HAL
open science

Spin-Foam Models and the Physical Scalar Product

Emanuele Alesci, Karim Noui, Francesco Sardelli

▶ **To cite this version:**

Emanuele Alesci, Karim Noui, Francesco Sardelli. Spin-Foam Models and the Physical Scalar Product. Physical Review D, 2008, 78. hal-00350214

HAL Id: hal-00350214

https://hal.science/hal-00350214v1

Submitted on 5 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:0807.3561v1 [gr-qc] 22 Jul 2008

Spin-Foam Models and the Physical Scalar Product

Emanuele Alesci[1ab], Karim Noui[2c] , Francesco Sardelli[3cd]

[a] *Laboratoire de Physique, ENS Lyon, CNRS UMR 5672, 46 Allée d'Italie, 69007 Lyon, France EU*
[b] *Centre de Physique Théorique de Luminy*, Université de la Méditerranée, F-13288 Marseille EU*
[c] *Laboratoire de Mathématiques et de Physique Théorique*, Parc de Grammont, 37200 Tours EU*
[d] *Dipartimento di Fisica Enrico Fermi, Università di Pisa, 56127 Pisa EU*

October 24, 2018

Abstract

This paper aims at clarifying the link between Loop Quantum Gravity and Spin-Foam models in four dimensions. Starting from the canonical framework, we construct an operator P acting on the space of cylindrical functions $\mathrm{Cyl}(\Gamma)$, where Γ is the 4-simplex graph, such that its matrix elements are, up to some normalization factors, the vertex amplitude of Spin-Foam models. The Spin-Foam models we are considering are the topological model, the Barrett-Crane model and the Engle-Pereira-Rovelli model. The operator P is usually called the "projector" into physical states and its matrix elements gives the physical scalar product. Therefore, we relate the physical scalar product of Loop Quantum Gravity to vertex amplitudes of some Spin-Foam models. We discuss the possibility to extend the action of P to any cylindrical functions on the space manifold.

Introduction

Finding the physical scalar product is certainly one of the most important question of Loop Quantum Gravity [1, 2]. This is somehow equivalent to the problem of finding solutions of the remaining scalar constraint which is, so far, still an open issue. Two main and very active directions have been followed to tackle the problem: (i) formulating consistently the scalar constraint as a well-defined operator acting on the kinematical Hilbert space; (ii) making sense of the covariant quantization to compute physical transitions amplitudes between states of quantum geometry. The former has been explored mainly by Thiemann[3] and collaborators: very tricky and very nice regularizations of the scalar constraints have been found; the important question is now to extract physical solutions out of it. The master constraint program [4] has been considered to that aim. Spin-Foam models [5] are the covariant alternative attempt to solve the problem: they propose a way to "compute" the path integral of gravity where space-time appears as a combinatorial foam which can be understood as a covariant generalization of the notion of spin-networks. Then a spin-foam is somehow interpreted as the structure which encodes the "time evolution" of a state of quantum gravity. Spin-Foam models have been studied intensively these last years to answer some fundamental questions they have raised, two of the most important being the following: What is the precise link between Spin-Foam models and the path integral of quantum gravity? Can we establish an explicit link between Spin-Foam models and Loop Quantum Gravity as in the three dimensional case [6] ?

To understand the meanning of the first question, it is worth recalling that Spin-Foam models are only ansatz for the path integral of quantum gravity. The ansatz is based on the Plebanski formulation of general relativity [7] where gravity appears as a topological BF theory supplemented

*Unité mixte de recherche (UMR 6207) du CNRS et des Universités de Provence (Aix-Marseille I), de la Mediterranée (Aix-Marseille II) et du Sud (Toulon-Var); laboratoire affilié à la FRUMAM (FR 2291).

†UMR/CNRS 6083, Fédération Denis Poisson

e-mail: [1]alesci@fis.uniroma3.it , [2]noui@lmpt.univ-tours.fr , [3]francesco.sardelli@yahoo.it

with simplicity constraints on the B field. The path integral of a (Euclidean) BF theory is a topological invariant which can be reformulated "exactly" as a Spin-Foam model which is called, in a more mathematical language, a state sum model. The natural idea is to try to impose the simplicity constraints at the level of the path integral to get a Spin-Foam model for gravity. Barrett and Crane (BC) [8] proposed a first model: it was studied a lot but recently it was shown not to reproduce expected behavior at the semi-classical limit [9] while computing the 2-points functions of gravity in the context of LQG propagator calculations [10]. It was then realized that the way Barrett and Crane had imposed the simplicity constraints at the level of the Spin-Foam would have been, in a sense, too strong. Engle, Pereira and Rovelli (EPR) have proposed a new model [11] which seems a more promising candidate: in a subsequent paper with Livine [12], they have proposed a way to impose the simplicity constraints using the "master constraint" techniques introduced in the context of canonical quantization by Thiemann. One can incorporate the Immirzi parameter in the new model and it is possible to extend it to the Lorenzian case[13]. In the meanwhile another model from Freidel and Krasnov (FK) [14] has appeared. The FK model instead imposes the constraints using the coherent states techniques introduced by Livine and Speziale [15]. All these new models are under study at this moment [16] in order to see, in particular, if they behave correctly in the classical and semi-classical limits [17].

The second question concerning the link between canonical and covariant quantizations of gravity has been quite problematic for a long time: the Lorentzian BC model seemed incompatible with Loop Quantum Gravity because it is known that the spectra of the area operator are not identical in the two approaches. Covariant Loop Quantum Gravity [18] was introduced to repair this problem modifying (in a covariant way) the canonical quantization: the obtained theory is unfortunately too cumbersome to be useful for the moment. Instead of modifying the canonical quantization, one could consider standard Loop Quantm Gravity as the good framework for the canonical quantization of gravity and think about finding a Spin-Foam model consistent with this approach. This is exactly what the new EPR model is doing: the projected states of the new model are the standard spin-network states; the spectrum of the area operators in the covariant quantization is the same as the one in the canonical quantization. Therefore, the EPR model seems to be a good candidate to test if Spin-Foam models can explicitely realize a "projection" (in the sense of Loop Quantum Gravity) into physical states. Indeed, we expect the physical scalar product between two spin-network states to be given by the Spin-Foam amplitude associated to a graph whose boundaries are the two given spin-networks.

This article aims at clarifying this relation with a simple example. We consider Euclidean Spin-Foam models associated to the group $G = SU(2) \times SU(2)$. It is characterized by its vertex amplitude V: the vertex amplitude is the weight associated to a 4-simplex; it is therefore a function $V(I_{ij}, \omega_i)$ of the G-representations I_{ij} coloring the 10 faces of the 4-simplex and of the G-intertwiners ω_i associated to the 5 tetrahedra of the 4-simplex. The index i runs from 1 to 5 and labels the five tetrahedra in the boundary of the 4-simplex. We want to interpret this vertex amplitude V as the physical scalar product between two spin-network states: the 1-tetrahedron state τ_1 and the 4-tetrahedra state τ_4 associated to spin-networks respectively dual to one tetrahedron and to four tetrahedra as illustrated in the figure (1). The free ends of these spin-networks coincinde and therefore τ_1 and τ_4 are particular cylindrical functions of the same graph, denoted $\tilde{\Gamma}$, as illustrated in the figure (9) in the core of the paper. The graph $\tilde{\Gamma}$ is the union of the 4-simplex graph Γ with four free edges and it was introduced to take into account the free ends of the states τ_1 and τ_4.

More precisely, we construct on operator P acting on the space of cylindrical functions $\mathrm{Cyl}(\tilde{\Gamma})$ such that its matrix elements are related to the vertex amplitude of Spin-Foam models as follows:

$$\langle \tau_4, P\tau_1 \rangle \;=\; N\, V(I_{ij}, \omega_i) \tag{1}$$

where N is an eventual normalization factor. In that sense, the matrix element $\langle \tau_4, P\tau_1 \rangle$ would be the physical scalar product between the kinematical states τ_1 and τ_4. In fact, the bra-ket notation for the physical scalar product might be misleading because mathematically P is a linear form on the space $\mathrm{Cyl}(\tilde{\Gamma})$, i.e. $P \in \mathrm{Cyl}(\tilde{\Gamma})^*$, abusively called a "projector", and the physical scalar

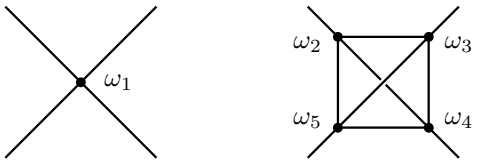


Figure 1: Illustration of the 1-tetrahedron state τ_1 on the left and the 4-tetrahedron state τ_4 on the right. Vertices, labelled by $i \in \{0, 5\}$, are colored with intertwiners ω_i and edges ℓ_{ij} with representations I_{ij}. The 4 free ends are colored with representations I_{1i}.

product is $\langle \tau_4, P\tau_1 \rangle = P(\overline{\tau_4}\tau_1)$. In the context of Gelfand-Naimark-Segal theory (see the Ashtekar-Lewandowski review [1] and references therein), P, if it satisfies some additional properties, would be a state and would allow to construct the whole physical Hilbert space in principle.

We find a solution for the projector P for different Spin-Foam models: the topological $SU(2)$ BF model whose vertex V_{BF} is the 15j symbol of $SU(2)$ (this system has no physical relevence); the BC model whose vertex V_{BC} is the well-known 10j symbol; the new model whose vertex V_{EPR} has been defined recently and also the FK model whose vertex construction is a direct extension of the EPR one (in this paper we concentrate only on the vertex amplitude without discussing the measure factors associated to the FK model, see [14]). The projector P_{BF} associated to the topological model is a multiplicative operator which acts only on the edges of the spin-networks and imposes that the connection is flat. The projectors P_{BC} and P_{EPR} respectively associated to the BC and the EPR models act both on the vertices (as derivative operators, in the sense that it involves left and right invariant derivatives) and on the edges of spin-networks. Note that we construct one solution of P and we do not precisely address the question of the unicity in this article.

The plan of this article is the following. In Section 1, we propose a simple and general integral formula of the vertex amplitudes of Euclidean 4 dimensional Spin-Foam models. It is quite an universal formula for it contains as particular cases the vertices of all the known models as the topological, the BC and the EPR models. In Section 2, we make use of this formula to construct physical operators for each model in a way similar to the 3 dimensional case. More precisely, we find a solution to the equation (1) for each model and we discuss the properties of these solutions. We conclude with some perspectives.

1 The vertex of a Spin-Foam model

In this Section, we present some properties concerning the vertex amplitude of several Spin-Foam models. The notion of vertex amplitude is defined in the first part where we give a very brief introduction on Spin-Foam models. In a second part, we propose a general and rather simple integral formula for the vertex amplitude which will be useful in the next Section to make a link with the canonical quantization. In the last part of this Section, we illustrate this formula in the particular models we are interested in, namely the topological, the BC and the EPR models. Furthermore, we underline that the BC and the EPR models are particular cases of a large class of Spin-Foam models. We present the construction of this class of Spin-Foam models and we show that their vertex amplitude admits an integral formulation of the same type.

1.1 A brief introduction on Spin-Foam models

A Spin-Foam model is basicly the assignment of a complex amplitude $\mathcal{A}(\mathcal{T})$ to any triangulation $\mathcal{T}$ of a given four dimensional manifold $\mathcal{M}$. The triangulation consists in the union $\cup_{i=2}^{4}\mathcal{T}_i$ of the set of its faces $\mathcal{T}_2$, the set of its tetrahedra $\mathcal{T}_3$ and the set of its 4-simplices $\mathcal{T}_4$. The amplitude $\mathcal{A}$ is constructed from the representation theory of a given Lie group G that we assume compact for simplicity. To do so, one first colors each face $f \in \mathcal{T}_2$ with an unitary irreducible representation

(UIR) j_f of G and each tetrahedron $t \in \mathcal{T}_3$ with intertwiners ι_t between representations coloring its four faces. Then, one associates an amplitude $\mathcal{A}_2(j_f)$ to each face f, an amplitude $\mathcal{A}_3(\omega_t, j_{f_t})$ to each tetrahedron t which depends on the intertwiner ω_t and on the representations coloring its 4 faces f_t, and an amplitude $V(\omega_{t_s}, j_{f_s})$ to each 4-simplex s which depends on the representations j_{f_s} and ω_{t_s} coloring its 10 faces f_s and 5 tetrahedra t_s. Finally, the spin-foam amplitude is formally defined by the series

$$\mathcal{A}(\mathcal{T}) \equiv \sum_{\{j_f\},\{\omega_t\}} \prod_{f\in\mathcal{T}_2} \mathcal{A}_2(j_f) \prod_{t\in\mathcal{T}_3} \mathcal{A}_3(\omega_t, j_{f_t}) \prod_{s\in\mathcal{T}_4} V(\omega_{t_s}, j_{f_s}) \tag{2}$$

where the sum runs into a certain subset of UIR and intertwiners of G. The sum is a priori infinite and therefore the amplitude is only defined formally at this stage unless it is convergent. Notice that in all the models that have been studied in the literature, the amplitude $\mathcal{A}_3$ is assumed to depends on the intertwiners ω_t only. The function V is precisely the vertex amplitude of the Spin-Foam model. To finish with this brief introduction of Spin-Foam models, let us mention that the previous construction could be generalized to the case where G is non-compact and to the case where G is replaced by a quantum group. Spin-Foam models can also be defined for any dimensional manifold $\mathcal{M}$.

In this paper, we consider exclusively the case where $\mathcal{M}$ is 4-dimensional and we study some properties of the vertex amplitude V only. Therefore, we will not mention the amplitudes $\mathcal{A}_2$ and $\mathcal{A}_3$ when we discuss the Spin-Foam models in the sequel; as a result, we will omit any discussion concerning the amplitude $\mathcal{A}$ and a fortiori the question of its convergence. We hope to study these aspects in the future. Furthermore, we will consider Euclidean Spin-Foam models only which are associated to the compact Lie groups $G = SU(2)$ (for the topological model) or $G = SU(2)\times SU(2)$ (for the BC and EPR models). Letters $I, J, \cdots$ label unitary irreducible representations of the group G and the associated vector spaces are denoted $U_I, U_J, \cdots$. When $G = SU(2)$, I is a half-interger whereas it is a couple of half-integers when $G = SU(2)\times SU(2)$. Due to the compactness of G, each representation I is finite dimensional and associates to any $g \in G$ a finite dimensional matrix which will be denoted $R^I(g)$ when $G = SU(2) \times SU(2)$ and $D^I(g)$ in the other case. To a representation I is associated a contragredient (or a dual) representation I^* such that $R^{I^*}(g) = {}^tR^I(g^{-1})$ and the same for the $SU(2)$ representations D^{I^*}; it is common to identify $U_I^* \equiv U_{I^*}$ to U_I. More precision concerning the representation theory of the groups G will be given later.

The vertex $V(I_{ij}, \omega_i)$ is then a function of the 5 intertwiners ω_i coloring the 5 tetrahedra (which are ordered and labelled by $i \in \{1, 5\}$) of a 4-simplex and of the 10 representations $(I_{ij})_{i<j}$ of G coloring the 10 faces at the intersections of the tetrahedra i and j; $\omega_i : \otimes_{j>i} U_{I_{ij}} \to \otimes_{j<i} U_{I_{ji}}$ is an intertwiner between the representations I_{ij} "meeting" at the tetrahedron i. In the next part, we are going to show that the vertex amplitude of all the models we consider can be written as an integral over 10 copies of the 3-sphere S^3 as follows:

$$V(I_{ij}, \omega_i) = \int \big(\prod_{i<j} dx_{ij}\big)\, C(x_{ij})\, \mathcal{V}(I_{ij}, \omega_i; x_{ij}) \tag{3}$$

where $C(x_{ij})$ is a universal function, in the sense that it is model independent, which reads

$$C(x_{ij}) \equiv \int \big(\prod_{i=1}^{5} dx_i\big)\, \delta(x_{ij}^{-1} x_i x_j^{-1})\ . \tag{4}$$

$\mathcal{V}$ is a model dependent function of the variables x_{ij}. As we will see in the next Section, such a formula will be crucial to link Spin-Foam models with Loop Quantum Gravity.

1.2 A General expression of the vertex

There exists many equivalent ways to define the vertex amplitude of a Spin-Foam models. For our purposes, it is convenient to view the vertex amplitude as a "Feynman graph" evaluation of a

closed oriented graph which is dual to a 4-simplex. The dual of a 4-simplex Γ is in fact topologically equivalent to a 4-simplex and then consists in a set of 5 vertices linked by 10 edges: we endow the set of vertices with a linear ordering such that the vertices are labelled with an integer $i \in \{1, 5\}$; this ordering induces a natural orientation on the links, indeed the link ℓ_{ij} between the edges i and j is oriented from i to j if $i < j$. One associates a complex amplitude to this graph using the following "Feynman" rules: each oriented link ℓ_{ij}, with $i < j$, is associated to a UIR of G denoted I_{ij} (the opposite link ℓ_{ji} is associated to the contragredient representation denoted for simplicity $I_{ji} = I_{ij}^*$); each vertex i is associated to an intertwiner $\omega_i : \otimes_{j>i} U_{I_{ij}} \to \otimes_{j<i} U_{I_{ji}}$. As a result, the "Feynman evaluation" of such a graph is the scalar obtained by contracting the 10 propagators with the 5 intertwiners and gives the vertex amplitude which formally reads:

$$V(I_{ij}, \omega_i) \;=\; \langle \bigotimes_{i=1}^{5} \omega_i \rangle \;\equiv\; \sum_{\{e_{ij}\}} \prod_{i=1}^{5} \langle \bigotimes_{j<i} e_{ji} | \omega_i | \bigotimes_{j>i} e_{ij} \rangle \tag{5}$$

where e_{ij} runs over the finite set of a given orthonormal basis of $U_{I_{ij}}$ and we have used the standard bra-ket notation to denote the vectors $|e_{ij}\rangle$ of $U_{I_{ij}}$ and the dual vectors $\langle e_{ij}|$. In the language of Loop Quantum Gravity, we would say that $V(I_{ij}, \omega_i)$ is simply the evaluation of the spin-network associated to the colored graph $(\Gamma, \{I_{ij}, \omega_i\})$ when the connection is flat.

In order to have a more useful formula, it will be convenient to trivially identify ω_i with an element of $\mathrm{Hom}(\otimes_{j\neq i} U_{I_{ij}}, \mathbb{C})$ and then to notice that ω_i is completely caracterized by a vector $v_i \in \otimes_{j\neq i} U_{I_{ij}}^*$. These vectors can be written in the form $v_i = \sum_{(a_{ij})} \alpha_i^{(a_{ij})} \otimes_{j\neq i} v_{a_{ij}}$ where $(a_{ij})_{j\neq i}$ is a set whose elements label vectors $v_{a_{ij}} \in U_{I_{ij}}$, $\alpha_i^{(a_{ij})}$ are complex numbers and the sum is finite. The explicit relation between ω_i and v_i is the following:

$$\omega_i \;=\; \langle v_i | \int dg \bigotimes_{j\neq i} R^{I_{ij}}(g) \;\in\; \mathrm{Hom}(\bigotimes_{j\neq i} U_{I_{ij}}, \mathbb{C}) \tag{6}$$

where we have used the $SU(2) \times SU(2)$ notations for the representations and $\int dg$ is the Haar measure of G. As a result, the vertex amplitude can be reformulated as a multi-integral over G according to the formula:

$$V(I_{ij}, \omega_i) \;=\; \sum_{(a_{ij})} \prod_{i=1}^{5} \alpha_i^{(a_{ij})} \int (\prod_{i=1}^{5} dg_i) \, \langle \otimes_{i<j} v_{a_{ij}} | \bigotimes_{i<j} R^{I_{ij}}(g_i g_j^{-1}) | \otimes_{i>j} v_{a_{ij}} \rangle \tag{7}$$

which can be written in the following more compact well-known form

$$V(I_{ij}, \omega_i) \;=\; \int (\prod_{i=1}^{5} dg_i) \, (\otimes_{i=1}^{5} v_i) \cdot (\bigotimes_{i<j} R^{I_{ij}}(g_i g_j^{-1})) \tag{8}$$

where the dot $\cdot$ denotes the appropriate contraction between the vectors v_i and the matrices of the representations. This vertex amplitude is in fact rather general and caracterizes partially a large class of Spin-Foam models. It is general because we have for the moment a total freedom in the choice of the representations and the intertwiners; it is nonetheless only partial because we do not consider the amplitudes associated to faces and tetrahedra. To go further in the study of this amplitude, we need to recall some basic results on the representation theory of $SU(2) \times SU(2)$.

1.2.1 Representation theory of G: basic results

Let us start with the group $SU(2)$: its representations are labelled by a half-integer, the spin I; they are finite dimensional of dimension $d_I = 2I + 1$ and we denote by $|I, i\rangle$ with $i \in [-I, I]$ the vector of an orthonormal basis of U_I. The group $G = SU(2) \times SU(2)$ is the double cover of $SO(4)$; it is also known as the spin group Spin(4). Any of its elements g can be written as a couple (g_L, g_R) of two $SU(2)$ group elements. Its Unitary Irreducible Representations (UIR) are

labelled by a couple of (integers or half-integers) spins (I,J): they are finite dimensional and the vector space $U_{IJ}=U_I\otimes U_J$ of the representation (I,J) is the tensor product of the two $SU(2)$ representations vector spaces U_I and U_J. Therefore, the family of vectors $(|I,i\rangle\otimes|J,j\rangle)_{IJij}$ form an orthonormal basis of U_{IJ}. The action of $g\in G$ in this basis is simply given by:

$$R^{IJ}(g)|I,i\rangle\otimes|J,j\rangle \;=\; R^{IJ}(g_L,g_R)|I,i\rangle\otimes|J,j\rangle \;=\; D^I(g_L)|I,i\rangle\otimes D^J(g_R)|J,j\rangle\;. \tag{9}$$

The $SU(2)$ matrix elements $\langle I,i|D^I|I,j\rangle$ are the Wigner functions.

The space U_{IJ} admits another natural basis which will be useful in the sequel. This other basis is constructed from the remark that the vector space U_{IJ} decomposes into $SU(2)$ UIR vector spaces U_K as follows:

$$U_{IJ}\;\simeq\;\bigoplus_{K=|I-J|}^{I+J}U_K\;. \tag{10}$$

This decomposition provides indeed another orthonormal basis of U_{IJ}, given by the family of vectors $(|K,k\rangle)_{Kk}$ where $K\in[|I-J|,I+J]$ and $k\in[-K,K]$ as usual. The changing basis formulae are given in terms of the Clebsch-Gordan coefficients $\langle Kk|IiJj\rangle$ as follows:

$$|Ii\rangle\otimes|Jj\rangle=\sum_{K,k}\langle Kk|IiJi\rangle\,|Kk\rangle\quad\text{and}\quad|Kk\rangle=\sum_{IJij}\langle Kk|IiJi\rangle\,|Ii\rangle\otimes|Jj\rangle\;. \tag{11}$$

To write the action of G on the basis elements $|Kk\rangle$, it is convenient to find the subgroup $H\subset G$ which leaves the subspaces U_K of the decomposition (10) invariant and then to identify G with the space $G\simeq H\times(G/H)$. In fact, it is immediate to see that $H\simeq SU(2)$, the coset G/H is isomorphic to the sphere S^3 and then we identify G with $SU(2)\times S^3$. Notice that the identification we have just mentionned is not canonical because G admits many $SU(2)$ subgroups; therefore, to make this identification well defined, one has to precise which $SU(2)$ subgroup one is talking about. In our case, the $SU(2)$ subgroup is the diagonal one, i.e. it is the group of the elements (g_L,g_R) where $g_L=g_R$. As a result, the explicit mapping between G and $SU(2)\times S^3$ is:

$$G\longrightarrow SU(2)\times S^3\qquad(g_L,g_R)=(u,ux)\longmapsto(u,x)=(g_L,g_L^{-1}g_R)\,. \tag{12}$$

This mapping is of course invertible and its inverse is trivially given by:

$$SU(2)\times S^3\longrightarrow G\qquad(u,x)\longmapsto(u,ux)\,. \tag{13}$$

The multiplication law $(g_L,g_R)(g'_L,g'_R)=(g_Lg'_L,g_Rg'_R)$ induces the multiplication rule

$$(u,x)(u',x')\;=\;(uu',u'^{-1}xu'x') \tag{14}$$

in the $SU(2)\times S^3$ representation of G. In particular, the inverse of the element (u,x) is given by $(u,x)^{-1}=(u^{-1},ux^{-1}u^{-1})$. The diagonal terms $u\equiv(u,1)$ and the pure spherical terms $x\equiv(1,x)$ will be relevant in the following construction.

Let us now come back to the action of G on the the vectors $|K,k\rangle$ of the vector space U_{IJ}; this action is best written and simpler using the factorization $SU(2)\times S^3$ of G. Indeed, a simple calculation shows that

$$\begin{aligned}R^{IJ}_{KkL\ell}(u)&=R^{IJ}_{KkL\ell}(u,u)=\sum_{m_1,m_2}\langle Kk|Im_1Jm_2\rangle D^I_{m_1,n_1}(g_L)D^J_{m_2,n_2}(u)\langle In_1Jn_2|L\ell\rangle=\delta_{K,L}D^K_{k\ell}(u)\\ R^{IJ}_{KkL\ell}(x)&=R^{IJ}_{KkL\ell}(1,x)=\sum_{ijj'}\langle Kk|IiJj\rangle\,\langle IiJj'|L\ell\rangle\,D^J_{jj'}(x)\;.\end{aligned} \tag{15}$$

where we have introduced the notation $R^{IJ}_{KkL\ell}(g)\equiv\langle Kk|R^{IJ}(g)|L\ell\rangle$ for the $SU(2)\times SU(2)$ matrix elements. As expected, we see that $u\in SU(2)$ leaves any $SU(2)$ representation spaces U_K of the decomposition (10) invariant whereas x moves the vectors from one $SU(2)$ representation space to another. This closes the brief review on $SU(2)\times SU(2)$ representations theory.

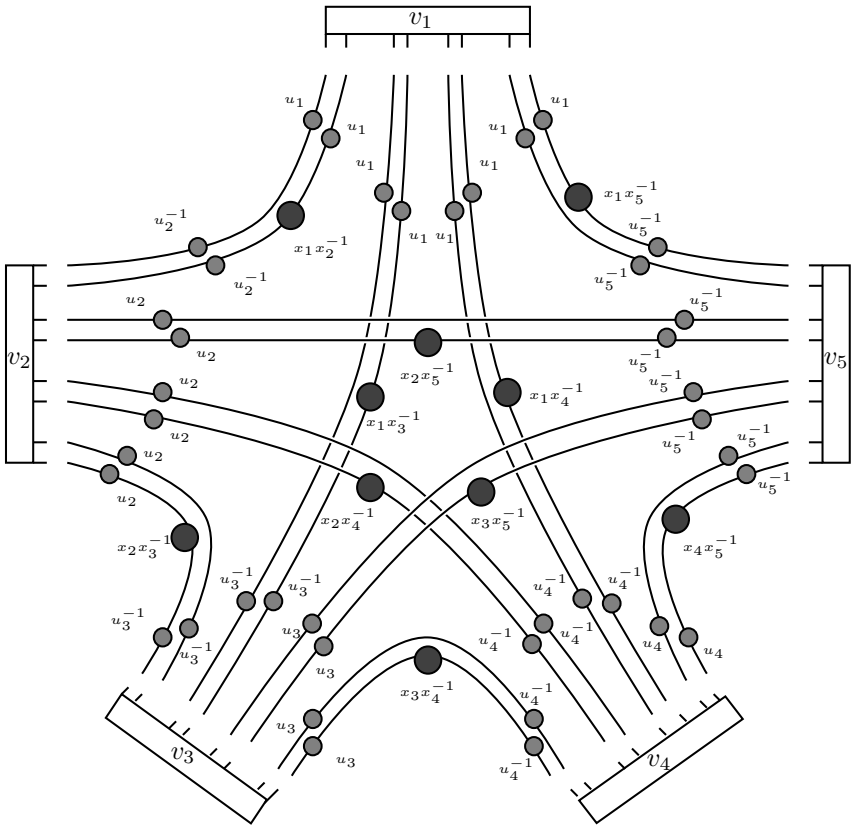


Figure 2: This picture is a graphical representation of the integrand in the formula (16) defining the vertex amplitude. Each line are doubled because it carries a representation of $SU(2) \times SU(2)$ and the single lines in the pair colored with (I, J) are colored by I and J separately. Furthermore, the single lines are endowed with bullets that represent the insertion of $SU(2)$ group elements: the small ones are associated to diagonal elements $u_i \in SU(2)$ whereas the big ones are associated to spherical elements $x_i x_j^{-1} \in S^3$. The vectors v_i are represented by boxes and they are contracted with the free ends of the graph.

1.2.2 The vertex amplitude as an integral over several copies of S^3

We make use of the basic properties on representations theory recalled above to write the general formula of the vertex amplitude (8) in the form (3). To do so, one splits the integrations over the group variables $g_i \in G$ in the formula (8) into integrations over the $x_i \in S^3$ variables and integrations over the $u_i \in SU(2)$ variables using the isomorphism (12) and one obtains:

$$V(I_{ij}, \omega_i) \;=\; \int (\prod_{i=1}^{5} dx_i)(\prod_{i=1}^{5} du_i)\,(\otimes_{i=1}^{5} v_i) \cdot (\bigotimes_{i<j} R^{I_{ij}}(u_i) R^{I_{ij}}(x_i x_j^{-1}) R^{I_{ij}}(u_j^{-1})) \qquad (16)$$

where $R^I(u) \equiv R^I(u,1)$ (resp. $R^I(u,u)$) and $R^I(x) \equiv R^I(1,x)$ (resp. $R^I(1,x)$) are the matrices of $SU(2) \times SU(2)$ representations I in the $SU(2) \times S^3$ (resp. $SU(2) \times SU(2)$) formulations. To have a "geometrical" intuition of this formula, we give a graphical representation of the integrand in the figure (2) below. In the models we are going to consider explicitely in the sequel, we can perform the integrations over the u_i variables; therefore we formally perform the integration over the u_i's in the general formula (16) and we obtain a formula for the vertex amplitude as an integral over 5

copies of S^3 only:

$$V(I_{ij},\omega_i) \;=\; \int (\prod_{i=1}^{5} dx_i)\,(\otimes_{i=1}^{5}\nu_i)\cdot(\bigotimes_{i<j} R^{I_{ij}}(x_i x_j^{-1}))\,. \tag{17}$$

The integrations over the five $SU(2)$ variables u_i have been hidden in the following definition of the vectors $\nu_i \in \otimes_{j\neq i} U^*_{I_{ij}}$:

$$\nu_i \;\equiv\; \sum_{(a_{ij})} \alpha_i^{(a_{ij})} \int du\;(\otimes_{j>i}\langle v_{a_{ij}}|R^{I_{ij}}(u_i))\;\otimes\;(\otimes_{j<i} R^{I_{ji}}(u)^{-1}|v_{a_{ij}}\rangle) \tag{18}$$

where we have used the explicit decomposition of the vectors $v_i \in \otimes_{j\neq i} U^*_{I_{ij}}$ given in the introductive part of Section 1.2. This formula will be much more explicit when we consider the particular Spin-Foam models we are interested in. For the moment, for pedagogical purposes, we propose a picturial representation in the figure (3) of the argument of the previous integral (18) when $i = 1$.

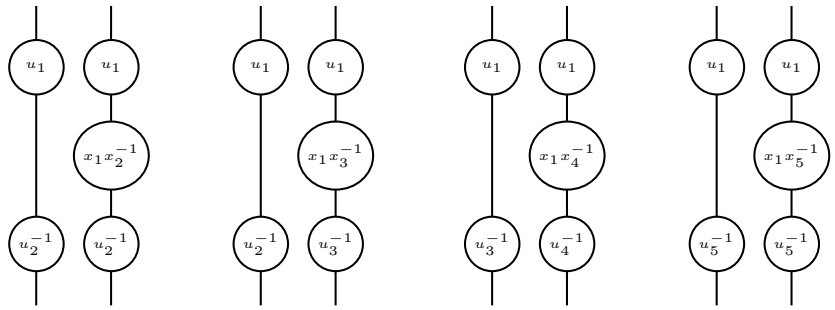


Figure 3: Structure of the node $i = 1$. Four pairs of edges are attached at each node of the graph: each edge are colored with a $SU(2)$ representation. The bullets illustrate the inclusions of $SU(2)$ variables u_i or S^3 variables $x_i x_j^{-1}$. Notice that, in the $SU(2)\times SU(2)$ formulation, each pair of lines is associated to the element (g_L, g_R), g_L corresponding to the left line and g_R to the right one.

Before considering specific examples, let us add one more important remark. The vertex amplitude can be trivially reformulated as an integral over 10 copies of G as follows:

$$V(I_{ij},\omega_i) \;=\; \int (\prod_{i<j} dx_{ij})\,C(x_{ij})\,(\otimes_{i=1}^{5}\nu_i)\cdot(\bigotimes_{i<j} R^{I_{ij}}(x_{ij})) \tag{19}$$

where the contraint $C(x_{ij})$ is a distribution which imposes, rougthly speaking, x_{ij} to be a "coboundary", i.e. of the form $x_i x_j^{-1}$. An explicit formula for $C(x_{ij})$ is simply given by the integral:

$$C(x_{ij}) \;=\; \int (\prod_{i=1}^{5} dx_i)\,\prod_{i\neq j}\delta(x_{ij}^{-1} x_i x_j^{-1}) \tag{20}$$

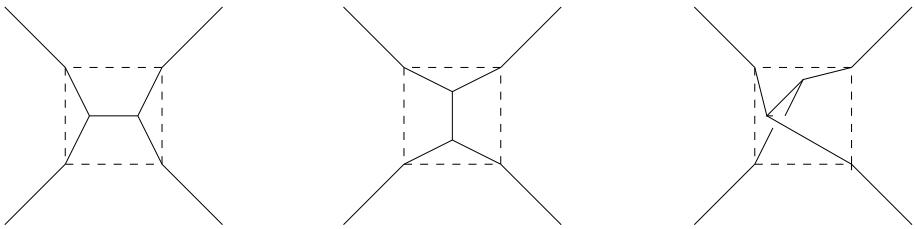
where δ is the $SU(2)$ delta distribution. It is possible to perform the above integration whose result is simply given by the product of five delta distributions:

$$C(x_{ij}) \;=\; \delta(x_{123})\,\delta(x_{234})\,\delta(x_{345})\,\delta(x_{451})\,\delta(x_{512}) \tag{21}$$

where $x_{ijk} = x_{ij}x_{jk}x_{ki}$ and, by convention, $x_{ij} = x_{ji}^{-1}$. The interpretation of the constraint $C(x_{ij})$ will become clear in the last Section where we make the link with the canonical quantization. To conclude, we underline that we have finally found the desired formula (3) for the vertex amplitude with the anounced expression of the distribution $C(x_{ij})$ and the model dependent function $\mathcal{V}(I_{ij},\omega_i;x_{ij}) = (\otimes_{i=1}^{5}\nu_i)\cdot(\bigotimes_{i<j} R^{I_{ij}}(x_{ij}))$ is a particular contraction of five $SU(2)$ matrices.

1.3 Vertices of particular models

This part is devoted to study some aspects of the vertex amplitude (17) for the topological model, the BC model and the EPR model. In fact, these models differ only by the choice of the intertwiners ω_i or equivalently the vectors v_i which are their building blocks. Thus, to understand the construction of these models and their differences, one has to understand the definition of their associated intertwiners. For that purpose, let us start by recalling basic properties of intertwiners. First of all, in Spin-Foam models, we are interested in 4-valent intertwiners only. The 4-valent intertwiners between four given representations form a (normed) vector space of finite dimension. In the case where $G = SU(2)$, one can exhibit three canonical (natural) orthogonal basis (labelled by an index $\epsilon \in \{+, -, 0\}$ that indicates the "coupling channel") presented in the figure (4). Whatever the basis we choose, any of its element is completely characterized by the representation appearing in the intermediate channel in the tensor product decomposition. Therefore, one often identifies the element of each basis with a representation. We will use the notations $\iota_\epsilon(\alpha)$ to denote the $SU(2)$ intertwiner in the basis ϵ with intermediate representation α. One can make used of these results to construct basis of $SU(2) \times SU(2)$ 4-valent intertwiners. In particular, one can naturally exhibit nine "tensor product" basis labelled by a couple (ϵ, ϵ'). However, we will consider in the sequel only the three basis of the type (ϵ, ϵ) which will be labelled by a single ϵ for simplicity: elements of the basis ϵ are denoted $\iota_\epsilon(\alpha)$ as in the $SU(2)$ case but with the difference that α is now a couple of $SU(2)$ representations.



$\iota_+(\alpha)$ $\iota_-(\alpha)$ $\iota_0(\alpha)$

Figure 4: The three canonical basis of the space of 4-valent intertwiners. The intermediate channel is endowed with the representation α.

Now, we are ready to define the intertwiner ω_i for the model we are interested in. Afterwards, we are going to make the general abstract formula of the vertex amplitude more concrete and more useful for studying its properties.

1.3.1 The topological model

We start with the simplest, certainly the more mathematically precise but non-physical model. The topological model is closely related to BF theory with gauge group $SU(2)$. More precisely, given a triangulation $\mathcal{T}$ of a 4-dimensional manifold $\mathcal{M}$, one can discretize the BF action to be well-defined on this triangulation and the path integral $\mathcal{Z}_{BF}(\mathcal{T})$ of the discretized action can be formulated as a state sum or equivalently a Spin-Foam model:

$$\mathcal{Z}_{BF}(\mathcal{T}) = \sum_{\{j_f\},\{\omega_t\}} \prod_{f\in\mathcal{T}^2} \dim(j_f) \prod_{t\in\mathcal{T}^3} \dim(\omega_t)^{-1} \prod_{s\in\mathcal{T}^4} V_{BF}(\omega_{t_s}, j_{f_s}) \tag{22}$$

where we have used notations of (2); we have identified the intertwiners ω_t with the associated representation and V_{BF} is the vertex amplitude completely defined by the graph (5). This amplitude is known as a 15j symbol and can be formulated as a finite sum of products of 6j symbols

$$\begin{aligned} V_{BF}(I_{ij}, \omega_i) &= \sum_K \frac{1}{d_{\omega_1} d_{\omega_5} d_K^2} \left\{ \begin{array}{ccc} \omega_1 & I_{12} & I_{13} \\ \omega_2 & I_{25} & K \end{array} \right\} \left\{ \begin{array}{ccc} \omega_2 & \omega_3 & I_{13} \\ I_{23} & I_{13} & K \end{array} \right\} \\ & \left\{ \begin{array}{ccc} I_{35} & I_{24} & I_{34} \\ \omega_3 & \omega_4 & K \end{array} \right\} \left\{ \begin{array}{ccc} \omega_1 & I_{14} & I_{15} \\ I_{25} & \omega_5 & K \end{array} \right\} \left\{ \begin{array}{ccc} I_{45} & \omega_5 & I_{34} \\ I_{14} & \omega_4 & K \end{array} \right\}. \end{aligned} \tag{23}$$

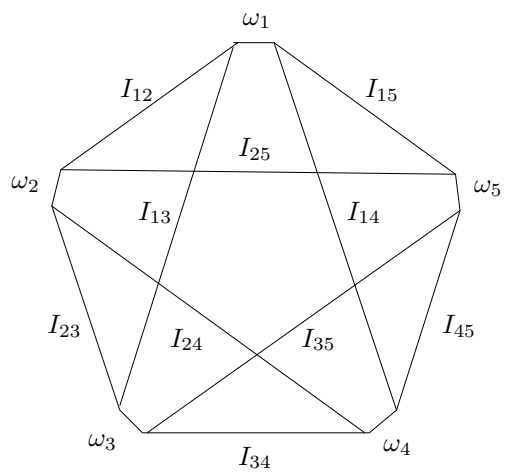


Figure 5: Pictorial representation of a 15j symbol: vertices are labelled by representations ω_i and edges by representations I_{ij}.

The 6j symbols are the totally symmetrized 6j symbols defined, for example, in the chapter 6 of the book [19]. Note that the sum is finite and then the vertex amplitude is well-defined. However, the state sum is generally divergent; it can be made convergent by gauge fixing or by turning classical groups into quantum groups. The state sum is a (formal) PL invariant, i.e. invariant under homeomorphisms.

We have voluntarily not given neither the interwiners ω_i^{BF} nor the vectors v_i defining the model according to the previous Section. Indeed, such a formulation is not very useful for the topological model and the description of the previous Section is naturally adpated for $SU(2) \times SU(2)$ Spin-Foam models and not really for $SU(2)$ Spin-Foam models.

1.3.2 The Barrett-Crane model

The Barrett-Crane model has been constructed as a step towards the covariant quantization of four dimensional pure Euclidean or Lorentzian gravity à la Plebanski. Here, we consider exclusively the Euclidean case. The BC model is then a state sum associated to a triangulation $\mathcal{T}$ of a 4-manifold $\mathcal{M}$ which is supposed to reproduce the path integral $\mathcal{Z}_{Pl}(\mathcal{T})$ of a discretized version of the Plebanski action. However, the link between the BC model and gravity is somehow misleading. Indeed, the BC state sum has been constructed heuristically as a modification of the $SU(2)\times SU(2)$ topological state sum according to the following rules: representations coloring the faces of the 4-simplex are supposed to be simple, i.e. of the form (I_{ij}, I_{ij}); the intertwiners ω_i^{BC} associated to the tetrahedra are also called simple or BC intertwiners we will recall the definition in the sequel; the vertex amplitude V_{BC} associated to the 4-simplices are the so-called 10j symbols whose definition will also be recalled later. The BC model does not say anything concerning the amplitudes $\mathcal{A}_2$ and $\mathcal{A}_3$ associated to the faces and the tetrahedra of the triangulation. However, many arguments lead to certain expressions of $\mathcal{A}_2$ and $\mathcal{A}_3$ and the corresponding state sums have been numerically tested [22]. Anyway, we will not consider these amplitudes in this paper.

Let us concentrate on the construction of the vertex amplitude V_{BC} whose basic ingredient is the simple intertwiner. A simple n-valent intertwiner is such that any of its decompositions into 3-valent intertwiners introduce only simple representations in the intermediate channel. The simple intertwiner has been studied intensively in the literature; in particular it was shown to be unique up to a global normalization [20]. This property makes clear that the vertex amplitude of the BC model is a function $V(I_{ij}, \omega_i^{BC})$ of only 10 representations and it is called a 10j symbol. To precisely define the simple intertwiner ω_i^{BC}, it is more convenient to start with the formula (6) which shows that ω_i^{BC} is completely determined by the choice of a "simple" vector $v_i^{BC} \in \otimes_{j\neq i} U^*_{I_{ij}J_{ij}}$ where (I_{ij}, J_{ij}) is a $SU(2) \times SU(2)$ UIR . If (I_{ij}, J_{ij}) is a simple representation, i.e. $I_{ij} = J_{ij}$, then the

associated vector space admits an unique normalized (diagonal) $SU(2)$ invariant vector w (or $|w\rangle$) which we identify with its dual $\langle w| \in V^*_{I_{ij}}$. In that case, indeed, the decomposition (10) of $U_{I_{ij}J_{ij}}$ into $SU(2)$ representations contains the space U_0 which is the one dimensional space of diagonal $SU(2)$ invariant states. The simple vector is in fact the tensor product of these invariant vectors: $\omega_i^{BC} = w^{\otimes 4}$. As a result, the expression of the simple intertwiner in the tensor product basis reads:

$$\omega_i^{BC} \;=\; \frac{1}{\prod_{j\neq i}\sqrt{d_{I_{ij}}}}\,\sum_\alpha d_\alpha\,\iota_\epsilon(\alpha) \tag{24}$$

where the sum runs over simple representations $\alpha \equiv (\alpha,\alpha)$ only and is finite. An important property is that the previous sum is independent on the choice of the basis ϵ. Using this formula of the simple intertwiner, one finds immediately the vertex amplitude of the BC model

$$V_{BC}(I_{ij},\omega_i^{BC}) \;=\; \frac{1}{\prod_{i\neq j} d_{I_{ij}}}\,\sum_\alpha d_\alpha\, V_{BF}(I_{ij},\alpha)^2 \tag{25}$$

as a sum of BF amplitudes V_{BF} which are $SU(2)$ 15j symbols. The sum runs over simple representations only and is independent on the choice of the intertwiners defining the 15j symbol. Such a formula is too cumbersome to be useful and one prefers to use the integral formulation (19) of the amplitude to study its physical properties. This integral formula simplifies indeed drastically because the $SU(2)$ integral defining ν_i (18) becomes trivial due to the $SU(2)$ invariance of the vectors v_i, and reads

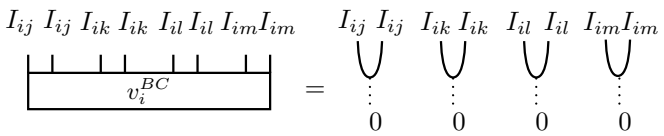
$$V_{BC}(I_{ij},\omega_i) \;=\; \int (\prod_{i=1}^5 dx_i)\,\langle w^{\otimes 10}|\bigotimes_{i<j} R^{I_{ij}}(1,x_ix_j^{-1})\,|w^{\otimes 10}\rangle. \tag{26}$$

Using the second equations in (15), one obtains the following integral formula for the 10j symbol:

$$V_{BC}(I_{ij},\omega_i^{BC}) \;=\; \int \prod_{i\neq j} dx_{ij}\,\frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}}\,C(x_{ij}) \;=\; \int \prod_{i=1}^5 dx_i \prod_{i<j}\frac{\chi_{I_{ij}}(x_ix_j^{-1})}{d_{I_{ij}}} \tag{27}$$

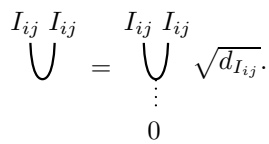
where $\chi_I(x)$ is the $SU(2)$ character of x in the representation I. Up to some normalization factors, the previous formula coincides with the Euclidean 10j symbols. This integral formulation was very useful to study the classical behavior of the Euclidean BC model. Let us finish this brief presentation of the BC model with two important remarks.

Remark 1. The previous calculation can be done in a completely graphical way. Indeed, the "black" boxes representing the vectors v_i^{BC} in (2) reduce to the following form



(28)

where the dashed lines represent spin 0 representation. We see explicitely that v_i^{BC} project into diagonal $SU(2)$ invariant vectors. Furthermore, the 3j vectors involving a spin 0 representation are proportional to the "identity" according to the following pictorial rule



(29)

As a result, one immediately obtains the pictorial representation of the BC vertex amplitude which is given by the product of the normalization factor $\prod_{i<j} d_{I_{ij}}^{-1}$ and the graph in Figure (6). The graph consists in 10 disconnected loops colored by representations I_{ij} which makes obvious that the vertex amplitude integrand is, up to a normalization, the product of 10 characters $\chi_{I_{ij}}(x_{ij})$.

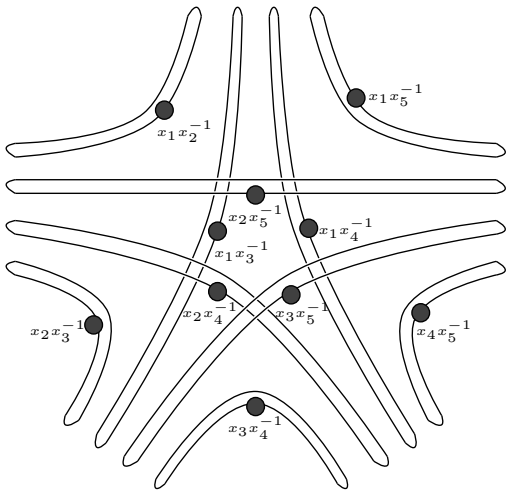


Figure 6: Pictorial representation of the BC vertex integrant up to the normalization factor $\prod_{i<j} d_{I_{ij}}^{-1}$. The graph is made of 10 disconnected unknots colored with representations I_{ij}. In each loop is inserted a S^3 element of the form $x_i x_j^{-1}$.

Remark 2. There is another equivalent expression for the vertex amplitude which was very useful to study the classical behavior of the vertex amplitude found by Freidel and Louapre [21]. This formula will not be used in this paper but it is still interesting to mention it at least to ask the question whether a similar formula exists for the EPR model. This formula is based on the simple fact that the character $\chi_I(x)$ depends only on the conjugacy class $\theta \in [0,\pi]$ of $x = \Lambda h(\theta)\Lambda^{-1}$: $\Lambda \in SU(2)/U(1)$ and $h(\theta)$ is in the Cartan torus of $SU(2)$. This fact leads after some calculations to an expression of the vertex amplitude as an integral over the conjugacy classes:

$$V_{BC}(I_{ij}, \omega_i^{BC}) \;=\; \int (\prod_{i\neq j} d\theta_{ij}\, \frac{\sin(d_{I_{ij}}\theta_{ij})}{d_{I_{ij}}})\, \widetilde{C}(\theta_{ij})\;. \tag{30}$$

The notation $\widetilde{C}$ holds for the "Fourier transform" of the distribution C; it is a distribution as well given by:

$$\widetilde{C}(\theta_{ij}) \;\equiv\; \frac{2^{10}}{\pi^{10}} \int (\prod_{i<j} \sin\theta_{ij}\, d\Lambda_{ij})\; C(\Lambda_{ij} h(\theta_{ij}) \Lambda_{ij}^{-1}) \;=\; \delta(G[\cos(\theta_{ij})]) \tag{31}$$

where G holds for the Gramm matrix. Such a relation is in fact a particular example of a much more general duality relation [25].

1.3.3 The Engle-Pereira-Rovelli model

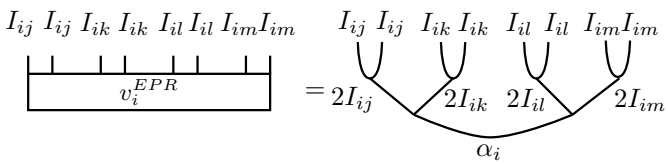
The BC model has been considered as the most promising Spin-Foam model for a long time: its definition is simple, it has a quite appealing physical interpretation and admits the good classical limit [21, 22, 23] in the sense that the associated vertex amplitude tends to the Regge action in the classical limit, apart from a term due to degenerate contributions, and it was also successful in reproducing the correct asymptotic behavior of the diagonal components of the graviton propagator [10, 24]. Nevertheless, it has been recently realized that the model does not satisfy the required properties to reproduce at the semi-classical limit the non-diagonal components of the propagator [9]. The reasons of this failure have been deeply investigated and a quest for a new model have been started. Recent researches have led to the so-called EPR model which has been argued to

be a serious candidate. This Section is devoted to recall the basis of this model in the Euclidean sector with no Immirzi-Barbero parameter $\gamma = 0$.

As in the BC framework, Engle, Pereira and Rovelli have proposed a formula for the vertex amplitude V_{EPR} only. To construct V_{EPR}, one starts by coloring the faces of the 4-simplex by simple representations and the tetrahedra i by specific intertwiners denoted ω_i^{EPR}. We propose to define ω_i^{EPR} throught its associated vector v_i^{EPR} according to the formula (6). To do so, to each simple representation (I_{ij}, I_{ij}), we associate the projector $\mathbb{I}_{2I_{ij}} : U_{I_{ij}I_{ij}} \to U_{2I_{ij}}$ from the $SU(2) \times SU(2)$ representation's vector space $U_{I_{ij}I_{ij}}$ into the vector space of the $SO(3)$ representation of spin $2I_{ij}$. In the standard bra-ket notation, the projector reads $\mathbb{I}_{2I_{ij}} = \sum_m |m\, 2I_{ij}\rangle\langle m\, 2I_{ij}|$; it is clear that it can be trivially identified to its dual $\mathbb{I}^*_{2I_{ij}} = \mathbb{I}_{2I_{ji}}$. Then, the vector v_i is constructed from this projector as follows:

$$v_i^{EPR} \;\equiv\; \iota_\epsilon(\alpha_i)\,(\bigotimes_{j\neq i} \mathbb{I}_{2I_{ij}}) \tag{32}$$

where $\iota_\epsilon(\alpha_i)$ is a $SO(3)$ intertwiner, viewed as an element of the tensor product $\otimes_{j\neq i} V_{2I_{ij}}*$, caracterized by $\epsilon \in \{0, +, -\}$ and the $SO(3)$ representation α_i as illustrated in the figure (4). As the vector v_i^{EPR} is totally determined by a $SO(3)$ representation α_i and a choice of basis ϵ, we will identify in the sequel the vector v_i^{EPR} with the couple (α_i, ϵ). The pictorial representation of v_i is the following:



(33)

Note that we made a particular choice for ϵ to draw the picture; another choice would lead to a different contraction of the four edges colored by the representations $2I_{ij}$. Contrary to the BC model, the EPR intertwiner between four given representations I_{ij} is not unique for it depends on α_i and ε, both belonging to a finite set.

Now, it is possible to decompose the EPR intertwiner in any tensor product basis of the space of 4-valent $SU(2) \times SU(2)$ intertwiners. We are interested in its decomposition in the basis of the type (ϵ, ϵ) whose elements are denoted $\iota_\epsilon(\alpha)$ After some simple calculation, we recover the following expression of the EPR intertwiner given in the literature:

$$\omega_i^{EPR} \;=\; \sum_\alpha f(\omega_i, I_{ij}, \iota_\epsilon(\alpha))\, \iota_\epsilon(\alpha) \tag{34}$$

where the coefficient f is graphically "represented" in the Figure (7) and the sum is finite and runs over $SU(2) \times SU(2)$ representations α with a fixed chosen basis ϵ. In the notation of Engle-Pereira-Rovelli, α is denoted (i_+, i_-) and the representation defining ω_i is denoted i. Note that the sum (34) is not restricted to simple representations.

Now, we have all the ingredients to compute the vertex amplitude $V_{EPR}(I_{ij}, \omega_i^{EPR})$ for the EPR model. ¿From the expression (34), we show immediately that:

$$V_{EPR}(I_{ij}, \omega_i^{EPR}) \;=\; \sum_{\alpha=(i_+,i_-)} f(\omega_i, I_{ij}, \iota_\epsilon(i_+, i_-))\; V_{BF}(I_{ij}, i_+)\; V_{BF}(I_{ij}, i_-) \tag{35}$$

where $V_{BF}(I_{ij}, i_\pm)$ are the $SU(2)$ 15j symbols which depends on the representations I_{ij} and α but also on the choice of the basis ϵ which has not been explicitely written. The sums runs over $SU(2) \times SU(2)$ representations α with a fixed ϵ. Such a formula is rather complicated and one might prefer working instead with an integral formula of the form (5). To obtain such a formula, one has to separate in the integral (8) the variables u_i from the variables x_i as in (16) and then to perform the integration over the variables u_i. These last integrations are very simple to compute: the integration over u_3 is trivial and those over the remaining variables u_i give a simple normalisation factor $N = (d_{2I_{12}} d_{2I_{45}} d_{\omega_1} d_{\omega_5})^{-1}$.

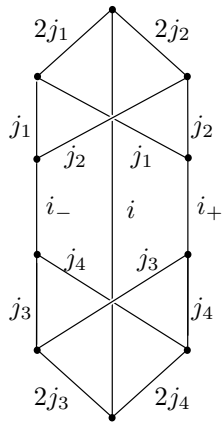


Figure 7: EPR fusion coefficients. The edges are colored with $SU(2)$ representations and the vertices with symmetric $SU(2)$ 3j symbols. The picture illustrates the coefficient $f(\omega_i, I_{ij}, \iota_\epsilon(\alpha))$ for $I_{ij} = \{j_1, j_2, j_3, j_4\}$, ω_i is caracterized by i (and some ϵ) and $\alpha = (i_+, i_-)$.

Afterwards, the vertex amplitude reduces to the formula:

$$V_{EPR}(I_{ij}, \omega_i) \;=\; N \int \prod_{i \neq j} dx_{ij}\, C(x_{ij})\, \mathcal{V}(I_{ij}, \omega_i; x_{ij}) \tag{36}$$

where the amplitude $\mathcal{V}$ is a function of the 10 variables x_{ij} and is graphically represented in the Figure (8). This formula is the EPR counterpart of the formula (27) for the BC model. It will appear very useful in the next Section to make a contact with Loop Quantum Gravity. It might also be useful to study the classical and semi-classical properties of the EPR model as it is the case for the BC model.

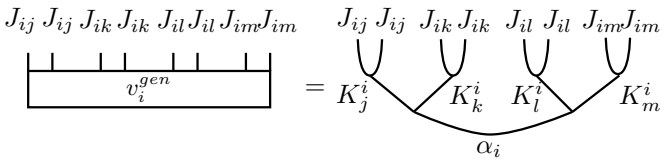
1.3.4 A direct generalization: the Freidel-Krasnov models

This Section is devoted to present a very direct generalization of the EPR model. This generalization leads to a large class of Spin-Foam models to which belong both the EPR and the BC models.

To motivate the construction of FK models, let us recall that the vector v_i^{EPR}, necessary to define the EPR intertwiner ω_i^{EPR}, has been constructed making use of a projector $\mathbb{I}_{2I_{ij}}$ from the vector space of the $SU(2) \times SU(2)$ simple representation (I_{ij}, I_{ij}) into the SO(3) vector space representation $U_{2I_{ij}}$. A direct generalization would be to define a vector v_i^{gen} using instead, at each vertex i, projectors $\mathbb{I}_{K_j^i}$ from $V_{I_{ij} I_{ij}}$ into the SO(3) representation $U_{K_j^i}$ for any representation $K_j^i \in [0, 2I_{ij}]$. The formal expression of the general vector is then the following:

$$v_i^{gen} \;\equiv\; \iota_\epsilon(\alpha_i)\, (\bigotimes_{j \neq i} \mathbb{K}_j^i)\,. \tag{37}$$

The vector v_i^{gen} so defined depends on the choice of the intertwiner $\iota_\epsilon(\alpha_i)$ and on the representations K_j^i. It is represented by the following diagram



(38)

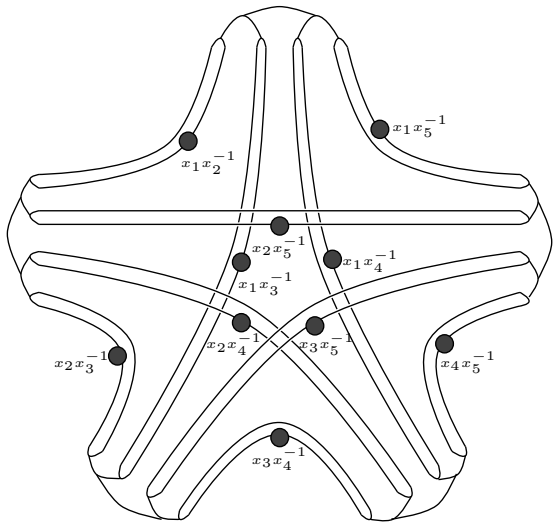


Figure 8: Picturial representation of the EPR argument in the integral formula: vertices are labelled by $i = 1, \cdots, 5$ where $i = 1$ is the top vertex and the others are enumerated according to the anti-clockwise orientation; edges are then oriented and are labelled by (ij) with $i < j$. The doubled lines are colored with simple representations (I_{ij}, I_{ij}). The lines (ij) in the same pair are linked to a line colored with the representation $2I_{ij}$. At each vertex, the four single lines are linked with a line of representation ω_i.

This leads to a vertex amplitude very similar to the EPR one. In particular, its integral formula takes the same form of (36) where the normalization factor is changed into $N = (d_{I_2^1} d_{I_5^4} d_{\omega_1} d_{\omega_1})^{-1}$ and the function $\mathcal{V}$ is represented by the same graph drawn in the Figure (8) with different spin labels.

As a consequence, we get a large class of Spin-Foam models vertex amplitudes V_i^{gen} which depends not only on the 10 representations I_{ij} coloring the faces of the 4-simplex but also depends on 5 other representations per tetrahedron i which have been denoted α_i, K^i_j. Up to now, only special cases of such models have been studied: the BC model where $K^i_j = \alpha_i = 0$, the EPR model where $K^i_j = K^j_i = 2I_{ij}$ and α_i is a free parameter. Thus, either we choose to project into the trivial representation either into the hightest representation. The FK model consists in another choice of the representations K^i_j and α_i.

Many arguments lead to the fact that the EPR intertwiners define the good physical model, namely the one which should reproduce the discretized path integral of the Euclidean Plebanski theory.

2 The vertex and the physical scalar product

In this Section we are proposing a link between (covariant) Spin-Foam models and (canonical) Loop Quantum Gravity. To explain our strategy, we start by recalling some needed basic results of LQG. One of the main points of LQG is the assumption that physical states can be constructed from the so-called kinematical Hilbert space $\mathcal{H}_{kin}$ which consists in the space of cylindrical functions endowed with the kinematical scalar product $\langle , \rangle$ defined from the $SU(2)$ Haar measure. The Spin-network states form an orthonormal basis of $\mathcal{H}_{kin}$. Then, the idea is basicly to impose the constraints of gravity to extract physical states out of the kinematical space. So far, we know how to impose the Gauss constraint and the space-diffeomorphisms constraints and this leads to the construction of the the diffeomorphism invariant states: they form the space $\mathcal{H}_{diff}$ which

is endowed with the Ashtekar-Lewandowski measure [26]. The physical Hilbert space $\mathcal{H}_{phys}$ is still unknown but expected to be constructed from the Ashtekar-Lewandowski measure. Up to now, we do not how to solve the remaining Hamiltonian constraint. Spin-Foam models have been introduced as an alternative to find physical states and the physical scalar product in the sense that the amplitude of a Spin-Foam models should reproduce the physical scalar product between the states at the boundary of the Spin-Foam. This Section aims precisely at clarifying this last point in a simple case.

More precisely, we consider the Spin-Foam associated to the 4-simplex graph denoted Γ. Its amplitude is given, up to some eventual irrelevant normalization factors, by the vertex amplitude V. From the general boundary (covariant) formulation point of view, Γ is viewed as a graph interpolating between two kinematical boundary states which are τ_1 and τ_4 as schematically depicted in the figure (9). In fact, as shown in the figure (9), τ_1 and τ_4 belong to the space $\mathrm{Cyl}(\widetilde{\Gamma})$ where $\widetilde{\Gamma}$ is the union of Γ with four free ends. These free ends have been added for technical purposes only. Notice that Γ can be equivalently interpreted as the graph interpolating between two different graphs that would be denoted τ_2 (with two vertices) and τ_3 (with three vertices). For that, one would need to introduce also some free ends at the graph Γ. From the canonical point of

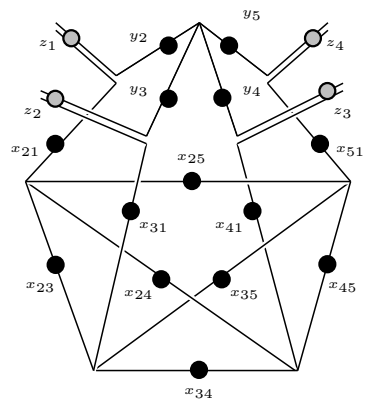


Figure 9: Representation of the graph $\widetilde{\Gamma}$. The subgraphs associated to τ_1 and τ_4 have been underlined and the group variables associated to each edge have been emphasized.

view, the states τ_1 and τ_4 are considered schematically as cylindrical functions on the graph $\widetilde{\Gamma}$. Therefore, one naturally asks the question whether it exists a "physical projector" P acting on the space $\mathrm{Cyl}(\widetilde{\Gamma})$ such that its matrix element $\langle \tau_4, P\tau_1 \rangle$ constructed from the kinematical scalar product gives the vertex amplitude. The notation $\langle \tau_4, P\tau_1 \rangle$ can be misleading because P has in fact to be viewed as a state in the sense of Gelfand-Naimark-Segal (GNS), i.e. P is a linear form on $\mathrm{Cyl}(\widetilde{\Gamma})$ and the physical scalar product reads $\langle \tau_4, P\tau_1 \rangle = P(\overline{\tau_4}\tau_1)$. We abusively use the same notation for the projector viewed as a " matricial operator" or a linear form. To be interpreted as a GNS state, P has to satisfy additional properties, like the positivity, that we will not discuss here. We show that it is possible to construct explicitely such an operator P for the topological, the BC and the EPR models. The "projector" for the FK model can also be obtained immediately generalizing the construction in the EPR case. We will use the obvious notations P_{BF}, P_{BC} and P_{EPR} to denote the physical "projector" in the different cases.

There are two important points to clarify. The first one is the issue of uniqueness of the solution: we find one (class of) solution(s) for P in each model but we do not know if it is unique (in some precise sense of course). Second we work in the kinematical Hilbert space and we expect P to behave correctly with respect to diffeomorphisms invariance in order to extend it to $\mathcal{H}_{diff}$. We hope to address these important mathematical issues in the future.

2.1 The topological model

The topological model is the simplest case to consider. Even if it is not of a great physical interest, it is a good toy model to test the possibility of constructing a "physical projector" P. Furthermore, we will see that this construction will be useful to study the other more physical cases. Let us emphasize that the construction of P_{BF} is very similar to the construction of the projector into physical states in three dimensions as expected from the topological nature of the model.

As we said in the introduction of this Section, the boundary states τ_1 and τ_4 are elements of $\mathrm{Cyl}(\widetilde{\Gamma})$: τ_1 is a function of the eight group variables y_h, z_k, with $k = 1, \cdots, 4$ and $h = 2, \cdots, 5$, as shown in the figure (10); τ_4 is a function of fourteen group variables, ten of them are denoted x_{ij} with $i, j = 1, \cdots, 5$ and $i \neq j$, and the four remaining are the z_k variables as shown in the figure 10. Note that the z_k group variables are those associated to the free ends of $\tilde{\Gamma}$ which are common to the spin-network graphs associated to τ_1 and τ_5.

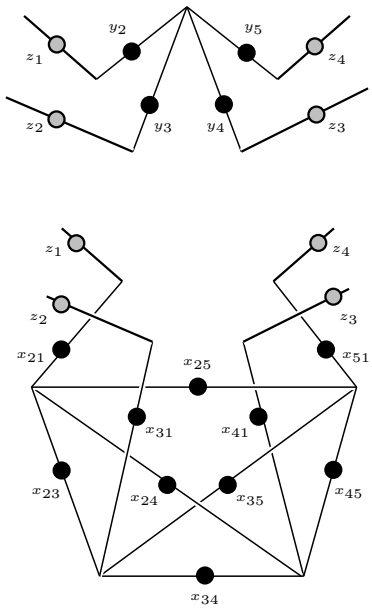


Figure 10: Pictorial representation of the graph associated to τ_1 and τ_4 separately. The free edges are oriented from the vertices to the free ends; the internal edges are oriented according to the order on the vertices. The variables associated to the free ends are denoted z_k for the two graphs; those associated to the internal edges of τ_4 are denoted x_{ij} with $i, j = 2, \cdots, 5$; those associated to the internal edges of τ_1 are denoted y_h.

We now address the concrete question of finding the projector P_{BC} such that $\langle \tau_4, P_{BC}\tau_1 \rangle$ is, up to some eventual irrelevant normalization factors, the vertex amplitude V_{BF}. Of course, we have implicitely assumed that τ_1 and τ_4 are spin-network states i.e. there are associated to a colorating of the edges and the vertices of their associated graphs. Concerning τ_1, its vertex is colored with an intertwiner denoted ω_1 and each edge associated to the variables y_k are colored with a representation denoted J_k. Concerning τ_4, its vertices i are colored with intertwiners ω_i and each edge associated to the variables x_{ij} are colored with representations I_{ij}.

The operator P_{BC} has to be a discretization of the flatness condition on the connection: it is a cylindrical distribution on Γ which imposes that the holonomies around the closed faces of Γ are trivial. One candidate which realizes such a requirement is given by:

$$P_{BC} \;=\; \delta(x_{123})\delta(x_{234})\delta(x_{345})\delta(x_{451})\delta(x_{512}) \tag{39}$$

with the notation of (21). We need only five delta distributions to impose the flatness condition on the ten faces of the 4-simplex. Furthermore, we see that P_{BC} is nothing but the distribution

$C(x_{ij})$ we have previously introduced (21). To show that this operator is indeed a solution of our problem, let us compute its matrix element between the states τ_1 and τ_4 making use of the kinematical scalar product:

$$\begin{aligned}\langle \tau_4, P_{BF}\tau_1\rangle &= \int (\prod_{i\neq j}^{5} dx_{ij})\,(\prod_{k=2}^{5} dy_k)\,(\prod_{\ell=1}^{4} dz_\ell)\,\overline{\tau_4(x,z_\ell)}\,C(x,y)\tau_1(y,z) \\ &= \prod_{k=1}^{4} \frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}} \int (\prod_{i\neq j} dx_{ij})\,C(x_{ij})\,\tau_5(x_{ij}) \end{aligned} \tag{40}$$

where $\tau_5(x_{ij})$ is the spin-network state associated to the 4-simplex graph. To obtain this result, we have performed the integration over the z variables first, then we have absorbed the y variables using the invariance of the Haar measure to get as a final result an integral involving only the variables x_{ij}. At this point, it is immediate to see that the previous integral simplifies and we have:

$$\langle \tau_4, P_{BF}\tau_1\rangle \;=\; \left(\prod_{k=1}^{4} \frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\right) V_{BF}(I_{ij},\omega_i)\;. \tag{41}$$

Up to a renormalization factor, the physical scalar product gives exactly the desired vertex amplitude of the topological model. Therefore, we found a projector P into the physical states of the topological model.

Let us finish the study of this case with some remarks. First, the construction of P_{BC} can be easily generalized to the space of all cylindrical functions: we only have to impose the flatness condition aroung the closed loops of the spin-netwoks but taking into account the fact that one has to avoid redundant delta distributions in order to have a finite amplitude. Second, as we have already said, the projector P_{BC} has a clear physical interpretation in the sense that it is a discretization of the first class constraints of the BF theory. For that reason, one can suppose that the solution we found is unique. As a final remark, let us emphasize that, even if the topological model is not physically interesting, it will appear very useful to understand the gravitational models, namely the BC and the EPR models. Indeed, the three models admit the same kinematical Hilbert space and, as we will see, the operators P_{BC} and P_{EPR} are constructed from the operator P_{BF} we have just constructed. In other words, the physical scalar products of the gravitational models are obtained from the physical scalar product of the topological model. This aspect will be precisely described in next Section.

2.2 The Barrett-Crane model

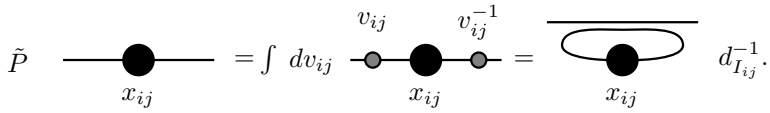
This Section is devoted to the construction of the operator P_{BC}. For that purpose, we use the same notations as in the previous Section concerning the space of cylindrical functions $\mathrm{Cyl}(\widetilde{\Gamma})$, in particular concerning the states τ_1 and τ_4. This makes sense because the topological and the BC models possess the same kinematical Hilbert space. Thus, we look for an operator P_{BC} acting on the space of cylindrical functions $\mathrm{Cyl}(\widetilde{\Gamma})$ such that

$$\langle \tau_4, P_{BC}\tau_1\rangle \;=\; N\left(\prod_{k=1}^{4} \delta_{J_k,I_{1k}}\right) V_{BC}(I_{ij},\omega_{BC}) \tag{42}$$

where N is an eventual normalization factor. We propose a solution where the projector is the product $P_{BC} = P_{BF}\tilde{P}_{BC}$ of the projector $P_{BF} = C(x_{ij})$ of the topological model and another operator $\tilde{P}_{BC}$ we are going to define. First, $\tilde{P}_{BC}$ has a non-trivial action on $\mathrm{Cyl}(\Gamma)$ but can be trivially extended to the space $\mathrm{Cyl}(\widetilde{\Gamma})$. Then, its action on any function $F \in \mathrm{Cyl}(\Gamma)$ is explicitely given by:

$$(\tilde{P}_{BC}F)(x_{ij}) \;=\; \int (\prod_{i<j} dv_{ij})\,F(v_{ij}x_{ij}v_{ij}^{-1}) \tag{43}$$

where we used the obvious notation x_{ij} for the group variable associated to the oriented edge (ij) of Γ. Thus, $\tilde{P}_{BC}$ acts non-trivially on the internal edges of $\tilde{\Gamma}$; this action can be graphically represented as follows:



(44)

Let us now see that P_{BC} reproduces the physical scalar product in the sense of the equation (42). Indeed, an immediate calculation leads to the result:

$$\langle \tau_4, P_{BC}\tau_1 \rangle = \left(\prod_{k=1}^{4} \frac{\delta_{J_k, I_{1k}}}{d_{I_{1k}}} \right) \int (\prod_{i<j} dx_{ij}) \, (P_{BC}\tau_5)(x_{ij}) \tag{45}$$

$$= \left(\prod_{k=1}^{4} \frac{\delta_{J_k, I_{1k}}}{d_{I_{1k}}} \right) \tau_5(1) \int (\prod_{i\neq j} dx_{ij}) \, \frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}} \, C(x_{ij}) \tag{46}$$

where $\tau_5(1)$ is the spin-network τ_5 evaluated at the identity $x_{ij} = 1$, then it is the vertex amplitude of the topological model, i.e. a $SU(2)$ 15j symbol. Thus, the previous equation can recasted the as follows:

$$\frac{\langle \tau_4, P_{BC}\tau_1 \rangle}{\langle \tau_4, P_{BF}\tau_1 \rangle} = \left(\prod_{k=1}^{4} \frac{\delta_{J_k, I_{1k}}}{d_{I_{1k}}} \right) V_{BC}(I_{ij}, \omega_i^{BC}) \, . \tag{47}$$

Up to some normalization factor, the operator P_{BC} reproduces the vertex amplitude of the BC model. Thus, P_{BC} can be interpreted as a projection into physical states of the BC model.

The construction we are proposing rises many important remarks.

Remark 1. The operators P_{BF} and $\tilde{P}_{BC}$ do not commute and therefore the order of their product clearly matters. The operator P_{BF} is a multiplicative operator that impose the discrete analoguous of the flatness of the Ashtekar connection and then it can be interpreted as a projector into space-diffeomorphism invariant states. This interpretation is based on the fact that, in three dimensions, the flatness constraint on the connection generates diffeomorphisms. The operator $\tilde{P}_{BC}$ is a kind of "derivative" operator for its action involves $SU(2)$ right and left derivatives. Its physical interpretation is not clear.

Remark 2. If one believes that the BC model is related to gravity, then it is clear that P_{BF} is the projection into $\mathcal{H}_{diff}$ and $\tilde{P}_{BC}$ should contain the projection into the kernel of the Hamiltonian constraint. This is far from being obvious and that conjecture is even false if the BC model is not the one that discretizes gravity as it is suspected. Let us notice that, in our construction, $\tilde{P}_{BC}$ acts first and then acts P_{BC} which is contrary to what one usually does in LQG where the projection into $\mathcal{H}_{diff}$ arises before the projection into the kernel of the Hamiltonian constraint.

Remark 3. The operator $\tilde{P}_{BC} : \mathrm{Cyl}(\Gamma) \to C(SU(2))_{Ad}^{\times 10}$ is in fact a projector from the space of cylindrical functions to ten copies of the space of functions on the conjugacy classes $C(SU(2))_{Ad}$ of the group $SU(2)$ where $F \in C(SU(2))_{Ad}$ if and only if $F(gxg^{-1}) = F(x)$ for any x and g in $SU(2)$. Its action on a τ_5 spin-network state is given by:

$$(\tilde{P}_{BC}\tau_5)(x_{ij}) = \tau_5(1) \prod_{i<j} \frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}} \tag{48}$$

where I_{ij} are the representations coloring the edges (ij) of the graph Γ. It is straightforward to check that $\tilde{P}_{BC}^2 = \tilde{P}_{BC}$. As a consequence, for the definition of P_{BC} to make sense, one has to extend P_{BF} as an operator acting on $C(SU(2))_{Ad}^{\times 10}$ which is trivial.

Remark 4. In fact, the decomposition of P_{BC} as the product of P_{BF} and $\tilde{P}_{BC}$ is not canonical. Our construction provides an equivalent class of functions $\tilde{P}_{BC}$ according to the trivial relation

$\tilde{P}_{BC} \sim \tilde{Q}_{BC}$ if and only if $P_{BF}\tilde{P}_{BC} = P_{BF}\tilde{Q}_{BC}$. Another natural choice for the derivative operator is $\tilde{Q}_{BC}$ defined by its following action on τ_5 spin-network states:

$$(\tilde{Q}_{BC}\tau_5)(x_{ij}) \;\equiv\; \tau_5(x_{ij}) \prod_{i<j} \frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}} \;. \tag{49}$$

This representative is clearly a multiplicative operator.

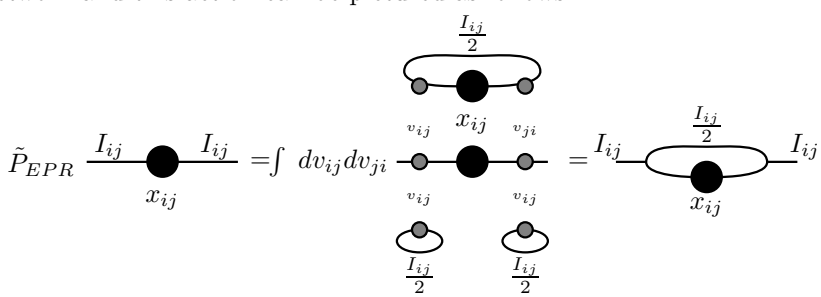
Remark 5. As a last remark, let us underline that the physical scalar product between two states in the BC model (47) can be viewed as the matrix element of the operator $\tilde{P}_{BC}$ with respect to the physical scalar product of the topological model up to the "norm" $\langle \tau_4, P_{BF}\tau_1 \rangle$. In that sense, the BC model is very closely related to the topological model.

2.3 The Engle-Pereira-Rovelli model

In this Section, we propose an operator P_{EPR} which reproduces the vertex amplitude of the EPR model. The construction of P_{EPR} is very similar to the construction of P_{BC}. As for the BC model, P_{EPR} is the product of the non-commuting operators, $P_{EPR} = P_{BF}\tilde{P}_{EPR}$, one of them being the projetor of the topological model as well. The operator $\tilde{P}_{EPR}$ is defined by its action on spin-network states $\tau_5(x_{ij})$ explicitely given by:

$$(\tilde{P}_{EPR}\tau_5)(x_{ij}) \;=\; \int (\prod_{i<j} dv_{ij}dv_{ji}) \left[\prod_{i<j} \chi_{\tilde{I}_{ij}}(v_{ij}x_{ij}v_{ji})\chi_{\tilde{I}_{ij}}(v_{ij})\chi_{\tilde{I}_{ij}}(v_{ji}) \right] \tau_5(v_{ij}x_{ij}v_{ji}) \tag{50}$$

where we have introduced the notation $\tilde{I}_{ij} = I_{ij}/2$. As in the BC model, $\tilde{P}_{EPR}$ acts on each edge of the spin-network and this action can be pictured as follows:



(51)

In this figure, the closed loops represent $SU(2)$ characters. The last equality has been obtained after integrating over the v_{ij} variables. Using this pictorial representation, it is quite easy to compute the matrix elements of P_{EPR} between the states τ_1 and τ_4:

$$\langle \tau_4, P_{EPR}\tau_1 \rangle \;=\; \left(\prod_{k=1}^{4} \frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}} \right) \int \prod_{i<j} dx_{ij}\, (P_{EPR}\tau_5)(x_{ij}) \tag{52}$$

$$= \left(\prod_{k=1}^{4} \frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}} \right) N^{-1}\, V_{EPR}(I_{ij}, \omega_i^{EPR}) \tag{53}$$

where N is the normalization factor introduced in (36). As a consequence, we claim that the physical scalar product reproduces the vertex amplitude of the EPR model. Let us now finish this Section with some important remarks.

Remark 1. The previous remarks 1 and 2 concerning the BC model can be transposed to the EPR model. In particular, if the EPR model is a discretization of the path integral of gravity, $\tilde{P}_{EPR}$ should be closely related to the Hamiltonian constraint. If this is true, our formula could give some hints about the regularization of the Hamiltonian constraint. Furthermore, we can easily generalize the construction to any cylindrical functions with no restriction on the underlying graph.

Remark 2. The operator $\tilde{P}_{EPR}$ is constructed making use of an integration over 20 variables v_{ij} with $i \neq j$ because $v_{ij} \neq v_{ji}$ in the formula (50). Contrary to the BC model, $\tilde{P}_{EPR}$ is not a projector neither an operator from Cyl(Γ) to the space of functions on the $SU(2)$ conjugacy classes. The integral (50) can be reduced to an integral over only 10 variables v_{ij} with $i < j$ as follows:

$$(\tilde{P}_{EPR}\tau_5)(x_{ij}) \;=\; \int (\prod_{i<j} dv_{ij}) \left[\prod_{i<j} \frac{\chi_{\tilde{I}_{ij}}(v_{ij})\chi_{\tilde{I}_{ij}}(x_{ij}^{-1}v_{ij})}{d_{\tilde{I}_{ij}}} \right] \tau_5(v_{ij}) \; . \tag{54}$$

To obtain such a formula, we have first integrated over the variables v_{ij} with $i > j$ and then we have performed some changing of variables.

Remark 3. Our construction can be generalized immediately to the FK models presented in Section 1.3.4. The resulting operator P_{FK} would take exactly the same form as P_{EPR} with some differences in the representations of the characters in the integrand of (50).

Remark 4. Concerning the unicity of $\tilde{P}_{EPR}$, we can make the same remark 4 as in the BC model, namely our construction provides a certain equivalent class of solutions for $\tilde{P}_{EPR}$ and the decomposition of P_{EPR} as a product of P_{BF} and $\tilde{P}_{EPR}$ is not canonical.

Conclusions and perspectives

On the first hand, this article opens one way towards the understanding of an eventual link between Loop Quantum Gravity and Spin-Foam models. We have shown that the vertex amplitudes of some Spin-Foam models can be precisely interpreted as a "physical" scalar product between two spin-networks. This makes a clear relation between the canonical and covariant quantizations of 4 dimensional Euclidean gravity. It is indeed possible to construct operators P acting on the space Cyl(Γ) of cylindrical functions on the (extended) 4-simplex graph $\tilde{\Gamma}$ such that its matrix elements between spin-networks states gives, up to some eventual normalization, the vertex amplitudes for Spin-Foam models. In a formal language, we have shown that

$$\langle s, Ps' \rangle \;=\; \mathcal{A}(s, s') \tag{55}$$

where $\langle , \rangle$ is the kinematical scalar product; s and s' belongs to Cyl($\tilde{\Gamma}$) and $\mathcal{A}(s, s')$ is the Spin-Foam amplitude of a graph interpolating between s and s' which is, here, proportional to the vertex amplitude. The construction works for the topological model, the Barrett-Crane model, the Engle-Pereira-Rovelli model and their direct generalizations, namely the Freidel-Krasnov models.

On the other hand, the same article opens questions that certainly deserve to be investigated. The first one concerns the possibility to extend our construction to the case where the spin-networks s and s' (55) are any cylindrical functions and not restricted to Cyl($\tilde{\Gamma}$) as this was the case in this article. It is clear that the action of the operators P we have constructed can be easily extended to any spin-networks with no assumption on the underlying graph defining the spin-networks. It would be very nice to first compute the matrix elements of P between these general states and to check if the result is related to a Spin-Foam amplitude associated to a graph interpolating between the two associated spin-network graphs. We hope to study this very exciting problem in the close future.

The second question concerns the link between the operators P we have constructed and the regularization of the Hamiltonian constraint à la Thiemann. Indeed, one would expect that, if the Spin-Foam models are a discretized version of the path integral of gravity, then P should be related to the Hamiltonian constraint. It is interesting to remark that the projector P_{EPR} of the EPR model have some "similarities" with the Thiemann constraint: for example, it acts on the nodes of the spin-networks. It is nonetheless intriguing to notice an important difference between the ways the constraints are imposed in LQG and in the Spin-Foam models throught the operators P: indeed, in LQG, one imposes the vectorial constraint before imposing the scalar constraint whereas the operator $P = P_{BF}\tilde{P}$ is the non-commutative product of two operators, the second one P_{BC}

imposes clearly the space-diffeomorphisms invariance and "projects" into the vectorial constraint kernel. Of course, it is too early to conclude anything but its seems to have a quite important discrepencie between the two approaches. To understand more precisely these aspects, one could start by understanding the link between the projector P and the classical constraints of gravity.

The third question is more mathematical: is P a GNS state? Indeed, it is quite misleading to view P as an operator acting on cylindrical functions for it is a linear form on $\mathrm{Cyl}(\tilde{\Gamma})$. Thus it seems that the GNS theory is the good mathematical framework to study P. But, if one wants to interpret P has a GNS state, one has to check that it satisfies all the required property, among other the positivity.

We finish this conclusion by mentionning the possibility that our work could give some hints to study the classical and semi-classical behaviors of the EPR model.

Aknowledgments

We are very grateful to Carlo Rovelli for many things: his interest and his enthusiasm to this subject, his reading of the paper and for leading us to study this problem. The work was partially supported by the ANR (BLAN06-3_139436 LQG-2006). E.Alesci wish to thank the support by Della Riccia Foundation.

References

[1] C. Rovelli, "Quantum Gravity", (Cambridge University Press, Cambridge, 2004).

T. Thiemann, "Modern canonical quantum general relativity", (Cambridge University Press, Cambridge, UK, 2007).

A. Ashtekar, J. Lewandowski, "Background independent quantum gravity: A status report", Class. Quant. Grav. 21 (2004) R53.

[2] C. Rovelli, L. Smolin, "Knot theory and quantum gravity" *Phys Rev Lett* **61** (1988) 1155-1158.

C. Rovelli, L. Smolin, "Loop space representation for quantum general relativity", *Nuclear Physics* **B331** (1990) 80-152.

A. Ashtekar, C. Rovelli, L. Smolin, "Weaving a classical geometry with quantum threads", *Phys Rev Lett* **69**, 237 (1992).

C. Rovelli, L. Smolin, "Discreteness of Area and Volume in Quantum Gravity", *Nucl Phys* **B442** (1995) 593-619;

Nucl Phys **B456**, 734 (1995).

A. Ashtekar, J. Lewandowski, "Quantum Theory of Geometry I: Area Operators" *Class Quantum Grav* **14** (1997) A55-A82; "II : Volume Operators", *Adv Theo Math Phys* **1** (1997) 388-429

[3] T. Thiemann. Anomaly-free formulation of non-perturbative, four-dimensional lorentzian quantum gravity. *Phys. Lett. B*, 380:257, 1996.

T. Thiemann. Quantum spin dynamics (QSD). *Class. Quant. Grav.*, 15:839–873, 1998.

T. Thiemann, Quantum spin dynamics (QSD) II, Class. Quant. Grav., 15:875- 905, 1998, gr-qc/9606090;

T. Thiemann, QSD III: Quantum constraint algebra and physical scalar product in quantum general relativity, Class. Quant. Grav., 15:1207-1247, 1998, grqc/ 9705017;

[4] T. Thiemann, The Phoenix project: Master constraint programme for loop quantum gravity, Class. Quant. Grav., 23:2211-2248, 2006, gr-qc/0305080

B. Dittrich, T. Thiemann, Testing the master constraint programme for loop quantum gravity. I: General framework, Class. Quant. Grav., 23:1025-1066, 2006, grqc/ 0411138.

T.Thiemann, Quantum spin dynamics. VIII: The master constraint, Class. Quant. Grav.,23:2249-2266, 2006, gr-qc/0510011;

[5] M. P. Reisenberger and C. Rovelli. Sum over Surfaces form of Loop Quantum Gravity. Phys.Rev. D, 56:34903508, 1997.

C. Rovelli. "The Projector on Physical States in Loop Quantum Gravity" Phys.Rev. D, 59:104015, 1999.

J.C. Baez, "Spin foam models" *Class Quant Grav* 15 1827 (1998), gr-qc/9709052

A Perez, "Spin Foam Models for Quantum Gravity", Class and Quantum Grav 20 (2003) R43

D. Oriti. Spacetime geometry from algebra: Spin foam models for non- perturbative quantum gravity. *Rept. Prog. Phys.*, 64:1489–1544, 2001.

[6] K. Noui, A. Perez, "Three Dimensional Loop Quantum Gravity: Spin-Foam models and the Physical Scalar Product", Class. Quant. Grav. 22 (2005) 1739-1762, gr-qc/0402110

[7] J. F. Plebanski, "On the separation between Einsteinien substuctures", J. Math. Phys. 12, 2511 (1977)

[8] J. W. Barrett, L. Crane, "Relativistic spin networks and quantum gravity", J. Math. Phys. 39 (1998) 3296–3302.

R. De Pietri, L. Freidel, K. Krasnov, C. Rovelli, "Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space", Nucl. Phys. B 574 (2000) 785. [arXiv:hep-th/9907154].

D. Oriti, R. M. Williams, "Gluing 4-simplices: a derivation of the Barrett-Crane spinfoam model for Euclidean quantum gravity", Phys. Rev. D 63(2001) 024022.

A. Perez, C. Rovelli, "A spin foam model without bubble divergences", Nucl. Phys. B 599 (2001) 255.

[9] E. Alesci, C.Rovelli, "The complete LQG propagator: I. Difficulties with the Barrett-Crane vertex " Phys.Rev.D 76, 104012 (2007), arXiv:0708.0883

E. Alesci, C. Rovelli "The complete LQG propagator: II. Asymptotic behavior of the vertex " Phys.Rev.D 77, 044024 (2008) arXiv:0711.1284

E. Alesci, "Tensorial Structure of the LQG graviton propagator" in Proceedings of the II Stueckelberg Workshop, Int.J.Mod.Phys.A 23:1209-1213, (2008), arXiv:0802.1201

[10] L. Modesto, C. Rovelli, "Particle scattering in loop quantum gravity", Phys. Rev. Lett. 95 (2005), 191301.

C. Rovelli, "Graviton propagator from background–independent quantum gravity", Phys. Rev. Lett. 97 (2006), 151301.

E. Bianchi, L. Modesto, C. Rovelli, S. Speziale, "Graviton propagator in loop quantum gravity", Class. Quant. Grav. 23 (2006), 6989-7028.

[11] J. Engle, R. Pereira, C. Rovelli, "The loop-quantum-gravity vertex amplitude", Phys. Rev. Lett., 99 (2007) 161301.

J. Engle, R. Pereira, C. Rovelli, "Flipped spinfoam vertex and loop gravity", Nucl.Phys.B798, 251-290 (2008), arXiv:0708.1236.

[12] J. Engle, E. Livine, R. Pereira, C. Rovelli, "LQG vertex with finite Immirzi parameter", Nucl.Phys.B799:136-149,2008 arXiv:0711.0146.

[13] R. Pereira, "Lorentzian LQG vertex amplitude" , Class.Quant.Grav.25, 085013, (2008), arXiv:0710.5043

[14] L. Freidel, K. Krasnov, "A New Spin Foam Model for 4d Gravity", arXiv:0708.1595.

[15] E. Livine, S. Speziale, "A New spinfoam vertex for quantum gravity", arXiv:0705.0674.

E. Livine, S. Speziale, "Consistently Solving the Simplicity Constraints for Spinfoam Quantum Gravity", arXiv:0708.1915.

[16] F. Conrady, L. Freidel, "Path integral representation of spin foam models of 4d gravity" , arXiv:0806.4640

[17] E. Magliaro, C. Perini, C. Rovelli , "Numerical indications on the semiclassical limit of the flipped vertex", Class.Quant.Grav.25: 095009, (2008), arXiv:0710.5034

E. Alesci, E. Bianchi, E. Magliaro, C. Perini, "Intertwiner dynamics in LQG flipped vertex" to appear

[18] S. Alexandrov, E.Livine, "SU(2) loop quantum gravity seen from covariant theory", Phys. Rev. D 67:044009, 2003.

[19] A.R. Edmonds, "Angular Momentum in Quantum Mechanics", Princeton Univ. Press. (1957).

[20] M. P. Reisenberger. "On Relativistic Spin Network Vertices" J.Math.Phys., 40:2046 2054, 1999. gr-qc/9809067

[21] J. W. Barrett, R. M. Williams, "The asymptotics of an amplitude for the 4-simplex", Adv. Theor. Math. Phys. 3 (1999) 209–215.

J. C. Baez, J. D. Christensen, G. Egan, "Asymptotics of 10j symbols", Class. Quant. Grav. 19 (2002) 6489.

J. W. Barrett, C. M. Steele, "Asymptotics of relativistic spin networks", Class. Quant. Grav. 20 (2003) 1341–1362.

L. Freidel, D. Louapre, "Asymptotics of 6j and 10j symbols", Class. Quant. Grav. 20 (2003) 1267–1294.

[22] J. C. Baez, J. D. Christensen, "Positivity of spin foam amplitudes," Class. Quant. Grav. 19 (2002) 2291.

J. D. Christensen, G. Egan, "An efficient algorithm for the Riemannian 10j symbols," Class. Quant. Grav. 19 (2002) 1185.

J. D. Christensen, "Finiteness of Lorentzian 10j symbols and partition functions," Class. Quant. Grav. 23 (2006) 1679.

I. Khavkine, J. D. Christensen, "q-Deformed spin foam models of quantum gravity", Class. Quant. Grav. 24 (2007) 3271-3290.

[23] E. Bianchi, L. Modesto, "The perturbative Regge-calculus regime of Loop Quantum Gravity", arXiv:0709.2051.

[24] J.D. Christensen, E. Livine, S. Speziale, "Numerical evidence of regularized correlations in spin foam gravity", arXiv:0710.0617

[25] L. Freidel, K.Noui, P.Roche, "6J symbols duality relations", J. Math. Phys. 48:113512, 2007.

[26] A. Ashtekar, J. Lewandowski, "Projective techniques and functional integration" J. Math. Phys., 36:2170, 1995.