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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender et al. conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry et al. [1] proposed an algorithm of running time $O(nm)$ per separator\textsuperscript{1} that uses the concept of generating new minimal separators only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].
separators from a previous minimal separator $S$ by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph $G$. In order to avoid the problem of recalculation, we define the set $S_a(S, O)$ of the $a, b$-minimal separators $S'$ for some $b$ such that the connected component of $a$ in $G \setminus S'$ contains the connected component of $a$ in $G \setminus S$ but avoids the set $O$. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets $A$ and $B$ are disjoint, their union is denoted by $A \cup B$.

A set $S \subseteq V$ is a separator if $G \setminus S$ has at least two connected components, an $a, b$-separator if $a$ and $b$ are in different connected components of $G \setminus S$, an $a, b$-minimal separator if no proper subset of $S$ is an $a, b$-separator. The connected component of $a$ in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. For an $a, b$-minimal separator $S$, both $C_a(S)$ and $C_b(S)$ are full. A set $S$ is a minimal separator if there exist $a$ and $b$ such that $S$ is an $a, b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a, \ast$-minimal separator of a graph $G = (V, E)$ is an $a, b$-minimal separator of $G$ for some $b \in V$. The set of the $a, \ast$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

It is possible to order the $a, \ast$-minimal separators in the following way:

$$S_1 \preceq_a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator $S_1$ is closer to $a$ than $S_2$. The set of $a, b$-minimal separators is a lattice for the relation $\preceq_a$[4] but we only need the following weaker lemma:

**Lemma 1** Let $C$ be a set of vertices of a graph $G$ inducing a connected subgraph of $G$, $a$ be a vertex of $C$ and $b$ be a vertex of $G \setminus (C \cup N(C))$. 

2
The neighbour $S$ of $C_b\left(C \cup N(C)\right)$ is an $a,b$-minimal separator such that $C$ is a subset of $C_a(S)$ that is closer to $a$ than any $a,b$-minimal separator $S'$ such that $C$ is a subset of $C_a(S')$.

**Proof.** By construction, $C$ is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since $S$ is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that $S$ is an $a,b$-minimal separator.

Let $S'$ be an $a,b$-minimal separator such that $C$ is a subset of $C_a(S')$. Let $p$ be a path in $C_b(S')$ with $b$ as one of its ends. The vertices of $S'$ are at least at distance 1 of $C$ so the vertices of $p$ are at least at distance 2 of $C$. Since $S$ is a subset of $N(C)$, $p \cap S = \emptyset$. In other words $p$ is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ i.e. $S$ is closer to $a$ than $S'$. □

For $S$ an $a,\ast$-minimal separator and $O \subseteq V$, the set $S_a(S,O)$ is the set of the $a,\ast$-minimal separators $S'$ further from $a$ than $S$ and such that $O \cap C_a(S') = \emptyset$. If $x \in V$, the set $S^x_a(S,O)$ is the set of $S' \in S_a(S,O)$ such that $x \in C_a(S')$.

**Remark 2** If $x \in S$, then $S_a(S,O)$ is the disjoint union

$$S_a(S,O) = S_a(S,O \cup \{x\}) \bigcup S^x_a(S,O).$$

More precisely, if $(S_i)_{i \in I}$ are the elements of $S^x_a(S,O)$ closest to $a$, then

$$S_a(S,O) \subseteq S_a(S,O \cup \{x\}) \bigcup \left( \bigcup_{i \in I} S_a(S_i,O) \right).$$

This gives the skeleton of an algorithm to compute the set $S_a(S,O)$.

**Remark 3** If $S$ belongs to $S^x_a(S,O)$, then $S^x_a(S,O) = S_a(S,O)$.

The algorithm is based on remarks 2 and 3. To list $S_a$, the algorithm computes the sets $S_a(S,\emptyset)$ for every $S$ closest to $a$ in $S_a$. During this calculation, it computes $S_a(S,O)$ with $O \subseteq S$. To do so, it chooses $x$ in $S \setminus O$ and calculates $S^x_a(S,O)$ and $S_a(S,O \cup \{x\})$. The set $S^x_a(S,O)$ is itself a union of $S_a(S_i,O)$. But to obtain such a decomposition, one needs to find the elements of $S^x_a(S,O)$ closest to $a$, which is the following proposition does.

**Proposition 4** Let $G = (V,E)$ be a graph, $S$ an $a,\ast$-minimal separator, $O \subseteq S$, $x \in S \setminus O$ and $C = C_a(S) \cup \{x\}$

The elements of $S^x_a(S,O)$ closest to $a$ are exactly the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ that contain $O$ and that are maximal for inclusion.

3
PROOF. Let $S_1$ be an $a,b$-minimal separator of $S_a^r(S,O)$ closest to $a$. Let $S'$ be the neighbourhood of $C_b(N(C) \cup C)$. By lemma 1, $S'$ is an $a,b$-minimal separator such that $C$ is a subset of $C_a(S')$ and $S'$ is closer to $a$ than $S_1$. Moreover, since $C_a(S_1) \cap O = \emptyset$ and $S'$ is closer to $a$ than $S_1$, $C_a(S') \cap O \subseteq C_a(S_1) \cap O = \emptyset$. Thus $S'$ belongs to $S_a^r(S,O)$ and is closer to $a$ than $S_1$. This proves that $S_1 = S'$. Since $S_1$ cannot be a subset of another element of $S_a^r(S,O)$, $S_1$ is the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ which is maximal for inclusion.

Conversely, let $S_1$ be a neighbourhood of a connected component $D$ of $G \setminus \{N(C) \cup C\}$ that contains $O$ and that is maximal for inclusion. By lemma 1, $S_1$ is an element of $S_a^r(S,O)$ that is closer to $a$ than any $a,b$-minimal separator of $S_a^r(S,O)$ with $b$ in $D$. So if $S_2$ is an $a,b$-minimal separator of $S_a^r(S,O)$ strictly closer to $a$ than $S_1$, $S_1$ is not an $a,b$-minimal separator. Suppose for a contradiction that such an $a,b$-minimal separator exists. It follows from the first part of the proof that such an $a,b$-minimal separator $S_2$ closest to $a$ is the neighbourhood of $C_b(N(C) \cup C)$. The set $S_2$ is an element of $S_a^r(S,O)$ that is closer to $a$ than $S_1$ and $S_1$ is a subset of $S_2$ (because $S_1 \setminus S_2 \subseteq C_a(S_2) \setminus C_a(S_1)$) and $S_2$ is closer to $a$ than $S_1$ and therefore $S_1$ is a strict subset of $S_2$ contradicting the fact that $S_1$ is maximal for inclusion. \(\square\)

Proposition 4 gives us a way to find the minimal elements of $S_a^r(S,O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G \setminus \{N(C) \cup C\}$ and then choosing among the minimal separators found the ones that contain $O$ and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $S_a(S,O)$ that may look like:

Algorithm 1 _calc3_

\[
\begin{align*}
\text{begin} & \quad \text{if } S \setminus O = \emptyset \text{ then} \\
& \quad \quad \text{return}(\{S\}) \\
& \quad \text{else} \\
& \quad \quad \text{let } x \in S \setminus O \\
& \quad \quad S \leftarrow \_\text{calc3}_\_ (G, a, S, O \cup \{x\}) \\
& \quad \quad \text{for each } S_i \text{ in } \text{find\_closest\_elements}(G, a, x, S, O) \\
& \quad \quad \quad S \leftarrow S \cup \_\text{calc3}_\_ (G, a, S_i, O) \\
& \quad \text{return}(S) \\
\end{align*}
\]

However several problems need to be solved.
i. We do not know whether the sets \( S_a(S_i, O) \) are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.

ii. To implement the function \texttt{find closest elements}, proposition 4 states that we can start with a graph search of \( G \).

But if \( S_a(S, O) = \{S\} \), the recursive calls to the algorithm will try to find an element of \( S_a^x(S, O) \) closest to \( a \) for every \( x \in S \setminus O \). Each call to \texttt{find min elements} costs at least \( O(m) \) and finally, we would have spent at least \( O(nm) \) to realise that \( S_a(S, O) = \{S\} \).

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, i.e. if \( S_1 \) and \( S_2 \) are two minimal elements of \( S_a^x(S, O) \), the sets \( S_a(S_1, O) \) and \( S_a(S_2, O) \) are disjoint. Section 3.3 then shows how to determine whether \( S_a^x(S, O) \) is empty or not in an overall \( O(n) \).

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let \( \Sigma \) be the plane. A \textit{plane graph} \( G_\Sigma = (V_\Sigma, E_\Sigma) \) is a graph drawn on the plane, that is \( V_\Sigma \subset \Sigma \) and each \( e \in E_\Sigma \) is a simple curve of \( \Sigma \) between two vertices of \( V_\Sigma \) in such a way that the interiors of two distinct edges do not meet. We will denote by \( \tilde{G}_\Sigma \) the drawing of \( G_\Sigma \). A \textit{planar graph} is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A \textit{face} of \( G_\Sigma \) is a connected component of \( \Sigma \setminus \tilde{G}_\Sigma \).

3.1 Minimal separators of 3-connected planar graphs

\textbf{Proposition 5} In a 3-connected planar graph, minimal separators are minimal for inclusion.

\textbf{PROOF.} Suppose that \( S \subset S' \) are two minimal separators of a 3-connected planar graph.

Let \( a, b, c \) and \( d \) be vertices such that \( S' \) is an \( a, b \)-minimal separator and \( S \) is a \( c,d \)-minimal separator. Since \( S \) is not an \( a, b \)-minimal separator, either \( C_c(S') \) or \( C_d(S') \) is disjoint with \( C_a(S') \) and \( C_b(S') \). Suppose that \( C_c(S') \) is such a component. In this case, \( C_c(S) \) and \( N(C_c(S)) \) are respectively equal to \( C_c(S') \) and \( S \).
But then $G$ admits $K_{3,3}$ as a minor if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices $a'$, $b'$ and $c'$, all these vertices have $S$ in their neighbourhood and since $G$ is 3-connected, $|S| \geq 3$. This contradicts the fact that $G$ is planar. \hfill \Box

**Proposition 6** Let $G = (V, E)$ be a 3-connected planar graph, $a$ a vertex of $G$, $S$ an $a, \ast$-minimal separator, $O$ a subset of $S$ and $x$ a vertex of $S \setminus O$.

If $S_1$ and $S_2$ are two distinct elements of $S^x_a(S, O)$ that are closest to $a$, then

$$S_a(S_1, O) \cap S_a(S_2, O) = \emptyset.$$

**PROOF.** Let $C$ be $C_a(S) \cup \{x\}$ and suppose for a contradiction that $S_3$ is a minimal separator of $S_a(S_1, O) \cap S_a(S_2, O)$ with $S_1$ and $S_2$ two distinct elements of $S^x_a(S, O)$ closest to $a$. Let $b$ be a vertex such that $S_3$ is an $a, b$-minimal separator.

Since $S_3$ is further from $a$ than $S_1$ and $S_2$, both $S_1$ and $S_2$ are $a, b$-separators. There exists an $a, b$-minimal separator $S'$ included in $S_1$. By proposition 5, a minimal separator of $G$ is minimal for inclusion which proves that $S_1 = S'$ and $S_1$ is an $a, b$-minimal separator. By lemma 1, the neighbourhood $S_4$ of $C_b(N(C) \cup C)$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S_4)$ that is closer to $a$ than $S_1$. So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and $S_4$ is an element of $S^x_a(S, O)$ that is closer to $a$ than $S_1$. Similarly, $S_2$ is an $a, b$-minimal separator and $S_4$ is closer to $a$ than $S_2$ which contradicts the fact that $S_1$ and $S_2$ are two distinct elements of $S^x_a(S, O)$ closest to $a$. \hfill \Box

### 3.2 The intermediate graph

**Definition 7** Let $G_{\Sigma} = (V_{\Sigma}, E_{\Sigma})$ be a 3-connected plane graph. Let $F$ be the set of its faces. In each face $f \in F$ pick up one point $v_f$. Let $R_F$ be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_{\Sigma} \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex $v$ is incident to the face $f$.

For $G'$ a subgraph of $G_I$, the set $G' \cap V_{\Sigma}$ will be denoted by $V(G')$.

**Proposition 8** Let $\mu$ be a cycle of $G_I$ such that the curve $\hat{\mu}$ separates at least two vertices $a$ and $b$ of $V_{\Sigma}$.

The set $V(\mu)$ is an $a, b$-separator of $G_{\Sigma}$.

**PROOF.** Let $p$ be a path in $G_{\Sigma}$ from $a$ to $b$. Since $a$ and $b$ are not in the
same connected component of $\Sigma \backslash \tilde{\mu}$, $\tilde{p}$ intersect $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from $a$ to $b$ meets $V(\mu)$ and so $V(\mu)$ is an $a, b$-separator. □

**Proposition 9** Let $S$ be an $a, b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve defined by $\mu$ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

**PROOF.** Let $C$ be the connected component of $a$ in $G \backslash S$. Let us contract $C$ into a supervertex $v_C$ to build the graph $G/C$. There is a cycle $\mu/C$ in $(G/C)_I$ such that $V(\mu/C)$ is the neighbourhood of $v_C$ in $G/C$. Therefore, the neighbourhood of $C$ in $G_I$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \backslash \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates $a$ and $b$. However, proposition 8 shows that $V(\mu')$ which is a strict subset of $S$ is an $a, b$-separator. This contradicts the fact that $S$ is a $a, b$-minimal separator. □

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the $G_I$, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

### 3.3 Ordered separators

**Definition 10** An ordered separator of $G$ is a sequence of distinct vertices $(v_0, \ldots, v_{p-1})$, $(p > 2)$ such that

i. there exists a face to which $v_i$ and $v_{i+1}[p]$ are incident;
ii. $v_i$ and $v_j$ are incident to a common face only if $i = j+1[p]$ or $j = i+1[p]$;
iii. if $p = 3$, no face is incident to $v_i$, $v_{i+1}[p]$ and $v_{i+2}[p]$.

The notation $i[p]$ means $i$ modulo $p$.

A set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.
If $S = (v_0, \ldots, v_{p-1})$ is an ordered separator of $G$, then $S$ is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

**Lemma 11** Every minimal separator $S$ of $G$ is ordered.

**PROOF.** Let $S$ be an $a, b$-minimal separator of $G$.

Proposition 9 states that there exists a simple cycle of $G_I$

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to $S$.

i. The construction of $T$ ensures that $v_i$ and $v_{i+1}$ are incident to a common face $(f_i)$.

ii. Suppose that $v_i$ and $v_j$ are incident to a common face $f$ and that $i + 1 \neq j \in [p]$ and $j + 1 \neq i \in [p]$.

$$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$$

and

$$\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$$

are laces of $G_I$. Moreover, since either $\mu_1$ or $\mu_2$ separates $a$ and $b$, there exists an $a, b$-separator strictly included in $S$ which is absurd.

iii. Suppose that $p = 3$ and that $v_0, v_1$ and $v_2$ are all incident to a common face $f$. If we add a vertex $f$ to $G$ connected to the vertices $v_0$, $v_1$ and $v_2$, the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of $a$, the connected component of $b$ and the vertex $f$ are all incident to $v_0$, $v_1$ and $v_2$ which builds up a $K_{3,3}$.

The sequence $T$ is an ordered separator corresponding to $S$. $\square$

Conversely,

**Lemma 12** Every ordered separator of $G$ is a minimal separator of $G$.

**PROOF.** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of $G$.

First, $S$ is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases, all the vertices of $S$ would be incident to a common face.

Let $S'$ be a minimal separator included in $S$. By lemma 11, $S'$ is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator $S$ is a minimal separator. $\square$
From lemmata 11 and 12, we have the following proposition:

**Proposition 13** A set \( S \subseteq V \) is a minimal separator of a 3-connected planar graph \( G = (V, E) \) if and only if it corresponds to an ordered separator of \( G \).

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether \( S_x (S, O) \) is empty or not (\( O \subseteq S \) and \( x \in S \setminus O \)).

**Proposition 14** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered \( a, \ast \)-separator of a 3-connected planar graph \( G = (V, E) \) and \( O = (v_0, \ldots, v_i) \), \( (i < p - 1) \) be an initial sequence of \( S \).

If there exists a face that is incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) with \( 0 < j < i \), then \( S_x (S, O) \) is empty.

**PROOF.** Let \( b \) be such that \( S \) is an \( a, b \)-minimal separator and suppose that \( y \in N(v_{i+1}) \) and \( v_j \) with \( 0 < j < i \) are both incident to a face \( f \). Since \( S \) is an ordered separator, there exists a cycle \( (v_0, f_0, \ldots, v_k, f_k) \) of \( G_f \) corresponding to a Jordan curve \( \tilde{\mu} \). Let \( \Sigma_b \) be the connected component of \( \Sigma \setminus \tilde{\mu} \) that contains \( b \). Since \( y \) and \( v_j \) are incident to \( f \), there exists a path \( (v_{i+1}, g, y, f, v_j) \) that corresponds to a curve \( \tilde{\nu} \) that cuts \( \Sigma_b \) in two parts \( \Sigma_1 \) and \( \Sigma_2 \) whose borders are \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) respectively. Since \( 0 < j < i \), neither \( V(\tilde{\mu}_1) \) nor \( V(\tilde{\mu}_2) \) contains \( O \).

Suppose that \( S' \) is an element of \( S_x (S, O) \) closest to \( a \). Let \( c \) be such that \( S' \) is an \( a, c \)-minimal separator. The vertex \( c \) belongs to \( \Sigma_b \). We may suppose that \( c \) belongs to \( \Sigma_1 \). By proposition 4, \( S' \) is the neighbourhood of \( C_a(S \cup N(v_{i+1})) \) i.e. \( S' = V(\tilde{\mu}_1) \), but \( O \) is not a subset of \( S' \) which is absurd. \( \square \)

Conversely,

**Proposition 15** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered \( a, \ast \)-separator of a 3-connected planar graph \( G = (V, E) \) and \( O = (v_0, \ldots, v_i) \), \( (i < p - 1) \) be an initial sequence of \( S \).

If there is no face incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) \( (0 < j < i) \), then there is an ordered separator in \( S \cup N(v_{i+1}) \setminus C_a(S) \) that contains \( O \).

**PROOF.** The neighbours \( (y_1, \ldots, y_l) \) of \( v_{i+1} \) taken in clockwise order are such that \( y_i \) and \( y_{i+1} \) are incident to a common face. Moreover, since \( v_{i+1} \) and \( v_i \) are both incident to a face \( f_1 \) and since \( v_{i+1} \) and \( v_{i+2} \) are both incident to a face \( f_2 \), there is a sequence \( P = (v_i, x_1, \ldots, x_k, v_0) \) such that there exists a
face incident to any two consecutive vertices of $P$ and such that $P$ uses only vertices of $N(v_i+1)\setminus C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is $(v_1, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0)$.

Let $P$ be such a sequence between $v_i$ and $v_0$ of minimal length. Together with $(v_1, \ldots, v_{i-1})$, $P$ forms an ordered separator of $G$ as required. \hfill \square

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $S_a(S, O)$ with $O \subseteq S$.

Algorithm 2 calc3_aux

input:
- $G$ a 3-connected planar graph
- $a$ a vertex of $G$
- $S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
- $O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of $S$

The vertices that have an incident face in common with $v_l$ ($l \neq 0$) are tagged $l$ unless they can be tagged $j$ ($1 \leq j \leq l-1$). These vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged “$C_a(S)$”.

output:
- $S_a(S, O)$ the set of $a, b$-minimal separators $S'$ further from $a$ than $S$ such that $C_a(S') \cap O = \emptyset$.

begin
  if $i = p-1$ then
    return($\{S\}$)
  else
    $x \leftarrow v_{i+1}$
    $S \leftarrow \text{calc3_aux}(G, a, S, (v_0, \ldots, v_i, x))$
    for each $y \in N(x)$ not tagged “$C_a(S)$”
      if $y$ is tagged $j < i$ then
        return($S$)
    for each $S'$ in find_closest_elements($G, a, x, S, O$)
      tag the vertices according to $S'$
      $S' \leftarrow S' \cup \text{calc3_aux}(G, a, S', (v_0, \ldots, v_i))$
  end

Proposition 16 The algorithm calc3_aux is correct. It computes the set
$S_a(S, O)$ of a 3-connected planar graph.

**PROOF.** The algorithm is an application of remark 2 and proposition 13, 14 and 15. □

**Proposition 17** The algorithm can be implemented to compute the set $S_a(S, O)$ in time $O(n|S_a(S, O)|)$.

**PROOF.** The algorithm `_calc_3_aux` is a recursive version of the for loop below:

```
for l from i + 1 to p - 1
    empty ← FALSE
    for each $y \in N(v_l)$ not tagged "C_a(S)"
        if $y$ is tagged $j < l - 1$ then
            empty ← TRUE
        if not empty then
            for each $S'$ in find_closest_elements($G, a, v_l, S, (v_0, \ldots, l - 1)$)
                tag the vertices according to $S'$
                $S ← S \cup _calc_3_aux(G, a, S', (v_0, \ldots, l))$
    return($S$)
```

For each minimal separator $S$, the algorithm performs the following operations:

i. the function `find_closest_elements` produces $S$;
ii. the vertices of $G$ are tagged;
iii. the for loop is executed in the recursive call to `calc3_aux`
iv. $S$ is returned.

The function `find_closest_elements` can be implemented in linear time. Computing the neighbourhoods of the connected component of $G \setminus \{N(C) \cup C\}$ that contain $O$ can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood $S$ was a strict subset of some other neighbourhood $S'$ then $S'$ would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs $O(n + m)$.

The for loop tests the neighbours of $v_l$ to check if they are forbidden. Since the vertex $v_l$ is always different, this costs at most $O(m)$. 

11
In a planar graph, the number \( m \) of edges satisfies \( 0 \leq m \leq 3n - 6 \), so the time spent on each minimal separator is \( O(n) \), which gives an overall time complexity of \( O(n|S_a(S, O)|) \). \( \square \)

The following algorithm uses the function \( \text{calc3_aux} \) to compute the set of all minimal separators of a planar graph \( G \).

**Algorithm 3 all_min_sep3**

**input:**
\( G \) a 3-connected planar graph

**output:**
the set of \( a, * \)-minimal separators of \( G \)

**begin**
\( S \leftarrow \emptyset \)

find \( a \in V \) with \( d(a) < 6 \)

for each minimal separator \( S \subseteq N(a) \)
\( S \leftarrow S \cup \text{calc3_aux}(G, a, S, \emptyset) \)

for each \( y \in N(a) \)

for each \( a, * \)-minimal separator \( S \subseteq N(y) \)
\( S \leftarrow S \cup \text{calc3_aux}(G, y, S, \emptyset) \)

return(\( S \))

**end**

**Theorem 18** Algorithm all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time \( O(n|S(G)|) \)

**PROOF.** Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex \( a \), \( S \in S(G) \) either belongs to \( S_a \) or runs through \( a \). In the second case, it is a \( b, * \)-minimal separator for a neighbour \( b \) of \( a \).

Moreover, there exists a vertex \( a \) of degree at most five in a planar graph. Let \( b_1, \ldots, b_p \) be its neighbours.

By computing \( S_a \cup \bigcup_{i \in [1..p]} S_{b_i} \), a minimal separator can be calculated no more than five times, which gives the claimed complexity. \( \square \)
5 Conclusion

This article confirms the feeling of Berry et al. [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to $n$ times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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References


