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To cite this version:

HAL Id: hal-00351147
https://hal.science/hal-00351147
Submitted on 8 Jan 2009

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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender et al. conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry et al. [1] proposed an algorithm of running time $O(nm)$ per separator\(^1\) that uses the concept of generating new minimal separators.

\(^1\) The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].
separators from a previous minimal separator $S$ by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph $G$. In order to avoid the problem of recalculation, we define the set $S_a(S, O)$ of the $a, b$-minimal separators $S'$ for some $b$ such that the connected component of $a$ in $G \setminus S'$ contains the connected component of $a$ in $G \setminus S$ but avoids the set $O$. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets $A$ and $B$ are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a separator if $G \setminus S$ has at least two connected components, an $a, b$-separator if $a$ and $b$ are in different connected components of $G \setminus S$, an $a, b$-minimal separator if no proper subset of $S$ is an $a, b$-separator. The connected component of $a$ in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. For an $a, b$-minimal separator $S$, both $C_a(S)$ and $C_b(S)$ are full. A set $S$ is a minimal separator if there exist $a$ and $b$ such that $S$ is an $a, b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a, *$-minimal separator of a graph $G = (V, E)$ is an $a, b$-minimal separator of $G$ for some $b \in V$. The set of the $a, *$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

It is possible to order the $a, *$-minimal separators in the following way:

$S_1 \leq_a S_2$ if $C_a(S_1) \subseteq C_a(S_2)$.

The minimal separator $S_1$ is closer to $a$ than $S_2$. The set of $a, b$-minimal separators is a lattice for the relation $\leq_a[4]$ but we only need the following weaker lemma:

**Lemma 1** Let $C$ be a set of vertices of a graph $G$ inducing a connected subgraph of $G$, $a$ be a vertex of $C$ and $b$ be a vertex of $G \setminus (C \cup N(C))$. 


The neighbour $S$ of $C_b(C \cup N(C))$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S)$ that is closer to $a$ than any $a, b$-minimal separator $S'$ such that $C$ is a subset of $C_a(S')$.

**Proof.** By construction, $C$ is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since $S$ is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that $S$ is an $a, b$-minimal separator.

Let $S'$ be an $a, b$-minimal separator such that $C$ is a subset of $C_a(S')$. Let $p$ be a path in $C_b(S')$ with $b$ as one of its ends. The vertices of $S'$ are at least at distance 1 of $C$ so the vertices of $p$ are at least at distance 2 of $C$. Since $S$ is a subset of $N(C)$, $p \cap S = \emptyset$. In other words $p$ is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ i.e. $S$ is closer to $a$ than $S'$. □

For $S$ an $a, *$-minimal separator and $O \subseteq V$, the set $S_a(S, O)$ is the set of the $a, *$-minimal separators $S'$ further from $a$ than $S$ and such that $O \cap C_a(S') = \emptyset$. If $x \in V$, the set $S_a^x(S, O)$ is the set of $S' \in S_a(S, O)$ such that $x \in C_a(S')$.

**Remark 2** If $x \in S$, then $S_a(S, O)$ is the disjoint union

$$S_a(S, O) \cup S_a^x(S, O).$$

More precisely, if $(S_i)_{i \in I}$ are the elements of $S_a^x(S, O)$ closest to $a$, then

$$S_a(S, O) = S_a(S, O \cup \{x\}) \cup \left( \bigcup_{i \in I} S_a(S_i, O) \right).$$

This gives the skeleton of an algorithm to compute the set $S_a(S, O)$.

**Remark 3** If $S$ belongs to $S_a^x(S, O)$, then $S_a^x(S, O) = S_a(S, O)$.

The algorithm is based on remarks 2 and 3. To list $S_a$, the algorithm computes the sets $S_a(S, \emptyset)$ for every $S$ closest to $a$ in $S_a$. During this calculation, it computes $S_a(S, O)$ with $O \subseteq S$. To do so, it chooses $x$ in $S \setminus O$ and calculates $S_a^x(S, O)$ and $S_a(S, O \cup \{x\})$. The set $S_a^x(S, O)$ is itself a union of $S_a(S_i, O)$. But to obtain such a decomposition, one needs to find the elements of $S_a^x(S, O)$ closest to $a$, which the following proposition does.

**Proposition 4** Let $G = (V, E)$ be a graph, $S$ an $a, *$-minimal separator, $O \subseteq S$, $x \in S \setminus O$ and $C = C_a(S) \cup \{x\}$

The elements of $S_a^x(S, O)$ closest to $a$ are exactly the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ that contain $O$ and that are maximal for inclusion.
PROOF. Let $S_1$ be an $a, b$-minimal separator of $S^*_a(S, O)$ closest to $a$. Let $S'$ be the neighbourhood of $C_b(N(C) \cup C)$. By lemma 1, $S'$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S')$ and $S'$ is closer to $a$ than $S_1$. Moreover, since $C_a(S_1) \cap O = \emptyset$ and $S'$ is closer to $a$ than $S_1$, $C_a(S') \cap O \subseteq C_a(S_1) \cap O = \emptyset$. Thus $S'$ belongs to $S^*_a(S, O)$ and is closer to $a$ than $S_1$. This proves that $S_1 = S'$. Since $S_1$ cannot be a subset of another element of $S^*_a(S, O)$, $S_1$ is the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ which is maximal for inclusion.

Conversely, let $S_1$ be a neighbourhood of a connected component $D$ of $G \setminus \{N(C) \cup C\}$ that contains $O$ and that is maximal for inclusion. By lemma 1, $S_1$ is an element of $S^*_a(S, O)$ that is closer to $a$ than any $a, b$-minimal separator of $S^*_a(S, O)$ with $b$ in $D$. So if $S_2$ is an $a, b$-minimal separator of $S^*_a(S, O)$ strictly closer to $a$ than $S_1$, $S_1$ is not an $a, b$-minimal separator. Suppose for a contradiction that such an $a, b$-minimal separator exists. It follows from the first part of the proof that such an $a, b$-minimal separator $S_2$ closest to $a$ is the neighbourhood of $C_b(N(C) \cup C)$. The set $S_2$ is an element of $S^*_a(S, O)$ that is closer to $a$ than $S_1$ and $S_1$ is a subset of $S_2$ (because $S_1 \setminus S_2 \subseteq C_a(S_2) \setminus C_a(S_1)$) and $S_2$ is closer to $a$ then $S_1$ and therefore $S_1$ is a strict subset of $S_2$ contradicting the fact that $S_1$ is maximal for inclusion. □

Proposition 4 gives us a way to find the minimal elements of $S^*_a(S, O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ and then choosing among the minimal separators found the ones that contain $O$ and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $S_a(S, O)$ that may look like:

Algorithm 1 _calc3_

begin
  if $S \setminus O = \emptyset$ then
    return($\{S\}$)
  else
    let $x \in S \setminus O$
    $S \leftarrow _calc3_\_a(G, a, S, O \cup \{x\})$

    for each $S_i$ in find_closest_elements$(G, a, x, S, O)$
      $S \leftarrow S \cup _calc3_\_a(G, a, S_i, O)$
    return($S$)
end

However several problems need to be solved.
i. We do not know whether the sets $S_a(S, O)$ are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.

ii. To implement the function $\text{find\ closest\ elements}$, proposition 4 states that we can start with a graph search of $G$. But if $S_a(S, O) = \{ S \}$, the recursive calls to the algorithm will try to find an element of $S_a^x(S, O)$ closest to $a$ for every $x \in S \setminus O$. Each call to $\text{find\ min\ elements}$ costs at least $O(m)$ and finally, we would have spent at least $O(nm)$ to realise that $S_a(S, O) = \{ S \}$.

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, i.e. if $S_1$ and $S_2$ are two minimal elements of $S_a(S, O)$, the sets $S_a(S_1, O)$ and $S_a(S_2, O)$ are disjoint. Section 3.3 then shows how to determine whether $S_a^x(S, O)$ is empty or not in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let $\Sigma$ be the plane. A plane graph $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of $\Sigma$ between two vertices of $V_\Sigma$ in such a way that the interiors of two distinct edges do not meet. We will denote by $\tilde{G}_\Sigma$ the drawing of $G_\Sigma$. A planar graph is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of $G_\Sigma$ is a connected component of $\Sigma \setminus \tilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Proposition 5 In a 3-connected planar graph, minimal separators are minimal for inclusion.

PROOF. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let $a, b, c$ and $d$ be vertices such that $S'$ is an $a, b$-minimal separator and $S$ is a $c,d$-minimal separator. Since $S$ is not an $a, b$-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. In this case, $C_c(S)$ and $N(C_c(S))$ are respectively equal to $C_c(S')$ and $S$. 
But then $G$ admits $K_{3,3}$ as a minor for if we contract $C_a(S'), C_b(S')$ and $C_c(S')$ into the vertices $a', b'$ and $c'$, all these vertices have $S$ in their neighbourhood and since $G$ is 3-connected, $|S| \geq 3$. This contradicts the fact that $G$ is planar. □

**Proposition 6** Let $G = (V, E)$ be a 3-connected planar graph, $a$ a vertex of $G$, $S$ an $a, *$-minimal separator, $O$ a subset of $S$ and $x$ a vertex of $S \setminus O$.

If $S_1$ and $S_2$ are two distinct elements of $S^x_a(S, O)$ that are closest to $a$, then $S_a(S_1, O) \cap S_a(S_2, O) = \emptyset$.

**PROOF.** Let $C = C_a(S) \cup \{x\}$ and suppose for a contradiction that $S_3$ is a minimal separator of $S_a(S_1, O) \cap S_a(S_2, O)$ with $S_1$ and $S_2$ two distinct elements of $S^x_a(S, O)$ closest to $a$. Let $b$ be a vertex such that $S_3$ is an $a, b$-minimal separator.

Since $S_3$ is further from $a$ than $S_1$ and $S_2$, both $S_1$ and $S_2$ are $a, b$-separators. There exists an $a, b$-minimal separator $S'$ included in $S_1$. By proposition 5, a minimal separator of $G$ is minimal for inclusion which proves that $S_1 = S'$ and $S_1$ is an $a, b$-minimal separator. By lemma 1, the neighbourhood $S_1$ of $C_b(N(C) \cup C)$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S_1)$ that is closer to $a$ than $S_1$. So $C_a(S_1) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and $S_4$ is an element of $S^x_a(S, O)$ that is closer to $a$ than $S_1$. Similarly, $S_2$ is an $a, b$-minimal separator and $S_4$ is closer to $a$ than $S_2$ which contradicts the fact that $S_1$ and $S_2$ are two distinct elements of $S^x_a(S, O)$ closest to $a$. □

### 3.2 The intermediate graph

**Definition 7** Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let $F$ be the set of its faces. In each face $f \in F$ pick up one point $v_f$. Let $R_F$ be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex $v$ is incident to the face $f$.

For $G'$ a subgraph of $G_I$, the set $G' \cap V_\Sigma$ will be denoted by $V(G')$.

**Proposition 8** Let $\mu$ be a cycle of $G_I$ such that the curve $\mu$ separates at least two vertices $a$ and $b$ of $V_\Sigma$.

The set $V(\mu)$ is an $a, b$-separator of $G_\Sigma$.

**PROOF.** Let $p$ be a path in $G_\Sigma$ from $a$ to $b$. Since $a$ and $b$ are not in the
Proposition 9  Let $S$ be an $a, b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve defined by $\mu$ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

PROOF. Let $C$ be the connected component of $a$ in $G\setminus S$. Let us contract $C$ into a supervertex $v_C$ to build the graph $G_{/C}$. There is a cycle $\mu_{/C}$ in $(G_{/C})_I$ such that $V(\mu_{/C})$ is the neighbourhood of $v_C$ in $G_{/C}$. Therefore, the neighbourhood of $C$ in $G_I$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates $a$ and $b$. However, proposition 8 shows that $V(\mu')$ which is a strict subset of $S$ is an $a, b$-separator. This contradicts the fact that $S$ is a $a, b$-minimal separator. □

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the $G_I$, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

3.3 Ordered separators

Definition 10  An ordered separator of $G$ is a sequence of distinct vertices

$$(v_0, \ldots, v_{p-1}), \ (p > 2)$$

such that

i. there exists a face to which $v_i$ and $v_{i+1[p]}$ are incident;
ii. $v_i$ and $v_j$ are incident to a common face only if $i = j+1[p]$ or $j = i+1[p]$;
iii. if $p = 3$, no face is incident to $v_i$, $v_{i+1[p]}$ and $v_{i+2[p]}$.

The notation $i[p]$ means $i$ modulo $p$.

A set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.
If $S = (v_0, \ldots, v_{p-1})$ is an ordered separator of $G$, then $S$ is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

**Lemma 11** Every minimal separator $S$ of $G$ is ordered.

**PROOF.** Let $S$ be an $a, b$-minimal separator of $G$.

Proposition 9 states that there exists a simple cycle of $G_I$

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to $S$.

i. The construction of $T$ ensures that $v_i$ and $v_{i+1}$ are incident to a common face ($f_i$).

ii. Suppose that $v_i$ et $v_j$ are incident to a common face $f$ and that $i + 1 \neq j$ and $j + 1 \neq i$.

$$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$$

and

$$\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$$

are laces of $G_I$. Moreover, since either $\mu_1$ or $\mu_2$ separates $a$ and $b$, there exists an $a, b$-separator strictly included in $S$ which is absurd.

iii. Suppose that $p = 3$ and that $v_0$, $v_1$ et $v_2$ are all incident to a common face $f$. If we add a vertex $f$ to $G$ connected to the vertices $v_0$, $v_1$ and $v_2$, the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of $a$, the connected component of $b$ and the vertex $f$ are all incident to $v_0$, $v_1$ and $v_2$ which builds up a $K_{3,3}$.

The sequence $T$ is an ordered separator corresponding to $S$.  

Conversely,

**Lemma 12** Every ordered separator of $G$ is a minimal separator of $G$.

**PROOF.** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of $G$.

First, $S$ is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases, all the vertices of $S$ would be incident to a common face.

Let $S'$ be a minimal separator included in $S$. By lemma 11, $S'$ is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator $S$ is a minimal separator.  

From lemmata 11 and 12, we have the following proposition:

**Proposition 13** A set \( S \subseteq V \) is a minimal separator of a 3-connected planar graph \( G = (V, E) \) if and only if it corresponds to an ordered separator of \( G \).

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether \( S_a(S, O) \) is empty or not \((O \subseteq S \text{ and } x \in S \setminus O)\).

**Proposition 14** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered \( a, \ast \)-separator of a 3-connected planar graph \( G = (V, E) \) and \( O = (v_0, \ldots, v_i), (i < p - 1) \) be an initial sequence of \( S \).

If there exists a face that is incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) with \( 0 < j < i \), then \( S_{a+1}(S, O) \) is empty.

**PROOF.** Let \( b \) be such that \( S \) is an \( a, b \)-minimal separator and suppose that \( y \in N(v_{i+1}) \) and \( v_j \) with \( 0 < j < i \) are both incident to a face \( f \). Since \( S \) is an ordered separator, there exists a cycle \( (v_0, f_1, \ldots, v_k, f_k) \) of \( G_f \) corresponding to a Jordan curve \( \tilde{\mu} \). Let \( \Sigma_b \) be the connected component of \( \Sigma \setminus \tilde{\mu} \) that contains \( b \). Since \( y \) and \( v_j \) are incident to \( f \), there exists a path \( (v_{i+1}, y, f, v_j) \) that corresponds to a curve \( \tilde{\nu} \) that cuts \( \Sigma_b \) in two parts \( \Sigma_1^b \) and \( \Sigma_2^b \) whose borders are \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) respectively. Since \( 0 < j < i \), neither \( V(\tilde{\mu}_1) \) nor \( V(\tilde{\mu}_2) \) contains \( O \).

Suppose that \( S' \) is an element of \( S_{a+1}(S, O) \) closest to \( a \). Let \( c \) be such that \( S' \) is an \( a, c \)-minimal separator. The vertex \( c \) belongs to \( \Sigma_b \). We may suppose that \( c \) belongs to \( \Sigma_1^b \). By proposition 4, \( S' \) is the neighbourhood of \( C_c(S \cup N(v_{i+1})) \) i.e. \( S' = V(\tilde{\mu}_1) \), but \( O \) is not a subset of \( S' \) which is absurd. \( \square \)

Conversely,

**Proposition 15** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered \( a, \ast \)-separator of a 3-connected planar graph \( G = (V, E) \) and \( O = (v_0, \ldots, v_i), (i < p - 1) \) be an initial sequence of \( S \).

If there is no face incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) \((0 < j < i)\), then there is an ordered separator in \( S \cup N(v_{i+1}) \setminus C_a(S) \) that contains \( O \).

**PROOF.** The neighbours \((y_1, \ldots, y_l)\) of \( v_{i+1} \) taken in clockwise order are such that \( y_i \) and \( y_{i+1} \) are incident to a common face. Moreover, since \( v_{i+1} \) and \( v_i \) are both incident to a face \( f_1 \) and since \( v_{i+1} \) and \( v_{i+2} \) are both incident to a face \( f_2 \), there is a sequence \( P = (v_i, x_1, \ldots, x_k, v_0) \) such that there exists a
face incident to any two consecutive vertices of $P$ and such that $P$ uses only vertices of $N(v_{i+1}) \setminus C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is $(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0)$.

Let $P$ be such a sequence between $v_i$ and $v_0$ of minimal length. Together with $(v_1, \ldots, v_{i-1})$, $P$ forms an ordered separator of $G$ as required.  

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $S_a(S,O)$ with $O \subseteq S$.

Algorithm 2 calc3_aux

input:
$G$ a 3-connected planar graph
$a$ a vertex of $G$
$S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
$O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of $S$

The vertices that have an incident face in common with $v_l$ ($l \neq 0$) are tagged $l$ unless they can be tagged $j$ ($1 \leq j \leq l-1$).
The vertices are the forbidden vertices.
The vertices of $C_a(S)$ are also tagged “$C_a(S)$”.

output:
$S_a(S,O)$ the set of $a,b$-minimal separators $S'$ further from $a$ than $S$ such that $C_a(S') \cap O = \emptyset$.

begin
if $i = p-1$ then
    return($\{S\}$)
else
    $x \leftarrow v_{i+1}$
    $S \leftarrow$ calc3_aux($G, a, S, (v_0, \ldots, v_i, x)$)

    for each $y \in N(x)$ not tagged “$C_a(S)$”
        if $y$ is tagged $j < i$ then
            return($S$)
    for each $S'$ in find_closest_elements($G, a, x, S, O$)
        tag the vertices according to $S'$
        $S \leftarrow S \cup$ calc3_aux($G, a, S', (v_0, \ldots, v_i)$)
end

Proposition 16 The algorithm calc3_aux is correct. It computes the set
$S_a(S, O)$ of a 3-connected planar graph.

**PROOF.** The algorithm is an application of remark 2 and proposition 13, 14 and 15. □

**Proposition 17** The algorithm can be implemented to compute the set $S_a(S, O)$ in time $O(n|S_a(S, O)|)$.

**PROOF.** The algorithm \_calc\_3\_aux is a recursive version of the for loop below:

```plaintext
for l from i + 1 to p - 1
    empty ← FALSE
    for each y ∈ N(v_l) not tagged "C_a(S)"
        if y is tagged $j < l - 1$ then
            empty ← TRUE
        if not empty then
            for each $S'$ in find\_closest\_elements($G, a, v_l, S, (v_0, \ldots, l - 1))
                tag the vertices according to $S'$
            $S ← S \cup \text{calc3\_aux}(G, a, S', (v_0, \ldots, l))$

return($S$)
```

For each minimal separator $S$, the algorithm performs the following operations:

i. the function find\_closest\_elements produces $S$;
ii. the vertices of $G$ are tagged;
iii. the for loop is executed in the recursive call to calc3\_aux
iv. $S$ is returned.

The function find\_closest\_elements can be implemented in linear time. Computing the neighbourhoods of the connected component of $G \\backslash \{N(C) \cup C\}$ that contain $O$ can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood $S$ was a strict subset of some other neighbourhood $S'$ then $S'$ would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs $O(n + m)$.

The for loop tests the neighbours of $v_l$ to check if they are forbidden. Since the vertex $v_l$ is always different, this costs at most $O(m)$. 

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In a planar graph, the number $m$ of edges satisfies $0 \leq m \leq 3n - 6$, so the time spent on each minimal separator is $O(n)$, which gives an overall time complexity of $O(n|S_a(S,O)|)$.  

The following algorithm uses the function `calc3_aux` to compute the set of all minimal separators of a planar graph $G$.

**Algorithm 3 all_min_sep3**

**input:**

$G$ a 3-connected planar graph  

**output:**

the set of $a, \ast$-minimal separators of $G$

**begin**  

$S \leftarrow \emptyset$  

**find** $a \in V$ with $d(a) < 6$  

**for each** minimal separator $S \subseteq N(a)$  

$S \leftarrow S \cup \text{calc3_aux}(G, a, S, \emptyset)$

**for each** $y \in N(a)$  

**for each** $a, \ast$-minimal separator $S \subseteq N(y)$  

$S \leftarrow S \cup \text{calc3_aux}(G, y, S, \emptyset)$  

**return**($S$)

**end**

**Theorem 18** Algorithm all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time $O(n|S(G)|)$

**PROOF.** Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex $a$, $S \in S(G)$ either belongs to $S_a$ or runs through $a$. In the second case, it is a $b, \ast$-minimal separator for a neighbour $b$ of $a$.

Moreover, there exists a vertex $a$ of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $S_a \cup (\bigcup_{c \in \{1, \ldots, p\}} S_{b_c})$, a minimal separator can be calculated no more than five times, which gives the claimed complexity.  

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5 Conclusion

This article confirms the feeling of Berry et al. [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to \( n \) times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

Acknowledgement

We thank Vincent Bouchitté and Ioan Todinca for the fruitful discussions we have had on this topic.

References


