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# Treewidth of planar graphs: connection with duality

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## 1 Preliminaries

A graph is said to be *chordal* if each cycle with at least four vertices has a chord, that is an edge between two non-consecutive vertices of the cycle. Given an arbitrary graph  $G = (V, E)$ , a *triangulation* of  $G$  is a chordal graph  $H (= V, F)$  such that  $E \subseteq F$ . We say that  $H$  is a *minimal triangulation* of  $G$  if no proper subgraph of  $H$  is a triangulation of  $G$ . The **treewidth**  $\text{tw}(H)$  of a chordal graph is its maximum cliquesize minus one. The tree-width of an arbitrary graph  $G$  is the minimum, over all triangulations  $H$  of  $G$ , of  $\text{tw}(H)$ . When computing the treewidth of  $G$ , we can clearly restrict to minimal triangulations. Treewidth was introduced by Robertson and Seymour in connection with graph minors [5], but it has wide algorithmic applications since many NP-hard problems become polynomial when restricted to graphs of bounded treewidth.

Robertson and Seymour conjectures in [5] that the treewidth of a planar graph  $G$  and its dual  $G^*$  differ by at most one. This conjecture was recently proved by Lapoire [3], who gives a more general result, on hypergraphs of bounded genus. Nevertheless, the proof of Lapoire is rather long and technical. Here, we show that any minimal triangulation  $H$  of a planar graph  $G$  can be easily transformed into a triangulation  $H^*$  of  $G^*$  such that  $\text{tw}(H^*) \leq \text{tw}(H) + 1$ .

The *minimal separators* play a crucial role in the characterisation of the minimal triangulations of a graph. A subset  $S \subseteq V$  separates two non-adjacent vertices  $a, b \in V$  if  $a$  and  $b$  are in different connected components of  $G \setminus S$ .  $S$  is a *minimal  $a, b$ -separator* if it separates  $a$  and  $b$  and no proper subset of  $S$  separates  $a$  and  $b$ . We say that  $S$  is a *minimal separator* of  $G$  if there are two vertices  $a$  and  $b$  such that  $S$  is a minimal  $a, b$ -separator. Notice that a minimal separator can be strictly included into another. We denote by  $\Delta_G$  the set of all minimal separators of  $G$ . Two minimal separators  $S$  and  $T$  *cross* if  $T$  intersects at least two components of  $G \setminus S$ . Otherwise,  $S$  and  $T$  are *parallel*. Both relations are symmetric.

Let  $S \in \Delta_G$  be a minimal separator. We denote by  $G_S$  the graph obtained from  $G$  by *completing*  $S$ , i.e. by adding an edge between every pair of non-adjacent vertices of  $S$ . If  $\Gamma \subseteq \Delta_G$  is a set of separators of  $G$ ,  $G_\Gamma$  is the graph obtained by completing all the separators of  $\Gamma$ . The result of [2], concluded in [4], establish a

strong relation between the minimal triangulations of a graph and its minimal separators.

**Theorem 1.**  *$H$  is a minimal triangulation of  $G$  if and only if there is a maximal set of pairwise parallel separators  $\Gamma \subseteq \Delta_G$  such that  $H = G_\Gamma$ .*

Since it is easy to extend our results to simply connected or disconnected graphs, we will restrict to 2-connected graphs.

## 2 Minimal separators in planar graphs

Consider a 2-connected planar graph  $G = (V, E)$ . We fix an embedding of  $G$  in the plane  $\mathbb{R}^2$ . Let  $F$  be the set of faces of this embedding. Let  $F$  be the set of faces of this embedding. The *intermediate graph*  $G_I$  has vertex set  $V \cup F$ . We place an edge in  $G_I$  between an original vertex  $v \in V$  and a face  $f \in F$  whenever the corresponding vertex and face are incident in  $G$ . Notice that  $(G^*)_I = G_I$ .

Let  $\nu$  be a cycle of  $G_I$  (by “cycle” we will always mean a cycle which does not get through a same vertex twice). The drawing of  $\nu$  forms a Jordan curve in the plane  $\mathbb{R}^2$ , denoted  $\tilde{\nu}$ . It is easy to see that if  $\tilde{\nu}$  separates two original vertices  $x$  and  $y$  in the plane (i.e.  $x$  and  $y$  are in different regions of  $\mathbb{R}^2 \setminus \nu$ ), then  $\nu \cap V$  separates  $x$  and  $y$  in  $G$ . Therefore, the original vertices of  $\nu$  form a separator in  $G$ . Conversely, to each minimal separator  $S$  of  $G$ , we can associate a cycle  $\nu$  of  $G_I$  (see [1]).

**Proposition 1.** *Let  $S$  be a minimal separator of the planar graph  $G$ . Consider two connected components  $C$  and  $D$  of  $G \setminus S$ . There is a cycle  $\nu_S$  of  $G_I$  such that  $\tilde{\nu}$  separates  $C$  and  $D$  in the plane.*

This cycle is usually not unique. In the case of 3-connected planar graphs, notice that if  $S$  is a minimal separator, then  $G \setminus S$  has exactly two connected components  $C$  and  $D$ . For each couple of original vertices  $x$  and  $y$  incident to a same face, fix a unique face  $f(x, y)$  containing both  $x$  and  $y$ . We say that a cycle  $\nu$  of  $G_I$  is well-formed if, for any two consecutive original vertices  $x, y \in \nu$ , the face-vertex between them is  $f(x, y)$ . If  $G$  is a 3-connected planar graph, for any minimal separator  $S$ , there is a unique well-formed cycle of  $G_I$  separating  $C$  and  $D$  in the plane.

**In what follows,  $G$  denotes a 3-connected planar graph.** However, our main results can be easily extended to arbitrary planar graphs.

We say that two Jordan curves  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$  *cross* if  $\tilde{\nu}_1$  intersects the two regions defined by  $\tilde{\nu}_2$ . Otherwise, they are *parallel*. Two cycles  $\nu_1$  and  $\nu_2$  of  $G_I$  *cross* if and only if  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$  cross. Notice that the parallel and crossing relations between curves and cycles are symmetric.

**Proposition 2.** *Two minimal separators  $S$  and  $T$  of  $G$  are parallel if and only if the corresponding cycles  $\nu_S$  and  $\nu_T$  of  $G_I$  are parallel.*

Let  $\tilde{\nu}$  be a Jordan curve in the plane. Let  $R$  be one of the regions of  $\mathbb{R}^2 \setminus \tilde{\nu}$ . We say that  $(\tilde{\nu}, R) = \tilde{\nu} \cup R$  is a *one-block region* of the plane, *bordered* by  $\tilde{\nu}$ . Let  $\tilde{\mathcal{C}}$  be a set of curves such that for each  $\tilde{\nu} \in \tilde{\mathcal{C}}$ , there is a one-block region  $(\tilde{\nu}, R(\tilde{\nu}))$  containing all the curves of  $\tilde{\mathcal{C}}$ . We define the *region between* the elements of  $\tilde{\mathcal{C}}$  as  $RB(\tilde{\mathcal{C}}) = \bigcap_{\tilde{\nu} \in \tilde{\mathcal{C}}} (\tilde{\nu}, R(\tilde{\nu}))$ . A subset  $Br \subseteq \mathbb{R}^2$  of the plane is a *block region* if  $BR$  is a one-block region  $(\tilde{\nu}, R)$  or  $BR$  is the region between some set of curves  $\tilde{\mathcal{C}}$ .

### 3 Minimal triangulations of $G$ and $G^*$

Let  $G$  be a 3-connected planar graph and let  $H$  be a minimal triangulation of  $G$ . According to Theorem 1, there is a maximal set of pairwise parallel separators  $\Gamma \subseteq \Delta_G$  such that  $H = G_\Gamma$ . Let  $\mathcal{C}(\Gamma) = \{\nu_S \mid S \in \Gamma\}$  be the cycles of  $G_I$  associated to the minimal separators of  $\Gamma$  and let  $\tilde{\mathcal{C}}(\Gamma) = \{\tilde{\nu}_S \mid S \in \Gamma\}$  be the curves associated to these cycles. According to Proposition 2, the cycles of  $\mathcal{C}(\Gamma)$  are pairwise parallel. Thus, the curves of  $\tilde{\mathcal{C}}(\Gamma)$  split the plane into block regions. Consider the set of all the block regions bordered by some elements of  $\tilde{\mathcal{C}}$ . We show that any maximal clique  $\Omega$  of  $H$  corresponds to the original vertices contained in a minimal block regions defined by  $\tilde{\mathcal{C}}(\Gamma)$ .

**Theorem 2.** *Let  $G$  be a 3-connected planar graph and let  $H = G_\Gamma$  be a minimal triangulation of  $G$ .  $\Omega \subseteq V$  is a maximal clique of  $H$  if and only if there is a minimal block region  $BR$  defined by  $\tilde{\mathcal{C}}(\Gamma)$ . such that  $\Omega = BR \cap V$ .*

Let now  $\mathcal{C}$  be an arbitrary set of pairwise parallel cycles of  $G_I$ . This family  $\tilde{\mathcal{C}}$  of curves associated to these cycles splits the plane into block regions. Let  $G^*$  be the dual of  $G$ . The graph  $H^*(\mathcal{C}) = (F, E_H)$  has vertex set  $F$ . We place an edge between two face-vertices  $f$  and  $f'$  of  $H$  if and only if  $f$  and  $f'$  are in a same minimal block region defined by  $\tilde{\mathcal{C}}$ . Equivalently,  $f$  and  $f'$  are non-adjacent in  $H^*(\mathcal{C})$  if and only if there is a  $\tilde{\nu} \in \tilde{\mathcal{C}}$  separating  $f$  and  $f'$  in the plane.

**Theorem 3.**  *$H^*(\mathcal{C})$  is a triangulation of  $G^*$ . Moreover, any clique  $\Omega^*$  of  $H^*$  is contained in some minimal block region  $BR$  defined by  $\tilde{\mathcal{C}}$ .*

Let  $H = G_\Gamma$  be a minimal triangulation of  $G$ . Consider the cycles  $\mathcal{G}(\Gamma)$  associated to the minimal separators of  $\Gamma$  and the corresponding curves  $\tilde{\mathcal{C}}(\Gamma)$ . We could try to considerate the triangulation  $H^*(\mathcal{C}(\Gamma))$  of  $G^*$ , but unfortunately it does not satisfy  $\text{tw}(H^*) \leq \text{tw}(H) + 1$ .

Thus, we consider a maximal set of pairwise parallel cycles  $\mathcal{C}'$  of  $G_I$  such that  $\mathcal{C}(\Gamma) \subseteq \mathcal{C}'$ . Clearly, each minimal block region defined by  $\mathcal{C}'$  is contained in a minimal block region defined by  $\tilde{\mathcal{C}}(\Gamma)$ .

**Theorem 4.** *Let  $\mathcal{C}'$  be a maximal set of pairwise parallel cycles of  $G_I$ . Let  $BR$  be a minimal block region of  $\mathcal{C}'$ . Then  $BR \cap G_I$  is either formed by a cycle  $\tilde{\nu}$  and a path  $\tilde{\mu}$  joining two vertices of  $\tilde{\nu}$  or  $BR$  is a one-block region  $(\tilde{\nu}, R)$  and  $BR \cap G_I = \nu$  where  $\nu$  is the cycle of  $G_I$  associated to  $\tilde{\nu}$ . In particular,  $|BR \cap V^*| \leq |BR \cap V| + 1$ .*

According to theorem 3, each maximal clique  $\Omega^*$  of  $H^*$  is contained in some minimal block region  $BR$ , and by the previous theorem it has at most one more vertex than  $\Omega = BR \cap V$ . By theorem 2,  $\Omega$  is a clique of  $H$ . Hence,  $|\Omega^*| \leq |\Omega| + 1$  and thus  $\text{tw}(H^*) \leq \text{tw}(H) + 1$ . By considering an optimal triangulation  $H$  of  $G$ , we obtain a triangulation  $H^*$  of  $G^*$  of width at most  $\text{tw}(G) + 1$ . We conclude that  $\text{tw}(G^*) \leq \text{tw}(G) + 1$ .

So we can state:

**Theorem 5 (Main theorem).** *Let  $G = (V, E)$  be a planar graph.*

$$|\text{tw}(G) - \text{tw}(G^*)| \leq 1.$$

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