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KASHIWARA AND ZELEVINSKY INVOLUTIONS IN AFFINE TYPE A

NICOLAS JACON AND CÉDRIC LECOUEY

ABSTRACT. We first describe how the Kashiwara involution $*$ on crystals of affine type A is encoded by the combinatorics of aperiodic multisegments. This yields a simple relation between $*$ and the Zelevinsky involution τ on the set of simple modules for the affine Hecke algebras. We then give efficient procedures for computing $*$ and τ . Remarkably, these procedures do not use the underlying crystal structure. They also permit to match explicitly the Ginzburg and Ariki parametrizations of the simple modules associated to affine and cyclotomic Hecke algebras, respectively .

1. INTRODUCTION

The Kashiwara involution $*$ in affine type A is a fundamental anti-isomorphism of the quantum group \mathcal{U}_v associated to the affine root system $A_{e-1}^{(1)}$. It induces a subtle involution on $B_e(\infty)$, the Kashiwara crystal corresponding to the negative part \mathcal{U}_v^- of \mathcal{U}_v . The Zelevinsky involution yields an involution τ of the affine Hecke algebra of type A . When q is specialized to an e -th root of 1, τ also induces an involution on $B_e(\infty)$. In this paper, we show how the combinatorics of aperiodic multisegments encodes a simple relation between the involutions τ and $*$. We also provide efficient procedures for computing these involutions. In addition, our results permit to match explicitly the Ginzburg and Ariki parametrizations of the simple modules associated to affine and cyclotomic Hecke algebras respectively. All our computations can be made independent of the crystal structure on $B_e(\infty)$. Moreover, they do not require the determination of i -induction or i -restriction operations on simple modules. Let us now describe the context and the results of the paper more precisely.

The Zelevinsky involution first appeared in [29] in connection with the representation theory of the linear group $GL(n, \mathbb{F}_p)$ over the p -adic field \mathbb{F}_p . Works by Mœglin and Waldspurger [22] then permit to link it with a natural involution τ of the affine type A Hecke algebra $\mathcal{H}_n^a(q)$ over the field \mathbb{F} with generic parameter q . When $e \geq 2$ is an integer and q is specialized at ξ , a primitive e -root of 1, it was conjectured by Vigneras [28] that this involution should be related to the modular representation theory of $GL(n, \mathbb{F}_p)$. In the sequel we will refer to τ as the Zelevinsky involution of $\mathcal{H}_n^a(\xi)$ (see Section 3 for a complete definition).

The involution τ induces an involution on the set of simple $\mathcal{H}_n^a(\xi)$ -modules. There exist essentially two different parametrizations of these modules in the literature. In the geometric construction of Chriss and Ginzburg [6] and under the assumption $\mathbb{F} = \mathbb{C}$, the simple $\mathcal{H}_n^a(\xi)$ -modules are labelled by aperiodic multisegments. These simple modules can also be regarded as simple modules associated to Ariki-Koike algebras $\mathcal{H}_n^v(\xi)$. The Specht module theory developed by Dipper, James and Mathas then provides a labelling of the simple $\mathcal{H}_n^a(\xi)$ -modules by Kleshchev multipartitions. Both constructions permit to endow the set of simple $\mathcal{H}_n^a(\xi)$ -modules with the structure of a crystal isomorphic to $B_e(\infty)$. The Kashiwara crystal operators then yield the modular branching rules for the Ariki-Koike algebras and affine Hecke algebras of type A ([2], [3]).

In [10], Grojnowski uses i -induction and i -restrictions operators to define an abstract crystal structure on the set of simple $\mathcal{H}_n^a(\xi)$ -modules. He then proves that this crystal is in fact isomorphic to $B_e(\infty)$. This approach is valid over an arbitrary field \mathbb{F} and does not require the Specht module theory of Dipper James and Mathas. This notably permits to extend the methods of [10] to the representation theory of the cyclotomic Hecke-Clifford super-algebras [5]. Nevertheless, this approach does not match up the abstract crystal obtained with the labellings of the simple modules by aperiodic multisegments or Kleshchev multipartitions. Since the i -induction operation on simple modules is difficult to obtain in general, it is also not really suited to explicit computations.

By identifying \mathcal{U}_v^- with the composition subalgebra of the Hall algebra associated to the cyclic quiver of type $A_e^{(1)}$ (see § 2.1), we obtain in fact two different structures of crystal on the set of aperiodic multisegments. They both come from two different parametrizations of the canonical basis of \mathcal{U}_v^- which correspond under the anti-isomorphism ρ on \mathcal{U}_v^- switching the generators f_i and f_{-i} . In particular ρ provides an involution on the crystal $B_e(\infty)$ which can be easily computed. The use of the composition algebra also permits to describe explicitly the structure of Kashiwara crystal on the set of aperiodic multisegments. This was obtained in [20] by Leclerc, Thibon and Vasserot. In addition, these authors prove that the involution τ can be computed from the two fold symmetry \sharp on $B_e(\infty)$ which switches the sign of each arrow.

In this paper, we first establish that the two crystal structures on aperiodic multisegments obtained by identifying \mathcal{U}_v^- with the composition algebra correspond up to the conjugation by the Kashiwara involution $*$. This yields the very simple relation

$$* = \rho \circ \sharp$$

between the involutions \sharp, ρ and $*$. Observe that an equivalent identity can also be established by using results of [10] but, as mentioned above, it then requires subtle considerations on representation theory of $\mathcal{H}_n^a(\xi)$ and does not permit to compute $*, \rho$ and \sharp efficiently. In contrast our proof uses only

elementary properties of crystal graphs and yields efficient procedures for computing the involutions $\tau, *, \rho$ and \sharp . This notably permits us to generalize an algorithm of Moeglin and Waldspurger which gives the Zelevinsky involution when $e = \infty$.

As a consequence, extending works of Vazirani [27], we completely solve the following natural problem. Given a simple $\mathcal{H}_n^a(\xi)$ -module L_ψ (with ψ an aperiodic multisegment ψ), we find all the Ariki-Koike algebras $\mathcal{H}_n^v(q)$ and the simple $\mathcal{H}_n^v(q)$ -modules D^λ (with λ a Kleshchev multipartition) such that $D^\lambda \simeq L_\psi$ as $\mathcal{H}_n^a(\xi)$ -modules. The procedure yielding the Kashiwara involution also permits to compute the commutor of $A_e^{(1)}$ -crystals introduced by Kamnitzer and Tingley in [17].

The paper is organized as follows. In Section 2, we review the identification of \mathcal{U}_v^- with the composition algebra and the two structures of crystal it gives on the set of aperiodic multisegments. We also recall basic facts on the Kashiwara involution. Section 3 is devoted to the definition of the Zelevinsky involution on the set of simple $\mathcal{H}_n^a(\xi)$ -modules and to the results of [20]. In Section 4, we prove the identity $*$ = $\rho \circ \sharp$. The problem of determining the algebras $\mathcal{H}_n^v(\xi)$ and the simple $\mathcal{H}_n^v(\xi)$ -module isomorphic to a given simple $\mathcal{H}_n^a(\xi)$ -module is studied in Section 5. In the last two sections, we give a simple combinatorial procedure for computing the involutions $\tau, *, \rho$ and \sharp on $B_e(\infty)$. We prove in fact that all these computations can essentially be obtained from the Mullineux involution on e -regular partitions and the crystal isomorphisms described in [14]. We also investigate several consequences of our results.

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2. BACKGROUND ON QUANTUM GROUPS IN AFFINE TYPE A

2.1. \mathcal{U}_v and the Hall algebra. Let v be an indeterminate and $e \geq 2$ an integer. Write $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ for the quantum group of type $A_{e-1}^{(1)}$. This is an associative $\mathbb{Q}(v)$ -algebra with generators e_i, f_i, t_i, t_i^{-1} , $i \in \mathbb{Z}/e\mathbb{Z}$ and ∂ (see [26, §2.1] for the complete description of the relations satisfied by these generators). Write $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$ and $\{\alpha_0, \dots, \alpha_{e-1}\}$ respectively for the set of dominant weights and the set of simple roots associated to $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$. Let P be the weight lattice of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$. We denote by $\mathcal{U}_v = \mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ the subalgebra generated by e_i, f_i, t_i, t_i^{-1} , $i \in \mathbb{Z}/e\mathbb{Z}$. Then $\overline{P} = P/\mathbb{Z}\delta$ is the set of classical weights of \mathcal{U}_v . For any $i \in \mathbb{Z}/e\mathbb{Z}$, we also denote by Λ_i and α_i the restriction of Λ_i and $\alpha_i \in P$ to \overline{P}^\wedge . Let \mathcal{U}_v^- be the subalgebra of \mathcal{U}_v generated by the f_i 's with $i \in \mathbb{Z}/e\mathbb{Z}$.

The algebra \mathcal{U}_v^- has a concrete description. Let Γ be the cyclic quiver of length e . This is an oriented graph with vertices $\mathbb{Z}/e\mathbb{Z}$ and edges $\{(i, i+1), i \in \mathbb{Z}/e\mathbb{Z}\}$. Let $V = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} V_i$ be a finite dimensional $\mathbb{Z}/e\mathbb{Z}$ -graded

vector space. Put

$$E_V = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \text{Hom}_{\mathbb{C}}(V_i, V_{i+1}) \subseteq \text{End}_{\mathbb{C}}(V).$$

The elements of E_V are the *representations* of Γ on V . If V runs through all finite dimensional $\mathbb{Z}/e\mathbb{Z}$ -graded vector spaces, we obtain the category of finite dimensional representations of Γ . This corresponds to the category of finite dimensional $\mathbb{C}\Gamma$ -modules, where $\mathbb{C}\Gamma$ is the path algebra of Γ . The simple modules are then naturally labelled by $\mathbb{Z}/e\mathbb{Z}$. For each $i \in \mathbb{Z}/e\mathbb{Z}$, we define the corresponding simple module S_i by $V = V_i = \mathbb{C}$ and $X = 0$ in E_V .

If $X \in E_V$ is nilpotent as an endomorphism of V , we say that the associated representation is *nilpotent*. We denote by \mathcal{N}_V the subset of nilpotent representations in E_V . The vector

$$\underline{\dim} V = (\dim V_i)_{i \in \mathbb{Z}/e\mathbb{Z}}$$

is then called the *dimension vector* of the representation.

The group $G_V = \prod_{i \in \mathbb{Z}/e\mathbb{Z}} \text{GL}(V_i)$ acts on E_V and \mathcal{N}_V by conjugation and two representations are equivalent if and only if they belong to the same G_V -orbit.

Definition 2.1. Let $l \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}/e\mathbb{Z}$. The *segment of length l and head i* is the sequence of consecutive residues $[i, i+1, \dots, i+l-1]$. We denote it by $[i; l]$. Similarly, The *segment of length l and tail i* is the sequence of consecutive residues $[i-l+1, \dots, i-1, i]$. We denote it by $(l; i]$.

Definition 2.2. A collection of segments is called a *multisegment*. If the collection is the empty set, we call it the empty multisegment and it is denoted by \emptyset .

The indecomposable nilpotent $\mathbb{C}\Gamma$ -modules are uniquely determined by their lengths l and their heads S_i . We denote by $\mathbb{C}[i; l)$ the $\mathbb{C}\Gamma$ -module with length l and head S_i . Since every representation is a direct sum of indecomposables, the nilpotent representations are labelled by the set of multisegments. For each nilpotent Γ -module M , there exists a unique multisegment

$$\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i; l)} [i; l)$$

such that M is isomorphic to

$$M_\psi := \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} \mathbb{C}[i; l)^{\oplus m_{[i; l)}}.$$

We denote the corresponding G_V -orbit in \mathcal{N}_V by \mathcal{O}_ψ .

Let \mathbb{F}_q be a finite field with q elements. The classes of nilpotent $\mathbb{F}_q\Gamma$ -modules are again labelled by multisegments. Let V , T and W be $\mathbb{Z}/e\mathbb{Z}$ -graded vector spaces over \mathbb{F}_q such that

$$\underline{\dim} V = \underline{\dim} T + \underline{\dim} W.$$

Let φ_1, φ_2 and ψ be multisegments such that $\mathcal{O}_{\varphi_1} \subseteq \mathcal{N}_T$, $\mathcal{O}_{\varphi_2} \subseteq \mathcal{N}_W$ and $\mathcal{O}_{\psi} \subseteq \mathcal{N}_V$. If the number of submodules U of M_{ψ} satisfying $U \simeq M_{\varphi_2}$ and $M_{\psi}/U \simeq M_{\varphi_1}$ is a polynomial in $q = \text{card}(\mathbb{F}_q)$, then this polynomial is called the *Hall polynomial* and we denote it by $F_{\varphi_1, \varphi_2}^{\psi}(q)$. The existence of Hall polynomials in our case was proved by Jin Yun Guo [11, Theorem 2.7].

For a and b in \mathbb{Z}^e we define a bilinear form m by

$$m(a, b) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (a_i b_{i+1} + a_i b_i).$$

Lusztig's version of the *Hall algebra* associated to Γ is the $\mathbb{C}(v)$ -algebra with basis $\{u_{\psi} \mid \psi \text{ is a multisegment}\}$ and the product is given by

$$u_{\varphi_1} u_{\varphi_2} = v^{m(\underline{\dim} T, \underline{\dim} W)} \sum_{\psi} F_{\varphi_1, \varphi_2}^{\psi}(v^{-2}) u_{\psi}.$$

Recall that, for any $i \in \mathbb{Z}/e\mathbb{Z}$, $[i; 1)$ is the multisegment associated to the simple module S_i . The $\mathbb{C}(v)$ -subalgebra generated by these $u_{[i; 1)}$ is called the *composition algebra*. As proved by Ringel and Lusztig, the isomorphism $u_{[i; 1)} \mapsto f_i$ permits to identify this composition algebra with \mathcal{U}_v^- .

2.2. Two crystal structures on the set of aperiodic multisegments.

Definition 2.3. For each multisegment ψ , we define $E_{\psi} = v^{\dim \psi} u_{\psi}$. The set $\{E_{\psi} \mid \psi \text{ is a multisegment}\}$ is called the PBW basis of the Hall algebra.

Definition 2.4. A multisegment ψ is *aperiodic* if, for every $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that $(l; i]$ does not appear in ψ . Equivalently, a multisegment ψ is aperiodic if, for each $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that $[i; l)$ does not appear in ψ . We denote by Ψ_e the set of aperiodic multisegments.

Let $B_e(\infty)$ be the (abstract) crystal basis of \mathcal{U}_v^- . Write $\{G_v(b) \mid b \in B_e(\infty)\}$ for the canonical basis of \mathcal{U}_v^- . The following theorem which shows that $B_e(\infty)$ can be labelled by Ψ_e is due to Lusztig.

Theorem 2.5. *For each $b \in B_e(\infty)$, there exists $\psi \in \Psi_e$ such that*

$$G_v(b) = E_{\psi} + \sum_{\psi' \neq \psi} c_{\psi, \psi'}(v) E_{\psi'},$$

and such that $c_{\psi, \psi'}(v) \in \mathbb{C}(v)$ is regular at $v = 0$ and $c_{\psi, \psi'}(0) = 0$. We then denote

$$G_v(\psi) := G_v(b).$$

As a consequence,

$$\{G_v(\psi) \mid \psi \in \Psi_e\}.$$

is the canonical basis of \mathcal{U}_v^- .

This notably permits to regard the vertices of $B_e(\infty)$ as aperiodic multisegments. The corresponding crystal structure was described by Leclerc, Thibon and Vasserot in [20, Theorem 4.1]. We are going to see below that there are in fact two natural crystal structures on the set of aperiodic multisegments.

Let ψ be a multisegment and let $\psi_{\geq l}$ be the multisegment obtained from ψ by deleting the multisegments of length less than l , for $l \in \mathbb{Z}_{>0}$. Denote by $m_{[i;l]}$ the multiplicity of $[i;l]$ in ψ . For any $i \in \mathbb{Z}/e\mathbb{Z}$, set

$$\widehat{S}_{l,i} = \sum_{k \geq l} (m_{[i+1;k]} - m_{[i;k]}).$$

Let \widehat{l}_0 be the minimal value of l that attains $\min_{l > 0} \widehat{S}_{l,i}$.

Theorem 2.6. *Let ψ be a multisegment, $i \in \mathbb{Z}/e\mathbb{Z}$ and let \widehat{l}_0 be as above. Then we have*

$$\widehat{f}_i \psi = \psi_{\widehat{l}_0, i},$$

where the multisegment $\psi_{\widehat{l}_0, i}$ is defined as follows

$$\psi_{\widehat{l}_0, i} = \begin{cases} \psi + [i; 1] & \text{if } \widehat{l}_0 = 1, \\ \psi + [i; \widehat{l}_0] - [i+1; \widehat{l}_0 - 1] & \text{if } \widehat{l}_0 > 1. \end{cases}$$

The crystal structure on Ψ_e obtained from the action of the operators \widehat{f}_i , $i \in \mathbb{Z}/e\mathbb{Z}$ does not coincide with that initially obtained by Leclerc, Thibon and Vasserot. Indeed there exists another labelling of the canonical basis by aperiodic multisegments. Let $V = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} V_i$ be a graded vector space as before, and define its dual graded vector space by $V^* = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} V_i^*$ where $V_i^* = \text{Hom}_{\mathbb{C}}(V_{-i}, \mathbb{C})$. Then, by sending $X \in E_V$ to its transpose, we have a linear isomorphism

$$E_V \simeq E_{V^*} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \text{Hom}_{\mathbb{C}}(V_i^*, V_{i+1}^*).$$

Note that the G_V -action on this $E_V = E_{V^*}$ is the conjugation by the transpose inverse of $g \in G_V$, while the G_V -action on the original E_V is the conjugation by $g \in G_V$. Then, we have an isomorphism ρ of two G_V -varieties E_V so that the G_V -orbit \mathcal{O}_ψ in the original E_V corresponds to the G_V -orbit $\mathcal{O}_{\rho(\psi)}$ in the new E_V . For any multisegment $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i;l]} [i;l]$, we have:

$$\rho(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i;l]} (l; -i).$$

Thus there is a linear automorphism of the Hall algebras also denoted by ρ such that

$$\rho(E_\psi) = E_{\rho(\psi)} \quad \text{and} \quad \rho(G_v(\psi)) = G_v(\rho(\psi)) \quad \text{if } \psi \text{ is aperiodic.}$$

Hence, ρ induces a relabelling of the PBW basis and the canonical basis. However, if we take the algebra structure into account, ρ induces the

anti-automorphism of \mathcal{U}_v^- given by $f_i \mapsto f_{-i}$, which is clear from the definition of the multiplication of the Hall algebra. In particular, the crystal structure on the set of aperiodic multisegments is changed in this new labelling, and the Kashiwara operators \tilde{e}_i and \tilde{f}_i correspond to the Kashiwara operators \widehat{e}_{-i} and \widehat{f}_{-i} in this new crystal structure. More precisely, set $S_{l,i} = \sum_{k \geq l} (m_{(k;i-1]} - m_{(k;i]})$. Let l_0 be the minimal l that attains $\min_{l > 0} S_{l,i}$. Then, the crystal structure in the new labelling is given as follows. This is in fact the version stated in [20].

Theorem 2.7. *Let ψ be a multisegment and let $i \in \mathbb{Z}/e\mathbb{Z}$ and l_0 be as above. Then we have*

$$\tilde{f}_i \psi = \psi_{l_0,i},$$

where the multisegment $\psi_{l_0,i}$ is defined as follows

$$\psi_{l_0,i} = \begin{cases} \psi + (1; i] & \text{if } l_0 = 1, \\ \psi + (l_0; i] - (l_0 - 1; i - 1] & \text{if } l_0 > 1. \end{cases}$$

Let ψ be a multisegment. Then to compute $\tilde{e}_i \psi$, we proceed as follows. If $\min_{l > 0} S_{l,i} = 0$, then $\tilde{e}_i \psi = 0$. Otherwise, let l_0 be the maximal l that attains $\min_{l > 0} S_{l,i}$. Then, $\tilde{e}_i \psi$ is obtained from ψ by replacing $(l_0; i]$ with $(l_0 - 1; i - 1]$.

In the sequel, we identify $B_e(\infty)$ with the crystal structure obtained on Ψ_e by considering the operators $\tilde{f}_i, i \in \mathbb{Z}/e\mathbb{Z}$ (see also Remark 2.9). We denote by $\text{wt}(\psi)$ the weight of the aperiodic multisegment ψ considered as a vertex of the crystal $B_e(\infty)$. Recall we have then

$$(1) \quad \text{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (\varphi_i(\psi) - \varepsilon_i(\psi)) \Lambda_i$$

where $\varepsilon_i(u) = \max\{k \in \mathbb{N} \mid \tilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k \in \mathbb{N} \mid \tilde{f}_i^k(u) \neq 0\}$.

Then ρ induces an involution on $B_e(\infty)$. One can easily check that the crystal operators \tilde{f}_i and \widehat{f}_i satisfy the simple relation

$$(2) \quad \widehat{f}_i = \rho \circ \tilde{f}_{-i} \circ \rho.$$

2.3. The Kashiwara involution. The Kashiwara involution $*$ is the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -antiautomorphism such that $q^* = q$ and defined on the generators as follows:

$$(3) \quad e_i^* = e_i, \quad f_i^* = f_i, \quad t_i^* = t_i^{-1}.$$

Since $*$ stabilizes \mathcal{U}_v^- , it induces an involution (also denoted $*$) on $B_e(\infty)$ the crystal graph of \mathcal{U}_v^- . By setting for any vertex $b \in B_e(\infty)$ and any $i \in \mathbb{Z}/e\mathbb{Z}$

$$(4) \quad \tilde{e}_i^*(b) = \tilde{e}_i(b^*)^*, \quad \tilde{f}_i^*(b) = \tilde{f}_i(b^*)^*, \quad \varepsilon_i^*(b) = \varepsilon_i(b^*) \quad \text{and} \quad \varphi_i^*(b) = b^*$$

we obtain another crystal structure on $B_e(\infty)$ (see [18]).

Let $i \in \mathbb{Z}/e\mathbb{Z}$ and write B_i for the crystal with set of vertices $\{b_i(k) \mid k \in \mathbb{Z}\}$ and such that

$$\begin{aligned} \text{wt}(b_i(k)) = k\alpha_i, \quad \varepsilon_j(b_i(k)) = \begin{cases} -k & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases}, \quad \varphi_j(b_i(k)) = \begin{cases} k & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases} \\ \tilde{e}_j b_i(k) = \begin{cases} b_i(k+1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{and} \quad \tilde{f}_j b_i(k) = \begin{cases} b_i(k-1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \end{aligned}$$

Set $b_i = b_i(0)$.

Recall the action of \tilde{e}_i and \tilde{f}_i on the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$ of the crystals B and B' :

$$(5) \quad \tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v), \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v), \end{cases}$$

and

$$(6) \quad \tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v), \\ \tilde{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v). \end{cases}$$

The embedding of crystals $\theta_i : B_e(\infty) \hookrightarrow B_e(\infty) \otimes B_i$ which sends the highest weight vertex $b_{\mathbf{0}}$ of $B_e(\infty)$ on $b_{\mathbf{0}} \otimes b_i$ permits to compute the action of the operators \tilde{e}_i^* and \tilde{f}_i^* at least on a theoretical point of view.

Proposition 2.8. (see Proposition 8.1 in [18]) Consider $b \in B_e(\infty)$ and set $\varepsilon_i^*(b) = m$. Then we have

- (1) $\theta_i(b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^m b_i$,
- (2) $\theta_i(\tilde{f}_i^* b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^{m+1} b_i$ and
- (3) $\theta_i(\tilde{e}_i^* b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^{m-1} b_i$ if $m > 0$ and $\theta_i(\tilde{e}_i^* b) = 0$ if $m = 0$.

Remark 2.9.

- (1) By 2.2, Ψ_e is equipped with two crystal structures. One is obtained from the action of the crystal operators $\hat{f}_i, i \in \mathbb{Z}/e\mathbb{Z}$ and the other one is related to the operators $\tilde{f}_i, i \in \mathbb{Z}/e\mathbb{Z}$ and yields the Kashiwara crystal graph structure $B_e(\infty)$ on Ψ_e . We shall see in Section 4, that the actions of the operators \hat{f}_i and \tilde{f}_i^* with $i \in \mathbb{Z}/e\mathbb{Z}$ coincide.
- (2) Observe that Proposition 2.8 does not provide an efficient procedure for computing the involution $*$. Indeed, in order to obtain $\theta_i(b)$, we have first to determine a path from b to the highest weight vertex of $B_e(\infty)$. Moreover, computing a section of the embedding θ_i is difficult in general.

2.4. Crystals of highest weight \mathcal{U}_v -modules. Let $l \in \mathbb{N}$ and consider $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l$. \mathbf{v} is called a multicharge and l is by definition the level of \mathbf{v} . One can then associate to \mathbf{v} the abstract \mathcal{U}_v -irreducible module $V_e(\Lambda_{\mathbf{v}})$ with highest weight $\Lambda_{\mathbf{v}} = \Lambda_{v_0(\text{mode})} + \dots + \Lambda_{v_{l-1}(\text{mode})}$. There exist distinct realizations of $V_e(\Lambda_{\mathbf{v}})$ as an irreducible component of a Fock space

$\mathfrak{F}_e^{\mathbf{v}}$ whose structure depends on \mathbf{v} . As a $\mathbb{C}(v)$ -vector space, the Fock space $\mathfrak{F}_e^{\mathbf{v}}$ of level l admits the set of all l -partitions as a natural basis. Namely the underlying vector space is

$$\mathfrak{F}_e = \bigoplus_{n \geq 0} \bigoplus_{\lambda \in \Pi_{l,n}} \mathbb{C}(v)\lambda$$

where $\Pi_{l,n}$ is the set of l -partitions with rank n . Consider $\mathbf{v} = (v_0, \dots, v_{l-1}) \in (\mathbb{Z}/e\mathbb{Z})^l$. We write $\mathbf{v} \in \mathfrak{v}$ when $v_c \in \mathfrak{v}_c$ for any $c = 0, \dots, l-1$. As \mathcal{U}_v -modules, the Fock spaces $\mathfrak{F}_e^{\mathbf{v}}$, $\mathbf{v} \in \mathfrak{v}$ are all isomorphic but with distinct actions for \mathcal{U}_v . For each of these actions, the empty l -partition $\emptyset = (\emptyset, \dots, \emptyset)$ is a highest weight vector of highest weight $\Lambda_{\mathbf{v}}$. We denote by $V_e(\mathbf{v})$ the irreducible component with highest weight vector \emptyset in $\mathfrak{F}_e^{\mathbf{v}}$. The modules $V_e(\mathbf{v})$ when \mathbf{v} runs over \mathfrak{v} are all isomorphic to the abstract module $V_e(\Lambda_{\mathbf{v}})$. However, the actions of the Chevalley operators on these modules do not coincide in general.

The module $\mathfrak{F}_e^{\mathbf{v}}$ admits a crystal graph $B_e^{\mathbf{v}}$ labelled by l -partitions. Let us now recall the crystal structures on $B_e^{\mathbf{v}}$ and $B_e(\mathbf{v})$ the crystal associated to $V_e(\mathbf{v})$. We will omit the description of the \mathcal{U}_v -module structures on $\mathfrak{F}_e^{\mathbf{v}}$ and $V_e(\mathbf{v})$ which are not needed in our proofs (see [16] for a complete exposition). Let λ be an l -partition (identified with its Young diagram). Then, the nodes of λ are the triplets $\gamma = (a, b, c)$ where $c \in \{0, \dots, l-1\}$ and a, b are respectively the row and column indices of the node γ in $\lambda^{(c)}$. The content of γ is the integer $c(\gamma) = b - a + v_c$ and the residue $\text{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$(7) \quad \text{res}(\gamma) \equiv c(\gamma) \pmod{e}.$$

We say that γ is an i -node of λ when $\text{res}(\gamma) \equiv i \pmod{e}$. This node is removable when $\gamma = (a, b, c) \in \lambda$ and $\lambda \setminus \{\gamma\}$ is an l -partition. Similarly γ is addable when $\gamma = (a, b, c) \notin \lambda$ and $\lambda \cup \{\gamma\}$ is an l -partition.

The structure of crystal on $B_e^{\mathbf{v}}$ (and in fact, the structure of \mathcal{U}_v -module on $\mathfrak{F}_e^{\mathbf{v}}$ itself) is conditioned by the total order $\prec_{\mathbf{v}}$ on the set of addable and removable i -nodes of the multipartitions. Consider $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ two i -nodes in λ . We define the order $\prec_{\mathbf{v}}$ by setting

$$\gamma_1 \prec_{\mathbf{v}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2. \end{cases}$$

Starting from any l -partition λ , consider its set of addable and removable i -nodes. Let w_i be the word obtained first by writing the addable and removable i -nodes of λ in increasing order with respect to $\prec_{\mathbf{v}}$ next by encoding each addable i -node by the letter A and each removable i -node by the letter R . Write $\tilde{w}_i = A^r R^q$ for the word derived from w_i by deleting as many of the factors RA as possible. If $p > 0$, let γ be the rightmost addable i -node in \tilde{w}_i . When $\tilde{w}_i \neq \emptyset$, the node γ is called the good i -node.

Proposition 2.10. *The crystal graph $B_e^{\mathbf{v}}$ of $\mathfrak{F}_e^{\mathbf{v}}$ is the graph with*

- (1) vertices: the l -partitions,
- (2) edges: $\lambda \xrightarrow{i} \mu$ if and only if μ is obtained by adding to λ its good i -node.
- (3) for any $i \in \mathbb{Z}/e\mathbb{Z}$, $\varepsilon_i(\lambda) = q$ and $\varphi_i(\lambda) = r$.

Since $V_e(\mathbf{v})$ is the irreducible module with highest weight vector \emptyset in $\mathfrak{F}_e^{\mathbf{v}}$, its crystal graph $B_e(\mathbf{v})$ can be realized as the connected component of highest weight vertex \emptyset in B_e^s . The vertices of $B_e(\mathbf{v})$ are labelled by l -partitions called Uglov l -partitions associated to \mathbf{v} .

Set

$$(8) \quad \mathcal{V}_l = \{\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l \mid v_0 \leq \dots \leq v_{l-1} \text{ and } v_{l-1} - v_0 < e\}.$$

Definition 2.11. Assume that $\mathbf{v} \in \mathcal{V}_l$. The l -partition $\lambda = (\lambda^0, \dots, \lambda^{l-1})$ is a FLOTW l -partition associated to \mathbf{v} if it satisfies the two following conditions :

- (1) for all $i = 1, 2, \dots$, we have :

$$\lambda_i^j \geq \lambda_{i+v_{j+1}-v_j}^{j+1} \text{ for all } j = 0, \dots, l-2 \text{ and } \lambda_i^{l-1} \geq \lambda_{i+e+v_0-v_{l-1}}^0.$$
- (2) for all $k > 0$, among the residues appearing in λ at the right ends of the length k rows , at least one element of $\{0, 1, \dots, e-1\}$ does not occur.

Proposition 2.12. [8] *When $\mathbf{v} \in \mathcal{V}_l$, the set of vertices of $B_e(\mathbf{v})$ coincides with the set of FLOTW l -partitions associated to \mathbf{v} .*

Let us denote by $\Phi_e(\mathbf{v})$ the set of FLOTW l -partitions associated to \mathbf{v} .

Consider $\mathbf{v} \in \mathcal{V}_l$ and $\lambda \in \Phi_e(\mathbf{v})$. We associate to each non zero part λ_i^c of λ the segment

$$(9) \quad [(1-i+v_c)(\text{mode}), (2-i+v_c)(\text{mode}), \dots, (\lambda_i^c - i + v_c)(\text{mode})].$$

The multisegment $f_{\mathbf{v}}(\lambda)$ is then the formal sum of all the segments associated to the parts λ_i^c of λ . Since $f_{\mathbf{v}}(\lambda)$ is aperiodic by (2) of Definition 2.11, the map:

$$f_{\mathbf{v}} : B_e(\mathbf{v}) \rightarrow \Psi_e$$

is well-defined.

Example 2.13. Let $e = 4$, we consider the FLOTW bipartition (2.1, 1) associated to $\mathbf{v} = (0, 1)$ then

$$f_{\mathbf{v}}(2.1, 1) = [0, 1] + [3] + [1].$$

Let $\mathbf{v} = (0, 1, 3)$ and consider the FLOTW 3-partition (2, 1, 1). We have :

$$f_{\mathbf{v}}(2, 1, 1) = [0, 1] + [1] + [3].$$

Let $T_{\Lambda} = \{t_{\Lambda}\}$ be the crystal defined by $\text{wt}(t_{\Lambda_{\mathbf{v}}}) = \Lambda$, $\varepsilon_i(t_{\Lambda_{\mathbf{v}}}) = \varphi_i(t_{\Lambda_{\mathbf{v}}}) = -\infty$ and $\tilde{e}_i t_{\Lambda_{\mathbf{v}}} = \tilde{f}_i t_{\Lambda_{\mathbf{v}}} = 0$. We have a unique crystal embedding $B_e(\mathbf{v}) \hookrightarrow B_e(\infty) \otimes T_{\Lambda}$. The following theorem has been established in [3].

Theorem 2.14. *For any $\mathbf{v} \in \mathcal{V}_l$, the map $f_{\mathbf{v}}$ coincides with the unique crystal embedding $B_e(\mathbf{v}) \hookrightarrow B_e(\infty) \otimes T_{\Lambda}$.*

According to Proposition 8.2 in [18], we have

$$f_{\mathbf{v}}(\Phi_e(\mathbf{v})) = \{\psi \in \Psi_e \mid \varepsilon_i(\psi^*) \leq r_i \text{ for any } i \in \mathbb{Z}/e\mathbb{Z}\}$$

where r_i is the number of coordinates in \mathbf{v} equal to i and ψ^* is the image of ψ under the Kashiwara involution of the crystal $B_e(\infty)$.

Given any $\psi \in \Psi_e$, write $\mathbf{v}(\psi)$ for the element of \mathcal{V}_l defined by the conditions

$$(10) \quad r_i = \varepsilon_i(\psi^*) = \varepsilon_i^*(\psi) \text{ for any } i \in \mathbb{Z}/e\mathbb{Z}.$$

Then, by the previous considerations, there exists a unique l -partition $\underline{\lambda}(\psi) = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e(\mathbf{v}(\psi))$ such that $f_{\mathbf{v}(\psi)}(\underline{\lambda}(\psi)) = \psi$.

3. THE ZELEVINSKY INVOLUTION OF $\mathcal{H}_n^a(q)$

3.1. Three natural involutions on $\mathcal{H}_n^a(q)$. Denote by $\mathcal{H}_n(q)$ the Hecke algebra of type A with parameter q over the field \mathbb{F} . This is the unital associative \mathbb{F} -algebra generated by T_1, \dots, T_{n-1} and the relations:

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (i = 1, \dots, n-2), \\ T_i T_j &= T_j T_i \quad (|j - i| > 1), \\ (T_i - q)(T_i + 1) &= 0 \quad (i = 1, \dots, n-1). \end{aligned}$$

The affine Hecke algebra $\mathcal{H}_n^a(q)$ is the \mathbb{F} -algebra which as an \mathbb{F} -module is isomorphic to

$$\mathcal{H}_n(q) \otimes_R \mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The algebra structure is obtained by requiring that $\mathcal{H}_n(q)$ and $\mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ are both subalgebras and for any $i = 1, \dots, n$

$$T_i X_i T_i = q X_{i+1}, \quad T_i X_j = X_j T_i \text{ if } i \neq j.$$

In the sequel, we assume that $q = \xi$ is a primitive e -th root of the unity and write $\mathcal{H}_n^a(\xi)$ for the affine Hecke algebra with parameter ξ a primitive e^{th} -root of the unity. We have three involutive automorphisms τ , \flat and \sharp on $\mathcal{H}_n^a(\xi)$. There are defined on the generators as follows :

$$(11) \quad \begin{aligned} T_i^\tau &= -\xi T_{n-i}^{-1}, & X_j^\tau &= X_{n+1-j}, \\ T_i^\flat &= T_{n-i}, & X_j^\flat &= X_{n+1-j}^{-1}, \\ T_i^\sharp &= -\xi T_i^{-1}, & X_j^\sharp &= X_j^{-1}. \end{aligned}$$

The involution \sharp has been considered by Iwahori and Matsumoto [12] and the involution τ , which is called the Zelevinsky involution, by Moeglin and Waldspurger [22]. One can easily check that they are connected as follows:

$$\forall x \in \mathcal{H}_n^a(\xi), \quad x^\sharp = (x^\flat)^\tau.$$

3.2. The involution \sharp on $B_e(\infty)$. We denote by Mod_n^a the category of finite-dimensional $\mathcal{H}_n^a(\xi)$ -modules such that for $j = 1, \dots, n$ the eigenvalues of X_j are power of ξ . For any multisegment $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i]}(l; i]$, we write $|\psi| = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} l m_{(l;i]}$. The geometric realization of $\mathcal{H}_n^a(\xi)$ due to Ginzburg permits to label the simple $\mathcal{H}_n^a(\xi)$ -modules in Mod_n^a by the aperiodic multisegments such that $|\psi| = n$. We do not use Ginzburg's construction in the sequel and just refer to [6] (see also [3] and [20]) for a complete exposition or a short review. Let L_ψ be the simple $\mathcal{H}_n^a(\xi)$ -module corresponding to ψ under this parametrization.

The three involutions τ, \flat, \sharp on $\mathcal{H}_n^a(\xi)$ induce involutions on the set of simple $\mathcal{H}_n^a(\xi)$ -modules that we will denote in the same way. This yields involutions on the set of aperiodic multisegments (also denoted by τ, \flat and \sharp) satisfying

$$L_\psi^\tau = L_{\psi^\tau}, \quad L_\psi^\flat = L_{\psi^\flat}, \quad L_\psi^\sharp = L_{\psi^\sharp}$$

for each aperiodic multisegment ψ . Thus we have three involutions on the vertices of $B_e(\infty)$.

By [20, §2.4], for each aperiodic multisegment $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i]}(l; i]$, we have $\psi^\flat = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i]}(l; -i]$. Hence it is equivalent to describe τ or \sharp . The following proposition makes explicit the involution \sharp on $B_e(\infty)$.

Theorem 3.1. ([20]) *Let ψ be an aperiodic multisegment. Then ψ^\sharp is the aperiodic multisegment obtained from ψ by the 2-fold symmetry $i \leftrightarrow -i$ in the graph $B_e(\infty)$.*

4. LINK BETWEEN THE INVOLUTIONS $*$ AND \sharp

4.1. More on the crystal operators \tilde{f}_i and \hat{f}_i . In [3], we have obtained an alternative description of the action of the crystal operators on Ψ_e . Consider $\psi \in \Psi_e$ and $i \in \mathbb{Z}/e\mathbb{Z}$. We encode the segments in ψ with *tail* i (resp. $i - 1$) by the symbol R (resp. by the symbol A). For any nonnegative integer l , write $w_{i,l} = R^{m_{(l;i]}} A^{m_{(l;i-1]}}$ where $m_{(l;i]}$ and $m_{(l;i-1]}$ are respectively the number of segments $(l; i]$ and $(l; i - 1]$ in ψ . Set $w_i = \prod_{l \geq 1} w_{i,l}$. Write $\tilde{w}_i = A^{a_i(\psi)} R^{r_i(\psi)}$ for the word derived from w_i by deleting as many of the factors RA as possible. If $a_i(\psi) > 0$, we denote by $l_{0,i}(\psi) > 0$ the length of the rightmost segment A in \tilde{w}_i . If $a_i(\psi) = 0$, set $l_{0,i}(\psi) = 0$. When there is no risk of confusion, we simply write l_0 instead of $l_{0,i}(\psi)$.

Lemma 4.1. [3] *With the above notation we have*

- (1) $\varepsilon_i(\psi) = r_i(\psi)$
- (2)

$$\tilde{f}_i \psi = \begin{cases} \psi + (l_0; i] - (l_0 - 1, i - 1] & \text{if } a_i(\psi) > 0, \\ \psi + (1, i] & \text{if } a_i(\psi) = 0. \end{cases}$$

We can compute similarly the action of the crystal operators \hat{f}_i (with $i \in \mathbb{Z}/e\mathbb{Z}$) on ψ . We encode the segments in ψ with *head* i (resp. $i + 1$) by the symbol \hat{R} (resp. by the symbol \hat{A}). For any nonnegative integer l , write

$\widehat{w}_{i,l} = \widehat{R}^{m_{[i;l]}} \widehat{A}^{m_{[i+1;l]}}$ where $m_{[i;l]}$ and $m_{[i+1;l]}$ are respectively the number of segments $[i;l]$ and $[i+1;l]$ in ψ . Set $\widehat{w}_i = \prod_{l \geq 1} \widehat{w}_{i,l}$. Write $\overline{w}_i = \widehat{A}^{\widehat{a}_i(\psi)} \widehat{R}^{\widehat{r}_i(\psi)}$ for the word derived from \widehat{w}_i by deleting as many of the factors $\widehat{R}\widehat{A}$ as possible. If $\widehat{a}_i(\psi) > 0$, let $\widehat{l}_{0,i}(\psi) > 0$ be length of the rightmost segment \widehat{A} in \overline{w}_i . If $\widehat{a}_i(\psi) = 0$, set $\widehat{l}_{0,i}(\psi) = 0$. When there is no risk of confusion, we also simply write \widehat{l}_0 instead of $\widehat{l}_{0,i}(\psi)$.

Lemma 4.2. *With the above notation, we have*

- (1) $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$ where $\widehat{\varepsilon}_i(\psi) = \max\{p \mid \widehat{e}_i^p(\psi) \neq 0\}$.
 (2)

$$(12) \quad \widehat{f}_i \psi = \begin{cases} \psi + [i; \widehat{l}_0] - [i+1, \widehat{l}_0 - 1] & \text{if } \widehat{a}_i(\psi) > 0, \\ \psi + [i, 1] & \text{if } \widehat{a}_i(\psi) = 0. \end{cases}$$

Remark 4.3. By Theorem 9.13 of [10], for any $i \in \mathbb{Z}/e\mathbb{Z}$, the integer $\varepsilon_i(\psi) = r_i(\psi)$ (resp. $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$) gives the maximal size of a Jordan block with eigenvalue ξ^i corresponding to the action of the generator X_n (resp. X_1) on the simple $\mathcal{H}_n^\alpha(\xi)$ -module L_ψ .

4.2. Equality of the crystal operators \widetilde{f}_i^* and \widehat{f}_i . The purpose of this paragraph is to establish the equality

$$(13) \quad \widetilde{f}_i^*(\psi) = \widehat{f}_i(\psi) \text{ for any } \psi \in \Psi_e.$$

This is achieved by showing that the relations $\widetilde{f}_i^* \widetilde{f}_j \psi = \widetilde{f}_j \widetilde{f}_i^* \psi$ and $\widehat{f}_i \widehat{f}_j \psi = \widehat{f}_j \widehat{f}_i \psi$ are both equivalent to a very simple condition on ψ .

Lemma 4.4. *Put $i \in \mathbb{Z}/e\mathbb{Z}$.*

- (1) *Consider $\psi, \chi \in \Psi_e$ such that $\psi = \widehat{f}_i \chi$ and put $j \in \mathbb{Z}/e\mathbb{Z}$. Then we have:*

$$l_{0,j}(\chi) \neq l_{0,j}(\psi) \iff i = j, \widehat{a}_i(\chi) = 0 \text{ and } a_i(\chi) = 1.$$

- (2) *Consider $\psi, \chi \in \Psi_e$ such that $\psi = \widetilde{f}_i \chi$ and put $j \in \mathbb{Z}/e\mathbb{Z}$. Then we have:*

$$\widehat{l}_{0,j}(\chi) \neq \widehat{l}_{0,j}(\psi) \iff i = j, a_i(\chi) = 0 \text{ and } \widehat{a}_i(\chi) = 1.$$

Proof. 1: Assume first $\widehat{l}_{0,i}(\chi) = \widehat{l}_0 > 1$. Hence $\widehat{a}_i(\chi) > 0$ and $\psi = \chi - [i+1, i + \widehat{l}_0 - 1] + [i, i + \widehat{l}_0 - 1]$. If $j \notin \{(i + \widehat{l}_0 - 1)(\text{mode}), (i + \widehat{l}_0)(\text{mode})\}$, neither $[i+1, i + \widehat{l}_0 - 1]$ or $[i, i + \widehat{l}_0 - 1]$ are segments A or R for j . We have $w_j(\psi) = w_j(\chi)$ and then $l_{0,j}(\chi) = l_{0,j}(\psi)$. Thus we can restrict ourselves to the cases $j \in \{(i + \widehat{l}_0 - 1)(\text{mode}), (i + \widehat{l}_0)(\text{mode})\}$. We write

$$\begin{aligned} \widehat{w}_i(\chi) = \cdots [i, i + \widehat{l}_0 - 2]^{m_{[i, i + \widehat{l}_0 - 2]}} [i + 1, i + \widehat{l}_0 - 1]^{m_{[i+1, i + \widehat{l}_0 - 1]}} \\ [i, i + \widehat{l}_0 - 1]^{m_{[i, i + \widehat{l}_0 - 1]}} [i + 1, i + \widehat{l}_0]^{m_{[i+1, i + \widehat{l}_0]}} \cdots \end{aligned}$$

where we have only pictured the segments of length $\widehat{l}_0 - 1$ and \widehat{l}_0 of $\widehat{w}_i(\chi)$. Since $\psi = \widehat{f}_i\chi$, we have

$$\widehat{w}_i(\psi) = \cdots [i, i + \widehat{l}_0 - 2]^{m_{[i, i + \widehat{l}_0 - 2]}} [i + 1, i + \widehat{l}_0 - 1]^{m_{[i + 1, i + \widehat{l}_0 - 1]}} \cdots \\ [i, i + \widehat{l}_0 - 1]^{m_{[i, i + \widehat{l}_0 - 1]} + 1} [i + 1, i + \widehat{l}_0]^{m_{[i + 1, i + \widehat{l}_0]}} \cdots$$

In particular, by (12), we must have $m_{[i, i + \widehat{l}_0 - 2]} < m_{[i + 1, i + \widehat{l}_0 - 1]}$ and $m_{[i, i + \widehat{l}_0 - 1]} \geq m_{[i + 1, i + \widehat{l}_0]}$.

When $j = (i + \widehat{l}_0 - 1)(\text{mode})$, $[i + 1, i + \widehat{l}_0 - 1]$ and $[i, i + \widehat{l}_0 - 1]$ are of type R for j . Hence, by considering only the segments of lengths $\widehat{l}_0 - 1$ and \widehat{l}_0 , we can write

$$w_j(\chi) = \cdots [i + 1, i + \widehat{l}_0 - 1]^{m_{[i + 1, i + \widehat{l}_0 - 1]}} [i, i + \widehat{l}_0 - 2]^{m_{[i, i + \widehat{l}_0 - 2]}} \\ [i, i + \widehat{l}_0 - 1]^{m_{[i, i + \widehat{l}_0 - 1]}} [i - 1, i + \widehat{l}_0 - 2]^{m_{[i - 1, i + \widehat{l}_0 - 2]}} \cdots$$

and

$$w_j(\psi) = \cdots [i + 1, i + \widehat{l}_0 - 1]^{m_{[i + 1, i + \widehat{l}_0 - 1]} - 1} [i, i + \widehat{l}_0 - 2]^{m_{[i, i + \widehat{l}_0 - 2]}} \\ [i, i + \widehat{l}_0 - 1]^{m_{[i, i + \widehat{l}_0 - 1]} + 1} [i - 1, i + \widehat{l}_0 - 2]^{m_{[i - 1, i + \widehat{l}_0 - 2]}} \cdots$$

Since $m_{[i, i + \widehat{l}_0 - 2]} < m_{[i + 1, i + \widehat{l}_0 - 1]}$, the cancellation procedures of the factors RA in $w_j(\chi)$ and $w_j(\psi)$ yield the same final word. Hence $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ and we have also $l_{0,j}(\chi) = l_{0,j}(\psi) = 1$.

When $j = (i + \widehat{l}_0)(\text{mode})$, $[i + 1, i + \widehat{l}_0 - 1]$ and $[i, i + \widehat{l}_0 - 1]$ are of type A for j . We obtain also $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ by considering the segments of lengths $\widehat{l}_0 - 1$ and \widehat{l}_0 . Thus $l_{0,j}(\chi) = l_{0,j}(\psi)$.

Observe that we have always $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ for any $j \in \mathbb{Z}/e\mathbb{Z}$ when $\widehat{l}_0 > 1$. In particular

$$(14) \quad \widehat{a}_i(\chi) > 0 \implies a_j(\chi) = a_j(\widehat{f}_i\chi) \text{ for any } j \in \mathbb{Z}/e\mathbb{Z}.$$

Now assume $\widehat{l}_0 = 1$, that is $\psi = \chi + [i]$. Write

$$\widehat{w}_i(\chi) = [i]^{m_{[i]}} [i + 1]^{m_{[i + 1]}} \cdots \text{ and } \widehat{w}_i(\psi) = [i]^{m_{[i]} + 1} [i + 1]^{m_{[i + 1]}} \cdots$$

with $m_{[i]} \geq m_{[i + 1]}$.

When $j = (i + 1)(\text{mode})$, $[i]$ is of type A for j and $[i + 1]$ is of type R . Thus we can write

$$w_j(\chi) = [i + 1]^{m_{[i + 1]}} [i]^{m_{[i]}} \cdots \text{ and } w_j(\psi) = [i + 1]^{m_{[i + 1]}} [i]^{m_{[i]} + 1} \cdots$$

Since $m_{[i]} \geq m_{[i + 1]}$, the rightmost segments A in $\widetilde{w}_j(\chi)$ and $\widetilde{w}_j(\psi)$ are the same and we have yet $l_{0,j}(\chi) = l_{0,j}(\psi)$.

When $j = i(\text{mode})$, $[i]$ is of type R for j . Observe that $\widehat{a}_i(\chi) = 0$. Set $\widetilde{w}_i(\chi) = A^{a_i(\chi)} R^{r_i(\chi)}$. Then $\widetilde{w}_i(\psi)$ is obtained by applying the cancellation procedure of the factors RA to the word $w = RA^{a_i(\chi)} R^{r_i(\chi)}$. Clearly, $l_{0,j}(\chi) \neq l_{0,j}(\psi)$ if and only if $a_i(\chi) = 1$ for in this case we have $l_{0,j}(\chi) > 1$ and $l_{0,j}(\psi) = 1$. This proves assertion 1.

2: The arguments are similar to those used in the proof of 1. \square

Proposition 4.5. *For any $\chi \in \Psi_e$ and $i, j \in \mathbb{Z}/e\mathbb{Z}$, we have $\widehat{f}_i \widehat{f}_j \chi \neq \widehat{f}_j \widehat{f}_i \chi \iff i = j$ and $a_i(\chi) + \widehat{a}_i(\chi) = 1$.*

Proof. Assume $i \neq j$ or, $i = j$ and $a_i(\chi) + \widehat{a}_i(\chi) > 1$. Then by assertions 1 and 2 of the previous lemma, we have $\widehat{l}_{0,j}(\chi) = \widehat{l}_{0,j}(\widehat{f}_i \chi) = \widehat{l}_0$ and $l_{0,i}(\chi) = l_{0,i}(\widehat{f}_j \chi) = l_0$. Hence

$$\widehat{f}_i \widehat{f}_j \chi = \chi + [j; \widehat{l}_0] + (l_0; i] - [j + 1; \widehat{l}_0 - 1] - (l_0 - 1; i - 1] = \widehat{f}_j \widehat{f}_i \chi$$

with $[j + 1; \widehat{l}_0 - 1] = \emptyset$ if $\widehat{l}_0 = 1$ and $(l_0 - 1; i - 1] = \emptyset$ if $l_0 = 1$.

Now, assume $i = j$, $a_i(\chi) = 1$ and $\widehat{a}_i(\chi) = 0$. We have

$$\widehat{f}_i \widehat{f}_i \chi = \chi + 2[i] \text{ and } \widehat{f}_i \widetilde{f}_i \chi = \chi + [i] + [i - l_0 + 1, i] - [i - l_0 + 1, i - 1]$$

with $l_0 = l_{0,i}(\chi) > 1$. Similarly, if we assume $i = j$, $a_i(\chi) = 1$ and $\widehat{a}_i(\chi) = 0$, we obtain

$$\widehat{f}_i \widetilde{f}_i \chi = \chi + 2[i] \text{ and } \widetilde{f}_i \widehat{f}_i \chi = \chi + [i] + [i + 1, i + \widehat{l}_0 - 1] - [i, i + \widehat{l}_0 - 1]$$

with $\widehat{l}_0 = \widehat{l}_{0,i}(\chi) > 1$. In both cases, $\widehat{f}_i \widehat{f}_j \chi \neq \widehat{f}_i \widetilde{f}_i \chi$ which completes the proof. Observe that we then have

$$(15) \quad \widehat{f}_i \widehat{f}_i \chi = (\widehat{f}_i)^2 \chi \text{ and } \widehat{f}_i \widetilde{f}_i \chi = (\widetilde{f}_i)^2 \chi.$$

\square

Proposition 4.6. *Consider $\psi \in \Psi_e$ and $i, j \in \mathbb{Z}/e\mathbb{Z}$.*

- (1) *If $i \neq j$, we have $\widetilde{f}_i \widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \psi$.*
- (2) *If $i = j$, set $m = \varepsilon_j^*(\psi)$. Then $\widetilde{f}_i \widetilde{f}_i^* \psi \neq \widetilde{f}_i^* \widetilde{f}_i \psi \iff \varphi_i((\widetilde{e}_i^*)^m \psi) = \varepsilon_i^*(\psi) + 1$.*

Proof. 1: This is a classical property of crystals. Write $\theta_j(\psi) = (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^m b_j$ where $m = \varepsilon_j^*(\psi)$. Then by (5), we have $\theta_j(\widetilde{f}_i \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^m b_j$ for $i \neq j$. By Proposition 2.8, we obtain $\theta_j(\widetilde{f}_j^* \widetilde{f}_i \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$. We have also $\theta_j(\widetilde{f}_j^* \psi) = (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$ and since $i \neq j$, this yields $\theta_j(\widetilde{f}_i \widetilde{f}_j^* \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$. Hence $\theta_j(\widetilde{f}_i \widetilde{f}_j^* \psi) = \theta_j(\widetilde{f}_j^* \widetilde{f}_i \psi)$ and we have $\widetilde{f}_i \widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \psi$ because θ_j is an embedding of crystals.

2: We derive by using the same arguments

$$\theta_i(\widetilde{f}_i \widetilde{f}_i^* \psi) = \begin{cases} \widetilde{f}_i (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) > m + 1, \\ (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) \leq m + 1. \end{cases}$$

We have also

$$\theta_i(\widetilde{f}_i^* \widetilde{f}_i \psi) = \begin{cases} \widetilde{f}_i (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) > m, \\ (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) \leq m. \end{cases}$$

Thus we obtain $\theta_i(\tilde{f}_i \tilde{f}_i^* \psi) = \theta_i(\tilde{f}_i^* \tilde{f}_i \psi)$ except when $\varphi_i((\tilde{e}_i^*)^m \psi) = m + 1$. Observe that we have in this case

$$(16) \quad \tilde{f}_i \tilde{f}_i^* \psi = (\tilde{f}_i^*)^2 \psi \neq (\tilde{f}_i)^2 \psi = \tilde{f}_i^* \tilde{f}_i \psi.$$

□

Lemma 4.7. *Consider $\psi \in \Psi_e$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Set $\text{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \text{wt}_i(\psi) \Lambda_i$. Then we have*

- (1) $\text{wt}_i(\psi) = a_i(\psi) - r_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi)$,
- (2) $\varphi_i(\psi) = a_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi)$.

Proof. 1: Set

$$\psi = \sum_{l \geq 1} m_{(l; i]}(l; i] = \sum_{l \geq 1} m_{[i; l]}[i; l).$$

During the cancellation procedure described in § 4.1, pairs of segments (R, A) or $(\widehat{R}, \widehat{A})$ are deleted. Thus assertion 1 is equivalent to the equality $\text{wt}_i(\psi) = \Delta_i(\psi)$ where

$$(17) \quad \Delta_i(\psi) = \sum_{l \geq 1} m_{(l; i-1]} - m_{(l; i]} + \sum_{l \geq 1} m_{[i+1; l]} - m_{[i; l]}.$$

We proceed by induction on $|\psi|$. For $\psi = \emptyset$, (17) is satisfied. Now assume the equalities (17) hold for any $i \in \mathbb{Z}/e\mathbb{Z}$ with $|\psi| = n$. Set $\psi' = \tilde{f}_j \psi$. We have $\text{wt}(\psi') = \text{wt}(\psi) - \alpha_j$. Since $\alpha_j = 2\Lambda_j - \Lambda_{j+1} - \Lambda_{j-1}$, this gives

$$(18) \quad \text{wt}_i(\psi') = \begin{cases} \text{wt}_i(\psi) & \text{if } i \notin \{j-1, j, j+1\}, \\ \text{wt}_i(\psi) - 2 & \text{if } i = j, \\ \text{wt}_i(\psi) + 1 & \text{if } i \in \{j-1, j+1\}. \end{cases}$$

The multisegment ψ' is obtained by adding the segments $[j]$ to ψ or by replacing a segment $(l-1; j-1]$ in ψ by the segment $(l, j]$. This shows that the relations (18) are also satisfied by the $\Delta_i(\psi')$'s. Hence $\Delta_i(\psi') = \text{wt}_i(\psi')$ for any $i \in \mathbb{Z}/e\mathbb{Z}$.

2: By (1), we have $\text{wt}_i(\psi) = \varphi_i(\psi) - \varepsilon_i(\psi)$. Lemma 4.1 then gives $\text{wt}_i(\psi) = \varphi_i(\psi) - r_i(\psi)$. Thus $\varphi_i(\psi) = a_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi)$ by 1. □

To prove (13), we are going to proceed by induction on $n = |\psi|$. We easily check that $\tilde{f}_i^*(\emptyset) = \widehat{f}_i(\emptyset) = [i]$ for the empty multisegment \emptyset . Now assume that $\tilde{f}_i^*(\psi) = \widehat{f}_i(\psi)$ holds for any multisegment $\psi \in \Psi_e$ such that $|\psi| \leq n$.

Proposition 4.8. *Under the previous induction hypothesis we have for any $\chi \in \Psi_e$ such that $|\chi| \leq n$*

$$(19) \quad \tilde{f}_i \widehat{f}_j \chi \neq \widehat{f}_j \tilde{f}_i \chi \iff \tilde{f}_i \tilde{f}_j^* \chi \neq \tilde{f}_j^* \tilde{f}_i \chi \iff i = j \text{ and } a_i(\chi) + \widehat{a}_i(\chi) = 1.$$

Proof. Note first that the proposition does not directly follow from the induction hypothesis for $|\tilde{f}_j \chi| = n + 1$. By this induction hypothesis, we

have $(\widehat{e}_i^*)^{\varepsilon_i^*(\chi)}\chi = (\widehat{e}_i)^{\widehat{r}_i(\chi)}\chi$. Set $\chi' = (\widehat{e}_i)^{\widehat{r}_j(\chi)}\chi$. Assertion 2 of Lemma 4.7 gives

$$\varphi_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi') - \widehat{r}_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi) + \widehat{r}_i(\chi)$$

for we have $\widehat{r}_i(\chi') = 0$ and $\widehat{a}_i(\chi') = \widehat{a}_i(\chi) + \widehat{r}_i(\chi)$. Observe that $\varepsilon_i^*(\chi) = \widehat{r}_i(\chi)$ be the induction hypothesis. Moreover, we have $a_i(\chi) = a_i(\chi')$ by (14) since $\widehat{a}_i(\varphi) > 0$ for any $\varphi = (\widehat{e}_i)^a\chi$ with $a \in \{1, \dots, \widehat{r}_j(\chi)\}$. This gives the equivalences

$$\varphi_i(\chi') = \varepsilon_i^*(\chi) + 1 \iff a_i(\chi') + \widehat{a}_i(\chi) = 1 \iff a_i(\chi) + \widehat{a}_i(\chi) = 1.$$

Now Propositions 4.5 and 4.6 yields (19). \square

We are now able to prove the main result of this section.

Theorem 4.9. *For any multisegment $\psi \in \Psi_e$ and any $j \in \mathbb{Z}/e\mathbb{Z}$, we have $\widetilde{f}_j^*(\psi) = \widehat{f}_j(\psi)$.*

Proof. We argue by induction on $n = |\psi|$. We already know that for all $j \in \mathbb{Z}/e\mathbb{Z}$, we have $\widetilde{f}_j^*(\emptyset) = \widehat{f}_j(\emptyset) = [j]$. Now assume $\widetilde{f}_j^*\chi = \widehat{f}_j\chi$ for any $j \in \mathbb{Z}/e\mathbb{Z}$ and any $\chi \in \Psi_e$ such that $|\chi| \leq n$. Consider $\psi \in \Psi_e$ such that $|\psi| = n + 1$. There exists $i \in \mathbb{Z}/e\mathbb{Z}$ and $\chi \in \Psi_e$ such that $\psi = \widetilde{f}_i\chi$ and $|\chi| = n$.

When $i \neq j$ or $a_i(\chi) + \widehat{a}_i(\chi) > 1$, we have by Proposition 4.8 $\widetilde{f}_j^*\psi = \widetilde{f}_j^*\widetilde{f}_i\chi = \widetilde{f}_i\widetilde{f}_j^*\chi$. By our induction hypothesis, we can thus write $\widetilde{f}_j^*\psi = \widetilde{f}_i\widehat{f}_j\chi$. Since $a_i(\chi) + \widehat{a}_i(\chi) > 1$, this finally gives $\widetilde{f}_j^*\psi = \widehat{f}_j\widetilde{f}_i\chi = \widehat{f}_j\psi$.

When $i = j$ and $a_i(\chi) + \widehat{a}_i(\chi) = 1$, we obtain $\widetilde{f}_i^*\psi = \widetilde{f}_i^*\widetilde{f}_i\chi = \widetilde{f}_i^2\psi$ by (16). Similarly, we have $\widehat{f}_i\psi = \widehat{f}_i\widetilde{f}_i\chi = \widehat{f}_i^2\psi$ by (15). Thus $\widetilde{f}_i^*\psi = \widehat{f}_i\psi$ which completes the proof. \square

Corollary 4.10. *The involutions $*$ and \sharp satisfy the following simple relation*

$$* = \rho \circ \sharp.$$

Proof. Let $\psi \in \Psi_e$. Then, there exists i_1, \dots, i_n in $\mathbb{Z}/e\mathbb{Z}$ such that

$$\psi = \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} \cdot \emptyset$$

Hence, we obtain

$$\begin{aligned} \psi^* &= \widetilde{f}_{i_1}^* \dots \widetilde{f}_{i_n}^* \cdot \emptyset \\ &= \widehat{f}_{i_1} \dots \widehat{f}_{i_n} \cdot \emptyset \end{aligned}$$

Using (2), this gives

$$\begin{aligned} \psi^* &= \rho(\widetilde{f}_{-i_1} \dots \widetilde{f}_{-i_n} \cdot \emptyset) \\ &= (\rho \circ \sharp)(\psi). \end{aligned}$$

\square

Remark 4.11. Theorem 4.9 and Proposition 4.8 notably implies the equivalence

$$\tilde{f}_i \tilde{f}_j^* \psi = \tilde{f}_j^* \tilde{f}_i \psi \Leftrightarrow i \neq j \text{ or } a_i(\psi) + \widehat{a}_i(\psi) > 1$$

for any $\psi \in \Psi_e$.

5. AFFINE HECKE ALGEBRA OF TYPE A AND ARIKI-KOIKE ALGEBRAS

5.1. Identification of simple modules. Let $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathcal{V}_l$. The Ariki-Koike algebra $\mathcal{H}_n^{\mathbf{v}}(\xi)$ is the quotient $\mathcal{H}_n^a(\xi)/I_{\mathbf{v}}$ where $I_{\mathbf{v}} = \langle P_{\mathbf{v}} = \prod_{i=0}^{l-1} (X_1 - \xi^{v_i}) \rangle$. Then each simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module is isomorphic to a simple $\mathcal{H}_n^a(\xi)$ -module of Mod_n^a . By the Specht module theory developed by Dipper, James and Mathas [7], the simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules are parametrized by certain l -partitions of n called Kleshchev multipartitions. Let $\Phi_e^K(\mathbf{v})$ be the set of Kleshchev l -partitions. Given μ in $\Phi_e^K(\mathbf{v})$, write D^μ for the simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module associated to μ under this parametrization. In fact, we shall need in the sequel the parametrization of the simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules by FLOTW l -partitions. The correspondence between the parametrizations by Kleshchev and FLOTW l -partitions has been detailed in [14]. In particular, the bijection $\Gamma : \Phi_e(\mathbf{v}) \rightarrow \Phi_e^K(\mathbf{v})$ is an isomorphism of \mathcal{U}_v -crystals which can easily be made explicit. This means that, given any λ in $\Phi_e(\mathbf{v})$, we can compute $\Gamma(\lambda)$ directly from λ without using the crystal structures on $\Phi_e(\mathbf{v})$ and $\Phi_e^K(\mathbf{v})$. We then set $\tilde{D}^\lambda = D^{\Gamma(\lambda)}$. This gives the natural labelling

$$\text{Irr}(\mathcal{H}_n^{\mathbf{v}}(\xi)) = \{\tilde{D}^\lambda \mid \lambda \in \Phi_e(\mathbf{v})\}$$

which coincides with the parametrization of the simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules in terms of Geck-Rouquier canonical basic set obtained in [13].

The simple $\mathcal{H}_n^a(\xi)$ -module L_ψ with $\psi \in \Psi_e$ isomorphic to \tilde{D}^λ is given by the following theorem (see [3, Thm 6.2]).

Theorem 5.1. *Let $\lambda \in \Phi_e(\mathbf{v})$ then*

$$\tilde{D}^\lambda \simeq L_{f_{\mathbf{v}}(\lambda)}$$

where $f_{\mathbf{v}}$ is the crystal embedding of Theorem 2.14.

Conversely, given any simple $\mathcal{H}_n^a(\xi)$ -module L_ψ , it is natural to search for the Ariki-Koike algebras $\mathcal{H}_n^{\mathbf{v}}(\xi)$ with \mathbf{v} in \mathcal{V}_l and the simple $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module \tilde{D}^λ such that $\tilde{D}^\lambda \simeq L_\psi$. This problem turns out to be more complicated. Indeed we have first to determine all the multicharges \mathbf{v} such that $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$ and next we need to compute the l -partition λ satisfying $f_{\mathbf{v}}(\lambda) = \psi$. Note that λ is necessarily unique for a given \mathbf{v} since $f_{\mathbf{v}}$ is injective. We will then say that \mathbf{v} is an *admissible multicharge* with respect to ψ when $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$. Then $\lambda = f_{\mathbf{v}}^{-1}(\psi)$ is its corresponding *admissible multipartition*. In the next paragraphs, we shall completely solve the problem of determining all the admissible multicharges and FLOTW multipartitions associated to an aperiodic multisegment ψ . To obtain the corresponding Kleshchev multipartition, it then suffices to apply Γ .

5.2. Admissible multicharges. Let $\psi \in \Psi_e$. To find a multicharge \mathbf{v} such that $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$, we compute $\varepsilon_i^*(\psi)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ by using the equality $\varepsilon_i^*(\psi) = \widehat{r}_i(\psi)$ established in Theorem 4.9. For a multicharge \mathbf{v} in \mathcal{V}_l and $i \in \mathbb{Z}/e\mathbb{Z}$, let $\kappa_i(\mathbf{v})$ be the nonnegative integers such that

$$\mathbf{v} = (\underbrace{0, \dots, 0}_{\kappa_0(\mathbf{v})}, \underbrace{1, \dots, 1}_{\kappa_1(\mathbf{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\kappa_{e-1}(\mathbf{v})}).$$

Then we have

$$f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset \iff \forall i \in \mathbb{Z}/e\mathbb{Z}, \kappa_i(\mathbf{v}) \geq \varepsilon_i^*(\psi).$$

Observe that the multicharge $\mathbf{v}(\psi)$ with $\kappa_i(\mathbf{v}(\psi)) = \varepsilon_i^*(\psi)$ (defined at the end of §2.4) is the multicharge of minimal level among all the admissible multicharges. It is of particular interest for the computation of the involution \sharp as we shall see in §5.2.

5.3. Admissible multipartitions. Consider $\psi \in \Psi_e$, $l \in \mathbb{N}$ and an admissible multicharge $\mathbf{v} \in \mathcal{V}_l$ with respect to ψ . The aim of this section is to give a simple procedure for computing the admissible l -partition $\boldsymbol{\lambda} \in \Phi_e(\mathbf{v})$ associated to \mathbf{v} (i.e. such that $f_{\mathbf{v}}(\boldsymbol{\lambda}) = \psi$).

We begin with a general lemma on FLOTW l -partitions. Consider $\mathbf{v} \in \mathcal{V}_l$ and $\boldsymbol{\lambda} \in \Phi_e(\mathbf{v})$ a non-empty l -partition. Let m be the length of the minimal non zero part of $\boldsymbol{\lambda}$. Let $\boldsymbol{\mu}$ be the l -partition obtained by deleting in $\boldsymbol{\lambda}$ the parts of length m .

Lemma 5.2. *The l -partition $\boldsymbol{\mu}$ belongs to $\Phi_e(\mathbf{v})$.*

Proof. Assume that $\boldsymbol{\mu} \notin \Phi_e(\mathbf{v})$. Then one of the following situations happens.

- (i) There exists $c \in \{0, 1, \dots, l-1\}$ and $i \in \mathbb{N}$ such that $\mu_i^c < \mu_{i+v_{c+1}-v_c}^{c+1}$. This implies in particular that $\mu_{i+v_{c+1}-v_c}^{c+1} \neq 0$. Since $\boldsymbol{\lambda}$ belongs to $\Phi_e(\mathbf{v})$, we have $\lambda_i^c \geq \lambda_{i+v_{c+1}-v_c}^{c+1}$. Thus $\lambda_{i+v_{c+1}-v_c}^{c+1} = \mu_{i+v_{c+1}-v_c}^{c+1}$, $\lambda_i^c = m$ and $\mu_i^c = 0$. We have $\lambda_{i+v_{c+1}-v_c}^{c+1} \neq 0$ and $\lambda_{i+v_{c+1}-v_c}^{c+1} \leq m$. This contradicts the fact that $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by deleting the minimal nonzero parts.
- (ii) There exists $i \in \mathbb{N}$ such that $\mu_i^{l-1} < \mu_{i+v_0-v_{l-1}+e}^0$. We obtain a contradiction similarly.

□

Consider the total order \triangleleft on the segments defined by

$$[k; l] \triangleleft [k'; l'] \iff \begin{cases} l < l', \\ l = l' \text{ and } k < k' \text{ as integers of } \{0, \dots, e-1\}. \end{cases}$$

Write

$$(20) \quad \psi = \sum_{i=1}^r a_i [k_i; l_i],$$

where the nonnegative integers $a_i \in \mathbb{N}$ are defined by requiring that

$$(21) \quad [k_r; l_r] \triangleleft \cdots \triangleleft [k_2; l_2] \triangleleft [k_1; l_1].$$

Set $\psi_0 = \emptyset$ the empty multisegment and

$$\psi_t = \sum_{i=1}^t a_i [k_i; l_i] \text{ for } 1 \leq t \leq r.$$

Set $\lambda[r] = \lambda$. Let $\lambda[t]$, $t = 0, \dots, r-1$ be the l -partitions obtained by deleting successively the parts of lengths l_r, \dots, l_{t+1} in λ . By Lemma 5.2, the l -partitions $\lambda[t]$, $t = 0, \dots, r-1$ all belong to $\Phi_e(\mathbf{v})$. Since $f_{\mathbf{v}}(\lambda) = \psi$ we must also have

$$f_{\mathbf{v}}(\lambda[t]) = \psi_t \text{ for any } t = 0, \dots, r$$

by definition of the map $f_{\mathbf{v}}$ (see (9)). To compute λ by induction from the empty l -partition, it thus suffices to explain how $\lambda[t+1] \in \Phi_e(\mathbf{v})$ can be obtained from $\lambda[t] \in \Phi_e(\mathbf{v})$. The l -partition $\lambda[t+1]$ is constructed by adding a_{t+1} parts of lengths l_{t+1} to $\lambda[t]$ such that the parts added give segments $[k_{t+1}; l_{t+1}]$ in the correspondence (9). Since the nonzero parts $\lambda[t]$ are greater or equal to l_{t+1} , these new parts can only appear on the bottom of the partitions belonging to $\lambda[t]$. This leads us to consider, for $c = 0, \dots, l-1$ the integers

$$i_c = \min\{a \in \mathbb{N} \mid \lambda[t]_a^c = 0\}.$$

Since $f_{\mathbf{v}(\psi)}^{-1}(\psi_{t+1}) \neq \emptyset$, there must exist by (9) integers $c_1, c_2, \dots, c_p \in \{0, \dots, l-1\}$ such that

$$k_{t+1} \equiv 1 - i_{c_1} + v_{c_1} \equiv \cdots \equiv 1 - i_{c_p} + v_{c_p} \pmod{e},$$

with $p \geq a_{t+1}$. Now our problem reduces to determine a_{t+1} partitions among the partitions $\lambda[t]^{c_f}$, $f = 1, \dots, p$ which, once completed with a part l_{t+1} , yield an l -partition of $\Phi_e(\mathbf{v})$.

We define a total order \blacktriangleleft on $\{c_1, \dots, c_p\}$ such that

$$(22) \quad c_a \blacktriangleleft c_b \iff \begin{cases} \text{(i) : } v_{c_a} - i_{c_a} < v_{c_b} - i_{c_b} \text{ or} \\ \text{(ii) : } v_{c_a} - i_{c_a} = v_{c_b} - i_{c_b} \text{ and } c_a < c_b \text{ as integers.} \end{cases}$$

Without loss of generality, we can assume $c_1 \blacktriangleleft \cdots \blacktriangleleft c_p$. Set $S[t+1] = \{(c_1, i_{c_1}), \dots, (c_{a_{t+1}}, i_{c_{a_{t+1}}})\}$. Let $\widehat{\lambda}[t+1]$ be the l -partition defined by

$$\widehat{\lambda}[t+1]_i^c = \begin{cases} \lambda[t]_i^c & \text{if } (c, i) \notin S[t+1], \\ l_{t+1} & \text{if } (c, i) \in S[t+1]. \end{cases}$$

Lemma 5.3. *With the above notation, $\widehat{\lambda}[t+1] = \lambda[t+1]$.*

Proof. It suffices to prove that $\widehat{\lambda}[t+1]$ belongs to $\Phi_e(\mathbf{v})$. Indeed, this will give $f_{\mathbf{v}}(\widehat{\lambda}[t+1]) = f_{\mathbf{v}}(\lambda[t+1]) = \psi_{t+1}$ and thus, $\widehat{\lambda}[t+1] = \lambda[t+1]$ since $f_{\mathbf{v}}$ is an embedding. The second condition to be a FLOTW l -partition is clearly satisfied for the multisegment ψ_{t+1} is aperiodic. We have to check that the first condition also holds.

Assume that $\widehat{\lambda}[t+1]$ does not satisfy condition 1 of Definition 2.11. Suppose first we have $\widehat{\lambda}[t+1]_i^s < \widehat{\lambda}[t+1]_{i+v_{s+1}-v_s}^{s+1}$ where $s \in \{1, \dots, l-1\}$ and i is a nonnegative integer. Since $\lambda[t] \in \Phi_e(\mathbf{v})$, we have $\widehat{\lambda}[t+1]_i^s = \lambda[t]_i^s = 0$ and $\lambda[t]_{i+v_{s+1}-v_s}^{s+1} = 0$, $\widehat{\lambda}[t+1]_{i+v_{s+1}-v_s}^{s+1} = l_{t+1}$. Thus $(s+1, i+v_{s+1}-v_s) \in S[t+1]$. We have two cases to consider.

- Assume $i+v_{s+1}-v_s > 1$ and $\widehat{\lambda}[t+1]_{i+v_{s+1}-v_s-1}^{s+1} = \lambda[t]_{i+v_{s+1}-v_s-1}^{s+1} > 0$. Then $i=1$ or $\widehat{\lambda}[t+1]_{i-1}^s \neq 0$. Indeed we must have $\widehat{\lambda}[t+1]_{i-1}^s = \widehat{\lambda}[t]_{i-1}^s \geq \lambda[t]_{i+v_{s+1}-v_s-1}^{s+1}$ because $\lambda[t]$ belongs to $\Phi_e(\mathbf{v})$. We have $(s+1, i+v_{s+1}-v_s) \in S[t+1]$. In particular

$$k_{t+1} \equiv v_{s+1} - (i+v_{s+1}-v_s) + 1 \equiv v_s - i + 1 \pmod{e}.$$

Since $\lambda[t]_i^s = 0$, this means that $(s, i) \in S[t+1]$. But this is a contradiction. Indeed by condition (ii) of (22), we should have $\widehat{\lambda}[t+1]_i^s = l_{t+1} \neq 0$.

- Assume $i+v_{s+1}-v_s = 1$ then $i=1$ and we have $v_{s+1} = v_s$. Thus $(s, i) \in S[t+1]$ and we derive a contradiction similarly.

Now suppose we have $\widehat{\lambda}[t+1]_i^{l-1} > \widehat{\lambda}[t+1]_{i+v_0-v_{l-1}+e}^0$. The proof is analogue. We obtain that $(l-1, i) \in S[t+1]$ and $(0, i+v_0-v_{l-1}+e) \in S[t+1]$. This contradicts condition (i) of (22). \square

By using the above procedure, we are now able to compute the l -partitions $\lambda[t]$, $t = 1, \dots, r$ from ψ and from its associated admissible multicharge \mathbf{v} . This thus gives a recursive algorithm for computing the admissible l -partition λ from ψ .

5.4. Example. Let $e = 4$. We consider the following aperiodic multisegment

$$\psi = [1, 5] + 2[0; 4] + [3; 3] + [2; 2] + [3; 1].$$

We have $\widehat{w}_0(\psi) = \widehat{R}\widehat{R}\widehat{A}$, $\widehat{w}_1(\psi) = \widehat{A}\widehat{R}$, $\widehat{w}_2(\psi) = \widehat{A}\widehat{R}\widehat{A}$ and $\widehat{w}_3(\psi) = \widehat{R}\widehat{R}\widehat{A}\widehat{A}$. This gives

$$\varepsilon_0^*(\psi) = 1, \quad \varepsilon_1^*(\psi) = 1, \quad \varepsilon_2^*(\psi) = 0, \quad \varepsilon_3^*(\psi) = 0.$$

Thus the multicharge $(0, 1)$ is an admissible multicharge. Actually this is the one with minimal level. We now use the above algorithm to compute the associated admissible l -partition λ . Using the same notation as above, we successively obtain

$$\lambda[0] = (\emptyset, \emptyset), \quad \lambda[1] = (\emptyset, 5), \quad \lambda[2] = (4, 5.4),$$

$$\lambda[3] = (4.3, 5.4), \quad \lambda[4] = (4.3.2, 5.4), \quad \lambda[5] = (4.3.2, 5.4.1).$$

Hence $\lambda = (4.3.2, 5.4.1)$ is the admissible bipartition associated to the multicharge $(0, 1)$ and ψ . We easily check that

$$f_{(0,1)}(4.3.2, 5.4.1) = \psi.$$

This means that the modules L_ψ and \widetilde{D}^λ are isomorphic.

The multicharge $(0, 0, 1, 2, 3, 3, 3)$ is another example of an admissible multicharge (with level 7) with associated admissible multipartition $(4, 4, 5, 2, 3, 1, \emptyset)$.

6. COMPUTATION OF THE INVOLUTION \sharp

6.1. The generalized Mullineux involution. The two fold symmetry $i \longleftrightarrow -i$ defines a skew crystal isomorphism from $B_e(\mathbf{v})$ to $B_e(\mathbf{v}^\sharp)$ where $\mathbf{v} = (v_0, \dots, v_{l-1})$ and $\mathbf{v}^\sharp = (-v_{l-1}, \dots, -v_0)$ belong to \mathcal{V}_l (see (8)). Given $\lambda \in \Phi_e(\mathbf{v})$, write $m_l^\mathbf{v}(\lambda) \in \Phi_e(\mathbf{v}^\sharp)$ for the image of λ under this skew isomorphism. In [9], Ford and Kleshchev proved that for $l = 1$, the map $m_l^\mathbf{v}$ reduces to the Mullineux involution m_1 on e -restricted partitions. Thus we call $m_l^\mathbf{v}$ the generalized Mullineux involution.

By § 5.1, the set $\Phi_e^K(\mathbf{v})$ of Kleshchev l -partitions has also the structure of an affine crystal isomorphic to $B_e(\mathbf{v})$. In particular the two fold symmetry $i \longleftrightarrow -i$ also defines a bijection $m_l^{\mathbf{v},K}$ from $\Phi_e^K(\mathbf{v})$ to $\Phi_e^K(\mathbf{v}^\sharp)$. In [15], we gave an explicit procedure yielding $m_l^{\mathbf{v},K}$. Given $\lambda = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e^K(\mathbf{v})$, the l -partition $\mu = m_l^{\mathbf{v},K}(\lambda)$ is obtained by computing first

$$\nu = (m_1(\lambda^0), \dots, m_1(\lambda^{l-1}))$$

i.e. the l -partition obtained by applying the Mullineux map to each partition of λ . The l -partition ν does not belong to $\Phi_e^K(\mathbf{v}^\sharp)$ in general and we have then to apply a straightening algorithm (detailed in [15] §4.3) to obtain μ . As already noted in §5.1, we have a bijection (in fact a crystal isomorphism)

$$\Gamma : \Phi_e(\mathbf{v}) \rightarrow \Phi_e^K(\mathbf{v})$$

which can be made explicit by using the results of [14]. This permits to compute the map $m_l^\mathbf{v}$ since

$$(23) \quad m_l^\mathbf{v} = \Gamma^{-1} \circ m_l^{\mathbf{v},K} \circ \Gamma.$$

Remark 6.1.

- (1) The previous procedure yielding the generalized Mullineux map $m_l^\mathbf{v}$ can be optimized. In particular the conjugation by the map Γ can be avoided. Nevertheless, the pattern of the computation remains essentially the same : it uses the original Mullineux map m_1 and the results of [14] on affine crystal isomorphisms. Since it requires some technical combinatorial developments which are not essential for our purposes, we have chosen to omit it here.
- (2) Note also that in the case $e = \infty$, the map Γ is the identity and m_1 is simply the conjugation operation on the partitions. As observed in [15, §4.4], the algorithm for computing $m_l^\mathbf{v} = m_l^{\mathbf{v},K}$ then considerably simplifies.

6.2. The algorithm. Let $\psi \in \Psi_e$ then, to compute ψ^\sharp , we first determine an admissible multicharge \mathbf{v} with respect to ψ and the associated admissible multipartition $\boldsymbol{\lambda}$. Then we apply the above algorithm to compute $m_l^\mathbf{Y}(\boldsymbol{\lambda})$. It turns out that the complexity of this algorithm considerably increases with the level of \mathbf{v} . Hence, the use of the admissible multicharge $\mathbf{v}(\psi)$ with minimal level is preferable. Let us summarize the different steps of the procedure we have to apply to compute ψ^\sharp :

- (1) For $i \in \mathbb{Z}/e\mathbb{Z}$, we compute $\varepsilon_i^*(\psi)$. To do this, we use Theorem 4.9 which gives the equalities $\varepsilon_i^*(\psi) = \widehat{r}_i(\psi)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. We then put

$$\mathbf{v}(\psi) = (\underbrace{0, \dots, 0}_{\varepsilon_0^*(\mathbf{v})}, \underbrace{1, \dots, 1}_{\varepsilon_1^*(\mathbf{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\varepsilon_{e-1}^*(\mathbf{v})}).$$

By Theorem 4.9, $\mathbf{v}(\psi)$ is an admissible multicharge in \mathcal{V}_l .

- (2) Using §5.3, we compute the admissible FLOTW multipartition $\underline{\boldsymbol{\lambda}}(\psi)$ with respect to $\mathbf{v}(\boldsymbol{\lambda})$ and ψ .
- (3) Using §6.1, we compute the image $m_l^\mathbf{Y}(\underline{\boldsymbol{\lambda}}(\psi))$ of $\underline{\boldsymbol{\lambda}}(\psi)$ under the generalized Mullineux involution,
- (4) We finally obtain the aperiodic multisegment $\psi^\sharp = f_{\mathbf{v}(\psi)}(m_l^\mathbf{Y}(\underline{\boldsymbol{\lambda}}(\psi)))$. using §2.4.

Remark 6.2. In the case where $e = \infty$, our algorithm for computing the Zelevinsky involution is essentially equivalent to that described by Moeglin and Waldspurger in [22] except we use multipartitions rather than multisegments.

7. FURTHER REMARKS

7.1. Computation of the Kashiwara involution. We have established in Section 4, that the crystal operators \widetilde{f}_i^* and \widehat{f}_i coincide for any $i \in \mathbb{Z}/e\mathbb{Z}$. Given $\psi \in \Psi_e$, we can thus compute ψ^* by determining a path $\psi = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \emptyset$ in $B_e(\infty)$ from the empty multisegment to ψ . We have then $\psi^* = \widehat{f}_{i_1} \cdots \widehat{f}_{i_n} \emptyset$.

By combining the algorithm described in Section 6 for computing the involution \sharp with the relation $* = \sharp \circ \rho$ of Corollary 4.10, we obtain another procedure determining $*$. This procedure is more efficient than the previous one since it does not required the determination of a path in the crystal $B_e(\infty)$.

Example 7.1. Assume $e = 2$, the involution \sharp is nothing but the identity and, thus the Kashiwara involution coincide with ρ .

7.2. Crystal commutor for $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$. In [17], Kamnitzer and Tingley introduced a crystal commutor for any symmetrizable Kac-Moody algebra. Recall that a crystal commutor for $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ is a crystal isomorphism

$$\sigma_{\mathbf{v}, \mathbf{v}'} : B_e(\mathbf{v}) \otimes B_e(\mathbf{v}') \rightarrow B_e(\mathbf{v}') \otimes B_e(\mathbf{v}).$$

This isomorphism is unique if and only if $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ does not contain two isomorphic connected components that is, if the decomposition of the corresponding tensor product is without multiplicity. Such a crystal commutor is defined by specifying the images of the highest weight vertices of $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$. It is easy to verify by using (6) that the highest weight vertices of $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ are precisely the vertices of the form $\emptyset \otimes \lambda$ with $\lambda \in B_e(\mathbf{v}')$ such that $\varepsilon_i(\lambda) \leq r_i$ for any $i \in \mathbb{Z}/e\mathbb{Z}$ (r_i is the number of coordinates in \mathbf{v} equal to i). Denote by $\mathcal{H}_{\mathbf{v},\mathbf{v}'}$ the set of highest weight vertices in $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$.

For any $\lambda \in B_e(\mathbf{v}')$, write for short $\lambda^* = \underline{\lambda}(f_{\mathbf{v}'}(\lambda)^*)$ (see the definition of $\underline{\lambda}$ below (10)). Since $*$ is an involution, we have $\lambda^* \in B_e(\mathbf{w})$ where $\mathbf{w} \in \mathcal{V}_l$ is the multicharge having $\varepsilon_i(\lambda)$ coordinates equal to i and level $l = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \varepsilon_i(\lambda)$. The condition $\varepsilon_i(\lambda) \leq r_i$ for any $i \in \mathbb{Z}/e\mathbb{Z}$ then implies that $\lambda^* \in B_e(\mathbf{v})$. We have the following theorem which is the main result of [17].

Theorem 7.2.

- (1) Assume $\emptyset \otimes \lambda \in \mathcal{H}_{\mathbf{v},\mathbf{v}'}$. Then $\emptyset \otimes \lambda^* \in \mathcal{H}_{\mathbf{v}',\mathbf{v}}$.
- (2) The map

$$\sigma_{\mathbf{v},\mathbf{v}'} : \begin{cases} \mathcal{H}_{\mathbf{v},\mathbf{v}'} \rightarrow \mathcal{H}_{\mathbf{v}',\mathbf{v}} \\ \emptyset \otimes \lambda \mapsto \emptyset \otimes \lambda^* \end{cases}$$

defines a crystal commutor for $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$.

The results established in Sections 4 and 6 then permit to compute the crystal commutor of Kamnitzer and Tingley for affine type A crystals.

Example 7.3. Assume $e = 2$. Then the crystal commutor $\sigma_{\mathbf{v},\mathbf{v}'}$ satisfies $\sigma_{\mathbf{v},\mathbf{v}'}(\emptyset \otimes \lambda) = (\emptyset \otimes \underline{\lambda}(\rho \circ f_{\mathbf{v}'}(\lambda)))$ for any $\emptyset \otimes \lambda \in \mathcal{H}_{\mathbf{v},\mathbf{v}'}$.

REFERENCES

- [1] S. ARIKI, Representations of quantum groups and combinatorics of Young tableaux, A. M. S. University Lecture Series **26** (2002).
- [2] S. ARIKI, Proof of the modular branching rule for Ariki-Koike algebras, Journal of Algebra **306** (2006), 290-300.
- [3] S. ARIKI, N. JACON AND C. LECOUCVEY, The modular branching rule for affine Hecke algebras of type A , preprint, <http://arxiv.org/abs/0808.3915>.
- [4] C. BESSENRODT AND J.B. OLSSON, On residue symbols and the Mullineux conjecture. J. Algebraic Combin. **7** (1998), no. 3, 227-251.
- [5] J. BRUNDAN AND A. KLESHCHEV, Hecke-Clifford superalgebras, crystals of type $A_{2\ell}^{(2)}$ and modular branching rule for \widehat{S}_n , Representation Theory **5** (2001), 317-403.
- [6] N. CHRISS AND V. GINZBURG, Representation Theory and Complex Geometry, Birkhäuser, 1997.
- [7] R. DIPPER, G. JAMES, A. MATHAS, Cyclotomic q -Schur Algebras, Mathematische Zeitschrift **229**, (1998), 385-416.
- [8] O. FODA, B. LECLERC, M. OKADO, J.-Y. THIBON AND T. WELSH, Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras, Advances in Math. **141** (1999), 322-365.
- [9] B. FORD AND A. KLESHCHEV, A proof of the Mullineux conjecture. Math. Z. **226** (1997), no. 2, 267-308.

- [10] I. GROJNOWSKI, Affine \widehat{sl}_p controls the modular representation theory of the symmetric group and related Hecke algebras, <http://arXiv:math/9907129>.
- [11] JIN YUN GUO, The Hall polynomials of a cyclic serial algebra, *Comm. Algebra* **23** (1995), 743–751.
- [12] N. IWAHORI AND H. MATSUMOTO, On some Bruhat decomposition of the Hecke rings of p -adic Chevalley groups, *Publ. Math. I.H.E.S.*, **25** (1965), 237–280.
- [13] N. JACON, On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity. *J. Math. Kyoto Univ.* **44** (2004), no. 4, 729–767.
- [14] N. JACON AND C. LECOUEVEY, Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{sl}_e)$ -modules of higher level, to appear in *Algebras and Rep. Theory*, <http://arxiv.org/abs/0706.0680>.
- [15] N. JACON AND C. LECOUEVEY, On the Mullineux involution for Ariki-Koike algebras, to appear in *Journal of Algebra (sec. Comp. Algebra)*, <http://arxiv.org/abs/0804.0478>.
- [16] M. JIMBO, K. C. MISRA, T. MIWA AND M. OKADO, Combinatorics of representations of $\mathcal{U}_q(\widehat{sl}(n))$ at $q = 0$, *Communication in Mathematical Physics* **136** (1991), 543–566.
- [17] J. KAMNITZER, P. TINGLEY, A definition of the crystal commutor using Kashiwara’s involution, to appear in the *Journal of Algebraic Combinatorics*.
- [18] M. KASHIWARA, On crystal bases, *Canadian Mathematical Society, Conference Proceedings*, **16** (1995), 155–197.
- [19] H. KNIGHT AND A. ZELEVINSKY, Representations of quivers of type A and the multisegment duality. *Adv. Math.* **117** (1996), no. 2, 273–293.
- [20] B. LECLERC, J.-Y. THIBON, E. VASSEROT, Zelevinsky’s involution at roots of unity. *J. Reine Angew. Math.* **513** (1999), 33–51.
- [21] G. LUSZTIG, Canonical basis and Hall algebras, *Lectures at the summer school on representation theory and algebraic geometry, Montreal (1997)*.
- [22] C. MOEGLIN AND J.-L. WALDSPURGER, Sur l’involution de Zelevinsky, *J. Reine Angew. Math.* **372** (1986) 136–177.
- [23] G. MULLINEUX, Bijections of p -regular partitions and p -modular irreducibles of the symmetric groups. *J. London Math. Soc. (2)* **20** (1979), no. 1, 60–66.
- [24] M. RICHARDS, Some decomposition numbers for Hecke algebras of general linear groups. *Math. Proc. Cambridge Philos. Soc.* **119** (1996), no. 3, 383–402.
- [25] C. RINGEL, The composition algebra of a cyclic quiver, *Proc. London Math. Soc.* **66** (1993) 507–537.
- [26] D. UGLOV, Canonical bases of higher-level q -deformed Fock spaces and Kazhdan-Lusztig polynomials. *Physical combinatorics (Kyoto, 1999)*, 249–299, *Progr. Math.*, **191**, Birkhäuser Boston, Boston, MA, 2000.
- [27] M. VAZIRANI Parametrizing Hecke algebra modules: Bernstein-Zelevinsky multisegments, Kleshchev multipartitions, and crystal graphs, *Transform. Groups* **7** (2002), 267–303.
- [28] M-F. VIGNERAS, A propos d’une conjecture de Langlands modulaire, in *Finite reductive groups: related structures and representations*, Cabanes, Marc (ed.), Birkhauser 1997.
- [29] A. ZELEVINSKY Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$. *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 2, 165–210.