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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender et al. conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry et al. [1] proposed an algorithm of running time $O(nm)$ per separator\footnote{The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].} that uses the concept of generating new minimal
separators from a previous minimal separator $S$ by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph $G$. In order to avoid the problem of recalculation, we define the set $S_a(S, O)$ of the $a, b$-minimal separators $S'$ for some $b$ such that the connected component of $a$ in $G \setminus S'$ contains the connected component of $a$ in $G \setminus S$ but avoids the set $O$. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

## 2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets $A$ and $B$ are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a separator if $G \setminus S$ has at least two connected components, an $a, b$-separator if $a$ and $b$ are in different connected components of $G \setminus S$, an $a, b$-minimal separator if no proper subset of $S$ is an $a, b$-separator. The connected component of $a$ in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. For an $a, b$-minimal separator $S$, both $C_a(S)$ and $C_b(S)$ are full. A set $S$ is a minimal separator if there exist $a$ and $b$ such that $S$ is an $a, b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a, \ast$-minimal separator of a graph $G = (V, E)$ is an $a, b$-minimal separator of $G$ for some $b \in V$. The set of the $a, \ast$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

It is possible to order the $a, \ast$-minimal separators in the following way:

$$S_1 \preceq a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator $S_1$ is closer to $a$ than $S_2$. The set of $a, b$-minimal separators is a lattice for the relation $\preceq a \mid 4]$ but we only need the following weaker lemma:

**Lemma 1** Let $C$ be a set of vertices of a graph $G$ inducing a connected subgraph of $G$, $a$ be a vertex of $C$ and $b$ be a vertex of $G \setminus (C \cup N(C))$. 

2
The neighbour $S$ of $C_b\left(C \cup N(C)\right)$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S)$ that is closer to $a$ than any $a, b$-minimal separator $S'$ such that $C$ is a subset of $C_a(S')$.

**Proof.** By construction, $C$ is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since $S$ is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that $S$ is an $a, b$-minimal separator.

Let $S'$ be an $a, b$-minimal separator such that $C$ is a subset of $C_a(S')$. Let $p$ be a path in $C_b(S')$ with $b$ as one of its ends. The vertices of $S'$ are at least at distance 1 of $C$ so the vertices of $p$ are at least at distance 2 of $C$. Since $S$ is a subset of $N(C)$, $p \cap S = \emptyset$. In other words $p$ is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ i.e. $S$ is closer to $a$ than $S'$. $\square$

For $S$ an $a, *$-minimal separator and $O \subseteq V$, the set $S_a(S, O)$ is the set of the $a, *$-minimal separators $S'$ further from $a$ than $S$ and such that $O \cap C_a(S') = \emptyset$. If $x \in V$, the set $S^x_a(S, O)$ is the set of $S' \in S_a(S, O)$ such that $x \in C_a(S')$.

**Remark 2** If $x \in S$, then $S_a(S, O)$ is the disjoint union
$$S_a\left(S, O \cup \{x\}\right) \bigcup S^x_a(S, O).$$

More precisely, if $(S_i)_{i \in I}$ are the elements of $S^x_a(S, O)$ closest to $a$, then
$$S_a(S, O) = S_a\left(S, O \cup \{x\}\right) \bigcup \left( \bigcup_{i \in I} S_a(S_i, O) \right).$$

This gives the skeleton of an algorithm to compute the set $S_a(S, O)$.

**Remark 3** If $S$ belongs to $S^x_a(S, O)$, then $S^x_a(S, O) = S_a(S, O)$.

The algorithm is based on remarks 2 and 3. To list $S_a$, the algorithm computes the sets $S_a(S, \emptyset)$ for every $S$ closest to $a$ in $S_a$. During this calculation, it computes $S_a(S, O)$ with $O \subseteq S$. To do so, it chooses $x$ in $S \setminus O$ and calculates $S^x_a(S, O)$ and $S_a(S, O \cup \{x\})$. The set $S^x_a(S, O)$ is itself a union of $S_a(S_i, O)$.

But to obtain such a decomposition, one needs to find the elements of $S^x_a(S, O)$ closest to $a$, which the following proposition does.

**Proposition 4** Let $G = (V, E)$ be a graph, $S$ an $a, *$-minimal separator, $O \subseteq S$, $x \in S \setminus O$ and $C = C_a(S) \cup \{x\}$

The elements of $S^x_a(S, O)$ closest to $a$ are exactly the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ that contain $O$ and that are maximal for inclusion.
PROOF. Let $S_1$ be an $a, b$-minimal separator of $S_a^r(S, O)$ closest to $a$. Let $S'$ be the neighbourhood of $C_b(N(C) \cup C)$. By lemma 1, $S'$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S')$ and $S'$ is closer to $a$ than $S_1$. Moreover, since $C_a(S_1) \cap O = \emptyset$ and $S'$ is closer to $a$ than $S_1$, $C_a(S') \cap O \subseteq C_a(S_1) \cap O = \emptyset$. Thus $S'$ belongs to $S_a^r(S, O)$ and is closer to $a$ than $S_1$. This proves that $S_1 = S'$. Since $S_1$ cannot be a subset of another element of $S_a^r(S, O)$, $S_1$ is the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ which is maximal for inclusion.

Conversely, let $S_1$ be a neighbourhood of a connected component $D$ of $G \setminus \{N(C) \cup C\}$ that contains $O$ and that is maximal for inclusion. By lemma 1, $S_1$ is an element of $S_a^r(S, O)$ that is closer to $a$ than any $a, b$-minimal separator of $S_a^r(S, O)$ with $b$ in $D$. So if $S_2$ is an $a, b$-minimal separator of $S_a^r(S, O)$ strictly closer to $a$ than $S_1$, $S_1$ is not an $a, b$-minimal separator. Suppose for a contradiction that such an $a, b$-minimal separator exists. It follows from the first part of the proof that such an $a, b$-minimal separator $S_2$ closest to $a$ is the neighbourhood of $C_b(N(C) \cup C)$. The set $S_2$ is an element of $S_a^r(S, O)$ that is closer to $a$ than $S_1$ and $S_1$ is a subset of $S_2$ (because $S_1 \setminus S_2 \subseteq C_a(S_2) \setminus C_a(S_1)$) and $S_2$ is closer to $a$ then $S_1$ and therefore $S_1$ is a strict subset of $S_2$ contradicting the fact that $S_1$ is maximal for inclusion. 

Proposition 4 gives us a way to find the minimal elements of $S_a^r(S, O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ and then choosing among the minimal separators found the ones that contain $O$ and that are maximal by inclusion.

Using the skeleton of remark 2, we can construct an algorithm to compute the set $S_a(S, O)$ that may look like:

**Algorithm 1 calc3.**

begin
    if $S \setminus O = \emptyset$ then
        return ({$S$})
    else
        let $x \in S \setminus O$
        $S \leftarrow$ _calc3_($G$, $a$, $S$, $O \cup \{x\}$)
    for each $S_i$ in _find_closest_elements_($G$, $a$, $x$, $S$, $O$)
        $S \leftarrow S \cup$ _calc3_($G$, $a$, $S_i$, $O$)
    return ($S$)
end

However several problems need to be solved.
i. We do not know whether the sets \( S_a(S_i, O) \) are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.

ii. To implement the function \texttt{find closest elements}, proposition 4 states that we can start with a graph search of \( G \).

But if \( S_a(S, O) = \{ S \} \), the recursive calls to the algorithm will try to find an element of \( S_a^x(S, O) \) closest to \( a \) for every \( x \in S \setminus O \). Each call to \texttt{find min elements} costs at least \( O(m) \) and finally, we would have spent at least \( O(nm) \) to realise that \( S_a(S, O) = \{ S \} \).

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, i.e. if \( S_1 \) and \( S_2 \) are two minimal elements of \( S_a^x(S, O) \), the sets \( S_a(S_1, O) \) and \( S_a(S_2, O) \) are disjoint. Section 3.3 then shows how to determine whether \( S_a^x(S, O) \) is empty or not in an overall \( O(n) \).

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let \( \Sigma \) be the plane. A \textit{plane graph} \( G_{\Sigma} = (V_{\Sigma}, E_{\Sigma}) \) is a graph drawn on the plane, that is \( V_{\Sigma} \subset \Sigma \) and each \( e \in E_{\Sigma} \) is a simple curve of \( \Sigma \) between two vertices of \( V_{\Sigma} \) in such a way that the interiors of two distinct edges do not meet. We will denote by \( \tilde{G}_{\Sigma} \) the drawing of \( G_{\Sigma} \). A \textit{planar graph} is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A \textit{face} of \( G_{\Sigma} \) is a connected component of \( \Sigma \setminus \tilde{G}_{\Sigma} \).

3.1 Minimal separators of 3-connected planar graphs

**Proposition 5** In a 3-connected planar graph, minimal separators are minimal for inclusion.

**PROOF.** Suppose that \( S \subset S' \) are two minimal separators of a 3-connected planar graph.

Let \( a, b, c \) and \( d \) be vertices such that \( S' \) is an \( a, b \)-minimal separator and \( S \) is a \( c,d \)-minimal separator. Since \( S \) is not an \( a, b \)-minimal separator, either \( C_c(S') \) or \( C_d(S') \) is disjoint with \( C_a(S') \) and \( C_b(S') \). Suppose that \( C_c(S') \) is such a component. In this case, \( C_c(S) \) and \( N(C_c(S)) \) are respectively equal to \( C_c(S') \) and \( S \).
But then $G$ admits $K_{3,3}$ as a minor if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices $a'$, $b'$ and $c'$, all these vertices have $S$ in their neighbourhood and since $G$ is 3-connected, $|S| \geq 3$. This contradicts the fact that $G$ is planar. □

**Proposition 6** Let $G = (V, E)$ be a 3-connected planar graph, $a$ a vertex of $G$, $S$ an $a, *$-minimal separator, $O$ a subset of $S$ and $x$ a vertex of $S \setminus O$.

If $S_1$ and $S_2$ are two distinct elements of $S_a^x(S,O)$ that are closest to $a$, then

$S_a(S_1, O) \cap S_a(S_2, O) = \emptyset$.

**PROOF.** Let $C$ be $C_a(S) \cup \{x\}$ and suppose for a contradiction that $S_3$ is a minimal separator of $S_a(S_1, O) \cap S_a(S_2, O)$ with $S_1$ and $S_2$ two distinct elements of $S_a^x(S,O)$ closest to $a$. Let $b$ be a vertex such that $S_3$ is an $a, b$-minimal separator. Since $S_3$ is further from $a$ than $S_1$ and $S_2$, both $S_1$ and $S_2$ are $a, b$-separators. There exists an $a, b$-minimal separator $S'$ included in $S_1$. By proposition 5, a minimal separator of $G$ is minimal for inclusion which proves that $S_1 = S'$ and $S_1$ is an $a, b$-minimal separator. By lemma 1, the neighbourhood $S_4$ of $C_b(N(C) \cup C)$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S_4)$ that is closer to $a$ than $S_1$. So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and $S_4$ is an element of $S_a^x(S,O)$ that is closer to $a$ than $S_1$. Similarly, $S_2$ is an $a, b$-minimal separator and $S_4$ is closer to $a$ than $S_2$ which contradicts the fact that $S_1$ and $S_2$ are two distinct elements of $S_a^x(S,O)$ closest to $a$. □

### 3.2 The intermediate graph

**Definition 7** Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let $F$ be the set of its faces. In each face $f \in F$ pick up one point $v_f$. Let $R_F$ be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_1 = (V_1, E_1)$ is a plane graph whose vertex set is $V_1 = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex $v$ is incident to the face $f$.

For $G'$ a subgraph of $G_1$, the set $G' \cap V_\Sigma$ will be denoted by $V(G')$.

**Proposition 8** Let $\mu$ be a cycle of $G_1$ such that the curve $\mu$ separates at least two vertices $a$ and $b$ of $V_\Sigma$.

The set $V(\mu)$ is an $a, b$-separator of $G_\Sigma$.

**PROOF.** Let $p$ be a path in $G_\Sigma$ from $a$ to $b$. Since $a$ and $b$ are not in the
same connected component of $\Sigma \setminus \tilde{\mu}$, $\tilde{p}$ intersect $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_{\Sigma}$. This implies that every path from $a$ to $b$ meets $V(\mu)$ and so $V(\mu)$ is an $a, b$-separator. □

**Proposition 9** Let $S$ be an $a, b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve defined by $\mu$ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

**Proof.** Let $C$ be the connected component of $a$ in $G \setminus S$. Let us contract $C$ into a supervertex $v_C$ to build the graph $G_{/C}$. There is a cycle $\mu_{/C}$ in $(G_{/C})_I$ such that $V(\mu_{/C})$ is the neighbourhood of $v_C$ in $G_{/C}$. Therefore, the neighbourhood of $C$ in $G_I$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates $a$ and $b$. However, proposition 8 shows that $V(\mu')$ which is a strict subset of $S$ is an $a, b$-separator. This contradicts the fact that $S$ is a $a, b$-minimal separator. □

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the $G_I$, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

### 3.3 Ordered separators

**Definition 10** An ordered separator of $G$ is a sequence of distinct vertices $(v_0, \ldots, v_{p-1})$, $(p > 2)$ such that

- $i$. there exists a face to which $v_i$ and $v_{i+1[p]}$ are incident;
- $ii$. $v_i$ and $v_j$ are incident to a common face only if $i = j+1[p]$ or $j = i+1[p]$;
- $iii$. if $p = 3$, no face is incident to $v_i$, $v_{i+1[p]}$ and $v_{i+2[p]}$.

The notation $i[p]$ means $i$ modulo $p$.

A set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.
If $S = (v_0, \ldots, v_{p-1})$ is an ordered separator of $G$, then $S$ is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

**Lemma 11** Every minimal separator $S$ of $G$ is ordered.

**PROOF.** Let $S$ be an $a, b$-minimal separator of $G$.

Proposition 9 states that there exists a simple cycle of $G_I$

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to $S$.

i. The construction of $T$ ensures that $v_i$ and $v_{i+1}$ are incident to a common face ($f_i$).

ii. Suppose that $v_i$ et $v_j$ are incident to a common face $f$ and that $i + 1 \not= j [p]$ and $j + 1 \not= i [p]$.

$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$

are laces of $G_I$. Moreover, since either $\mu_1$ or $\mu_2$ separates $a$ and $b$, there exists an $a, b$-separator strictly included in $S$ which is absurd.

iii. Suppose that $p = 3$ and that $v_0$, $v_1$ et $v_2$ are all incident to a common face $f$. If we add a vertex $f$ to $G$ connected to the vertices $v_0$, $v_1$ and $v_2$, the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of $a$, the connected component of $b$ and the vertex $f$ are all incident to $v_0$, $v_1$ and $v_2$ which builds up a $K_{3,3}$.

The sequence $T$ is an ordered separator corresponding to $S$. □

Conversely,

**Lemma 12** Every ordered separator of $G$ is a minimal separator of $G$.

**PROOF.** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of $G$.

First, $S$ is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases, all the vertices of $S$ would be incident to a common face.

Let $S'$ be a minimal separator included in $S$. By lemma 11, $S'$ is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator $S$ is a minimal separator. □
From lemmata 11 and 12, we have the following proposition:

**Proposition 13** A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of $G$.

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $S_{a+i}(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \setminus O$).

**Proposition 14** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, \ast$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of $S$.

If there exists a face that is incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and $v_j$ with $0 < j < i$, then $S_{a+i}(S, O)$ is empty.

**Proof.** Let $b$ be such that $S$ is an $a, b$-minimal separator and suppose that $y \in N(v_{i+1})$ and $v_j$ with $0 < j < i$ are both incident to a face $f$. Since $S$ is an ordered separator, there exists a cycle $(v_0, f_0, \ldots, v_k, f_k)$ of $G$ corresponding to a Jordan curve $\tilde{\mu}$. Let $\Sigma_b$ be the connected component of $\Sigma \setminus \tilde{\mu}$ that contains $b$. Since $y$ and $v_j$ are incident to $f$, there exists a path $(v_{i+1}, g, y, f, v_j)$ that corresponds to a curve $\tilde{\nu}$ that cuts $\Sigma_b$ in two parts $\Sigma_1^b$ and $\Sigma_2^b$ whose borders are $\tilde{\mu}_1$ and $\tilde{\mu}_2$ respectively. Since $0 < j < i$, neither $V(\tilde{\mu}_1)$ nor $V(\tilde{\mu}_2)$ contains $O$.

Suppose that $S'$ is an element of $S_{a+i}(S, O)$ closest to $a$. Let $c$ be such that $S'$ is an $a, c$-minimal separator. The vertex $c$ belongs to $\Sigma_b$. We may suppose that $c$ belongs to $\Sigma_1^b$. By proposition 4, $S'$ is the neighbourhood of $C_c(S \cup N(v_{i+1}))$ i.e. $S' = V(\tilde{\mu}_1)$, but $O$ is not a subset of $S'$ which is absurd. □

Conversely,

**Proposition 15** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, \ast$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of $S$.

If there is no face incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and $v_j$ ($0 < j < i$), then there is an ordered separator in $S \cup N(v_{i+1}) \setminus C_a(S)$ that contains $O$.

**Proof.** The neighbours $(y_1, \ldots, y_l)$ of $v_{i+1}$ taken in clockwise order are such that $y_i$ and $y_{i+1}$ are incident to a common face. Moreover, since $v_{i+1}$ and $v_i$ are both incident to a face $f_1$ and since $v_{i+1}$ and $v_{i+2}$ are both incident to a face $f_2$, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a
face incident to any two consecutive vertices of $P$ and such that $P$ uses only vertices of $N(v_{i+1}) \setminus C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is $(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0)$.

Let $P$ be such a sequence between $v_i$ and $v_0$ of minimal length. Together with $(v_1, \ldots, v_{i-1})$, $P$ forms an ordered separator of $G$ as required. □

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $S_a(S, O)$ with $O \subseteq S$.

Algorithm 2 calc3_aux

input:
- $G$ a 3-connected planar graph
- $a$ a vertex of $G$
- $S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
- $O = (v_0, \ldots, v_i)$ with $i \leq p - 1$ a subset of $S$

The vertices that have an incident face in common with $v_l$ ($l \neq 0$) are tagged $l$ unless they can be tagged $j$ ($1 \leq j \leq l - 1$).

These vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged “$C_a(S)$”.

output:
- $S_a(S, O)$ the set of $a, b$-minimal separators $S'$ further from $a$ than $S$ such that $C_a(S') \cap O = \emptyset$.

begin
  if $i = p - 1$ then
    return($\{S\}$)
  else
    $x \leftarrow v_{i+1}$
    $S \leftarrow$ calc3_aux($G, a, S, (v_0, \ldots, v_i, x)$)
    for each $y \in N(x)$ not tagged “$C_a(S)$”
      if $y$ is tagged $j < i$ then
        return($S$)
    for each $S'$ in find_closest_elements($G, a, x, S, O$)
      tag the vertices according to $S'$
      $S \leftarrow S \cup$ calc3_aux($G, a, S', (v_0, \ldots, v_i)$)
  end

Proposition 16 The algorithm calc3_aux is correct. It computes the set
$S_a(S,O)$ of a 3-connected planar graph.

**PROOF.** The algorithm is an application of remark 2 and proposition 13, 14 and 15. □

**Proposition 17** The algorithm can be implemented to compute the set $S_a(S,O)$ in time $O(n|S_a(S,O)|)$.

**PROOF.** The algorithm _calc_3_aux is a recursive version of the for loop below:

```plaintext
for l from \(i + 1\) to \(p - 1\)
    empty ← FALSE
    for each \(y \in N(v_l)\) not tagged \(“C'(S)”\)
        if \(y\) is tagged \(j < l - 1\) then
            empty ← TRUE
        if not empty then
            for each \(S'\) in find_closest_elements\((G, a, v_l, S, (v_0, \ldots, l - 1))\)
                tag the vertices according to \(S'\)
            \(S ← S \cup \text{calc3_aux}(G, a, S', (v_0, \ldots, l))\)
    return(S)
```

For each minimal separator \(S\), the algorithm performs the following operations:

i. the function find_closest_elements produces \(S\);  
ii. the vertices of \(G\) are tagged;  
iii. the for loop is executed in the recursive call to calc3_aux  
iv. \(S\) is returned.

The function find_closest_elements can be implemented in linear time. Computing the neighbourhoods of the connected component of \(G \setminus \{N(C) \cup C\}\) that contain \(O\) can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood \(S\) was a strict subset of some other neighbourhood \(S'\) then \(S'\) would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs \(O(n + m)\).

The for loop tests the neighbours of \(v_l\) to check if they are forbidden. Since the vertex \(v_l\) is always different, this costs at most \(O(m)\).
In a planar graph, the number \( m \) of edges satisfies \( 0 \leq m \leq 3n - 6 \), so the
time spent on each minimal separator is \( O(n) \), which gives an overall
time complexity of \( O(n|S_a(S, O)|) \).  

The following algorithm uses the function `calc3_aux` to compute the set of all
minimal separators of a planar graph \( G \).

**Algorithm 3 all_min_sep3**

**input:**

\( G \) a 3-connected planar graph

**output:**

the set of \( a, * \)-minimal separators of \( G \)

**begin**

\( S \leftarrow \emptyset \)

find \( a \in V \) with \( d(a) < 6 \)

for each minimal separator \( S \subseteq N(a) \)

\( S \leftarrow S \cup \text{calc3_aux}(G, a, S, \emptyset) \)

for each \( y \in N(a) \)

\( S \leftarrow S \cup \text{calc3_aux}(G, y, S, \emptyset) \)

**return**(\( S \))

**end**

**Theorem 18** Algorithm all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time \( O(n|S(G)|) \)

**PROOF.** Since in a 3-connected planar graph minimal separators are min-
imal for inclusion, given a vertex \( a \), \( S \in S(G) \) either belongs to \( S_a \) or runs
through \( a \). In the second case, it is a \( b, * \)-minimal separator for a neighbour \( b \)
of \( a \).

Moreover, there exists a vertex \( a \) of degree at most five in a planar graph. Let
\( b_1, \ldots, b_p \) be its neighbours.

By computing \( S_a \cup \left( \bigcup_{c \in [1..p]} S_{b_c} \right) \), a minimal separator can be calculated no
more than five times, which gives the claimed complexity.  

\( \square \)
5 Conclusion

This article confirms the feeling of Berry et al. [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to $n$ times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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References


