$

Here \inf^s means the simple, or point-wise, infimum of functions on Ω . By definition $\llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket \leq \llbracket \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rrbracket$ for all \mathbf{b} , and by the induction hypothesis for φ we have $\llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket \leq \langle \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rangle$. It follows that $\llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket \leq \langle \inf_x \varphi(x, \bar{\mathbf{a}}) \rangle$. Conversely, by Lemma 3.13, for every $\varepsilon > 0$ there exists \mathbf{b} such that $\llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket + \varepsilon \geq \llbracket \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rrbracket$. Using the induction hypothesis again we obtain:

$$\llbracket \inf_x \varphi(x, \bar{\mathbf{a}}) \rrbracket + \varepsilon \geq \langle \varphi(\mathbf{b}, \bar{\mathbf{a}}) \rangle \geq \langle \inf_x \varphi(x, \bar{\mathbf{a}}) \rangle.$$

Equality follows. ■_{3.14}

Corollary 3.15. *Let $\mathcal{M} \models T_0^{Ra}$ and assume its canonical representation is based on the family $\mathcal{M} = \{\mathcal{M}_\omega\}_{\omega \in \Omega}$. Then for every \mathcal{L} -sentence φ :*

$$\mathcal{M} \models \llbracket \varphi \rrbracket = 0 \iff \mathcal{M}_\omega \models \varphi \text{ for all } \omega \in \Omega.$$

Proof. Immediate from the fact that $\llbracket \varphi \rrbracket = \langle \varphi \rangle$ on Ω . ■_{3.15}

Definition 3.16. Let T be a set of \mathcal{L} -sentences. We define its randomisation T^{Ra} to be the \mathcal{L}^R -theory consisting of the base theory along with the translation of T (Keisler's *Transfer Axioms*):

$$T^{Ra} = T_0^{Ra} \cup \{\llbracket \varphi \rrbracket = 0\}_{\varphi \in T}.$$

Corollary 3.17. *Let T be arbitrary set of sentences, φ a sentence. Then $T \vdash \varphi \iff T^{Ra} \vdash \llbracket \varphi \rrbracket = 0$.*

Proof. Immediate. ■_{3.17}

Corollary 3.18 (Keisler's *Validity Axiom*). *Assume φ is a valid \mathcal{L} -sentence. Then $T_0^{Ra} \vdash \llbracket \varphi \rrbracket = 0$.*

3.4. A variant of Łoś's Theorem. Given a randomisation $(\mathbf{M}, \mathcal{A})$, let $[\cdot]: (\mathbf{M}, \mathcal{A}) \rightarrow (\widehat{\mathbf{M}}, \widehat{\mathcal{A}})$ denote the canonical mapping to the associated structure.

Theorem 3.19 (Łoś's Theorem for randomisation). *Let \mathcal{M}_Ω be a family of structures, $\mathbf{M} = \prod \mathcal{M}$, and let E an integration functional on $\mathcal{A} = [0, 1]^\Omega$. Let $(\widehat{\mathbf{M}}, \widehat{\mathcal{A}})$ denote the structure associated to the randomisation $(\mathbf{M}, \mathcal{A})$.*

Then $(\mathbf{M}, \mathcal{A})$ is full and for every formula $\varphi(\bar{x})$ and every $\bar{\mathbf{a}} \in \mathbf{M}^n$:

$$\llbracket \langle \varphi(\bar{\mathbf{a}}) \rangle \rrbracket = \llbracket \varphi([\bar{\mathbf{a}}]) \rrbracket.$$

Proof. Fullness is immediate. We claim that $\llbracket \langle \inf_y \varphi(\bar{\mathbf{a}}, y) \rangle \rrbracket = \inf_{\mathbf{b} \in \mathbf{M}} \llbracket \langle \varphi(\bar{\mathbf{a}}, \mathbf{b}) \rangle \rrbracket$ for every formula $\varphi(\bar{x}, y)$ and every $\bar{\mathbf{a}} \in \mathbf{M}^n$, where the infimum on the right hand side is in the sense of the lattice $\widehat{\mathcal{A}}$. Indeed, the inequality \leq is immediate. For \geq observe that using the Axiom of Choice, for every $\varepsilon > 0$ we can find $\mathbf{b} \in \mathbf{M}$ such that $\langle \inf_y \varphi(\bar{\mathbf{a}}, y) \rangle + \varepsilon \geq \langle \varphi(\bar{\mathbf{a}}, \mathbf{b}) \rangle$, whereby $\llbracket \langle \inf_y \varphi(\bar{\mathbf{a}}, y) \rangle \rrbracket + \varepsilon \geq \llbracket \langle \varphi(\bar{\mathbf{a}}, \mathbf{b}) \rangle \rrbracket$.

We now prove the main assertion. First of all, we may replace φ with an equivalent formula ψ . Indeed, on the left hand side we have immediately $\langle \varphi(\bar{\mathbf{a}}) \rangle = \langle \psi(\bar{\mathbf{a}}) \rangle$. For the right hand side, we have $|\llbracket \varphi \rrbracket - \llbracket \psi \rrbracket| = \llbracket |\varphi - \psi| \rrbracket$, whereby $T_0^{Ra} \vdash \llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$. We may therefore assume that φ is in prenex form. We now proceed by induction on the number of quantifiers. If φ is quantifier-free then $\llbracket \langle \varphi(\bar{\mathbf{a}}) \rangle \rrbracket = \llbracket \varphi([\bar{\mathbf{a}}]) \rrbracket$ by construction. For the induction step, recall that

$$\llbracket \langle \inf_y \varphi([\bar{\mathbf{a}}], y) \rangle \rrbracket = \inf_{\mathbf{y}} \llbracket \varphi([\bar{\mathbf{a}}], \mathbf{y}) \rrbracket = \inf_{\mathbf{b} \in \widehat{\mathbf{M}}} \llbracket \varphi([\bar{\mathbf{a}}], \mathbf{b}) \rrbracket = \inf_{\mathbf{b} \in \mathbf{M}} \llbracket \varphi([\bar{\mathbf{a}}], [\mathbf{b}]) \rrbracket.$$

We conclude using the claim and the induction hypothesis. ■_{3.19}

Let us go back to the ultra-product example (Example 3.5), where $\mathbf{M} = \prod \mathcal{M}$ and $\widehat{\mathcal{M}} = \prod_{\mathcal{U}} \mathcal{M}$. By construction $E[\inf_y \varphi([\bar{\mathbf{a}}], y)] = \inf_{\mathbf{b} \in \mathbf{M}} E[\varphi([\bar{\mathbf{a}}], [\mathbf{b}])]$. One also always has $E[\neg \varphi([\bar{\mathbf{a}}])] = \neg E[\varphi([\bar{\mathbf{a}}])]$, $E[\frac{1}{2} \varphi([\bar{\mathbf{a}}])] = \frac{1}{2} E[\varphi([\bar{\mathbf{a}}])]$. Since $E = E_{\mathcal{U}}$ is given by an ultra-filter, we have moreover $E[\varphi([\bar{\mathbf{a}}]) \div \psi([\bar{\mathbf{a}}])] = E[\varphi([\bar{\mathbf{a}}])] \div E[\psi([\bar{\mathbf{a}}])]$. Thus the truth value of $\varphi([\bar{\mathbf{a}}])$ in the ultra-product is precisely $E[\varphi([\bar{\mathbf{a}}])]$ in the sense of the randomised structure. Now the last item of Theorem 3.19 yields the classical version of Loś's Theorem:

$$\varphi([\bar{\mathbf{a}}]) = E_{\mathcal{U}}[\langle \varphi(\bar{\mathbf{a}}) \rangle] = \lim_{\mathcal{U}} \varphi(\mathbf{a}(\omega)).$$

Let us pursue a little further this analogy with classical ultra-products. Classical ultra-product constructions consist of fixing a family \mathcal{M}_{Ω} and a filter \mathcal{F} on Ω with certain desired properties, then extending this filter to an ultra-filter and taking the ultra-product. A filter on Ω can be viewed as a partial 0/1 measure: some sets have measure zero, some measure one, and for some the measure is not known. The $[0, 1]$ -valued analogue is a partial integration functional on $[0, 1]^{\Omega}$.

Definition 3.20. A *partial integration space* is a triplet $(\Omega, \mathcal{A}_0, E_0)$ where Ω is a set, $\mathcal{A}_0 \subseteq [0, 1]^{\Omega}$ is any subset, and $E_0: \mathcal{A}_0 \rightarrow [0, 1]$ is a functional satisfying that for every finite sequence $\{(X_i, m_i)\}_{i < \ell} \subseteq \mathcal{A}_0 \times \mathbb{Z}$ and $k \in \mathbb{Z}$:

$$\sum m_i X_i \geq k \implies \sum m_i E_0(X_i) \geq k.$$

In this case we say that E_0 is a *partial integration functional*.

Clearly every integration functional is a partial integration functional. Conversely,

Fact 3.21. Let $(\Omega, \mathcal{A}_0, E_0)$ be a partial integration space. Then E_0 can be extended to a total integration functional E on $\mathcal{A} = [0, 1]^{\Omega}$, rendering (Ω, \mathcal{A}, E) a (total) integration space.

Moreover, if $(\Omega, \mathcal{A}_0, E_0)$ is an integration space, and atomless as such, then (Ω, \mathcal{A}, E) is atomless as well.

Proof. See [BK, Section 5]. ■_{3.21}

Definition 3.22. A *partial randomisation* based on a family \mathcal{M}_{Ω} is a triplet $(\mathbf{M}, \mathcal{A}_0, E_0)$ satisfying all the properties of an ordinary (total) randomisation, with the exception that we do not require that $\langle P(\bar{\mathbf{a}}) \rangle \in \mathcal{A}_0$. We say that a partial randomisation is *atomless* if $(\mathcal{A}_0, E_0) \models ARV$.

By Fact 3.21 we may extend E_0 to an integration functional E on $\mathcal{A} = [0, 1]^{\Omega}$. We say that the (full) randomisation $(\mathbf{M}, \mathcal{A}) = (\mathbf{M}, \mathcal{A}, E)$ is a *totalisation* of $(\mathbf{M}, \mathcal{A}_0, E_0)$ and that the associated structure $(\widehat{\mathcal{M}}, \widehat{\mathcal{A}})$ is a structure associated to $(\mathbf{M}, \mathcal{A}_0, E_0)$. (It is *an* associated structure rather than *the* associated structure because of the arbitrary choices involved.)

Definition 3.23. We recall that a *random family of structures* $\mathcal{M}_{(\Omega, \mathcal{F}, \mu)}$ consists of a family of structures $\mathcal{M}_\Omega = \{\mathcal{M}_\omega\}_{\omega \in \Omega}$ indexed by a probability space $(\Omega, \mathcal{F}, \mu)$. To every such random family we associate a natural partial randomisation $(\mathbf{M}, \mathcal{A}_0, E_0)$ where $\mathbf{M} = \prod \mathcal{M}$ and $(\Omega, \mathcal{A}_0, E_0)$ is the integration space of \mathcal{F} -measurable functions on Ω . It is atomless if and only if $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space.

If $(\widehat{\mathbf{M}}, \widehat{\mathcal{A}})$ is a structure associated to $(\mathbf{M}, \mathcal{A}_0, E_0)$ then we also say that it is a structure associated to the random family $\mathcal{M}_{(\Omega, \mathcal{F}, \mu)}$.

Corollary 3.24. Let $\mathcal{M}_{(\Omega, \mathcal{F}, \mu)}$ be a random family of structures and let $(\mathcal{M}, \mathcal{A})$ be an associated structure. Then for every $\bar{\mathbf{a}}$ in $\prod \mathcal{M}$ and every formula $\varphi(\bar{x})$, if $\langle \varphi(\bar{\mathbf{a}}) \rangle \in [0, 1]^\Omega$ is \mathcal{F} -measurable then

$$E[\langle \varphi(\bar{\mathbf{a}}) \rangle]^{\mathbf{M}} = \int_{\Omega} \langle \varphi(\bar{\mathbf{a}}) \rangle d\mu$$

Proof. Immediate from Theorem 3.19 and the construction. ■_{3.24}

This can be improved to construct extensions containing elements with desired properties.

Definition 3.25. An embedding $\sigma: (\mathcal{M}, \mathcal{A}) \rightarrow (\mathcal{M}_1, \mathcal{A}_1)$ will be called a $[\cdot]$ -embedding if $\sigma[\langle \varphi(\bar{\mathbf{a}}) \rangle]^{\mathbf{M}} = [\langle \varphi(\sigma \bar{\mathbf{a}}) \rangle]^{\mathbf{M}_1}$ for every $\bar{\mathbf{a}} \in \mathbf{M}^n$ and formula $\varphi(\bar{x})$.

Definition 3.26. A *morphism* of integration spaces $\pi: (\Omega', \mathcal{A}', E') \rightarrow (\Omega, \mathcal{A}, E)$ is a projection $\pi: \Omega' \rightarrow \Omega$ such that $X \circ \pi \in \mathcal{A}'$ and $E'(X \circ \pi) = E(X)$ for all $X \in \mathcal{A}$.

Corollary 3.27. Let $(\mathcal{M}, \mathcal{A}) \models T_0^{Ra}$ with canonical representation $(\mathbf{M}, \mathcal{A})$ based on $\mathcal{M}_{(\Omega, \mathcal{F}, \mu)}$, so in particular $\mathcal{A} = C(\Omega, [0, 1])$.

Let $\pi: (\Omega', \mathcal{A}'_0, E'_0) \rightarrow (\Omega, \mathcal{A}, E)$ be a morphism of integration spaces and let $\mathcal{M}'_{\Omega'} = \{\mathcal{M}'_{\omega'}\}_{\omega' \in \Omega'}$ be a family of elementary extensions $\mathcal{M}_{\pi\omega'} \preceq \mathcal{M}'_{\omega'}$. Set $\mathbf{M}' = \prod \mathcal{M}'_{\Omega'}$, $\mathcal{A}' = [0, 1]^{\Omega'}$, and for $\mathbf{a} \in \mathbf{M}$ and $X \in \mathcal{A}$ define

$$\sigma \mathbf{a} = \mathbf{a} \circ \pi = (\omega' \mapsto \mathbf{a}(\pi\omega')) \in \mathbf{M}', \quad \sigma X = X \circ \pi \in \mathcal{A}'.$$

Let $(\widehat{\mathbf{M}}', \widehat{\mathcal{A}}')$ be an associated structure to the partial randomisation $(\mathbf{M}', \mathcal{A}'_0, E'_0)$, and let $[\sigma]: (\mathcal{M}, \mathcal{A}) \rightarrow (\widehat{\mathbf{M}}', \widehat{\mathcal{A}}')$ be the mapping $\mathbf{a} \mapsto [\sigma \mathbf{a}]$, $X \mapsto [\sigma X]$. Then

- (i) The mapping $\sigma: (\mathcal{M}, \mathcal{A}) \rightarrow (\widehat{\mathbf{M}}', \widehat{\mathcal{A}}') \models T_0^{Ra}$ is a $[\cdot]$ -embedding.
- (ii) For every $\bar{\mathbf{a}}$ in $\prod \mathcal{M}'$ and every formula $\varphi(\bar{x})$, if $\langle \varphi(\bar{\mathbf{a}}) \rangle \in \mathcal{A}'_0$ then

$$E[\langle \varphi(\bar{\mathbf{a}}) \rangle]^{\widehat{\mathbf{M}}'} = E'_0 \langle \varphi(\bar{\mathbf{a}}) \rangle.$$

Proof. For the first item it is easy to check that $[\sigma]$ is indeed an embedding. In order to see that $[\sigma]$ is a $[\cdot]$ -embedding let $\bar{\mathbf{a}} \in \mathbf{M}^n$ and let $\varphi(\bar{x})$ be a formula. Then $\langle \varphi(\bar{\mathbf{a}}) \rangle = [\langle \varphi(\bar{\mathbf{a}}) \rangle] \in \mathcal{A} \subseteq [0, 1]^\Omega$ by Theorem 3.14, so

$$[\sigma[\langle \varphi(\bar{\mathbf{a}}) \rangle]] = [\sigma \langle \varphi(\bar{\mathbf{a}}) \rangle] = [\langle \varphi(\sigma \bar{\mathbf{a}}) \rangle] = [\langle \varphi([\sigma \bar{\mathbf{a}}]) \rangle].$$

The second item is an immediate consequence of Theorem 3.19. ■_{3.27}

3.5. Quantifier elimination and types. Let T_0^R consist of T_0^{Ra} along with the atomlessness axiom ARV. In other words, T_0^R consists of the theory ARV for the auxiliary sort plus axioms R1-3. Similarly, we define $T^R = T^{Ra} + \text{ARV} = T_0^R \cup \{\llbracket \varphi \rrbracket = 0\}_{\varphi \in T}$.

Example 3.28. Let $\mathcal{M} \models T$ and let $(\Omega, \mathcal{F}, \mu)$ be any atomless probability space. Let $(\mathcal{M}, \mathcal{A})$ be an associated structure to the random family $\mathcal{M}^{(\Omega, \mathcal{F}, \mu)}$. Then $(\mathcal{M}, \mathcal{A}) \models T^{Ra}$ by Corollary 3.24 and \mathcal{A} is atomless, whereby $(\mathcal{M}, \mathcal{A}) \models T^R$.

Lemma 3.29. *The theories T^{Ra} and T^R are companions. Moreover, every model $(\mathcal{M}, \mathcal{A}) \models T^{Ra}$ admits a $\llbracket \cdot \rrbracket$ -embedding $\sigma: (\mathcal{M}, \mathcal{A}) \rightarrow (\mathcal{M}^1, \mathcal{A}^1) \models T^R$.*

Proof. It is enough to prove the moreover part. Let $([0, 1], \mathcal{B}, \lambda)$ denote the Lebesgue measure on $[0, 1]$. Apply Corollary 3.27 to $(\Omega', \mathcal{F}', \mu') = (\Omega, \mathcal{F}, \mu) \times ([0, 1], \mathcal{B}, \lambda)$ and $\mathcal{M}'_{\omega, r} = \mathcal{M}_\omega$. The resulting embedding $\sigma: (\mathcal{M}, \mathcal{A}) \rightarrow (\mathcal{M}^1, \mathcal{A}^1)$ is a $\llbracket \cdot \rrbracket$ -embedding and \mathcal{A}^1 is atomless. If $\varphi \in T$ is a sentence then $\llbracket \varphi \rrbracket^{\mathcal{M}^1} = \sigma \llbracket \varphi \rrbracket^{\mathcal{M}} = \sigma 0 = 0$. Thus $(\mathcal{M}^1, \mathcal{A}^1) \models T^R$, as desired. $\blacksquare_{3.29}$

Let us now fix an \mathcal{L} -theory T . As usual, $S_n(T)$ (or sometimes $S_{\bar{x}}(T)$) denotes the space of n -types of T . Similarly, $S_n(T^R)$ (or $S_{\bar{x}}(T^R)$) denotes the space of n -types of the \mathcal{L}^R -theory T^R .

Let us fix some additional notation. For a compact Hausdorff space X , let $\mathfrak{R}(X)$ denote the space of regular Borel probability measures on X . For $\varphi \in C(X, \mathbb{C})$ and $\mu \in \mathfrak{R}(X)$ let $\langle \varphi, \mu \rangle = \int \varphi d\mu$ and equip $\mathfrak{R}(X)$ with the weak topology, namely $\mu_s \rightarrow \mu$ if $\langle \varphi, \mu_s \rangle \rightarrow \langle \varphi, \mu \rangle$ for all φ . It is a classical (and easy) fact that this renders $\mathfrak{R}(X)$ a compact Hausdorff space as well.

Let $\mathbf{p}(\bar{x}) \in S_n(T^R)$. It is not difficult to verify (e.g., using the Riesz Representation Theorem) that there exists a unique regular Borel probability measure $\nu_{\mathbf{p}} \in \mathfrak{R}(S_n(T))$ characterised by the identity $E \llbracket \varphi(\bar{x}) \rrbracket^{\mathbf{p}} = \langle \varphi, \nu_{\mathbf{p}} \rangle$ for every \mathcal{L} -formula $\varphi(\bar{x})$. The mapping $\mathbf{p} \mapsto \nu_{\mathbf{p}}$ is continuous by definition of the topology on $\mathfrak{R}(S_n(T))$.

We next claim that $\mathbf{p} \mapsto \nu_{\mathbf{p}}$ is surjective. Indeed, let $\mu \in \mathfrak{R}(S_n(T))$. For each $p \in S_n(T)$ choose a model \mathcal{M}_p and a realisation $\bar{a}_p \in M_p^n$ of p (we do not assume that T is complete so \mathcal{M}_p may have to vary with p). Let $(\mathcal{M}, \mathcal{A})$ be a structure associated to the random family $\mathcal{M} = \mathcal{M}_{(S_n(T), \mu)} = \{\mathcal{M}_p\}_{p \in S_n(T)}$. Let $\bar{\mathbf{a}} \in \prod \mathcal{M}$ be given by $\bar{\mathbf{a}}(p) = \bar{a}_p$. By Corollary 3.24, for every formula $\varphi(\bar{x})$:

$$E \llbracket \varphi(\bar{\mathbf{a}}) \rrbracket = E \langle \varphi(\bar{\mathbf{a}}) \rangle = \langle \varphi, \mu \rangle.$$

In particular, if $\varphi \in T$ is a sentence then $E \llbracket \varphi \rrbracket = 0$, so $(\mathcal{M}, \mathcal{A}) \models T^{Ra}$. By Lemma 3.29 we can embed $(\mathcal{M}, \mathcal{A})$ in a model $(\mathcal{M}^1, \mathcal{A}^1) \models T^R$ without changing the values of $E \llbracket \varphi(\bar{\mathbf{a}}) \rrbracket$. Thus in \mathcal{M}^1 we have $\nu_{\mathbf{tp}(\bar{\mathbf{a}})} = \mu$. (Using the same trick as in Lemma 3.29 we could have arranged for $(\mathcal{M}, \mathcal{A})$ to be atomless directly, working with $\Omega = S_n(T) \times [0, 1]$.) We argued above for types in finitely many variables, but in exactly the same manner we associate to each $\mathbf{p} \in S_I(T^R)$ a regular Borel probability measure $\nu_{\mathbf{p}} \in \mathfrak{R}(S_I(T))$ and this mapping is surjective, for an arbitrary index set I .

We now wish to prove quantifier elimination. We shall require a modified version of the Hall Marriage Theorem:

Fact 3.30. *Let S be a finite set, $(\Omega, \mathcal{F}, \mu)$ an atomless probability space. For each $x \in S$ let us be given a weight $w_x \geq 0$ and an event $C_x \in \mathcal{F}$. For $T \subseteq S$ let $W_T = \sum_{x \in T} w_x$, $C_T = \bigcup_{x \in T} C_x$. Then the following are equivalent:*

- (i) *For all $T \subseteq S$: $\mu(C_T) \geq w_T$.*
- (ii) *There exists a family $\{D_x\}_{x \in S}$ such that $D_x \subseteq C_x$ and $\mu(D_x) = w_x$.*

Notice that the family $\{D_x\}_{x \in S}$ is necessarily disjoint. If $w_S = 1$ then $\{D_x\}_{x \in S}$ is a partition of Ω (up to null measure).

Proof. One direction is immediate, so let us prove the other by induction on $|S|$. There is no harm in assuming that $w_x > 0$ for every $x \in S$. Let $\Xi \subseteq \mathcal{F}^S$ consist of all families of events $\{D_x\}_{x \in S}$ which satisfy $\mu(D_T) \geq w_T$ for all $T \subseteq S$. We order Ξ by coordinate-wise inverse inclusion: $\{D_x\}_{x \in S} \leq \{D'_x\}_{x \in S}$ if $D_x \supseteq D'_x$ (up to null measure) for all $x \in S$. A strictly increasing sequence in Ξ is at most countable, and its coordinate-wise intersection is still in Ξ , so by Zorn's Lemma Ξ admits a maximal element $\{D_x\} \geq \{C_x\}$.

Let $r = \min\{\mu(D_T) - w_T : \emptyset \subsetneq T \subsetneq S\}$, and assume that $r > 0$. Choose $x_0 \in S$ arbitrarily. Then $\mu(D_{x_0}) \geq r$, so we may choose $B \subseteq D_{x_0}$, $\mu(B) = r$. Let $D'_{x_0} = D_{x_0}$, and for $x \neq x_0$ let $D'_x = D_x \setminus B$. Then $\{D'_x\} \in \Xi$, contradicting the maximality of $\{D_x\}$. Therefore $r = 0$, i.e., there is $\emptyset \subsetneq T_0 \subsetneq S$ such that $\mu(D_{T_0}) = w_{T_0}$. For $x \notin T_0$ let $D'_x = D_x \setminus D_{T_0}$. For every $T \subseteq S \setminus T_0$ we calculate:

$$\mu(D'_T) = \mu(D'_{T \setminus T_0}) = \mu(D_{T \cup T_0}) - \mu(D_{T_0}) \geq w_{T \cup T_0} - w_{T_0} = w_T.$$

May may now apply the induction hypothesis separately to $\{D_x\}_{x \in T_0}$ and to $\{D'_x\}_{x \in S \setminus T_0}$ and conclude. ■_{3.30}

Lemma 3.31. *Let $(\mathbf{M}, \mathcal{A}) \models T_0^R$ be \aleph_0 -saturated, $\bar{\mathbf{a}} \in \mathbf{M}^n$, and let $\nu_{\bar{\mathbf{a}}}$ be an abbreviation for $\nu_{\text{tp}(\bar{\mathbf{a}})}$. Let $\theta: S_{n+1}(\mathcal{L}) \rightarrow S_n(\mathcal{L})$ be the restriction to the first n variables. Then:*

- (i) *For every $\mathbf{b} \in \mathbf{M}$, $\nu_{\bar{\mathbf{a}}}$ is the image measure of $\nu_{\bar{\mathbf{a}}, \mathbf{b}}$ under θ .*
- (ii) *Conversely, let $\eta \in \mathfrak{R}(S_{n+1}(\mathcal{L}))$ by such that its image measure under θ is $\nu_{\bar{\mathbf{a}}}$. Then there is $\mathbf{b} \in \mathbf{M}$ such that $\eta = \nu_{\bar{\mathbf{a}}, \mathbf{b}}$.*

Proof. The first item is immediate. For the second, it is enough to show that for every finite family $\varphi_i(\bar{x}, y)$, $i < \ell$, and for every $\varepsilon > 0$, there is $\mathbf{b} \in \mathbf{M}$ such that $|\langle \varphi_i, \eta \rangle - E[\varphi_i(\bar{\mathbf{a}}, \mathbf{b})]| < \varepsilon$ for $i < \ell$.

Let $S = \{s_j\}_{j < k}$ be a partition of $[0, 1]^\ell$ into finitely many Borel subsets, $\text{diam}(s_j) < \varepsilon$. For $j < k$ let $w_j = \eta\{\bar{\varphi} \in s_j\}$. Choose also $\bar{t}_j \in s_j$ and let $\psi_j = \bigvee_{i < \ell} |\varphi_i - t_{j,i}|$. Notice that

$$\left| \langle \varphi_i, \eta \rangle - \sum_{j < k} w_j t_{j,i} \right| \leq \sum_{j < k} w_j \text{diam}(s_j) < \varepsilon.$$

Let $C_j \in \mathcal{F}$ be the event $\{\llbracket \inf_y \psi_j(\bar{\mathbf{a}}, y) \rrbracket < \varepsilon\}$. Following the notations of Fact 3.30, we claim that $\mu(C_T) \geq w_T$ for all $T \subseteq k$. Indeed, notice that $\{\bar{\varphi} \in s_j\} \subseteq \theta^{-1}\{\psi_j < \varepsilon\}$, whereby:

$$\begin{aligned} w_T &= \sum_{j \in T} \eta\{\bar{\varphi} \in s_j\} = \eta\left(\bigcup_{j \in T} \{\bar{\varphi} \in s_j\}\right) \leq \eta\left(\bigcup_{j \in T} \theta^{-1}\{\psi_j < \varepsilon\}\right) \\ &= \nu_{\bar{\mathbf{a}}}\left(\bigcup_{j \in T} \{\psi_j < \varepsilon\}\right) = \mu(C_T). \end{aligned}$$

By Fact 3.30 there are events $D_j \subseteq C_j$ such that $w_T = \mu(D_T)$ for all $T \subseteq k$. Since the total weight is one, $\{D_j\}_{j < k}$ is a partition. By Lemma 3.13 and saturation of \mathcal{M} there are $\mathbf{b}_j \in \mathbf{M}$ such that $\llbracket \inf_y \psi_j(\bar{\mathbf{a}}, y) \rrbracket = \llbracket \psi_j(\bar{\mathbf{a}}, \mathbf{b}_j) \rrbracket$. Notice that:

$$\|\varphi_i(\bar{\mathbf{a}}, \mathbf{b}_j) - t_{j,i}\| \mathbb{1}_{D_j} \leq \llbracket \inf_y \psi_j(\bar{\mathbf{a}}, y) \rrbracket \mathbb{1}_{C_j} < \varepsilon.$$

Let $\mathbf{b} = \langle D_0, \mathbf{b}_0, \langle D_1, \mathbf{b}_1, \dots \rangle \rangle$, i.e., $\mathbf{b}(\omega) = \mathbf{b}_j(\omega)$ when $\omega \in D_j$. Now:

$$\begin{aligned} \left| \sum_{j < k} w_j t_{j,i} - E\llbracket \varphi_i(\bar{\mathbf{a}}, \mathbf{b}) \rrbracket \right| &\leq \sum_{j < k} |w_j t_{j,i} - E(\llbracket \varphi_i(\bar{\mathbf{a}}, \mathbf{b}) \rrbracket \mathbb{1}_{D_j})| \\ &= \sum_{j < k} \left| E\left[(t_{j,i} - \llbracket \varphi_i(\bar{\mathbf{a}}, \mathbf{b}_j) \rrbracket) \mathbb{1}_{D_j}\right] \right| \\ &\leq E\left[\sum_{j < k} |t_{j,i} - \llbracket \varphi_i(\bar{\mathbf{a}}, \mathbf{b}_j) \rrbracket| \mathbb{1}_{D_j} \right] < \varepsilon. \end{aligned}$$

Thus $|\langle \varphi_i, \eta \rangle - E\llbracket \varphi_i(\bar{\mathbf{a}}, \mathbf{b}) \rrbracket| < 2\varepsilon$, which is good enough. ■_{3.31}

Theorem 3.32. (i) *The theories of the form T^R (and in particular T_0^R) eliminate quantifiers in the main sort down to formulae of the form $E\llbracket \varphi(\bar{\mathbf{x}}) \rrbracket$.*

(ii) *The mapping $\mathbf{p} \mapsto \nu_{\mathbf{p}}$ defined by $\langle \varphi, \nu_{\mathbf{p}} \rangle = E\llbracket \varphi \rrbracket^{\mathbf{p}}$ induces a homeomorphism $S_{\bar{\mathbf{x}}}(T^R) \simeq \mathfrak{R}(S_{\bar{\mathbf{x}}}(T))$.*

(iii) *Let $f: n \rightarrow m$ be any mapping. Let $f^*: S_m(T) \rightarrow S_n(T)$ be the mapping $\text{tp}(a_0, \dots, a_{m-1}) \mapsto \text{tp}(a_{f(0)}, \dots, a_{f(n-1)})$ and similarly $f^{*,R}: S_m(T^R) \rightarrow S_n(T^R)$. Let $\tilde{f}^*: \mathfrak{R}(S_m(T)) \rightarrow \mathfrak{R}(S_n(T))$ be the image measure mapping corresponding to \tilde{f}^* . Then the following diagram commutes:*

$$\begin{array}{ccc} S_m(T^R) & \xrightarrow{\cong} & \mathfrak{R}(S_m(T)) \\ \downarrow f^{*,R} & & \downarrow \tilde{f}^* \\ S_n(T^R) & \xrightarrow{\cong} & \mathfrak{R}(S_n(T)) \end{array}$$

- (iv) *The completions of T^R are in bijection with regular Borel probability measures on the space of completions of T . In particular, if T is complete then so is T^R .*

Proof. The first item follows from Lemma 3.31 via a standard back-and-forth argument. For the second item, we have already seen that the mapping $\mathbf{p} \mapsto \nu_{\mathbf{p}}$ is continuous and surjective. From the first item it follows that it is injective. Since both spaces are compact and Hausdorff, it is a homeomorphism. The third item is easily verified. The last item is a special case of the second item for 0-types. $\blacksquare_{3.32}$

Corollary 3.33. *Assume that T eliminates quantifiers. Then so does T^R , and it is the model completion of T^{Ra} . If T is merely model complete then T^R is model complete as well, and is the model companion of T^{Ra} .*

Proof. The case where T eliminates quantifiers is easy. The case where T is model complete requires a bit more attention to details which we leave to the reader. $\blacksquare_{3.33}$

We have described formulae and types on the main sort. In order to handle the auxiliary sort, add to \mathcal{L} a sort $S_{[0,1]}$ for the set $[0, 1]$, equipped with the tautological predicate $\text{id}: S_{[0,1]} \rightarrow [0, 1]$ and with the usual distance $d(r, s) = |\text{id}(r) - \text{id}(s)|$. This is a compact structure and therefore the unique model of its theory, and adding it to models of T as a new sort does not add any structure on the original sorts. Call the resulting signature \mathcal{L}_+ and the corresponding theory T_+ . It is easy to check that T_+ eliminates quantifiers if T does. Passing to T_+^R , $\llbracket \text{id} \rrbracket$ is an isometric bijection between the main sort $S_{[0,1]}^R$ of T_+^R and the auxiliary sort, so questions about types and definability in the auxiliary sort can be settled by applying Theorem 3.32 to the sort $S_{[0,1]}^R$.

Corollary 3.34. *Every \mathcal{L}^R -definable predicate on the auxiliary sort of T^R , possibly with parameters $\bar{\mathbf{a}}$ from the main sorts, is equivalent to one in the pure language of the auxiliary sort and with parameters in*

$$\sigma(\bar{\mathbf{a}}) = \sigma(\llbracket \varphi(\bar{\mathbf{a}}) \rrbracket)_{\varphi(\bar{x}) \in \mathcal{L}_{\omega, \omega}} \subseteq \mathcal{F}.$$

Consequently, the auxiliary sort is stable and stably embedded in models of T^R , and if \bar{X} is a tuple in the auxiliary sort then

$$\text{tp}(\bar{X}/\bar{\mathbf{a}}) \equiv \text{tp}(\bar{X}/\sigma(\bar{\mathbf{a}})).$$

Proof. We may assume that T eliminates quantifiers, in which case so does T_+ and therefore T_+^R . It is therefore enough to show that for a tuple of variables $\bar{\mathbf{r}}$ in the sort $S_{[0,1]}^R$ and for any possible additional parameters $\bar{\mathbf{a}}$, any atomic formula in $\bar{\mathbf{r}}$ and $\bar{\mathbf{a}}$ is equivalent to a formula entirely in $S_{[0,1]}^R$, possibly using parameters in $\llbracket \text{id} \rrbracket^{-1}(\sigma(\bar{\mathbf{a}}))$. Given the minimalistic structure we put on $S_{[0,1]}$, such an atomic formula can either involve precisely one free variable \mathbf{r}_i or some of the parameters but no free variable. In the first case we have $\llbracket \text{id}(\mathbf{r}_i) \rrbracket$ which is as desired. In the second we have $\llbracket \varphi(\bar{\mathbf{a}}) \rrbracket$ where $\varphi(\bar{x})$ is an atomic \mathcal{L} -formula. In this case let $X = \llbracket \text{id} \rrbracket^{-1} \circ \llbracket \varphi(\bar{\mathbf{a}}) \rrbracket \in S_{[0,1]}^R$, so $\llbracket \varphi(\bar{\mathbf{a}}) \rrbracket = \llbracket \text{id}(X) \rrbracket$, and the latter is again as desired. $\blacksquare_{3.34}$

3.6. Types in T^R when T is incomplete. Theorem 3.32 provides us with a complete description of types in T^R , whether T is complete or not. In various situations we shall encounter later on, this description turns out to be much more useful when T is complete. What follows here is a brief discussion of the general case and a reduction of sorts to the special case of a complete theory.

Let T be an incomplete theory and let $\mathbf{p} \in S_n(T^R)$. By Theorem 3.32 we may identify \mathbf{p} with a regular Borel probability measure $\nu_{\mathbf{p}} \in \mathfrak{R}(S_n(T))$. Let $\mathcal{A}_{\mathbf{p}} = L^1((S_n(T), \nu_{\mathbf{p}}), [0, 1])$ and let $(\Omega_{\mathbf{p}}, \mu_{\mathbf{p}})$ be its Stone space. We have a natural identification of $C(\Omega_{\mathbf{p}}, [0, 1])$ with $\mathcal{A}_{\mathbf{p}}$, so in particular every n -ary \mathcal{L} -formula $\varphi(\bar{x})$ gives rise to a continuous function $\varphi: S_n(T) \rightarrow [0, 1]$ with image $\bar{\varphi}_{\mathbf{p}} \in \mathcal{A}_{\mathbf{p}} = C(\Omega_{\mathbf{p}}, [0, 1])$. Thus for every $\omega \in \Omega_{\mathbf{p}}$ we may define a complete type $\pi_{\mathbf{p}}\omega \in S_n(T)$ by $\varphi^{\pi_{\mathbf{p}}\omega} = \bar{\varphi}_{\mathbf{p}}(\omega)$. We obtain a mapping $\pi_{\mathbf{p}}: (\Omega_{\mathbf{p}}, \mu_{\mathbf{p}}) \rightarrow (S_n(T), \nu_{\mathbf{p}})$ which is continuous and sends $\mu_{\mathbf{p}}$ to $\nu_{\mathbf{p}}$ (as an image measure). It follows that the image of $\pi_{\mathbf{p}}$ is precisely the support of $\nu_{\mathbf{p}}$ there, i.e., the smallest closed measure one set. This discussion holds in particular when $n = 0$, i.e., when $\mathbf{T} = \mathbf{p} \in \mathcal{T}_0(T^R)$ is a completion of T^R .

Let now \mathbf{T} be a completion of T^R and $\mathbf{p}(\bar{x}) \in S_n(\mathbf{T})$. There is a natural \mathcal{L}_{RV} -inclusion $\mathcal{A}_{\mathbf{T}} \subseteq \mathcal{A}_{\mathbf{p}}$ giving rise to a projection $(\Omega_{\mathbf{p}}, \mu_{\mathbf{p}}) \rightarrow (\Omega_{\mathbf{T}}, \mu_{\mathbf{T}})$ where $\mu_{\mathbf{T}}$ is the image of $\mu_{\mathbf{p}}$. As in [BK, Section 5] this projection gives rise to a conditional expectation mapping $\mathbb{E}[\cdot|\mathbf{T}]: \mathcal{A}_{\mathbf{p}} \rightarrow \mathcal{A}_{\mathbf{T}}$. In particular, to every formula $\varphi(\bar{x})$ we associated $\bar{\varphi}_{\mathbf{p}}$ which in turn gets sent to $\mathbb{E}[\bar{\varphi}_{\mathbf{p}}|\mathbf{T}] \in \mathcal{A}_{\mathbf{T}} = C(\Omega_{\mathbf{T}}, [0, 1])$. Let us fix $\omega \in \Omega_{\mathbf{T}}$. It is not difficult to check that $\varphi \mapsto \mathbb{E}[\bar{\varphi}_{\mathbf{p}}|\mathbf{T}](\omega)$ is an integration functional on $S_n(T)$. Therefore there exists a unique type $\mathbf{p}_{\omega} \in S_n(T^R)$ verifying for all $\varphi(\bar{x})$:

$$E[\varphi]^{\mathbf{p}_{\omega}} = \langle \varphi, \nu_{\mathbf{p}_{\omega}} \rangle = \mathbb{E}[\bar{\varphi}_{\mathbf{p}}|\mathbf{T}](\omega).$$

The mapping $\omega \mapsto \mathbf{p}_{\omega}$ has the following properties:

- (i) It is determined by \mathbf{p} (in particular, the completion \mathbf{T} is determined by \mathbf{p}).
- (ii) Conversely, it determines \mathbf{p} as follows:

$$E[\varphi]^{\mathbf{p}} = \int_{\Omega_{\mathbf{T}}} E[\varphi]^{\mathbf{p}_{\omega}} d\mu_{\mathbf{T}}(\omega).$$

- (iii) Let $\pi_{\mathbf{T}}: \Omega_{\mathbf{T}} \rightarrow S_0(T)$ be as in the previous paragraph, associating to each $\omega \in \Omega_{\mathbf{T}}$ a completion $\pi_{\mathbf{T}}\omega$ of T . Then $\mathbf{p}_{\omega} \in S_n((\pi_{\mathbf{T}}\omega)^R)$.
- (iv) The mapping $\omega \mapsto \mathbf{p}_{\omega}$ is continuous in the appropriate weak topology. Specifically, for every formula $\varphi(\bar{x})$ the mapping $\omega \mapsto \langle \varphi, \nu_{\mathbf{p}_{\omega}} \rangle$ is continuous on $\Omega_{\mathbf{T}}$.

We therefore write

$$\mathbf{p} = \int_{\Omega_{\mathbf{T}}} \mathbf{p}_{\omega} d\mu_{\mathbf{T}}(\omega),$$

saying this is an integral of a continuous family. (Conversely, every integral of a continuous or even measurable family of T^R -types gives rise to a T^R -type.)

A special case of this situation is a type over parameters. Let $(\mathcal{M}, \mathcal{A}) \models T^R$, $\mathbf{A} \subseteq \mathbf{M}$ and $\mathbf{p} \in S_n(\mathbf{A})$. Let us enumerate $\mathbf{A} = \{\mathbf{a}_{\alpha}\}_{\alpha \in I}$. Let $A = \{a_{\alpha}\}_{\alpha \in I}$ be a set of new

constant symbols and let $\mathcal{L}_A = \mathcal{L} \cup A$. Even if T is a complete \mathcal{L} -theory it is incomplete as an \mathcal{L}_A -theory. We view $(\mathcal{M}, \mathcal{A})$ as an \mathcal{L}_A^R -structure naming \mathbf{A} by A . It is then the model of a complete \mathcal{L}_A^R -theory \mathbf{T} and $\mathbf{p} \in S_n(\mathbf{T})$. Each $\omega \in \Omega_{\mathbf{T}}$ gives rise to an \mathcal{L}_A -completion $\pi_{\mathbf{T}\omega}$ of T . In other words, each ω determines, so to speak, the \mathcal{L} -type of the constants A . Let A_ω be an actual set in a model of T realising this type. Then $\mathbf{p}_\omega \in \mathfrak{R}(S_n(\pi_{\mathbf{T}\omega})) = \mathfrak{R}(S_n(A_\omega))$.

4. PRESERVATION AND NON-PRESERVATION RESULTS

4.1. Categoricity. The theory ARV is \aleph_0 -categorical but not uncountably categorical, so this is the most we can hope for from T^R . We shall use the following criterion for \aleph_0 -categoricity.

Fact 4.1 (Ryll-Nardzewski Theorem for metric structures). *A complete countable theory T is \aleph_0 -categorical if and only if $S_n(T)$ is metrically compact for all n , if and only if the logic topology on $S_n(T)$ coincides with the metric for all n .*

Proof. The was originally stated and proved by C. Ward Henson for Banach space structures. For the proof in continuous logic see [BU07, Fact 1.14]. Notice that it is customary to exclude the case of a complete theory with a compact model (which is its unique model) from the definition of \aleph_0 -categoricity, as well as from the statement of this theorem, but the theorem holds as stated if we do not. ■_{4.1}

Theorem 4.2. *Assume T is a complete \aleph_0 -categorical theory in a countable language. Then so it T^R .*

Proof. It is enough to show that $S_n(T^R)$ is totally bounded, i.e., that it can be covered by finitely many ε -balls for every $\varepsilon > 0$. Let us therefore fix $\varepsilon > 0$. By assumption we can cover $S_n(T)$ with finitely many ε -balls, say $S_n(T) = \bigcup_{i < k} B(p_i, \varepsilon)$. Fix $N > \frac{k}{\varepsilon}$, and let $R = \{\bar{m} \in \mathbb{N}^k : \sum m_i = N\}$. Then R is finite, and for every $\bar{m} \in R$ we may define $\mathbf{p}_{\bar{m}} = \sum \frac{m_i}{N} p_i \in S_n(T^R)$. Let also $C_i = B(p_i, \varepsilon) \setminus \bigcup_{j < i} B(p_j, \varepsilon)$, so $S_n(T) = \bigcup_i C_i$ is a partition of $S_n(T)$ into a finite disjoint union of Borel sets of diameter $\leq \varepsilon$.

Now let $\mathbf{q} \in S_n(T^R)$ be any type. Find a tuple $\bar{m} \in R$ such that $E = \|\bar{m}/N - (\nu_{\mathbf{q}}(C_i))_{i < k}\|_1$ is minimal. We can do this so that at each co-ordinate the difference is at most $\frac{1}{N}$, so $E < \frac{k}{N} < \varepsilon$. We claim that $d(\mathbf{q}, \mathbf{p}_{\bar{m}}) < 2\varepsilon$, which will conclude the proof.

Let $\mathbf{a} \in \mathcal{M}$ realise \mathbf{q} , and as usual let us identify \mathcal{M} with its canonical representation, based on $\mathcal{M}_{(\Omega, \mu)}$. Let $D_i = \{\omega \in \Omega : \text{tp}(\mathbf{a}(\omega)) \in C_i\}$, so $\mu(D_i) = \nu_{\mathbf{q}}(C_i)$, and $\Omega = \bigcup D_i$ is a partition of Ω into disjoint Borel sets. We can now choose another such partition $\Omega = \bigcup D'_i$ such that each D'_i is comparable with D_i (i.e., either $D_i \subseteq D'_i$ or $D'_i \subseteq D_i$) and $\mu(D'_i) = \frac{m_i}{N}$, so $\mu(D_i \Delta D'_i) = |\frac{m_i}{N} - \nu_{\mathbf{q}}(C_i)|$. For each $\omega \in D'_i$ choose $\mathcal{M}'_\omega \succeq \mathcal{M}_\omega$ and $\mathbf{b}(\omega) \in \mathcal{M}'_\omega$ realising p_i . If $\omega \in D_i \cap D'_i$ then we arrange that in addition $d(\mathbf{b}(\omega), \mathbf{a}(\omega)) < \varepsilon$. Apply Corollary 3.27 to obtain an elementary extension $\mathcal{M}' \succeq \mathcal{M}$ and $\mathbf{b} \in \mathcal{M}'$ such that $\text{tp}(\mathbf{b}) = \mathbf{p}_{\bar{m}}$ and $d(\mathbf{b}, \mathbf{a}) \leq \varepsilon(1 - E) + E < 2\varepsilon$, as desired. ■_{4.2}

Corollary 4.3. *Assume T is a countable theory, possibly incomplete, with countably many completions, all of which are \aleph_0 -categorical. Then every completion of T^R is \aleph_0 -categorical.*

Proof. Let $\{T_n\}_{n \in \alpha}$, denote the set of completions of T , where $\alpha \in \mathbb{N} \cup \{\mathbb{N}\}$. The completions of T are in bijection with measures on α , namely with sequences $\bar{\lambda} \in [0, 1]^\alpha$ such that $\sum \lambda_n = 1$. Every model \mathcal{M} of such a completion can be identified with a combination $\sum \lambda_n \mathcal{M}_n$ where $\mathcal{M}_n \models (T_n)^R$, and is uniquely determined by \mathcal{M} except where $\lambda_n = 0$. If \mathcal{M} is separable then so is \mathcal{M}_n (when $\lambda_n > 0$), whence the uniqueness of \mathcal{M} . ■_{4.3}

Of course, in this case T^R may admit continuum many completions, and yet it is not too difficult to see that every completion of $(T^R)^R$ is still \aleph_0 -categorical. On the other hand, there are theories T with uncountably many completions, all of which are \aleph_0 -categorical, such that T^R admits a non \aleph_0 -categorical completion. Indeed, let T be the classical theory saying that there exist precisely 2 elements, in a language with constants a and b_n for $n \in \mathbb{N}$. Using a as reference, a completion of T is determined by whether $b_n = a$ or not for each n , so the space of completions of T is homeomorphic to $2^\mathbb{N}$. Let \mathbf{T} be the completion of T^R saying that $\llbracket b_n = a \rrbracket$ are independent events all of measure $\frac{1}{2}$. Let $\mathbf{p}_n \in S_1(\mathbf{T})$ be the type $\mathbf{x} = b_n$. Then $d(\mathbf{p}_n, \mathbf{p}_m) = \frac{1}{2}$ for all $n \neq m$ and $S_1(\mathbf{T})$ is not metrically compact.

4.2. Stability. For all facts regarding stability in continuous logic, and in particular local stability, we refer the reader to Ben Yaacov & Usvyatsov [BU]. For topometric Cantor-Bendixson ranks see [Ben08].

When proving the preservation of stability in Ben Yaacov & Keisler [BK] we considered φ -types over arbitrary sets in models of T and of T^R , calculating averages over the finite set of non forking extensions of such types. In doing so we proved not only that the randomisation of a stable theory is stable, but also that in such a randomised theory all types over sets (in sorts of the original theory) were stationary.

In continuous logic the situation is, at least on the surface, much more complicated. Assume $A \subseteq M$, $p \in S_\varphi(A)$, and let $P \subseteq S_\varphi(M)$ be the set of non forking extensions of p . Rather than being a finite set, as in classical logic, P is merely known to be a transitive compact metric space (in the standard metric on $S_\varphi(M)$, namely $d(q, q') = \sup\{|\varphi(x, b)^q - \varphi(x, b)^{q'}| : b \in M\}$). By *transitive* we mean that the action of the isometry group of P is transitive, which leads to the existence of a canonical probability measure on P and thus to a canonical notion of an average value of a function on P . With this notion of average we could, in principle, translate the entire argument of [BK] to the case where T is continuous. However, calculating averages over a transitive compact metric space is significantly more involved than merely averaging over a finite set, rendering the translated argument quite difficult to follow.

We therefore choose to split the argument in two, and at a first time restrict our attention to types over models, in which case the non forking extension is unique and no

averaging is required. In Section 5 we prove quite independently that for any theory T (stable or not), types in T^R coincide with Lascar types. It follows that if T is stable then all types in T^R are stationary.

As in [BK] we shall use Shelah ranks, this time adapted to continuous logic. Let us fix for the time being a monster model \mathfrak{M} containing all the parameters under consideration. We define the (k, φ) -rank of a partial type $\pi(x)$, denoted $R_k(\pi, \varphi)$, and its *multiplicity at rank s* , denoted $M_k(\pi, \varphi, s)$:

- (i) If π is consistent then $R_k(\pi, \varphi) \geq 0$.
- (ii) Having defined when $R_k(\pi, \varphi) \geq s$ we define $M_k(\pi, \varphi, s)$. We say that $M_k(\pi, \varphi, s) \geq M$ if there are types $\pi(x) \subseteq \pi_n(x)$ for $n < M$ such that for every $n < m < M$ there exists b_{nm} for which

$$\pi_n(x) \cup \pi_m(x') \vdash |\varphi(x, b_{nm}) - \varphi(x', b_{nm})| \geq 2^{-k},$$

and in addition $R_k(\pi_n, \varphi) \geq s$ for all $i < M$.

- (iii) If $M_k(\pi, \varphi, s) = \infty$ then $R_k(\pi, \varphi) \geq s + 1$.

It is not difficult to see that if $[\pi]_\varphi$ denotes the closed set π defines in $S_\varphi(\mathfrak{M})$, then:

$$R_k(\pi, \varphi) = \text{CB}_{f, 2^{-k}}([\pi]_\varphi) = \text{CB}_{b, 2^{-k}}([\pi]_\varphi),$$

where $\text{CB}_{f, \varepsilon}$ and $\text{CB}_{b, \varepsilon}$ are the topometric Cantor-Bendixson ranks defined in [Ben08, Section 3]. (Or almost: these are the ranks we would obtain if we replaced there “ $\leq \varepsilon$ ” with “ $< \varepsilon$ ” and “ $> \varepsilon$ ” with “ $\geq \varepsilon$ ”. Since we consider ranks for all $\varepsilon > 0$ this makes no difference.)

Let W denote a possibly infinite tuple of variables $\pi(x, W)$ a partial type and $k, s \in \mathbb{N}$. Then $R_k(\pi(x, W), \varphi) \geq s$ is a property of W , holding for A (of the appropriate size) if $R_k(\pi(x, A), \varphi) \geq s$. We may think of $R_k(\cdot, \varphi(x, y))$ as a quantifier binding the variable x . Let also $R_k(x/W, \varphi) \geq s$ be the property of xW which holds for aA if $R_k(\text{tp}(a/A), \varphi) \geq s$.

Fact 4.4. *The properties $R_k(\pi(x, W), \varphi) \geq s$ and $R_k(x/W, \varphi) \geq s$ are type-definable (in W and in xW , respectively).*

Proof. Both are shown using a standard “there exists a tree such that . . .” argument. The second can be deduced from the first since it may be re-written as $R_k(x' \equiv_W x, \varphi(x', y)) \geq s$ where x' is the bound variable and xW the parameter variables. ■_{4.4}

For $\alpha \leq \omega$ let $R_{<\alpha}(\pi, \alpha)$ denote the (finite or infinite) sequence $(R_k(\pi, \varphi))_{k < \alpha}$. Given a sequence $\sigma \in \mathbb{N}^\alpha$ and a partial type $\pi(x)$ let

$$\begin{aligned} S_\varphi(\mathfrak{M})^{(\sigma)} &= \{q \in S_\varphi(\mathfrak{M}) : R_{<\alpha}(q, \varphi) \geq \sigma\} \\ &= \{q \in S_\varphi(\mathfrak{M}) : R_k(q, \varphi) \geq \sigma(k) \text{ for all } k < \alpha\}, \\ [\pi]_\varphi^{(\sigma)} &= [\pi]_\varphi \cap S_\varphi(\mathfrak{M})^{(\sigma)} = \{q \in S_\varphi(\mathfrak{M}) : R_{<\alpha}(q \cup \pi, \varphi) \geq \sigma\}. \end{aligned}$$

We observe that $S_\varphi(\mathfrak{M})^{(\sigma)}$ and therefore $[\pi]_\varphi^{(\sigma)}$ are closed sets (either directly or using properties of the topometric Cantor-Bendixson ranks).

Before going further let us recall that for a topological space X , a mapping $f: X \rightarrow \mathbb{R}$ is *upper (respectively, lower) semi-continuous* if $\inf f(A) = \inf f(\overline{A})$ (respectively, $\sup f(A) = \sup f(\overline{A})$) for every non empty $A \subseteq X$. The following topological fact probably appears in general topology textbooks.

Fact 4.5. *Let X be a compact Hausdorff space.*

- (i) *A function $f: X \rightarrow \mathbb{R}$ is lower semi-continuous if and only if it can be written as the point-wise supremum of a family of continuous functions on X .*
- (ii) *Assume that $f: X \rightarrow \mathbb{R}$ is lower semi-continuous, $g: X \rightarrow \mathbb{R}$ is upper semi-continuous, and $g < f$. Then there is a continuous function $h: X \rightarrow \mathbb{R}$ such that $g < h < f$.*

Proof. For the first item let $\mathcal{F} \subseteq C(X, \mathbb{R})$ consist of all continuous functions smaller than f . If $f = \sup \mathcal{F}$ point-wise then for all $\emptyset \neq A \subseteq X$

$$\sup f(A) = \sup_{g \in \mathcal{F}} \sup g(A) = \sup_{g \in \mathcal{F}} \sup g(\overline{A}) = \sup f(\overline{A}).$$

Conversely, assume f is lower semi-continuous, and it will be enough to show that $f \leq \sup \mathcal{F}$ point-wise. Let $x \in X$, $r < f(x)$, and let $K = \{x: f(x) \leq r\}$. It follows from the assumptions that $K = \overline{K}$ is compact. By Urysohn's Lemma there exists a continuous function $g: X \rightarrow [0, r]$ verifying $g \upharpoonright_K = 0$ and $g(x) = r$, so in particular $g \in \mathcal{F}$. Thus $\sup \mathcal{F}(x) \geq r$, which is enough.

For the second item let us first fix $x \in X$. Since $g(x) < f(x)$ there is a continuous function $f_x < f$ such that $g(x) < f_x(x)$. Let $U_x = \{y: g(y) < f_x(x)\}$. Notice that $f_x - g$ is lower semi-continuous so U_x is open. By compactness there is a finite family $\{x_i\}_{i < n} \subseteq X$ such that $X = \bigcup_{i < n} U_{x_i}$. Then $h = \bigvee_{i < n} f_{x_i}$ is as desired. ■_{4.5}

At this point let us fix $\varepsilon > 0$. For a finite sequence $\sigma \in \mathbb{N}^{<\omega}$ we define Ξ_σ to be the set of all formulae $\xi(x, \bar{w})$ such that for any $\bar{a} \in M$ the diameter of $[\xi(x, \bar{a}) < 1]^{(\sigma)} \subseteq S_\varphi(M)$ is smaller than ε . (This is analogous to $\Xi_{s,2}$ as defined in [BK].)

Lemma 4.6. *Let $\sigma \in \mathbb{N}^k$, $\xi(x, \bar{w}) \in \Xi_\sigma$. Then there exists a formula $\hat{\xi}_\sigma(y, \bar{w})$ such that:*

$$\{\xi(x, \bar{w}) \leq \tfrac{1}{2}\} \cup \{R_{<k}(x/\bar{w}y, \varphi) \geq \sigma\} \vdash |\hat{\xi}_\sigma(y, \bar{w}) - \varphi(x, y)| < \varepsilon.$$

Proof. Let $Y \subseteq S_{x,y,\bar{w}}(T)$ consist of all types $q(x, y, \bar{w})$ for which the left hand side holds. Let $X \subseteq S_{x,\bar{w}}(T)$ consist of all types verifying $\xi(x, \bar{w}) \leq \frac{1}{2}$ and $R_{<k}(x/\bar{w}, \varphi) \geq \sigma$. The restriction mapping $\pi: Y \rightarrow X$ is surjective.

For $p(x, \bar{z}) \in X$ let $a, \bar{c} \models p$ and define

$$\begin{aligned} \bar{f}(p) &= \max\{\varphi(x, y)^q: q \in \pi^{-1}(p)\}, \\ \underline{f}(p) &= \min\{\varphi(x, y)^q: q \in \pi^{-1}(p)\}. \end{aligned}$$

Let us make a few remarks regarding this definition. Since π is surjective the set $\pi^{-1}(p)$ is non empty and compact. The maximum and minimum are therefore attained and

$\underline{f}(p) \leq \overline{f}(q)$. Moreover, there are types (not necessarily uniquely determined) $\overline{p}(x), \underline{p}(x) \in [p(x, \bar{c})]^{(\sigma)}$ such that $\varphi(x, b)^{\overline{p}} = \overline{f}(p)$ and $\varphi(x, b)^{\underline{p}} = \underline{f}(p)$. By hypothesis $\xi(x, \bar{z})^p \leq \frac{1}{2}$, so $d(\overline{p}, \underline{p}) < \varepsilon$ and thus $\overline{f}(p) < \underline{f}(p) + \varepsilon$.

Letting p vary over X it is easy to check that $\overline{f}(p)$ is upper semi-continuous and similarly \underline{f} is lower semi-continuous. Thus there is a continuous function $h: X \rightarrow [0, 1]$ verifying $\overline{f} < h < \underline{f} + \varepsilon$. By Tietze's Extension Theorem there exists a continuous function $\tilde{h}: S_{x, \bar{w}}(T) \rightarrow [0, 1]$ extending h and we may identify \tilde{h} with a definable predicate $\hat{\xi}_\sigma(x, \bar{w})$ (or, if we insist on having an actual formula, we take $\hat{\xi}_\sigma(x, \bar{w})$ to be a formula close enough to h so that $\overline{f} < \hat{\xi}_\sigma(x, \bar{w}) < \underline{f} + \varepsilon$ on X).

It is left to show that $\hat{\xi}_\sigma(y, \bar{w})$ is as desired. Indeed, assume that $a, b, \bar{c} \models q \in Y$. Then $\underline{f}(q \upharpoonright_{x, \bar{w}}) \leq \varphi(a, b) \leq \overline{f}(q \upharpoonright_{x, \bar{w}})$, whereby

$$\varphi(a, b) \leq \overline{f}(q \upharpoonright_{x, \bar{w}}) < \hat{\xi}_\sigma(a, \bar{c}) < \underline{f}(q \upharpoonright_{x, \bar{w}}) + \varepsilon \leq \varphi(a, b) + \varepsilon. \quad \blacksquare_{4.6}$$

We now turn to showing that members of Ξ_σ are, in a sense, plenty enough.

Lemma 4.7. *Let \mathcal{M} be a model of T , $p \in S_x(M)$ a type, and let $\eta = R_{<\omega}(p, \varphi)$.*

- (i) *Let $\mathfrak{M} \succeq \mathcal{M}$ be a very homogeneous and saturated extension and let $[p]^{(\eta)} \subseteq S_\varphi(\mathfrak{M})$ be defined as above. Then $[p]^{(\eta)} = \{q\}$ where q is the unique non forking φ -extension of p .*
- (ii) *There are $k \in \mathbb{N}$, $\xi(x, \bar{w}) \in \Xi_{\eta \upharpoonright_k}$ and $\bar{c} \in M$ such that $\xi(x, \bar{c}) \in p$. Moreover, for any dense subset $M_0 \subseteq M$ we may arrange our choices so that $\bar{c} \subseteq M_0$.*

Proof. The argument for the first item essentially appears in [BU], although the Cantor-Bendixson ranks used there are different. It goes through the following steps. The set $[p]^{(\eta)}$ is topologically and therefore metrically closed. By construction it is non empty and totally bounded, and therefore metrically compact. Clearly it is also M -invariant, and it follows that every $q \in [p]^{(\eta)}$ is definable over $\text{acl}^{eq}(M) = M$. We conclude there is a unique such q which follows the definition of p .

For the second item consider the following partial type over M :

$$p(x) \cup p(x') \cup \{|\varphi(x, y) - \varphi(x', y)| \geq \varepsilon\} \\ \cup \{R_{<\omega}(x/My, \varphi) \geq \eta\} \cup \{R_{<\omega}(x'/My, \varphi) \geq \eta\}.$$

By the first item this type is contradictory. Let us re-write $p(x)$ as $p(x, M)$ where $p(x, W) \in S_{x, W}(T)$ is a complete type. Then the following is inconsistent:

$$p(x, W) \cup p(x', W) \cup \{|\varphi(x, y) - \varphi(x', y)| \geq \varepsilon\} \\ \cup \{R_{<\omega}(x/Wy, \varphi) \geq \eta\} \cup \{R_{<\omega}(x'/Wy, \varphi) \geq \eta\}.$$

Thus by compactness there are $k \in \mathbb{N}$ and $\xi_0(x, \bar{w}) \in p(x, W)$ such that the following is inconsistent:

$$\begin{aligned} & \{\xi(x, \bar{w}) \leq 2^{-k}\} \cup \{\xi(x', \bar{w}) \leq 2^{-k}\} \cup \{|\varphi(x, y) - \varphi(x', y)| \geq \varepsilon\} \\ & \cup \{R_{<k}(x/\bar{w}y, \varphi) \geq \eta \upharpoonright_k\} \cup \{R_{<k}(x'/\bar{w}y, \varphi) \geq \eta \upharpoonright_k\}. \end{aligned}$$

Let $\xi = \xi_0 \dot{+} \dots \dot{+} \xi_0$ (2^k many times) and let $\bar{c} \subseteq M$ correspond to $\bar{w} \subseteq W$. Then ξ , k and \bar{c} are as desired.

For the moreover part first notice that p is equivalent to its restriction to M_0 (where $M_0 \subseteq M$ is any dense subset), so the argument above holds just as well with M_0 in place of M . ■_{4.7}

Lemma 4.8. *Let T be any theory in a countable language. Let $\mathcal{M} \models T^R$ be a model based on (Ω, μ) and let $\mathcal{M}_0 \preceq \mathcal{M}$ be a countable elementary pre-sub-structure. For $\omega \in \Omega$ let $\mathbf{M}_0(\omega) = \{\mathbf{a}(\omega)\}_{\mathbf{a} \in \mathbf{M}_0} \subseteq \mathbf{M}(\omega)$. Then $\mathcal{M}_0(\omega) \preceq \mathcal{M}(\omega)$ as \mathcal{L} -pre-structures for all ω outside a null measure set.*

Proof. Let us fix a formula $\varphi(x, \bar{w})$ and $\bar{\mathbf{b}} \in \mathbf{M}_0$. By Theorem 3.14 we have $\sup_x \varphi(x, \bar{\mathbf{b}}(\omega))^{\mathcal{M}(\omega)} = \llbracket \sup_x \varphi(x, \bar{\mathbf{b}}) \rrbracket^{\mathcal{M}(\omega)}$ for all $\omega \in \Omega$, where $\llbracket \sup_x \varphi(x, \bar{\mathbf{b}}) \rrbracket^{\mathcal{M}}$ is viewed as a continuous function $\Omega \rightarrow [0, 1]$. On the other hand we have

$$\begin{aligned} \llbracket \sup_x \varphi(x, \bar{\mathbf{b}}) \rrbracket^{\mathcal{M}} &= (\sup_{\mathbf{x}} \llbracket \varphi(\mathbf{x}, \bar{\mathbf{b}}) \rrbracket^{\widehat{\mathcal{M}}_0}) \\ &= \sup^{L_1} \left\{ \llbracket \varphi(\mathbf{a}, \bar{\mathbf{b}}) \rrbracket^{\widehat{\mathcal{M}}_0} : \mathbf{a} \in \widehat{\mathbf{M}}_0 \right\} = \sup^{L_1} \left\{ \llbracket \varphi(\mathbf{a}, \bar{\mathbf{b}}) \rrbracket^{\mathcal{M}} : \mathbf{a} \in \mathbf{M}_0 \right\}. \end{aligned}$$

Thus we have outside a null measure set

$$\llbracket \sup_x \varphi(x, \bar{\mathbf{b}}) \rrbracket^{\mathcal{M}}(\omega) = \sup \left\{ \varphi(\mathbf{a}(\omega), \bar{\mathbf{b}}(\omega))^{\mathcal{M}(\omega)} : \mathbf{a} \in \mathbf{M}_0 \right\}.$$

There are countably many formulae $\varphi(x, \bar{\mathbf{b}})$ to be considered, so outside a null measure set the Tarski-Vaught Criterion holds and $\mathcal{M}_0(\omega) \preceq \mathcal{M}(\omega)$. ■_{4.8}

Theorem 4.9. *Let $\varphi(x, y)$ be a stable formula for a theory T . Then the formula $E[\varphi(\mathbf{x}, \mathbf{y})]$ is stable for T^R . If T is stable then so is T^R .*

Proof. We may assume that the language of T is countable (for if not, we may restrict to a sub-language containing just the symbols appearing in φ). It will therefore be enough to show that for every separable model $\mathcal{M} \models T^R$, every type $\mathbf{p} \in S_{\mathbf{x}}(\mathbf{M})$ is definable. For this purpose let $\mathbf{a} \in \mathcal{M}' \succeq \mathcal{M}$ realise \mathbf{p} . Let also $\mathbf{M}_0 = \{\mathbf{c}_n\}_{n \in \mathbb{N}} \subseteq \mathbf{M}$ be a dense pre-sub-structure. By Lemma 4.8 there is a measure one set $\Omega_0 \subseteq \Omega$ such that $\mathcal{M}_0(\omega) \preceq \mathcal{M}(\omega)$ for all $\omega \in \Omega_0$.

Let Υ consist of all triplets $(\sigma, \xi, \bar{\mathbf{c}})$ where $\sigma \in \mathbb{N}^{<\omega}$, $\xi(x, \bar{w}) \in \Xi_\sigma$ and $\bar{\mathbf{c}} \in \mathbf{M}_0$ with $|\bar{w}| = |\bar{\mathbf{c}}|$. For $(\sigma, \xi, \bar{\mathbf{c}}) \in \Upsilon$ define subsets of Ω as follows:

$$\begin{aligned} A_\sigma &= \{\omega : R_{<|\sigma|}(\mathbf{a}(\omega)/M_\omega, \varphi) \geq \sigma\}, \\ B_{\sigma, \xi, \bar{\mathbf{c}}} &= \{\omega \in A_\sigma : \xi(\mathbf{a}(\omega), \bar{\mathbf{c}}(\omega)) < 1\}. \end{aligned}$$

The set A_σ is closed and each $B_{\sigma,\xi,\bar{c}}$ is relatively open in A_σ , so in particular Borel. Moreover, by Lemma 4.7 every $\omega \in \Omega_0$ belongs to some $B_{\sigma,\xi,\bar{c}}$.

Given all our countability assumptions we may enumerate $\Upsilon = \{(\sigma^m, \xi^m, \bar{c}^m)\}_{m \in \mathbb{N}}$. Let us also write ξ^m explicitly as $\xi^m(x, \bar{w}^m)$. By Lemma 4.6 there is a formula $\hat{\xi}_{\sigma^m}^m(y, \bar{w}^m)$ such that

$$\{\xi^m(x, \bar{w}^m) < 1\} \cup \{R_{<|\sigma^m|}(x/\bar{w}^m y, \varphi) \geq \sigma^m\} \vdash |\hat{\xi}_{\sigma^m}^m(y, \bar{w}^m) - \varphi(x, y)| < \varepsilon.$$

For $m \in \mathbb{N}$ let $D_m = B_{\sigma^m, \xi^m, \bar{c}^m} \setminus \bigcup_{k < m} B_{\sigma^k, \xi^k, \bar{c}^k}$. Then $\{D_m\}_{m \in \mathbb{N}}$ is a family of disjoint Borel sets and $\mu(\bigcup D_m) = \mu(\Omega_0) = 1$. In addition, for all $\omega \in D_m \in B_{\sigma^m, \xi^m, \bar{c}^m}$ and $\mathbf{b} \in \mathbf{M}$ we have

$$|\hat{\xi}_{\sigma^m}^m(\mathbf{b}(\omega), \bar{c}^m(\omega)) - \varphi(\mathbf{a}(\omega), \mathbf{b}(\omega))| < \varepsilon.$$

For each m let $X_m = \mathbb{P}[D_m | \mathcal{F}^{\mathbf{M}}] \in \mathcal{A}^{\mathbf{M}}$ and let

$$\psi(\mathbf{y}) = \sum_m E[X_m \llbracket \hat{\xi}_{\sigma^m}^m(\mathbf{y}, \bar{c}^m) \rrbracket].$$

Since $\sum X_m = 1$ this infinite sum does converge to an \mathbf{M} -definable predicate. We now claim that ψ is ε -close to a $E[\varphi]$ -definition for \mathbf{p} . Indeed, let $\mathbf{b} \in \mathbf{M}$. Then $\llbracket \hat{\xi}_{\sigma^m}^m(\mathbf{y}, \bar{c}^m) \rrbracket \in \mathcal{A}^{\mathbf{M}}$ whereby

$$\psi(\mathbf{b}) = \sum_m \int_{D_m} \llbracket \hat{\xi}_{\sigma^m}^m(\mathbf{y}, \bar{c}^m) \rrbracket d\mu.$$

We obtain

$$|\psi(\mathbf{b}) - E[\varphi(\mathbf{a}, \mathbf{b})]| \leq \sum_m \int_{D_m} |\hat{\xi}_{\sigma^m}^m(\mathbf{b}, \bar{c}^m) - \varphi(\mathbf{a}, \mathbf{b})| d\mu < \varepsilon.$$

We have shown that the predicate $\mathbf{b} \mapsto E[\varphi(\mathbf{x}, \mathbf{b})]^{\mathbf{p}}$ is arbitrarily well approximated on \mathbf{M}_0 , and therefore on \mathbf{M}_0 , by an \mathbf{M}_0 -definable predicate. It follows that \mathbf{p} admits an $E[\varphi]$ -definition. Since this holds for every type \mathbf{p} over a model the formula $E[\varphi(\mathbf{x}, \mathbf{y})]$ is stable.

The second assertion follows from the first using quantifier elimination down to formulae of the form $E[\varphi]$ (Theorem 3.32), since continuous combinations of stable formulae are stable. ■_{4.9}

4.3. Dependence. Recall that a formula $\varphi(\bar{x}, \bar{y})$ is ε -independent in a theory T for some $\varepsilon > 0$ if for every n one can find in some model of T tuples \bar{a}_i for $i < n$ and \bar{b}_w for $w \subseteq n$ such that:

$$\bigvee_{i,w: i \in w} \varphi(\bar{a}_i, \bar{b}_w) + \varepsilon < \bigwedge_{i,w: i \notin w} \varphi(\bar{a}_i, \bar{b}_w).$$

The formula φ is *independent* if it independent for some $\varepsilon > 0$. The theory T is *dependent* if every formula is dependent, i.e., if every formula is ε -dependent for every $\varepsilon > 0$.

Theorem 4.10. *A theory T is dependent if and only if its randomisation T^R is.*

Proof. It is immediate to check that if $\varphi(\bar{x}, \bar{y})$ is ε -independent in T then $E[\llbracket\varphi(\bar{x}, \bar{y})\rrbracket]$ is ε -independent in T^R . The converse is [Bena, Theorem 5.3]. ■_{4.10}

Theorem 4.11. *Assume T is independent. Then T^R is not simple.*

Proof. For simplicity of notation we assume T is a classical first order theory but the same proof holds when T is continuous. In this case $\llbracket\varphi\rrbracket$ is always a characteristic function, which may be identified with an event, whose probability is $\mu[\llbracket\varphi\rrbracket]$. The auxiliary sort can then be taken to be a probability algebra \mathcal{F} rather than its space of random variable $L^1(\mathcal{F}, [0, 1])$. Let the independent formula be $\varphi(x, y)$.

In some model $\mathcal{M} \models T$ we may find $\{b_n\}_{n \in \mathbb{N}}$ and $\{a_w\}_{w \subseteq \mathbb{N}}$ such that $\varphi(a_w, b_n) \iff n \in w$. Let c, c' be distinct and we may assume they are named in the language. For every n and $w \subseteq \mathbb{N}$ choose also $d_{n,w}$ as follows: If $n \in w$ then $d_{n,w} = c$, otherwise $d_{n,w} = c'$. In particular $\varphi(a_w, b_n) \iff d_{n,w} = c$.

Let \mathcal{F} be a sufficiently saturated atomless probability algebra let $\mathcal{M} = \mathcal{M}^{\mathcal{F}}$ (i.e., $\mathcal{M}^{L^1(\mathcal{F}, [0, 1])}$) as in Example 3.7. Then $(\mathcal{M}, \mathcal{F}) \models T^R$. We shall identify $a \in M$ with the constant random variable $a \in \mathbf{M}$. Let $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ be a sequence of independent events of measure one half.

Let $m \in \mathbb{N}$ be fixed. For $w \subseteq m$ let $A_w^m = \bigwedge_{i < m} A_i^{i \in w}$ (where $A^T = A$, $A^F = \neg A$). Now define:

$$\begin{aligned} \mathbf{a}^m &= \langle A_{m, \emptyset}, a_{\emptyset}, \dots \langle A_w^m, a_w, \dots \rangle \dots \rangle, \\ \mathbf{d}_n^m &= \langle A_{m, \emptyset}, d_{n, \emptyset}, \dots \langle A_w^m, d_{n,w}, \dots \rangle \dots \rangle. \end{aligned}$$

In other words, if $\omega \in A_w^m$ then $\mathbf{a}^m(\omega) = a_w$ and $\mathbf{d}_n^m(\omega) = d_{n,w}$. For $n < m$ we have:

- $A_n = \llbracket\varphi(\mathbf{a}^m, b_n)\rrbracket = \llbracket\mathbf{d}_n^m = c\rrbracket$.
- $\sigma(b_{\leq n}, \mathbf{d}_{< n}^m) = \sigma(\mathbf{d}_{< n}^m) = \langle A_{< n} \rangle$, whereby $A_n \perp \sigma(b_{\leq n}, \mathbf{d}_{< n}^m)$ in \mathcal{F} .

By compactness we can find in a model of T^R a sequence $(\mathbf{a}, \mathbf{b}_i, \mathbf{d}_i)_{i < \kappa}$ of arbitrary length that for all $i < \kappa$:

- $\mu[\llbracket\varphi(\mathbf{a}, \mathbf{b}_i)\rrbracket] = \frac{1}{2}$.
- $\llbracket\varphi(\mathbf{a}, \mathbf{b}_i)\rrbracket = \llbracket\mathbf{d}_i = c\rrbracket$.
- $\llbracket\varphi(\mathbf{a}, \mathbf{b}_i)\rrbracket \perp \sigma(\mathbf{b}_{\leq i}, \mathbf{d}_{< i})$.

The last item is indeed expressible, saying that for every $\psi(y_{\leq i}, z_{< i})$ (in which only finitely many variable actually appear, of course):

$$\mu[\llbracket\varphi(\mathbf{a}, \mathbf{b}_i) \wedge \psi(\mathbf{b}_{\leq i}, \mathbf{d}_{< i})\rrbracket] = \frac{1}{2} \mu[\llbracket\psi(\mathbf{b}_{\leq i}, \mathbf{d}_{< i})\rrbracket].$$

If T^R were simple there would be $i < \kappa$ such that $\mathbf{a} \perp_{\mathbf{b}_{\leq i} \mathbf{d}_{< i}} \mathbf{d}_i$. Let $A = \llbracket\varphi(\mathbf{a}, \mathbf{b}_i)\rrbracket = \llbracket\mathbf{d}_i = c\rrbracket$, so $A \perp \sigma(\mathbf{b}_{\leq i} \mathbf{d}_{< i})$. Working entirely in \mathcal{F} , which is a stable structure, let $\{B_n\}_{n \in \mathbb{N}}$ be a Morley sequence in the type of A over $\sigma(\mathbf{b}_{\leq i} \mathbf{d}_{< i})$, i.e., a Morley sequence over \emptyset which is independent from $\sigma(\mathbf{b}_{\leq i} \mathbf{d}_{< i})$. By Corollary 3.34 we have $A \equiv_{\mathbf{b}_{\leq i} \mathbf{d}_{< i}} B_n$ in

$(\mathcal{M}, \mathcal{F})$ for all n . Possibly passing to an elementary extension we can find an indiscernible sequence $\{\mathbf{d}'_n\}_{n \in \mathbb{N}}$ in $\text{tp}(\mathbf{d}_i/\mathbf{b}_{\leq i} \mathbf{d}_{< i})$ such that $B_n = \llbracket \mathbf{d}'_n = c \rrbracket$.

Since $\mathbf{a} \perp_{\mathbf{b}_{\leq i} \mathbf{d}_{< i}} \mathbf{d}_i$ we should be able to find \mathbf{a}' such that $\mathbf{a}' \mathbf{d}'_n \equiv_{\mathbf{b}_{\leq i} \mathbf{d}_{< i}} \mathbf{a} \mathbf{d}_i$ for all n . In particular $\llbracket \varphi(\mathbf{a}', \mathbf{b}_i) \rrbracket = \llbracket \mathbf{d}'_n = c \rrbracket = B_n$, so the sequence $\{B_n\}$ is constant. This contradiction shows that T^R cannot be simple. $\blacksquare_{4.11}$

The idea behind the proof is that each \mathbf{d}_i introduces a new independent event $\llbracket \mathbf{d}_i = c \rrbracket$ into the set of parameters (this can be done with any theory T with two distinct elements) and every \mathbf{b}_i introduces a new tie between \mathbf{a} and $\llbracket \mathbf{d}_i = c \rrbracket$ (this requires that T be independent). Thus \mathbf{a} is tied to an arbitrary large set of independent parameters, so the local character fails.

Question 4.12. Say that a continuous theory T has the *strict order property (SOP)* if there exists a formula $\varphi(\bar{x}, \bar{y})$ which defines a continuous pre-ordering with infinite ε -chains for some $\varepsilon > 0$, i.e., satisfies:

- *Reflexivity:* $\varphi(\bar{a}, \bar{a}) = 0$.
- *Transitivity:* $\varphi(\bar{a}, \bar{c}) \leq \varphi(\bar{a}, \bar{b}) + \varphi(\bar{b}, \bar{c})$.
- *Infinite ε -chain:* There exists $\varepsilon > 0$ and a sequence $(\bar{a}_n)_{n \in \mathbb{N}}$ in a model of T such that:

$$\bigvee_{n < m} \varphi(\bar{a}_n, \bar{a}_m) + \varepsilon < \bigwedge_{n > m} \varphi(\bar{a}_n, \bar{a}_m).$$

One can show that T is unstable if and only if it is independent or has the strict order property. Indeed, a straightforward translation of the proof for classical first order theories, as can be found in Poizat [Poi85], would work, keeping in mind that every formula of the form $\varphi(x, x') = \sup_y (\psi(x, y) \dot{-} \psi(x', y))$ defines a continuous pre-ordering, in analogy with formulae of the form $\forall y (\psi(x, y) \rightarrow \psi(x', y))$ in classical logic.

- (i) Assume T is independent. Does T^R has the strict order property?
- (ii) Alternatively, is it true that if T does not have the strict order property then neither does T^R ?

5. LASCAR TYPES

Definition 5.1. Let a and b be two tuples, possibly infinite, in a structure \mathcal{M} . We say that $d^L(a, b) \leq 1$ if in some (every) sufficiently saturated elementary extension $\mathcal{N} \succeq \mathcal{M}$ there exists an elementary sub-structure $\mathcal{N}_0 \preceq \mathcal{N}$ such that $a \equiv_{\mathcal{N}_0} b$. We say that $d^L(a, b) \leq n$ if in some (every) sufficiently saturated elementary extension there exist $a_0 = a, a_1, \dots, a_n = b$ such that $d^L(a_k, a_{k+1}) \leq 1$ for $k < n$.

If $d^L(a, b) < \infty$, i.e., if $d^L(a, b) \leq n$ for some n , then we say that a and b have the same *Lascar type*, in symbols $a \equiv^L b$.

Fact 5.2. (i) *For every n , the relation $d^L(x, y) \leq n$ is a reflexive, symmetric type-definable relation.*

- (ii) The relation \equiv^L is the transitive closure of $d^L(x, y) \leq n$ for any $n > 0$. It is the finest bounded automorphism-invariant equivalence relation on the sort in question.
- (iii) If $d^L(a, b) \leq n$ in some sufficiently saturated structure \mathcal{M} then for every other tuple a' there exists b' such that $d^L(a'a, b'b) \leq n$.

Definition 5.3. Let $(\mathcal{M}, \mathcal{A}) \models T^R$. An \mathcal{A} -type in $(\mathcal{M}, \mathcal{A})$ is a complete type over a subset of \mathcal{A} which has a unique extension to a type over \mathcal{A} . We define the \mathcal{A} -type of $\bar{\mathbf{a}} \in \mathbf{M}^n$ to be $\text{tp}_{\mathcal{A}}(\bar{\mathbf{a}}) = \text{tp}(\bar{\mathbf{a}}/\sigma(\bar{\mathbf{a}}))$.

Lemma 5.4. A type $\mathbf{p}(\bar{\mathbf{x}})$ over a subset of \mathcal{A} is an \mathcal{A} -type if and only if it is equivalent to $\text{tp}_{\mathcal{A}}(\bar{\mathbf{a}})$ for some $\bar{\mathbf{a}}$, if and only if it determines $\llbracket \varphi(\bar{\mathbf{a}}) \rrbracket$ for every formula $\varphi(\bar{\mathbf{x}})$. It is then axiomatised by the set of all conditions of the form $\llbracket \varphi(\bar{\mathbf{x}}) \rrbracket = \llbracket \varphi(\bar{\mathbf{a}}) \rrbracket$. Moreover, $\bar{\mathbf{a}} \equiv_{\mathcal{A}} \bar{\mathbf{b}}$ if and only if, in the canonical representation, $\bar{\mathbf{a}}(\omega) \equiv \bar{\mathbf{b}}(\omega)$ for all ω .

Proof. Easy, using Corollary 3.34. ■_{5.4}

We may therefore write $\bar{\mathbf{a}} \equiv_{\mathcal{A}} \bar{\mathbf{b}}$ to say that $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ have the same \mathcal{A} -type. Similarly, if $\mathbf{p}(\bar{\mathbf{x}})$ is an \mathcal{A} -type we may write $\llbracket \varphi(\bar{\mathbf{x}}) \rrbracket^{\mathbf{p}} \in \mathcal{A}$ for the value of $\llbracket \varphi(\bar{\mathbf{x}}) \rrbracket$ as determined by \mathbf{p} .

Lemma 5.5. Let T be any theory, $(\mathcal{M}, \mathcal{A})$ a sufficiently saturated model of T^R . Let $\mathbf{a}, \mathbf{b} \in \mathbf{M}$, $\mathbf{a} \equiv \mathbf{b}$. Then there exists $\mathbf{c} \in \mathbf{M}$ such that $d^L(\mathbf{a}, \mathbf{c}) \leq 1$ (so $\mathbf{a} \equiv^L \mathbf{c}$) and $\mathbf{c} \equiv_{\mathcal{A}} \mathbf{b}$.

Proof. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the sub-structure generated by $\llbracket \varphi \rrbracket$ where φ varies over all sentences. Then $\mathcal{A}_0 \subseteq \text{dcl}(\emptyset)$ in $(\mathcal{M}, \mathcal{A})$. Let Φ be the collection of all formulae $\varphi(x)$ with the appropriate variable. For $\varphi \in \Phi$ let $X_\varphi = \llbracket \varphi(\mathbf{a}) \rrbracket$ and let $\bar{X} = (X_\varphi)_{\varphi \in \Phi}$. Define $\bar{Y} = (Y_\varphi)_{\varphi \in \Phi} = (\llbracket \varphi(\mathbf{b}) \rrbracket)_{\varphi \in \Phi}$ similarly. Then by assumption $\bar{X} \equiv \bar{Y}$ in $(\mathcal{M}, \mathcal{A})$, whereby $\bar{X} \equiv_{\mathcal{A}_0} \bar{Y}$.

Let $(\mathcal{M}_1, \mathcal{A}_1) \preceq (\mathcal{M}, \mathcal{A})$ be a small elementary sub-structure. Then necessarily $\mathcal{A}_0 \subseteq \mathcal{A}_1$. By our saturation assumption we may find $\mathcal{A}_2 \subseteq \mathcal{A}$ such that $\mathcal{A}_2 \equiv_{\mathcal{A}_0} \mathcal{A}_1$ and $\mathcal{A}_2 \downarrow_{\mathcal{A}_0} \bar{X}, \bar{Y}$, both in the sense of \mathcal{A} (as a model of *ARV*). By Corollary 3.34 we have $\mathcal{A}_2 \equiv_{\mathcal{A}_0} \mathcal{A}_1$ in the structure $(\mathcal{M}, \mathcal{A})$ so again by saturation there is $\mathcal{M}_2 \subseteq \mathcal{M}$ such that $(\mathcal{M}_2, \mathcal{A}_2) \equiv (\mathcal{M}_1, \mathcal{A}_1)$, and in particular $(\mathcal{M}_2, \mathcal{A}_2) \preceq (\mathcal{M}, \mathcal{A})$.

By construction we have $\bar{X} \equiv_{\mathcal{A}_2} \bar{Y}$ in the sense of \mathcal{A} and applying Corollary 3.34 again we obtain $\bar{X} \equiv_{(\mathcal{M}_2, \mathcal{A}_2)} \bar{Y}$. Thus $d^L(\bar{X}, \bar{Y}) \leq 1$. By Fact 5.2 there is \mathbf{c} such that $d^L(\mathbf{a}\bar{X}, \mathbf{c}\bar{Y}) \leq 1$. In particular $\mathbf{a}\bar{X} \equiv \mathbf{c}\bar{Y}$ whereby $Y_\varphi = \llbracket \varphi(\mathbf{c}) \rrbracket$ for all $\varphi(x)$. Thus $\mathbf{c} \equiv_{\mathcal{A}} \mathbf{b}$ as desired. ■_{5.5}

We now turn to consider the case where $\mathbf{a} \equiv_{\mathcal{A}} \mathbf{b}$. We shall require an additional technical result.

Lemma 5.6. Let Ω be a set, $\tau: \Omega \rightarrow \Omega$ a bijection. Then there exists an integration functional E on $\mathcal{A} = [0, 1]^\Omega$ which is moreover invariant under τ : $E[X] = E[X \circ \tau]$ for all $X \in \mathcal{A}$.

Proof. This is a special case of a general fact that if an amenable group G (in our case, $(\mathbb{Z}, +)$) acts on a space Ω then Ω admits a G -invariant probability integration functional. ■_{5.6}

Lemma 5.7. *Let T be any theory, $(\mathcal{M}, \mathcal{A}) \models T^R$. Let $\mathbf{a}, \mathbf{b} \in \mathbf{M}$, $\mathbf{a} \equiv_{\mathcal{A}} \mathbf{b}$. Then $d^L(\mathbf{a}, \mathbf{b}) \leq 1$ (so in particular $\mathbf{a} \equiv^L \mathbf{b}$).*

Proof. Let $(\mathbf{M}, \mathcal{A})$ be the canonical representation of $(\mathcal{M}, \mathcal{A})$, based on $\mathcal{M} = \{\mathcal{M}_\omega\}_{\omega \in \Omega}$. Then for every $\omega \in \Omega$ we have $\mathbf{a}(\omega) \equiv \mathbf{b}(\omega)$ in \mathcal{M}_ω , so there exists an elementary extension $\mathcal{M}'_\omega \succeq \mathcal{M}_\omega$ and $h_\omega \in G_\omega = \text{Aut}(\mathcal{M}'_\omega)$ such that $h_\omega \mathbf{a}(\omega) = \mathbf{b}(\omega)$. Let $\bar{G} = \prod G_\omega$, $\bar{h} = (h_\omega)_\omega \in \bar{G}$.

Let $\Omega' = \Omega \times \bar{G}$, and let $\pi: \Omega' \rightarrow \Omega$ be the projection on the first co-ordinate. For $(\omega, \bar{g}) \in \Omega'$ let $\mathcal{M}'_{(\omega, \bar{g})} = \mathcal{M}'_\omega$, thus obtaining a family $\mathcal{M}'_{\Omega'} = \{\mathcal{M}'_{\omega'}\}_{\omega' \in \Omega'}$ with $\mathcal{M}_{\pi\omega'} \preceq \mathcal{M}'_{\omega'}$. Let $\sigma_{\omega'}: \mathcal{M}_{\pi\omega'} \hookrightarrow \mathcal{M}'_{\omega'}$ denote this elementary inclusion. For $\omega' \in \Omega'$ we may write $\omega' = (\omega, \bar{g}) = (\omega, g_\zeta)_{\zeta \in \Omega}$, and define $\eta_{(\omega, \bar{g})} = g_\omega \circ \sigma_{(\omega, \bar{g})}: \mathcal{M}_\omega \hookrightarrow \mathcal{M}'_{(\omega, \bar{g})}$, which is another elementary embedding. With $\mathbf{M}' = \prod \mathcal{M}'_{\Omega'}$, we obtain two mappings $\sigma, \eta: \mathbf{M} \rightarrow \mathbf{M}'$, given by

$$\begin{aligned} (\sigma \mathbf{c})(\omega, \bar{g}) &= \sigma_{(\omega, \bar{g})}(\mathbf{c}(\omega)) = \mathbf{c}(\omega), \\ (\eta \mathbf{c})(\omega, \bar{g}) &= \eta_{(\omega, \bar{g})}(\mathbf{c}(\omega)) = g_\omega(\mathbf{c}(\omega)). \end{aligned}$$

We observe that for given $(\omega, \bar{g}) \in \Omega'$ and $\mathbf{c} \in \mathbf{M}$:

$$h_\omega(\sigma \mathbf{a})(\omega, \bar{g}) = (\sigma \mathbf{b})(\omega, \bar{g}), \quad h_\omega(\eta \mathbf{c})(\omega, \bar{g}) = (\eta \mathbf{c})(\omega, \bar{h}\bar{g}).$$

By Fact 3.21 there exists an integration functional E_1 on $\mathcal{A}_1 = [0, 1]^\Omega$ which extends integration of Borel functions. The left action of \bar{h} on \bar{G} is a bijection, so $[0, 1]^{\bar{G}}$ admits an integration functional E_G which is invariant by the left action of \bar{h} . Let $\mathcal{A}' = [0, 1]^{\Omega'}$, and for $X' \in \mathcal{A}'$ define $E'[X'] = E[\omega \mapsto E_G[X'(\omega, \cdot)]]$, which we may also write as $E^\omega[E_G^{\bar{g}}[X'(\omega, \bar{g})]]$ or simply $E[E_G[X]]$. Then E' is easily checked to be an integration functional.

We are now in the situation described in Corollary 3.27. In particular, the triplet $(\mathbf{M}', \mathcal{A}', E')$ is a randomisation, and we obtain *two* $[[\cdot]]$ -embeddings, $[\sigma], [\eta]: (\mathcal{M}, \mathcal{A}) \rightarrow (\widehat{\mathcal{M}'}, \widehat{\mathcal{A}'})$, where $[\sigma \mathbf{c}]$ and $[\eta \mathbf{c}]$ are the images in $\widehat{\mathcal{M}'}$ of $\sigma \mathbf{c}$ and $\eta \mathbf{c}$ defined above, and $[\sigma X] = [\eta X] = [X \circ \pi]$. By quantifier elimination for T^R these are in fact elementary embeddings.

We claim that $\sigma \mathbf{a} \equiv_{\eta \mathbf{M}} \sigma \mathbf{b}$. Indeed, let $\bar{\mathbf{c}} \in \mathbf{M}$, $\varphi(x, \bar{y})$ any formula, and let $X = \langle \varphi(\sigma \mathbf{a}, \eta \bar{\mathbf{c}}) \rangle$ and $Y = \langle \varphi(\sigma \mathbf{b}, \eta \bar{\mathbf{c}}) \rangle$, both members of $[0, 1]^{\Omega'}$. Fix $\omega \in \Omega$, and let $\bar{g} \in \bar{G}$

vary freely. Then:

$$\begin{aligned}
 X(\omega, \bar{g}) &= \varphi(\mathbf{a}(\omega), \eta_{(\omega, \bar{g})} \bar{\mathbf{c}}(\omega)) \\
 &= \varphi(h_\omega \mathbf{a}(\omega), h_\omega \eta_{(\omega, \bar{g})} \bar{\mathbf{c}}(\omega)) \\
 &= \varphi(\mathbf{b}(\omega), \eta_{(\omega, \bar{h}\bar{g})} \bar{\mathbf{c}}(\omega)) \\
 &= Y(\omega, \bar{h}\bar{g}).
 \end{aligned}$$

Since E_G was chosen invariant under the left action of \bar{h} on \bar{G} we obtain that $E_G[X(\omega, \cdot)] = E_G[Y(\omega, \cdot)]$ for all ω , whereby $E'[X] = E'[Y]$. We obtain

$$E[\varphi([\sigma \mathbf{a}], [\eta \bar{\mathbf{c}}])] = E'\langle \varphi(\sigma \mathbf{a}, \eta \bar{\mathbf{c}}) \rangle = E'\langle \varphi(\sigma \mathbf{b}, \eta \bar{\mathbf{c}}) \rangle = E[\varphi([\sigma \mathbf{b}], [\eta \bar{\mathbf{c}}])],$$

proving our claim. Since η is an elementary embedding we have $d^L([\sigma \mathbf{a}], [\sigma \mathbf{b}]) \leq 1$, and since σ is an elementary embedding we conclude that $d^L(\mathbf{a}, \mathbf{b}) \leq 1$. $\blacksquare_{5.7}$

Theorem 5.8. *Let T be any theory, $(\mathcal{M}, \mathcal{A}) \models T^R$, $\mathbf{a}, \mathbf{b} \in \mathbf{M}$, and let $\mathbf{A} \subseteq \mathbf{M}$ be any set of parameters. Then the following are equivalent:*

- (i) $\mathbf{a} \equiv_{\mathbf{A}}^L \mathbf{b}$.
- (ii) $\mathbf{a} \equiv_{\mathbf{A}} \mathbf{b}$.
- (iii) $d_{\mathbf{A}}^L(\mathbf{a}, \mathbf{b}) \leq 2$, where $d_{\mathbf{A}}^L$ is defined as d^L , over parameters in \mathbf{A} .

Proof. First of all we may name \mathbf{A} in the language (at no point did we assume that T was complete), so we may assume that $\mathbf{A} = \emptyset$. For the implication (ii) \implies (iii), just apply Lemma 5.5 followed by Lemma 5.7. The rest is standard and holds in arbitrary structures. $\blacksquare_{5.8}$

Corollary 5.9. *Let $\mathcal{M} \models T^R$, $\mathbf{A} \subseteq \mathbf{M}$. Let $\text{dcl}^{eq,R}$ denote the definable closure in the sense of $(T^R)^{eq}$, and similarly for $\text{acl}^{eq,R}$. Then $\text{dcl}^{eq,R}(\mathbf{A}) = \text{acl}^{eq,R}(\mathbf{A})$ in \mathcal{M} .*

Notice that even though $\text{dcl}^{eq,R}(\mathbf{A})$ and $\text{acl}^{eq,R}(\mathbf{A})$ may contain imaginary elements in the sense T^R , the set \mathbf{A} is required to consist of real elements, i.e., elements coming from sorts of T .

Corollary 5.10. *For every theory T , the theory T^R is G -compact, which means that for every set of parameters \mathbf{A} and for every tuple length α , the relation $\bar{\mathbf{a}} \equiv_{\mathbf{A}}^L \bar{\mathbf{b}}$ between tuples of length α is type-definable over \mathbf{A} .*

Proof. The relation $d_{\mathbf{A}}^L(\bar{x}, \bar{y}) \leq n$ is always type-definable, for any fixed n . $\blacksquare_{5.10}$

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