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Effective particle kinematics from Quantum Gravity

J. Kowalski–Glikman*, and A. Starodubtsev†

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Abstract

Particles propagating in de Sitter spacetime can be described by the topological BF $\mathsf{SO}(4,1)$ theory coupled to point charges. Gravitational interaction between them can be introduced by adding to the action a symmetry breaking term, which reduces the local gauge symmetry down to $\mathsf{SO}(3,1)$, and which can be treated as a perturbation. In this paper we focus solely on topological interactions which corresponds to zeroth order in this perturbative expansion. We show that in this approximation the system is effectively described by the $\mathsf{SO}(4,1)$ Chern-Simons theory coupled to particles and living on the 3 dimensional boundary of spacetime. Then, using Alekseev–Malkin construction we find the effective theory of particles kinematics. We show that the particles action contains standard kinetic terms and the deformation shows up in the presence of interaction terms. The strength of the interactions is proportional to deformation parameter, identified with Planck mass scale.

1 Introduction

It is well known that in 3 dimensions gravity is described by a topological field theory, and therefore has a finite number of topological degrees of freedom reflecting the topology of spacetime [1]. If $3d$ gravity is coupled to point particles, which can be modeled as charged punctures of space manifold, these topological degrees of freedom can be "integrated out" leading to effective, deformed particle kinematics [2], [4], [3]. This effective system of deformed particles is of Doubly Special Relativity type (see [5], [6], [8] for the original proposal and [9], [10] for reviews), being characterized by two scales, velocity of light c and Planck mass κ. Similarly it turns out that integrating out topological gravitational degrees of freedom in the case of gravity coupled to fields leads to effective field theory on non-commutative spacetime [11], [12]. As a result spacetime symmetries of

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deformed systems become quantum symmetries, being described by quantum groups, instead of the standard Lie groups.

The question arises if something similar could happen in the case of physical, 4-dimensional gravity coupled to particles and/or fields? Certainly, in $4d$ gravity possesses local degrees of freedom exhibited by Newtonian interactions and gravitational waves, for example. However, as stressed in the context of recent investigations in quantum gravity phenomenology (see [13] for recent review and references to earlier works) our best hope to see signals of quantum gravity is to look for high-energetic events events (like scattering of ultra high energy cosmic rays) in description of whose local degrees of freedom of gravity play no role. If we expect to see some quantum gravity imprints there, to describe them we must look for "no local gravity limit of gravity", hoping that in this limit the effective theory behaves like gravity in $3d$, effectively deforming particles and fields kinematics. This is why we make use of DSR-like test theories, that predict deformation of spacetime-symmetries characterized by Planck-scale deformation parameter to describe quantum gravity signal that may be detected in foreseeable experiments.

It is therefore of interest to look for a formulation of gravity such that one keeps good control over the limit, in which local gravitational degrees of freedom are not present. In this limit gravity is described locally by its maximally symmetric vacuum state: Minkowski space for zero cosmological constant, and (anti) de Sitter space in the case of the (negative) positive one. Coupling such a theory to point particles and then taking the limit makes it possible investigate the effective behavior of the particles. One then can ask the question if the quantum gravity scale is still present in this effective theory, acting as a deformation? Naively the answer would be in the negative, and after taking a limit we would end up with the standard theory of particles moving on Minkowski space of special relativity (or on (anti) de Sitter space when the cosmological constant is non-zero.) On the other hand, the experience of 3d gravity suggests that even in the limit the effective theory might be deformed by the presence of Planck scale.

This is the problem that we would like to investigate in this paper.

As said above, in 4 dimensions gravity is certainly not a topological theory. However, it can be nevertheless described by a BF topological theory with de Sitter $\mathsf{SO}(4,1)$ gauge group, appended by a small term that breaks gauge symmetry down to Lorentz $\mathsf{SO}(3,1)$. The presence of this symmetry breaking term switches on the local degrees of freedom of gravity, while in the limit, in which this term vanishes the theory becomes topological, as in $3d$. Such formulation of gravitational field has its roots in the proposal of MacDowell and Mansouri [14] and has been recently investigated in depth in [15] and [16].

The construction presented in these works is as follows. The building blocks are the $\mathsf{SO}(4,1)$ connection A^{IJ} with curvature $F^{IJ} = dA^{IJ} + A^I{}_K \wedge A^{KJ}$ and the $\mathsf{so}(4,1)$ valued two-form field B^{IJ} $(I, J = 0, \ldots, 4)$, in terms of whose the

action takes the form

$$S = \int B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{1}{2} B^{IJ} \wedge B^{KL} \epsilon_{IJKLM} V^M \tag{1}$$

In this action a constant algebra element V^M enforces breaking the gauge symmetry group down to $\mathsf{SO}(3,1)$ (the subgroup of $\mathsf{SO}(4,1)$ that leaves V^M invariant.) Without loss of generality we can take this vector to be $V^M = (0,0,0,0,V^4) = (0,0,0,0,\frac{\alpha}{2})$ so that the action takes the form

$$S = \int B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} B^{IJ} \wedge B^{KL} \epsilon_{IJKL4} \tag{2}$$

The first two terms in the action describe the topological field theory, and in the limit $\alpha \to 0$ the only solution of equations of motion is de Sitter space, the topological vacuum of the full theory. Notice that both the topological lagrangian and the gauge breaking one are manifestly diffeomorphism invariant, and thus the perturbation theory in α, around topological vacuum, corresponding to topological theory at $\alpha = 0$, is going to be manifestly diffeomorphism invariant as well.

Remarkably, it can be shown [16] that if one decomposes the connection $A^{a4} = \frac{1}{\ell} e^a$, with ℓ being a constant of dimension of length, $A^{ab} = \omega^{ab}$, $a, b = 0 \ldots 3$, where e^a and ω^{ab} are tetrad and Lorentz connection one forms, respectively, after solving for B this action reproduces, up to topological terms, to the standard Einstein–Cartan one

$$S_P = -\frac{1}{2G} \int (R^{ab} \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d) \epsilon_{abcd} - \frac{2}{\gamma} R^{ab} \wedge e_a \wedge e_b \tag{3}$$

The last term in this action does not modify field equations if torsion vanishes. The physical constants: Newton's constant G, cosmological constant Λ and Immirzi parameter γ are related to the dimensional parameters of the original action (2) and the scale ℓ as follows

$$\gamma = \frac{\beta}{\alpha}, \qquad \frac{1}{\ell^2} = \frac{\Lambda}{3}, \qquad G = \frac{3\alpha(1-\gamma^2)}{\Lambda} = \frac{3\beta(1-\gamma^2)}{\Lambda\gamma} \tag{4}$$

One can couple gravity described by the action (2) to particles in a rather straightforward way [17] (see also [18] for recent discussion.) Each particle with (dimensionless) mass $\mu = \ell m$ and spin s at rest is described by an appropriate element of the $\mathsf{so}(4,1)$ algebra[1]

$$D = \mu T^{04} + s T^{23} \tag{5}$$

where T^{04} and T^{23} are "translational" "rotational" generators of $\mathsf{so}(4,1)$ algebra, respectively. Then the lagrangian describing the particle at rest is simply

$$S_{rest} = \int d\tau \mathrm{Tr}(D A_\tau), \quad A_\tau \equiv A_\mu \dot{z}^\mu(\tau) \tag{6}$$

[1]Strictly speaking it is an element of the dual algebra $\mathsf{so}(4,1)^*$. Here we describe it by a canonically conjugated element of the algebra.

where $z^\mu(\tau)$ is the particle worldline, and τ is the affine parameter. The action (6) breaks gauge invariance at the particle worldline. However $\mathsf{SO}(4,1)$ gauge transformations are just translations and Lorentz transformations, so acting by gauge transformations on the particle at rest just makes the particle moving. In this way the gauge degrees of freedom of gravity on worldline become dynamical degrees of freedom of the particle and the action describing arbitrarily moving particle has the form

$$S_{particle} = \int d\tau \mathrm{Tr}(DA^h_\tau), \quad A^h_\tau \equiv h^{-1}A_\tau h + h^{-1}\partial_\tau h, \quad h \in \mathsf{SO}(4,1) \tag{7}$$

It can be shown [17] that this action leads to correct equations of motion for the particle and generalized Einstein equations with mass and spin of the particle being the source of curvature and torsion, respectively.

Notice that the action (7) easily generalizes to the case of a finite number of particles in which case it reads

$$S_{particles} = \sum_i \int d\tau \mathrm{Tr}(D_i A^h_\tau), \tag{8}$$

For further discussion of this formulation of gravity coupled to particles see [17] and [19].

The action of gravity (2) coupled to particles (8) is a convenient starting point to address the question raised above. The point is that the presence of the parameter α in the action (2) makes the "no gravity limit" easy to control, and to set up a perturbative theory. In the next section we consider the quantum perturbative expansion in parameter α and we argue that the zeroth order of this expansion is described by a holographic theory living on the boundary of spacetime. Then, in the following two sections, we will show how such boundary theory reduces to effective particles dynamics. In the final section we discuss the obtained results.

2 From gravity in $4d$ to $3d$ Chern–Simons theory

The purpose of this section is to demonstrate that the zeroth order approximation of perturbation theory of quantum gravity with particles around BF topological quantum field theory is described by Chern-Simons theory coupled to point sources. We will not study higher order corrections here.

For shortness let us write the action (2) as

$$S = \int L_{BF} + \frac{\alpha}{4}\int L_I, \tag{9}$$

where now L_{BF} is a lagrangian of topological field theory possibly coupled to particles, and L_I is a symmetry breaking interaction term.

The general perturbative expression for the partition function coupled to arbitrary finite number of particles (where we neglect all the interactions except

gravitational) looks like

$$Z(\{g_{p_i}\},\{g_{p_f}\}) = \int \mathcal{D}A\mathcal{D}B \sum_n \frac{(i\alpha)^n}{n!}\Big(\int L_I(x)\Big)^n \exp\left[i\frac{1}{\beta}\int_M L_{BF}\right]$$
$$= \sum_n \frac{(i\alpha)^n}{n!} Z_n(\{g_{p_i}\},\{g_{p_f}\}), \qquad (10)$$

Here g_{p_i} and g_{p_f} are $SO(4,1)$ group elements labeling initial and final positions and orientations of p-th particle with respect to the selected reference point. They can be obtained as holonomies of connection A between corresponding points (see [19] for detailed discussion.)

The expression (10) is formal, of course, and we have to define it precisely. First we must specify the measure $\mathcal{D}A$ in the path integral. Since the action describes a system with gauge symmetry the conventional way to proceed would be to introduce a gauge-fixing term. An immediate problem with this approach has been pointed out in [20] for Yang-Mills theory. Namely, since the interaction term breaks the symmetry, higher order terms in perturbative expansion have less gauge symmetry than the free action. On the other hand, if we do not fix all the gauge symmetries of the free action we cannot construct a propagator because of non-invertibility of the quadratic form. The approach of [20] has been to introduce an auxiliary field that turns gauge degrees of freedom into physical ones already at zero order. Such procedure however trivially reduces the perturbative expansion to the standard one around fixed background. This is appropriate in the case of Yang-Mills theory, but cannot be applied in the case of a theory that is supposed to be background independent.

In this paper we consider a different approach which does not use a gauge fixing. To define a path integral without gauge fixing one has to explicitly construct the reduced phase space spanned by the complete set of gauge invariant observables and define a measure on it. In most situations the later is not possible for technical reasons, e.g. it is certainly impossible to construct all the diffeomorphism invariant observables of four dimensional General Relativity. Fortunately, this turns out to be possible for any finite order terms of the perturbation theory considered here. This happens because the starting point for the expansion is a topological field theory whose reduced phase space (moduli space) is finite dimensional. Then, in finite order of perturbative expansion the dimensionality of the moduli space is getting larger, but remains finite. Therefore, in any finite order of expansion the moduli space can be, explicitly constructed and the measure on it can be defined.

To show this let us replace the integral over the symmetry breaking lagrangian in (10) by its Riemann definition. Divide the manifold $\mathcal{M}$ into N cells $\mathcal{M}_i$ where each cell is sufficiently small so that every field ϕ on which the lagrangian depends can be considered constant within each cell

$$\phi(x)\Big|_{\mathcal{M}_i} = \phi_i$$

The integral is a sum of contributions from every cell

$$\int L_{Int}(\phi(x)) = \sum_i^N S_I(\phi_i),$$

where

$$S_I(\phi_i) = \int\limits_{\mathcal{M}_i} L_I(\phi_i).$$

Here we took into account that the lagrangian L_I is a density and the volume element is already contained in it.

Consider the contribution of a cell $\mathcal{M}_i$ to the partition function at the first order $Z_{1i} = \int \mathcal{D}\phi e^{iS_T} S_I(\phi_i)$ and compare it with the contribution Z_{1i} from another cell $\mathcal{M}_j$. One can always find a diffeomorphism $x \to x'$ such that $\mathcal{M}_i \to \mathcal{M}_j$. The fields also transform under this diffeomorphism $\phi \to \phi'$. Due to diffeomorpfism invariance of the interaction term

$$S_I(\phi_i) = S_I(\phi'_j).$$

The free action is also diffeomorphism invariant and we assume that we can define a diffeomorphism invariant measure of the path integral

$$S_T(\phi) = S_T(\phi'), \quad \mathcal{D}\phi = \mathcal{D}\phi'.$$

As a result, the contribution to the path integral from different cells is equal, $Z_{1i} = Z_{1j}$, and the sum over cells becomes trivial.

At higher order one has to distinguish the situations when the cells on which the interaction term is applied are the same or distinct.

$$Z_n = N^n \int \mathcal{D}\phi \prod_i S_I(\phi_i) e_+^{iS_T} ... + N \int \mathcal{D}\phi S_I(\phi_i)^n e^{iS_T}$$

In the limit of large number of cells the first term, where interaction is applied to the distinct cells, will be dominating.

The above argument is analogous to that of coordinates-independence of n-point functions of diffeomorphism invariant theories (see e.g. [21] and references therein).

So far we were using only diffeomorphism invariance of the free action. Its topological invariance allows us to give a complete definition of every term in the expansion of the the path integral. The free action can be replaced by its discretized version $S_T \to \sum\limits_i S_T(\phi_i)$, and the measure can be specified as $\mathcal{D}\phi \to \prod\limits_i d\phi_i$. The resulting path integral is finite dimensional and due to topological invariance of the free action is independent of the discretization.

Let us now specify the model in (9) to be (2) with particles coupling in the $\alpha \to 0$ limit, i.e. being defined, after integrating out B-field, by

$$L_{BF} = \frac{1}{\beta} F^{IJ} \wedge F_{IJ} + \sum_i D_i^{IJ} A_{0IJ} \delta^3(x - x_i(\tau)) \tag{11}$$

In the formula above $D_i = \mu_i T^{04} + s_i T^{23}$ is an algebra element defining mass and spin of i-th particle (both $\mu_i = m\ell$, where m is the physical mass and ℓ is the length scale of the model (4), and s_i are dimensionless here), $x_i(\tau)$ is a timelike particle trajectory which can be taken arbitrary due to topological invariance of the model. In the present paper we are studying only the zeroth order contribution, and therefore we do not consider the interaction term L_{Int} here.

Due to Bianchi identity the integral of the first term in (11) after integration by parts reduces to a Chern-Simons action on the boundary of the original manifold $\mathcal{M}$.

$$\frac{1}{\beta}\int_{\mathcal{M}} F^{AB}\wedge F_{AB} = \frac{1}{\beta}\int_{\partial\mathcal{M}} Y_{CS}(A) \tag{12}$$

Below we consider the boundary $\partial\mathcal{M}$ to be the direct product of a (punctured) sphere S^2_n with the real line R.

The second term in (11) breaks the gauge symmetry at the location of the particles thus promoting some of the gauge degrees of freedom to the physical ones. We can include the latter explicitly by substituting into the action the connection in an explicit gauge transformed form:

$$A \to h^{-1}dh + h^{-1}Ah \tag{13}$$

At the location of the particles $h(x_i) \equiv h_i$ become physical degrees of freedom.

To make a link with canonical formulation we will rewrite the resulting action explicitly, decomposing the connection into .spacelike and timelike components.

$$\begin{aligned} S_{BF}[A_S,A_0,h_i] = \int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi}\left\langle \partial_0 A_S \wedge A_S\right\rangle - \int_{\mathbb{R}} dx^0 \sum_{p=1}^{n} \left\langle D_i\,,\, h_i^{-1}\partial_0 h_i\right\rangle \\ + \int_{\mathcal{M}} d^4x \left\langle A_0\,,\ \frac{k}{2\pi}F_S\delta(\partial\mathcal{M}) - \sum_{p=1}^{n} T_i\delta^{(3)}(x-x_{(i)})dx^1\wedge dx^2\wedge dx^3\right\rangle \end{aligned} \tag{14}$$

Let us explain the notation used in the formula above. The Chern-Simon connection one-form A is decomposed into time and space part, to wit

$$A = A_0\, dx^0 + A_S, \quad F = dx^0\wedge(\partial_0 A_S - d_S A_0 + [A_0, A_S]) + F_S \tag{15}$$

and thus F_S is the space part of the curvature two-form. We take the Chern-Simon coupling constant to be $k/4\pi = 1/\beta$. D_i is the charge carried by the particle, which in our case will be just its mass $D_i = \mu_i\, T^{04}$, while T_i is the algebra element defined by

$$T_i = h_i D_i h_i^{-1} \tag{16}$$

Finally $< \cdot,\cdot >$ denotes the invariant, symmetric, bilinear form on the gauge group algebra, which below will be taken to be a trace of product of appropriate matrices, normalized such that $< \mathbf{1} >= 1$.

The one-dimensional delta function $\delta(\partial\mathcal{M})$ reducing the expression to the boundary is defined by

$$\int_{\mathcal{M}} \delta(\partial\mathcal{M})(\star) = \int_{\partial\mathcal{M}} (\star) \tag{17}$$

In the action (15) the first line is the kinetic term, while the second is the constraint

$$\frac{k}{2\pi} F_S \delta(\partial\mathcal{M}) - \sum_{i=1}^{n} T_i \delta^{(3)}(x - x_{(i)}) dx^1 \wedge dx^2 \wedge dx^3 = 0 \tag{18}$$

Notice that because of topological invariance of the model the deformation of the trajectory of a particle $x_{(i)} \to x'_i(x_{(i)})$ does not change the value of the physical degrees of freedom of the model. The later are encoded in the group elements h_i at the location of the particle and not in the position of the trajectory. In particular, by such deformation one can map the whole particle trajectory $x_{(i)}$ on the boundary $\partial\mathcal{M}$ of the manifold without changing h_i.

However, the constraint (18) distinguishes the boundary, and in fact, mapping the particle trajectories to the boundary is the only way to satisfy it for a nonzero value of T_i. Indeed, if some part of the particle trajectory does not belong to the boundary, at such points the first term in (18) is zero, while the second term is equal to a constant times T_i. In such situation the constraint (18) would force T_i to be zero. As due to Bianchi identity the charge T_i has to be conserved, $T_i = 0$ along some part of trajectory means $T_i = 0$ along the whole trajectory which in turn means that there is no particle. Thus, the only way to introduce a particle satisfying constraint (18) is to map the whole particle trajectory on the boundary.

Mapping particle trajectories on the boundary is analogous to introducing the Dirac string singularity for magnetic monopole in electromagnetism. Through the Dirac string the magnetic flux can reach from the boundary to the point at which the monopole is located. In our model the situation is even simpler. In topological field theory the position of a monopole with respect to a manifold coordinates has no physical relevance. Therefore connecting a monopole to the boundary with a string is the same as placing the monopole on the boundary.

We have seen that the equations of motion of the model, namely the constraint (18), force the particle trajectories to lie on the boundary $\partial\mathcal{M}$ of the manifold. So far our considerations were classical. But we should expect these results to hold also in quantum theory because the equations of motion which are constraints on initial data, such as (18) hold in quantum theory exactly. To see this let us substitute the action (14) into the path integral over the connection A and perform integration over A_0. Among the spatial components of connection A we will distinguish A_S – the components lying within the boundary and A_3 – the component transverse to the boundary. As the action does

not depend on A_3 we will not include integration over it. We obtain

$$W_0(\{g_{p_i}\},\{g_{p_f}\}) = \int \mathcal{D}A e^{iS_{BF}[A_S,A_0,h_i]} =$$

$$\int \mathcal{D}A_S e^{i\Big(\int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi}\langle \partial_0 A_S \wedge A_S\rangle - \int_{\mathbb{R}} dx^0 \sum_{p=1}^n \big\langle D_i\,,\, h_i^{-1}\partial_0 h_i\big\rangle\Big)}$$

$$\prod_{x\in\partial M} \delta\Big(\frac{k}{2\pi}F_{S,12} - \sum_{i=1}^n T_i \delta^{(2)}(x-x_{(i)})\Big)$$

$$\prod_{x\notin\partial M} \delta\Big(\sum_{i=1}^n T_i \delta^{(`3)}(x-x_{(i)})\Big) \tag{19}$$

The last factor in the path integral in (19),

$$\prod_{x\notin\partial M} \delta\Big(\sum_{p=1}^n T_i \delta^{(`3)}(x-x_{(i)})\Big)$$

forces the partition function to be zero whenever we have a particle away from the boundary. If all the particles are on the boundary this factor is (an infinite) constant which can be absorbed in the normalization of the partition function. Therefore the path integral (19) has only contributions from the particles sitting on the boundary where the last factor can be ignored. The resulting path integral is a path integral for the action

$$S[A_S,A_0,h_i] = \int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi}\langle \partial_0 A_S \wedge A_S\rangle - \int_{\mathbb{R}} dx^0 \sum_{i=1}^n \big\langle D_i\,,\, h_i^{-1}\partial_0 h_i\big\rangle \tag{20}$$
$$+ \int_{\mathbb{R}} dx^0 \int_{S^2_n} \Big\langle A_0\,,\, \frac{k}{2\pi}F_S - \sum_{i=1}^n T_i \delta^{(2)}(x-x_{(i)}) dx^1\wedge dx^2\Big\rangle$$

of Chern–Simons theory on the boundary coupled to point charges. We will study this action in the next section.

The relation between quantum gravity in the bulk and Chern–Simons theory on the boundary was first studied in [22] and extended to include translational degrees of freedom in [23]. For the amplitudes considered in this paper the correspondence is precise. This picture is also supported by spinfoam studies [24] where it was shown that the invariants of four dimensional Crane-Yetter model are equivalent to Turaev-Viro invariants on the three dimensional boundary.

3 From Chern–Simons to particle kinematics

Let us summarize what was achieved above. We showed that in the topological limit the action for gravity coupled to the particles is equivalent to the Chern-Simon action for the gauge group $\mathsf{SO}(4,1)$, coupled to particles, carrying the charges (masses and spins) of the same group. The 3 dimensional manifold

on which this theory is defined is assumed to be a product of a punctured 2-sphere S^2_n, with each puncture corresponding to the particle, with real line $\mathbb{R}$ representing time. Thus in the zeroth order of perturbative expansion, i.e., in the "no local gravity limit of quantum gravity" our theory is described by a holographic quantum $\mathsf{SO}(4,1)$ Chern–Simons theory with particles.

In what follows we recall the construction of Alekseev and Malkin [25] (see also [4], from which we borrowed the notation). In the case when the topology of the boundary of spacetime is simple (no handles) and thus reflects only the presence of particles exhibited by punctures, the topological degrees of freedom of gravity can be absorbed by particle's. Thus effectively we obtain a (possibly deformed) theory of particles kinematics. As stressed above this theory could only depend on gauge group elements, as positions in spacetime do not play any role.

In what follows we will consider only the classical theory. The symplectic form on the space of gauge field and particles configurations can be easily found from the action (20) and reads

$$\Omega = \frac{k}{4\pi}\int_{S^2_n} \langle \delta A_S \wedge \delta A_S \rangle + \sum_{i=1}^{n} \delta \left\langle D_i, h_i^{-1}\delta h_i \right\rangle \tag{21}$$

This symplectic form is subject to the constraint

$$\frac{k}{4\pi} F_S = \sum_{i=1}^{n} T_i \delta^{(2)}(x - x_{(i)}) dx^1 \wedge dx^2 \tag{22}$$

which makes the curvature zero everywhere except for the positions of the particles. It follows that the connection A_S takes simple form at appropriately defined submanifolds of S^2_n.

Alekseev and Malkin [25] define these submanifolds as follows. Take a point p_0 away from the punctures and construct loops m_i with this base point around each puncture (Figure 1). Along these loops we will later calculate holonomies. Now cut the surface along the loops and remove the so obtained discs. As a result we obtain n punctured discs Q_i and the polygon Q_0 with no punctures inside, whose boundary will contain exactly n edges which we denote m_i and vertices p_i $p_0 \overset{m_1}{\to} p_1 \cdots p_{n-1} \overset{m_n}{\to} p_0$ (Figure 2).

Now the first term in (21) can be decomposed as follows

$$\frac{k}{4\pi}\int_{S^2_n} \langle \delta A_S \wedge \delta A_S \rangle = \frac{k}{4\pi}\int_{Q_0} \langle \delta A_S \wedge \delta A_S \rangle + \frac{k}{4\pi}\sum_{i=1}^{n}\int_{Q_i} \langle \delta A_S \wedge \delta A_S \rangle \tag{23}$$

The virtue of the decomposition (23) is that on each region the form of the connection is quite simple. Consider the region Q_0 first. Since there are no punctures in this simply connected region the constraint (22) tells that connection A_S is trivial there

$$A_S|_{Q_0} = \gamma_0 d_S \gamma_0^{-1} \tag{24}$$

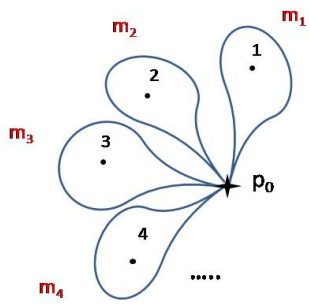

Figure 1:

Then by direct calculation one can convince oneself that the first integral in (23) reduces to the boundary one

$$\frac{k}{4\pi}\int_{Q_0}\langle\delta A_S\wedge\delta A_S\rangle=\frac{k}{4\pi}\int_{\partial Q_0}\left\langle\delta\gamma_0^{-1}\gamma_0,d(\delta\gamma_0^{-1}\gamma_0)\right\rangle \tag{25}$$

The contribution from the discs Q_i can be found by similar analysis. The only difference is that now we have to do with a region with puncture inside, carrying the charge, being an element of gauge algebra. Since the curvature must have the delta singularity at the puncture, the most general connections on the disc must be given by gauge transformations of the canonical ones. The latter are defined by

$$B_i=\frac{1}{k}\,\tilde{D}_i d\phi_i,\quad \tilde{D}_i=\frac{k}{2\pi}\,\mu_i T^{04}=\frac{k}{2\pi}\,D_i$$

where ϕ_i are angular coordinates, that along with the radial ones (defined such that the singularity corresponds to $r_i=0$) can be introduced on the disks globally. Thus

$$A_S|_{Q_i}=\frac{1}{k}\,\gamma_i\tilde{D}_i d\phi_i\gamma_i^{-1}+\gamma_i d\gamma_i^{-1} \tag{26}$$

It is easy to see that since $dd\phi_i=2\pi\delta(x-x_i)dx\wedge dy$ this connection solves the constraint (22) if $\gamma_i(x_i)=h_i$. Plugging (26) to one of the Q_i integrals in (23) we find

$$\int_{Q_i}\langle\delta A_S\wedge\delta A_S\rangle$$

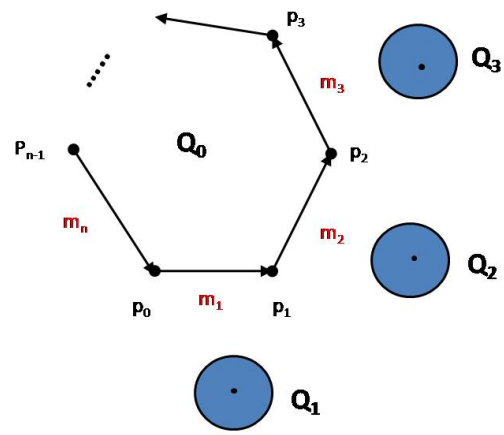

Figure 2:

$$= -\delta\left\langle \tilde{D}_i, h_i^{-1}\delta h_i\right\rangle + \frac{k}{4\pi}\int_{\partial Q_i}\left\langle \delta\gamma_i^{-1}\gamma_i d\left(\delta\gamma_i^{-1}\gamma_i\right)\right\rangle - \frac{1}{2\pi}\int_{\partial Q_i}\delta\left\langle \tilde{D}_i\delta\gamma_i^{-1}\gamma_i\right\rangle d\phi_i \tag{27}$$

Notice that the sum of the first terms in (27) cancels exactly the second term in (21). Thus the symplectic form is a sum of (25) and the last two terms in (27). Its form can be simplified further by observing that the connections on the segments of $\partial P_0 = \bigcup_i m_i$ must be equal to the connection on appropriate ∂P_i, in order to make connection continuous. We have therefore

$$A_S|_{m_i} = \gamma_0 d_S \gamma_0^{-1}\big|_{m_i} = \left(\frac{1}{k}\,\gamma_i\tilde{D}_i d\phi_i\gamma_i^{-1} + \gamma_i d\gamma_i^{-1}\right)\Big|_{\partial Q_i} = A_S|_{\partial Q_i} \tag{28}$$

This equation can be solved to give

$$\gamma_0^{-1}\big|_{m_i} = N_i \exp\left(\frac{1}{k}\,\tilde{D}_i\phi_i\right)\,\gamma_i^{-1}\big|_{m_i}\,, \quad dN_i = 0 \tag{29}$$

Substituting this expression to (25) and then, along with (27) to (23) will give us the final result.

Before doing this final step, let us consider again the polygon P_0. It has n vertices $p_0, p_1, \ldots, p_{n-1}$ connected by edges m_i. Since the connection is trivial on the polygon, the parallel transport along an edge is given simply by a product of the gauge parameters at the beginning and the end of the edge

$$PT_{p_{i-1}\to p_i} = \gamma_0(p_i)\gamma_0^{-1}(p_{i-1})$$

But because of the continuity of the connection the values of γ_0 at the vertices of the of the polygon P_0 are just the products of holonomies M_i of the connection of the disks

$$\gamma_0(p_i) \equiv K_i^{-1} = M_i M_{i-1} \cdots M_1, \quad \gamma_0(p_0) \equiv K_0^{-1} = 1 \tag{30}$$

Next, since the connection γ_i is single valued on each disk it is clear from (26) that each holonomy is of the form

$$M_i = g_i C_i^{-1} g_i^{-1}, \quad C_i = \exp\left(\frac{2\pi}{k}\,\tilde{D}_i\right), \quad g_i = \gamma_i(p_i) \tag{31}$$

Now we are ready to present the central result of this section, the Alekseev–Malkin theorem [25]: The symplectic form Ω (21) gets contributions only from vertices of the polygon Q_0 and reads

$$\Omega = \frac{k}{4\pi}\sum_{i=1}^{n}\left(\left\langle C_i g_i^{-1}\delta g_i C_i^{-1} \wedge g_i^{-1}\delta g_i\right\rangle - \left\langle \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1}\right\rangle\right) \tag{32}$$

The first term in this expression is a deformed kinetic term (a deformation of the standard free particle symplectic form $\sum \delta p_\mu \wedge \delta q^\mu$, and the second describes some "topological interaction" of the particles. Notice that as expected the final form of the symplectic structure does contain no trace of the connection; there is as many degrees of freedom as the number of particles, each described by a gauge group element. In this way the topological degrees of freedom of gravity has been "eaten" by the particle's ones.

We will discuss the theory of particles provided by this symplectic structure in the next section.

4 The deformed particle

Having obtained the symplectic form for single particle (32), let us try to understand in which sense it describes a deformed particle. To do that let us first compare it with with an analogous expression for free particle moving in 4 dimensional de Sitter spacetime in the limit of vanishing cosmological constant.

The Lagrangian for such particle is given by equation (7) and reads (following [17] we represent $\mathsf{SO}(4,1)$ generators by gamma matrices, $T^{IJ} = \gamma^{IJ} \equiv 1/2[\gamma^I, \gamma^J]$, $I, J = 0, \ldots, 4$ with γ^5 matrix denoted here γ^4)

$$L =< DA_\tau^h >= \left\langle \mu\gamma^{04}(h^{-1}A_\tau^{(0)}h + h^{-1}\dot{h})\right\rangle, \tag{33}$$

where $\mu = \ell m$ with ℓ being the cosmological length scale, related to the cosmological constant, cf. (4). Since de Sitter background is gauge equivalent to zero configuration we take $A_\tau^{(0)} = 0$ Then (33) takes the form

$$L = \left\langle \mu\gamma^{04}h^{-1}\dot{h}\right\rangle \tag{34}$$

Let us decompose the $\mathsf{SO}(4,1)$ group element into translational and Lorentz $\mathsf{SO}(3,1)$ parts (using Cartan decomposition)

$$h = \mathcal{T}\,\mathcal{L}, \quad \mathcal{T} = \mathbf{1} + \frac{q_a}{\ell}\,\gamma^{a4} + O\left(\frac{1}{\ell^2}\right), \tag{35}$$

Substituting this into (34) and keeping only the leading term after simple calculation we find (the $\mathcal{L}^{-1}\dot{\mathcal{L}}$ term is proportional to γ^{ab} and cancels under the trace)

$$L = \left\langle \frac{\mu}{\ell}\,\gamma^{04}\mathcal{L}^{-1}\gamma^{a4}\mathcal{L}\dot{q}_a \right\rangle, \quad a = 0,\ldots,3 \tag{36}$$

After some γ matrices algebra this expression gives

$$L = p_a\dot{q}^a, \quad p_a = m\left\langle \gamma^0\mathcal{L}^{-1}\gamma_a\mathcal{L}\right\rangle \tag{37}$$

and it follows that p_a defined above is to be identified with particle momentum.

Notice that it follows from (37) that

$$p_a\gamma^a = m\mathcal{L}\gamma_0\mathcal{L}^{-1} \tag{38}$$

This equation just says that there is one to one correspondence between momentum of a particle and the Lorentz transformation that boosts the particle from the rest to its actual velocity. It follows that the components of momenta on the left hand side are restricted to be on shell, $p^2 + m^2 = 0$. As usual we can treat these components as independent adding to the lagrangian the on shell constraint $p^2 + m^2$. Thus we conclude that the lagrangian (33) describes a free relativistic particle, as it should.

Returning to the starting point, eq. (33) it can be easily computed that the Lagrangian $L =< Dh^{-1}\dot{h} >$ corresponds to the symplectic form

$$\Omega_{free} =< D\delta h^{-1} \wedge \delta h >= -\frac{1}{2}\left\langle [D, h^{-1}\delta h] \wedge h^{-1}\delta h\right\rangle$$

Comparing this with the first term in (32) we see that as the result of deformation instead of commutator with algebra element $D = \mu\gamma^{04}$, in the deformed case we have to do with conjugation with group element

$$C = e^{2\pi D/k} = \exp(2\pi\mu\gamma^{04}/k) = \cosh\frac{2\pi\mu}{k}\,\mathbf{1} + \sinh\frac{2\pi\mu}{k}\,\gamma^{04}$$

It can be checked that the first "free" term in (32) reduces to

$$\Omega = \frac{k}{4\pi}\cosh\frac{2\pi\mu}{k}\,\sinh\frac{2\pi\mu}{k}\,\left\langle [\gamma^{04},\, g^{-1}\delta g] \wedge g^{-1}\delta g\right\rangle \tag{39}$$

Now we have to recall that the Chern–Simons coupling constant is related to the original coupling constant β: $k/4\pi = 1/\beta \sim \ell^2\kappa^2$, where $\kappa = \hbar/G$. Therefore the prefactor in Ω in (39) becomes, in the leading order in ℓ, $\frac{1}{2}\,m\ell$. One can then easily check using definition of momenta (38) and the expansion

(35) that (39) exactly reproduces (up to the sign and a prefactor depending on Immirzi parameter) the free particle symplectic structure. We conclude that the first term in (32) is just a sum of free particle actions.

Let us now turn to the second, interaction term, in the symplectic form (32). As an example consider the case of two particles. In this case the interaction term reads

$$-\frac{k}{4\pi}\left\langle \delta K_2 K_2^{-1} \wedge \delta K_1 K_1^{-1} \right\rangle, \quad K_1 = M_1^{-1}, \quad K_2 = M_1^{-1} M_2^{-1} \tag{40}$$

To calculate this let us first expand the holonomy in powers of ℓ, by making use of the Cartan decomposition of the group (35) and definition of momenta (38)

$$M^{-1} = gCg^{-1} = \mathbf{1} + \frac{1}{\kappa^2 \ell} p_a \gamma^a \gamma^4 + O\left(\frac{1}{\ell^2}\right) \tag{41}$$

Thus

$$-\frac{k}{4\pi}\left\langle \delta K_2 K_2^{-1} \wedge \delta K_1 K_1^{-1} \right\rangle =$$

$$-\frac{1}{\kappa^2}\left(\delta p^{(2)}{}_a + \delta p^{(1)}{}_a\right) \wedge \delta p^{(1)}{}_b \left\langle \gamma^a \gamma^4 \gamma^b \gamma^4 \right\rangle = \frac{1}{\kappa^2}\, \delta p^{(2)}{}_a \wedge \delta p^{(1)a} \tag{42}$$

Equation (42) can be easily generalized to an arbitrary number of particles, and knowing the symplectic form one can readily reproduce the lagrangian for the n-particles system. It is a sum of the standard kinetic terms and the on shell constraints for each particle along with the interaction terms

$$L = \sum_{i=1}^{n} \left(p^{(i)} \cdot \frac{d}{d\tau} q^{(i)} + \lambda^{(i)} \left[p^{(i)2} + m^{(i)2} \right] + \frac{1}{\kappa^2} p^{(i)} \cdot \sum_{j=1}^{i-1} \frac{d}{d\tau} p^{(j)} \right) \tag{43}$$

where τ is the affine parameter and $\lambda(\tau)$ is the Lagrange multiplier enforcing the mass shell constraints, and $\cdot$ is the standard Minkowski product.

Equation (43) is the final result of our paper. It shows that in the action of n-particles system, the actions of each particle is not deformed while the deformation arises in the form of the presence of the additional interaction terms. It is worth noticing that this deformation does not change equations of motion (because the momenta $p^{(i)}$ are constants of motion), while it certainly changes the Poisson brackets. In particular the brackets of positions will not vanish anymore. However, it is easy to see that by simple change of variables

$$q^{(i)} \longrightarrow \tilde{q}^{(i)} = q^{(i)} + \frac{1}{\kappa^2} \sum_{j=1}^{i-1} \frac{d}{d\tau} p^{(j)} \tag{44}$$

the lagrangian (43) can be cast into the form of the one of the standard system of free relativistic particles.

5 Discussion

The results of the preceding section can be regarded surprising. Building on the experience with $3d$ gravity one would expect that the deformation of the final particle action is essentially guaranteed, and the recent investigations reported in [27] suggested that the deformation was to be of κ-Poincarè form [28]. Instead what we got was just a standard undeformed relativistic particles action. Let us therefore try to understand this result.

To get some more insight let us try to investigate how the results above change if we go beyond the the leading order in large cosmological scale ℓ expansion. Our starting point is again the formula (39)

$$\Omega = \frac{k}{4\pi}\,\sinh\frac{4\pi\mu}{k}\,\left\langle \gamma^{04}\, g^{-1}\delta g \wedge g^{-1}\delta g\right\rangle \tag{45}$$

Using the definition of momenta and Cartan decomposition as above one computes

$$\Omega = \frac{k}{4\pi m}\,\sinh\frac{4\pi\mu}{k}\,\left(p_a\left\langle \gamma^{a4}\,\mathcal{T}^{-1}\delta\mathcal{T}\wedge\mathcal{T}^{-1}\delta\mathcal{T}\right\rangle - \delta p_a\wedge\left\langle\gamma^{a4}\,\mathcal{T}^{-1}\delta\mathcal{T}\right\rangle\right) \tag{46}$$

with momenta restricted to be on-shell $p^2+m^2=0$. It is now convenient to parametrize the translational part of the group as follows

$$\mathcal{T} = Q_4\,\mathbf{1} + Q_a\,\gamma^a\gamma^4 \tag{47}$$

with (dimensionless) positions Q belonging to de Sitter space

$$Q_4^2 + Q^2 = 1, \quad Q^2 \equiv -Q_0^2 + \vec{Q}^2 \tag{48}$$

Plugging this into the symplectic form above after straightforward calculations we find

$$\Omega = \frac{k}{4\pi m}\,\sinh\frac{4\pi\mu}{k}\,\delta\left(Q_4\,p_a\,\delta Q^a + \frac{1}{Q_4}\,p_a Q^a\,Q_b\,\delta Q^b\right) \tag{49}$$

It follows that the lagrangian is again up to the irrelevant prefactor just the one of a relativistic particle moving on de Sitter background: defining $q^a = \ell Q^a$ we have

$$L = \frac{1}{\ell}\,\sqrt{1-q^2/\ell^2}\,p_a\dot{q}^a + \frac{1}{\ell^3}\,\frac{1}{\sqrt{1-q^2/\ell^2}}\,p_a q^a\,q_b\dot{q}^b + \lambda\left[p^2+m^2\right] \tag{50}$$

Notice that again we see no trace of any deformation and both the symplectic structure and the lagrangian are linear in momenta, which makes the position space commutative.

However we were not able yet to calculate the second "interaction" terms of the symplectic structure beyond the leading term in large ℓ expansion, which may change the picture considerably. It is a general result of Alekseev and Malkin [25] that such interactions can be removed by an appropriate symplectic transformation. In this transformation the Borel subroup of the original gauge

group is known to play a role, and on the other hand it is directly related to κ-Poincarè algebra and κ-Minkowski space [29]. One should also remember that in order to get the κ-Poincarè algebra as a contraction limit of $\ell \to \infty$, a non-trivial rescaling of momenta is required (see [28] and [30] for details.) Although such rescaling is well motivated mathematically, it is not clear how to justify it physically. We are going to address all these questions in the forthcoming paper.

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