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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender *et al.* conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry *et al.* [1] proposed an algorithm of running time $O(nm)$ per separator¹ that uses the concept of generating new minimal

¹ The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].

separators from a previous minimal separator S by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph G . In order to avoid the problem of recalculation, we define the set $\mathfrak{S}_a(S, O)$ of the a, b -minimal separators S' for some b such that the connected component of a in $G \setminus S'$ contains the connected component of a in $G \setminus S$ but avoids the set O . Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets A and B are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a *separator* if $G \setminus S$ has at least two connected components, an *a, b -separator* if a and b are in different connected components of $G \setminus S$, an *a, b -minimal separator* if no proper subset of S is an a, b -separator. The connected component of a in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a *full connected component* if $N(C_a(S)) = S$. For an a, b -minimal separator S , both $C_a(S)$ and $C_b(S)$ are full. A set S is a *minimal separator* if there exist a and b such that S is an a, b -minimal separator or, which is equivalent, if it has at least two full connected components. An *$a, *$ -minimal separator* of a graph $G = (V, E)$ is an a, b -minimal separator of G for some $b \in V$. The set of the $a, *$ -minimal separators is denoted by \mathfrak{S}_a and the set of the minimal separators of G is denoted by $\mathfrak{S}(G)$.

It is possible to order the $a, *$ -minimal separators in the following way:

$$S_1 \preceq_a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator S_1 is *closer* to a than S_2 . The set of a, b -minimal separators is a lattice for the relation \preceq_a [4] but we only need the following weaker lemma:

Lemma 1 *Let C be a set of vertices of a graph G inducing a connected subgraph of G , a be a vertex of C and b be a vertex of $G \setminus (C \cup N(C))$.*

The neighbour S of $C_b(C \cup N(C))$ is an a, b -minimal separator such that C is a subset of $C_a(S)$ that is closer to a than any a, b -minimal separator S' such that C is a subset of $C_a(S')$.

PROOF. By construction, C is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since S is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that S is an a, b -minimal separator.

Let S' be an a, b -minimal separator such that C is a subset of $C_a(S')$. Let p be a path in $C_b(S')$ with b as one of its ends. The vertices of S' are at least at distance 1 of C so the vertices of p are at least at distance 2 of C . Since S is a subset of $N(C)$, $p \cap S = \emptyset$. In other words p is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ *i.e.* S is closer to a than S' . \square

For S an $a, *$ -minimal separator and $O \subseteq V$, the set $\mathfrak{S}_a(S, O)$ is the set of the $a, *$ -minimal separators S' further from a than S and such that $O \cap C_a(S') = \emptyset$. If $x \in V$, the set $\mathfrak{S}_a^x(S, O)$ is the set of $S' \in \mathfrak{S}_a(S, O)$ such that $x \in C_a(S')$.

Remark 2 If $x \in S$, then $\mathfrak{S}_a(S, O)$ is the disjoint union

$$\mathfrak{S}_a(S, O \cup \{x\}) \sqcup \mathfrak{S}_a^x(S, O).$$

More precisely, if $(S_i)_{i \in I}$ are the elements of $\mathfrak{S}_a^x(S, O)$ closest to a , then

$$\mathfrak{S}_a(S, O) = \mathfrak{S}_a(S, O \cup \{x\}) \sqcup \left(\bigcup_{i \in I} \mathfrak{S}_a(S_i, O) \right).$$

This gives the skeleton of an algorithm to compute the set $\mathfrak{S}_a(S, O)$.

Remark 3 If S belongs to $\mathfrak{S}_a^x(S, O)$, then $\mathfrak{S}_a^x(S, O) = \mathfrak{S}_a(S, O)$.

The algorithm is based on remarks 2 and 3. To list \mathfrak{S}_a , the algorithm computes the sets $\mathfrak{S}_a(S, \emptyset)$ for every S closest to a in \mathfrak{S}_a . During this calculation, it computes $\mathfrak{S}_a(S, O)$ with $O \subseteq S$. To do so, it chooses x in $S \setminus O$ and calculates $\mathfrak{S}_a^x(S, O)$ and $\mathfrak{S}_a(S, O \cup \{x\})$. The set $\mathfrak{S}_a^x(S, O)$ is itself a union of $\mathfrak{S}_a(S_i, O)$. But to obtain such a decomposition, one needs to find the elements of $\mathfrak{S}_a^x(S, O)$ closest to a , which the following proposition does.

Proposition 4 Let $G = (V, E)$ be a graph, S an $a, *$ -minimal separator, $O \subset S$, $x \in S \setminus O$ and $C = C_a(S) \cup \{x\}$

The elements of $\mathfrak{S}_a^x(S, O)$ closest to a are exactly the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ that contain O and that are maximal for inclusion.

PROOF. Let S_1 be an a, b -minimal separator of $\mathfrak{S}_a^x(S, O)$ closest to a . Let S' be the neighbourhood of $C_b(N(C) \cup C)$. By lemma 1, S' is an a, b -minimal separator such that C is a subset of $C_a(S')$ and S' is closer to a than S_1 . Moreover, since $C_a(S_1) \cap O = \emptyset$ and S' is closer to a than S_1 , $C_a(S') \cap O \subseteq C_a(S_1) \cap O = \emptyset$. Thus S' belongs to $\mathfrak{S}_a^x(S, O)$ and is closer to a than S_1 . This proves that $S_1 = S'$. Since S_1 cannot be a subset of another element of $\mathfrak{S}_a^x(S, O)$, S_1 is the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ which is maximal for inclusion.

Conversely, let S_1 be a neighbourhood of a connected component D of $G \setminus \{N(C) \cup C\}$ that contains O and that is maximal for inclusion. By lemma 1, S_1 is an element of $\mathfrak{S}_a^x(S, O)$ that is closer to a than any a, b -minimal separator of $\mathfrak{S}_a^x(S, O)$ with b in D . So if S_2 is an a, b -minimal separator of $\mathfrak{S}_a^x(S, O)$ strictly closer to a than S_1 , S_1 is not an a, b -minimal separator. Suppose for a contradiction that such an a, b -minimal separator exists. It follows from the first part of the proof that such an a, b -minimal separator S_2 closest to a is the neighbourhood of $C_b(N(C) \cup C)$. The set S_2 is an element of $\mathfrak{S}_a^x(S, O)$ that is closer to a than S_1 and S_1 is a subset of S_2 (because $S_1 \setminus S_2 \subseteq C_a(S_2) \setminus C_a(S_1)$ and S_2 is closer to a than S_1) and therefore S_1 is a strict subset of S_2 contradicting the fact that S_1 is maximal for inclusion. \square

Proposition 4 gives us a way to find the minimal elements of $\mathfrak{S}_a^x(S, O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ and then choosing among the minimal separators found the ones that contain O and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $\mathfrak{S}_a(S, O)$ that may look like:

Algorithm 1 `_calc3_`

```

begin
  if  $S \setminus O = \emptyset$  then
    return( $\{S\}$ )
  else
    let  $x \in S \setminus O$ 
     $\mathfrak{S} \leftarrow \text{\_calc3\_}(G, a, S, O \cup \{x\})$ 

    for each  $S_i$  in find_closest_elements( $G, a, x, S, O$ )
       $\mathfrak{S} \leftarrow \mathfrak{S} \cup \text{\_calc3\_}(G, a, S_i, O)$ 
    return( $\mathfrak{S}$ )
end

```

However several problems need to be solved.

- i. We do not know whether the sets $\mathfrak{S}_a(S_i, O)$ are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.
- ii. To implement the function `find_closest_elements`, proposition 4 states that we can start with a graph search of G .

But if $\mathfrak{S}_a(S, O) = \{S\}$, the recursive calls to the algorithm will try to find an element of $\mathfrak{S}_a^x(S, O)$ closest to a for every $x \in S \setminus O$. Each call to `find_min_elements` costs at least $O(m)$ and finally, we would have spent at least $O(nm)$ to realise that $\mathfrak{S}_a(S, O) = \{S\}$.

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, *i.e.* if S_1 and S_2 are two minimal elements of $\mathfrak{S}_a^x(S, O)$, the sets $\mathfrak{S}_a(S_1, O)$ and $\mathfrak{S}_a(S_2, O)$ are disjoint. Section 3.3 then shows how to determine whether $\mathfrak{S}_a^x(S, O)$ is empty or not in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by \tilde{G}_Σ the drawing of G_Σ . A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A *face* of G_Σ is a connected component of $\Sigma \setminus \tilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Proposition 5 *In a 3-connected planar graph, minimal separators are minimal for inclusion.*

PROOF. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let a, b, c and d be vertices such that S' is an a, b -minimal separator and S is a c, d -minimal separator. Since S is not an a, b -minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. In this case, $C_c(S)$ and $N(C_c(S))$ are respectively equal to $C_c(S')$ and S .

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a' , b' and c' , all these vertices have S in their neighbourhood and since G is 3-connected, $|S| \geq 3$. This contradicts the fact that G is planar. \square

Proposition 6 *Let $G = (V, E)$ be a 3-connected planar graph, a a vertex of G , S an $a, *$ -minimal separator, O a subset of S and x a vertex of $S \setminus O$.*

If S_1 and S_2 are two distinct elements of $\mathfrak{S}_a^x(S, O)$ that are closest to a , then

$$\mathfrak{S}_a(S_1, O) \cap \mathfrak{S}_a(S_2, O) = \emptyset.$$

PROOF. Let C be $C_a(S) \cup \{x\}$ and suppose for a contradiction that S_3 is a minimal separator of $\mathfrak{S}_a(S_1, O) \cap \mathfrak{S}_a(S_2, O)$ with S_1 and S_2 two distinct elements of $\mathfrak{S}_a^x(S, O)$ closest to a . Let b be a vertex such that S_3 is an a, b -minimal separator.

Since S_3 is further from a than S_1 and S_2 , both S_1 and S_2 are a, b -separators. There exists an a, b -minimal separator S' included in S_1 . By proposition 5, a minimal separator of G is minimal for inclusion which proves that $S_1 = S'$ and S_1 is an a, b -minimal separator. By lemma 1, the neighbourhood S_4 of $C_b(N(C) \cup C)$ is an a, b -minimal separator such that C is a subset of $C_a(S_4)$ that is closer to a than S_1 . So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and S_4 is an element of $\mathfrak{S}_a^x(S, O)$ that is closer to a than S_1 . Similarly, S_2 is an a, b -minimal separator and S_4 is closer to a than S_2 which contradicts the fact that S_1 and S_2 are two distinct elements of $\mathfrak{S}_a^x(S, O)$ closest to a . \square

3.2 The intermediate graph

Definition 7 *Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let F be the set of its faces. In each face $f \in F$ pick up one point v_f . Let R_F be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex v is incident to the face f .*

For G' a subgraph of G_I , the set $\tilde{G}' \cap V_\Sigma$ will be denoted by $V(G')$.

Proposition 8 *Let μ be a cycle of G_I such that the curve $\tilde{\mu}$ separates at least two vertices a and b of V_Σ .*

The set $V(\mu)$ is an a, b -separator of G_Σ .

PROOF. Let p be a path in G_Σ from a to b . Since a and b are not in the

same connected component of $\Sigma \setminus \tilde{\mu}$, \tilde{p} intersect $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a, b -separator. \square

Proposition 9 *Let S be an a, b -minimal separator of G . There exists a simple cycle μ of G_I such that the Jordan curve defined by μ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

PROOF. Let C be the connected component of a in $G \setminus S$. Let us contract C into a supervertex v_C to build the graph $G_{/C}$. There is a cycle $\mu_{/C}$ in $(G_{/C})_I$ such that $V(\mu_{/C})$ is the neighbourhood of v_C in $G_{/C}$. Therefore, the neighbourhood of C in G_I has the structure of a cycle μ .

Suppose $\tilde{\mu}$ is not a Jordan curve, the border μ' of the connected component of b in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates a and b . However, proposition 8 shows that $V(\mu')$ which is a strict subset of S is an a, b -separator. This contradicts the fact that S is a a, b -minimal separator. \square

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the G_I , it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of G_I correspond to no minimal separator of G .

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

3.3 Ordered separators

Definition 10 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \dots, v_{p-1}), (p > 2)$$

such that

- i. there exists a face to which v_i and $v_{i+1[p]}$ are incident;*
- ii. v_i and v_j are incident to a common face only if $i = j+1[p]$ or $j = i+1[p]$;*
- iii. if $p = 3$, no face is incident to v_i , $v_{i+1[p]}$ and $v_{i+2[p]}$.*

The notation $i[p]$ means i modulo p .

A set $S = \{v_0, \dots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \dots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \dots, v_{p-1})$ is an ordered separator of G , then S is naturally associated to the set $\{v_0, \dots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

Lemma 11 *Every minimal separator S of G is ordered.*

PROOF. Let S be an a, b -minimal separator of G .

Proposition 9 states that there exists a simple cycle of G_I

$$\mu = (v_0, f_0, \dots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \dots, v_{p-1})$ is an ordered separator corresponding to S .

- i. The construction of T ensures that v_i and v_{i+1} are incident to a common face (f_i).
- ii. Suppose that v_i et v_j are incident to a common face f and that $i + 1 \neq j [p]$ and $j + 1 \neq i [p]$.
 $\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \dots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \dots, v_i, f)$ are laces of G_I . Moreover, since either μ_1 or μ_2 separates a and b , there exists an a, b -separator strictly included in S which is absurd.
- iii. Suppose that $p = 3$ and that v_0, v_1 et v_2 are all incident to a common face f . If we add a vertex f to G connected to the vertices v_0, v_1 and v_2 , the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a , the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

The sequence T is an ordered separator corresponding to S . \square

Conversely,

Lemma 12 *Every ordered separator of G is a minimal separator of G .*

PROOF. Let $S = (v_0, \dots, v_{p-1})$ be an ordered separator of G .

First, S is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases, all the vertices of S would be incident to a common face.

Let S' be a minimal separator included in S . By lemma 11, S' is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator S is a minimal separator. \square

From lemmata 11 and 12, we have the following proposition:

Proposition 13 *A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G .*

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \setminus O$).

Proposition 14 *Let $S = (v_0, \dots, v_{p-1})$ be an ordered $a, *$ -separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \dots, v_i)$, ($i < p - 1$) be an initial sequence of S .*

If there exists a face that is incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and v_j with $0 < j < i$, then $\mathcal{S}_a^{v_{i+1}}(S, O)$ is empty.

PROOF. Let b be such that S is an a, b -minimal separator and suppose that $y \in N(v_{i+1})$ and v_j with $0 < j < i$ are both incident to a face f . Since S is an ordered separator, there exists a cycle $(v_0, f_0, \dots, v_k, f_k)$ of G_I corresponding to a Jordan curve $\tilde{\mu}$. Let Σ_b be the connected component of $\Sigma \setminus \tilde{\mu}$ that contains b . Since y and v_j are incident to f , there exists a path (v_{i+1}, g, y, f, v_j) that corresponds to a curve $\tilde{\nu}$ that cuts Σ_b in two parts Σ_b^1 and Σ_b^2 whose borders are $\tilde{\mu}_1$ and $\tilde{\mu}_2$ respectively. Since $0 < j < i$, neither $V(\tilde{\mu}_1)$ nor $V(\tilde{\mu}_2)$ contains O .

Suppose that S' is an element of $\mathcal{S}_a^{v_{i+1}}(S, O)$ closest to a . Let c be such that S' is an a, c -minimal separator. The vertex c belongs to Σ_b . We may suppose that c belongs to Σ_b^1 . By proposition 4, S' is the neighbourhood of $C_c(S \cup N(v_{i+1}))$ i.e. $S' = V(\tilde{\mu}_1)$, but O is not a subset of S' which is absurd. \square

Conversely,

Proposition 15 *Let $S = (v_0, \dots, v_{p-1})$ be an ordered $a, *$ -separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \dots, v_i)$, ($i < p - 1$) be an initial sequence of S .*

If there is no face incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and v_j ($0 < j < i$), then there is an ordered separator in $S \cup N(v_{i+1}) \setminus C_a(S)$ that contains O .

PROOF. The neighbours (y_1, \dots, y_l) of v_{i+1} taken in clockwise order are such that y_i and y_{i+1} are incident to a common face. Moreover, since v_{i+1} and v_i are both incident to a face f_1 and since v_{i+1} and v_{i+2} are both incident to a face f_2 , there is a sequence $P = (v_i, x_1, \dots, x_k, v_0)$ such that there exists a

face incident to any two consecutive vertices of P and such that P uses only vertices of $N(v_{i+1}) \setminus C_a(S)$ and v_{i+2}, \dots, v_{p-1} . One such sequence is

$$(v_i, y_j, y_{j+1}, \dots, y_k, v_{i+2}, \dots, v_{p-1}, v_0).$$

Let P be such a sequence between v_i and v_0 of minimal length. Together with (v_1, \dots, v_{i-1}) , P forms an ordered separator of G as required. \square

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $\mathfrak{S}_a(S, O)$ with $O \subseteq S$.

Algorithm 2 calc3_aux

input:

G a 3-connected planar graph

a a vertex of G

$S = (v_0, \dots, v_{p-1})$ an ordered separator such that $a \notin S$

$O = (v_0, \dots, v_i)$ with $i \leq p - 1$ a subset of S

The vertices that have an incident face in common with v_l ($l \neq 0$) are tagged l unless they can be tagged j ($1 \leq j \leq l - 1$).

These vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged " $C_a(S)$ ".

output:

$\mathfrak{S}_a(S, O)$ the set of a, b -minimal separators S' further from a than S such that $C_a(S') \cap O = \emptyset$.

begin

if $i = p - 1$ **then**
 return $\{S\}$

else

$x \leftarrow v_{i+1}$

$\mathfrak{S} \leftarrow \text{calc3_aux}(G, a, S, (v_0, \dots, v_i, x))$

for each $y \in N(x)$ not tagged " $C_a(S)$ "

if y is tagged $j < i$ **then**

return \mathfrak{S}

for each S' in $\text{find_closest_elements}(G, a, x, S, O)$

 tag the vertices according to S'

$\mathfrak{S} \leftarrow \mathfrak{S} \cup \text{calc3_aux}(G, a, S', (v_0, \dots, v_i))$

end

Proposition 16 *The algorithm calc3_aux is correct. It computes the set*

$\mathcal{S}_a(S, O)$ of a 3-connected planar graph.

PROOF. The algorithm is an application of remark 2 and proposition 13, 14 and 15. \square

Proposition 17 *The algorithm can be implemented to compute the set $\mathcal{S}_a(S, O)$ in time $O(n|\mathcal{S}_a(S, O)|)$.*

PROOF. The algorithm `_calc3_aux` is a recursive version of the **for** loop below:

```

for  $l$  from  $i + 1$  to  $p - 1$ 
    empty  $\leftarrow$  FALSE
    for each  $y \in N(v_l)$  not tagged " $C_a(S)$ "
        if  $y$  is tagged  $j < l - 1$  then
            empty  $\leftarrow$  TRUE
    if not empty then
        for each  $S'$  in find_closest_elements( $G, a, v_l, S, (v_0, \dots, l - 1)$ )
            tag the vertices according to  $S'$ 
             $\mathcal{S} \leftarrow \mathcal{S} \cup \text{calc3\_aux}(G, a, S', (v_0, \dots, l))$ 
return( $\mathcal{S}$ )

```

For each minimal separator S , the algorithm performs the following operations:

- i. the function `find_closest_elements` produces S ;
- ii. the vertices of G are tagged;
- iii. the **for** loop is executed in the recursive call to `calc3_aux`
- iv. S is returned.

The function `find_closest_elements` can be implemented in linear time. Computing the neighbourhoods of the connected component of $G \setminus \{N(C) \cup C\}$ that contain O can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood S was a strict subset of some other neighbourhood S' then S' would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs $O(n + m)$.

The **for** loop tests the neighbours of v_l to check if they are forbidden. Since the vertex v_l is always different, this costs at most $O(m)$.

In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n - 6$, so the time spent on each minimal separator is $O(n)$, which gives an overall time complexity of $O(n|\mathcal{S}_a(S, O)|)$. \square

The following algorithm uses the function `calc3_aux` to compute the set of all minimal separators of a planar graph G .

Algorithm 3 `all_min_sep3`

input:

G a 3-connected planar graph

output:

the set of $a, *$ -minimal separators of G

begin

$\mathcal{S} \leftarrow \emptyset$

find $a \in V$ with $d(a) < 6$

for each minimal separator $S \subseteq N(a)$

$\mathcal{S} \leftarrow \mathcal{S} \cup \text{calc3_aux}(G, a, S, \emptyset)$

for each $y \in N(a)$

for each $a, *$ -minimal separator $S \subseteq N(y)$

$\mathcal{S} \leftarrow \mathcal{S} \cup \text{calc3_aux}(G, y, S, \emptyset)$

return(\mathcal{S})

end

Theorem 18 Algorithm `all_min_sep3` computes the set of the minimal separators of a 3-connected planar graph in time $O(n|\mathcal{S}(G)|)$

PROOF. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex a , $S \in \mathcal{S}(G)$ either belongs to \mathcal{S}_a or runs through a . In the second case, it is a $b, *$ -minimal separator for a neighbour b of a .

Moreover, there exists a vertex a of degree at most five in a planar graph. Let b_1, \dots, b_p be its neighbours.

By computing $\mathcal{S}_a \cup \left(\bigcup_{i \in [1..p]} \mathcal{S}_{b_i} \right)$, a minimal separator can be calculated no more than five times, which gives the claimed complexity. \square

5 Conclusion

This article confirms the feeling of Berry *et al.* [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to n times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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