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# KASHIWARA AND ZELEVINSKY INVOLUTIONS IN AFFINE TYPE $A$

NICOLAS JACON AND CÉDRIC LECOUEY

ABSTRACT. We describe how the Kashiwara involution  $*$  on crystals of affine type  $A$  is encoded by the combinatorics of aperiodic multisegments. This permits to prove in an elementary way that  $*$  coincides with the Zelevinsky involution  $\tau$  on the set of simple modules for the affine Hecke algebras. We then give efficient procedures for computing  $*$  and  $\tau$ . Remarkably, these procedures do not use the underlying crystal structure. They also permit to match explicitly the Ginzburg and Ariki parametrizations of the simple modules associated to affine and cyclotomic Hecke algebras, respectively.

## 1. INTRODUCTION

The Kashiwara involution  $*$  in affine type  $A$  is a fundamental anti-isomorphism of the quantum group  $\mathcal{U}_v$  associated to the affine root system  $A_{e-1}^{(1)}$ . It induces a subtle involution on  $B_e(\infty)$ , the Kashiwara crystal corresponding to the negative part  $\mathcal{U}_v^-$  of  $\mathcal{U}_v$ . The Zelevinsky involution yields an involution  $\tau$  of the affine Hecke algebra of type  $A$ . When  $q$  is specialized to an  $e$ -th root of 1,  $\tau$  also induces an involution on  $B_e(\infty)$ . In this paper, we show by using the combinatorics of aperiodic multisegments that the Kashiwara and Zelevinsky involutions coincide on  $B(\infty)$ . We also provide efficient procedures for computing these involutions. In addition, our results permit to match explicitly the Ginzburg and Ariki parametrizations of the simple modules associated to affine and cyclotomic Hecke algebras respectively. All our computations can be made independent of the crystal structure on  $B_e(\infty)$ . Moreover, they do not require the determination of  $i$ -induction or  $i$ -restriction operations on simple modules. Let us now describe the context and the results of the paper more precisely.

The Zelevinsky involution first appeared in [29] in connection with the representation theory of the linear group  $GL(n, \mathbb{F}_p)$  over the  $p$ -adic field  $\mathbb{F}_p$ . Works by Mœglin and Waldspurger [22] then permit to link it with a natural involution  $\tau$  of the affine type  $A$  Hecke algebra  $\mathcal{H}_n^a(q)$  over the field  $\mathbb{F}$  with generic parameter  $q$ . When  $e \geq 2$  is an integer and  $q$  is specialized at  $\xi$ , a primitive  $e$ -root of 1, it was conjectured by Vigneras [28] that this involution should be related to the modular representation theory of  $GL(n, \mathbb{F}_p)$ . In the sequel we will refer to  $\tau$  as the Zelevinsky involution of  $\mathcal{H}_n^a(\xi)$  (see Section 3 for a complete definition).

The involution  $\tau$  induces an involution on the set of simple  $\mathcal{H}_n^a(\xi)$ -modules. There exist essentially two different parametrizations of these modules in the literature. In the geometric construction of Chriss and Ginzburg [6] and under the assumption  $\mathbb{F} = \mathbb{C}$ , the simple  $\mathcal{H}_n^a(\xi)$ -modules are labelled by aperiodic multisegments. These simple modules can also be regarded as simple modules associated to Ariki-Koike algebras  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ . The Specht module theory developed by Dipper, James and Mathas then provides a labelling of the simple  $\mathcal{H}_n^a(\xi)$ -modules by Kleshchev multipartitions. Both constructions permit to endow the set of simple  $\mathcal{H}_n^a(\xi)$ -modules with the structure of a crystal isomorphic to  $B_e(\infty)$ . The Kashiwara crystal operators then yield the modular branching rules for the Ariki-Koike algebras and affine Hecke algebras of type  $A$  ([2], [3]).

In [10], Grojnowski uses  $i$ -induction and  $i$ -restrictions operators to define an abstract crystal structure on the set of simple  $\mathcal{H}_n^a(\xi)$ -modules. He then proves that this crystal is in fact isomorphic to  $B_e(\infty)$ . This approach is valid over an arbitrary field  $\mathbb{F}$  and does not require the Specht module theory of Dipper James and Mathas. This notably permits to extend the methods of [10] to the representation theory of the cyclotomic Hecke-Clifford superalgebras [5]. Nevertheless, this approach does not match up the abstract crystal obtained with the labellings of the simple modules by aperiodic multisegments or Kleshchev multipartitions. Since the  $i$ -induction operation on simple modules is difficult to obtain in general, it is also not really suited to explicit computations.

The identification of  $\mathcal{U}_v^-$  with the composition subalgebra of the Hall algebra associated to the cyclic quiver of type  $A_e^{(1)}$  yields two different structures of crystal on the set of aperiodic multisegments. They both come from two different parametrizations of the canonical basis of  $\mathcal{U}_v^-$  which correspond under the anti-isomorphism  $\rho$  on  $\mathcal{U}_v^-$  switching the generators  $f_i$  and  $f_{-i}$ . In particular  $\rho$  provides an involution on the crystal  $B_e(\infty)$  which can be easily computed. The use of the composition algebra also permits to describe explicitly the structure of Kashiwara crystal on the set of aperiodic multisegments. This was obtained in [20] by Leclerc, Thibon and Vasserot. In addition, these authors prove that the involution  $\tau$  on  $B_e(\infty)$  satisfies the identity  $\tau = \sharp \circ \rho$  where  $\sharp$  is the two fold symmetry on  $B_e(\infty)$  which switches the sign of each arrow.

In this paper, we first establish that the two crystal structures on aperiodic multisegments obtained by identifying  $\mathcal{U}_v^-$  with the composition algebra correspond up to the conjugation by the Kashiwara involution  $*$ . This implies that

$$* = \tau \text{ on } B_e(\infty).$$

Observe that an equivalent identity can also be established by using results of [10] but, as mentioned above, it then requires subtle considerations on representation theory of  $\mathcal{H}_n^a(\xi)$  and does not permit to compute  $* = \tau$  efficiently. In contrast our proof uses only elementary properties of crystal

graphs and yields efficient procedures for computing the involution  $* = \tau$ . This notably permits us to generalize an algorithm of Moeclin and Waldspurger which gives the Zelevinsky involution when  $e = \infty$ .

As a consequence, extending works of Vazirani [27], we completely solve the following natural problem. Given a simple  $\mathcal{H}_n^a(\xi)$ -module  $L_\psi$  (with  $\psi$  an aperiodic multisegment), we find all the Ariki-Koike algebras  $\mathcal{H}_n^v(q)$  and the simple  $\mathcal{H}_n^v(q)$ -modules  $D^\lambda$  (with  $\lambda$  a Kleshchev multipartition) such that  $D^\lambda \simeq L_\psi$  as  $\mathcal{H}_n^a(\xi)$ -modules. The procedure yielding the Kashiwara involution also permits to compute the commutor of  $A_e^{(1)}$ -crystals introduced by Kamnitzer and Tingley in [17].

The paper is organized as follows. In Section 2, we review the identification of  $\mathcal{U}_v^-$  with the composition algebra and the two structures of crystal it gives on the set of aperiodic multisegments. We also recall basic facts on the Kashiwara involution. Section 3 is devoted to the definition of the Zelevinsky involution on the set of simple  $\mathcal{H}_n^a(\xi)$ -modules and to the results of [20]. In Section 4, we prove the identity  $* = \tau$ . The problem of determining the algebras  $\mathcal{H}_n^v(\xi)$  and the simple  $\mathcal{H}_n^v(\xi)$ -modules isomorphic to a given simple  $\mathcal{H}_n^a(\xi)$ -module is studied in Section 5. In the last two sections, we give a simple combinatorial procedure for computing the involutions  $\tau, \rho$  and  $\sharp$  on  $B_e(\infty)$ . We prove in fact that all these computations can essentially be obtained from the Mullineux involution on  $e$ -regular partitions and the crystal isomorphisms described in [14]. We also investigate several consequences of our results.

## 2. QUANTUM GROUPS AND CRYSTALS IN AFFINE TYPE A

**2.1. The quantum group  $\mathcal{U}_v$ .** Let  $v$  be an indeterminate and  $e \geq 2$  an integer. Write  $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$  for the quantum group of type  $A_{e-1}^{(1)}$ . This is an associative  $\mathbb{Q}(v)$ -algebra with generators  $e_i, f_i, t_i, t_i^{-1}$ ,  $i \in \mathbb{Z}/e\mathbb{Z}$  and  $\partial$  (see [26, §2.1] for the complete description of the relations satisfied by these generators). Write  $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$  and  $\{\alpha_0, \dots, \alpha_{e-1}\}$  respectively for the set of fundamental weights and the set of simple roots associated to  $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ . Let  $P$  be the weight lattice of  $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ . We denote by  $\mathcal{U}_v = \mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$  the subalgebra generated by  $e_i, f_i, t_i, t_i^{-1}$ ,  $i \in \mathbb{Z}/e\mathbb{Z}$ . Then  $\overline{P} = P/\mathbb{Z}\delta$  is the set of classical weights of  $\mathcal{U}_v$ . For any  $i \in \mathbb{Z}/e\mathbb{Z}$ , we also denote by  $\Lambda_i$  and  $\alpha_i$  the restriction of  $\Lambda_i$  and  $\alpha_i \in P$  to  $\overline{P}^\wedge$ . Let  $\mathcal{U}_v^-$  be the subalgebra of  $\mathcal{U}_v$  generated by the  $f_i$ 's with  $i \in \mathbb{Z}/e\mathbb{Z}$ .

## 2.2. Two crystal structures on the set of aperiodic multisegments.

**Definition 2.1.** Let  $l \in \mathbb{Z}_{>0}$  and  $i \in \mathbb{Z}/e\mathbb{Z}$ . The *segment of length  $l$  and head  $i$*  is the sequence of consecutive residues  $[i, i+1, \dots, i+l-1]$ . We denote it by  $[i; l]$ . Similarly, The *segment of length  $l$  and tail  $i$*  is the sequence of consecutive residues  $[i-l+1, \dots, i-1, i]$ . We denote it by  $(l; i]$ .

**Definition 2.2.** A collection of segments is called a *multisegment*. If the collection is the empty set, we call it the empty multisegment and it is denoted by  $\emptyset$ .

It is convenient to write a multisegment  $\psi$  on the form

$$\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i;l]} [i;l].$$

**Definition 2.3.** A multisegment  $\psi$  is *aperiodic* if, for every  $l \in \mathbb{Z}_{>0}$ , there exists some  $i \in \mathbb{Z}/e\mathbb{Z}$  such that  $(l; i]$  does not appear in  $\psi$ . Equivalently, a multisegment  $\psi$  is aperiodic if, for each  $l \in \mathbb{Z}_{>0}$ , there exists some  $i \in \mathbb{Z}/e\mathbb{Z}$  such that  $[i; l)$  does not appear in  $\psi$ . We denote by  $\Psi_e$  the set of aperiodic multisegments.

Let  $B_e(\infty)$  be the (abstract) crystal basis of  $\mathcal{U}_v^-$ . By results of Ringel and Lusztig, the algebra  $\mathcal{U}_v^-$  is isomorphic to the composition algebra of the Hall algebra associated to the cyclic quiver  $\Gamma_e$  of length  $e$ . This yields in particular a natural parametrization of the vertices of  $B_e(\infty)$  by  $\Psi_e$ . We can thus regard the vertices of  $B_e(\infty)$  as aperiodic multisegments. The corresponding crystal structure was described by Leclerc, Thibon and Vasserot in [20, Theorem 4.1]. In fact we shall need in the sequel two different structures of crystal on  $\Psi_e$ . They are linked by the involution  $\rho$  which negates all the segments of a given multisegment, that is such that

$$(1) \quad \psi^\rho = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i;l]} (l; -i]$$

for any multisegment  $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{[i;l]} [i;l]$ . The involution  $\rho$  as a natural algebraic interpretation since it also yields a linear automorphism of the Hall algebra associated to  $\Gamma_e$ . Since we do not use Hall algebras in this paper, we only recall below the two crystal structures relevant for our purpose.

Let  $\psi$  be a multisegment and let  $\psi_{\geq l}$  be the multisegment obtained from  $\psi$  by deleting the multisegments of length less than  $l$ , for  $l \in \mathbb{Z}_{>0}$ . Denote by  $m_{[i;l]}$  the multiplicity of  $[i;l)$  in  $\psi$ . For any  $i \in \mathbb{Z}/e\mathbb{Z}$ , set

$$\widehat{S}_{l,i} = \sum_{k \geq l} (m_{[i+1;k)} - m_{[i;k)}).$$

Let  $\widehat{l}_0$  be the minimal value of  $l$  that attains  $\min_{l > 0} \widehat{S}_{l,i}$ .

**Theorem 2.4.** *Let  $\psi$  be a multisegment,  $i \in \mathbb{Z}/e\mathbb{Z}$  and let  $\widehat{l}_0$  be as above. Then we have*

$$\widehat{f}_i \psi = \psi_{\widehat{l}_0, i},$$

where the multisegment  $\psi_{\widehat{l}_0, i}$  is defined as follows

$$\psi_{\widehat{l}_0, i} = \begin{cases} \psi + [i; 1) & \text{if } \widehat{l}_0 = 1, \\ \psi + [i; \widehat{l}_0) - [i + 1; \widehat{l}_0 - 1) & \text{if } \widehat{l}_0 > 1. \end{cases}$$

The crystal structure on  $\Psi_e$  obtained from the action of the operators  $\widehat{f}_i$ ,  $i \in \mathbb{Z}/e\mathbb{Z}$  does not coincide with that initially described by Leclerc, Thibon and Vasserot. The LTV crystal structure stated in [20] is obtained by using the crystal operators

$$(2) \quad \widetilde{f}_i = \rho \circ \widehat{f}_{-i} \circ \rho, \quad i \in \mathbb{Z}/e\mathbb{Z}$$

rather than the operators  $\widehat{f}_i$ . More precisely, set  $S_{l,i} = \sum_{k \geq l} (m_{(k;i-1]} - m_{(k;i]})$ . Let  $l_0$  be the minimal  $l$  that attains  $\min_{l > 0} S_{l,i}$ . Then, the crystal structure corresponding to the  $\widetilde{f}_i$ 's is given as follows.

**Theorem 2.5.** *Let  $\psi$  be a multisegment and let  $i \in \mathbb{Z}/e\mathbb{Z}$  and  $l_0$  be as above. Then we have*

$$\widetilde{f}_i \psi = \psi_{l_0,i},$$

where the multisegment  $\psi_{l_0,i}$  is defined as follows

$$\psi_{l_0,i} = \begin{cases} \psi + (1; i] & \text{if } l_0 = 1, \\ \psi + (l_0; i] - (l_0 - 1; i - 1] & \text{if } l_0 > 1. \end{cases}$$

Let  $\psi$  be a multisegment. Then to compute  $\widetilde{e}_i \psi$ , we proceed as follows. If  $\min_{l > 0} S_{l,i} = 0$ , then  $\widetilde{e}_i \psi = 0$ . Otherwise, let  $l_0$  be the maximal  $l$  that attains  $\min_{l > 0} S_{l,i}$ . Then,  $\widetilde{e}_i \psi$  is obtained from  $\psi$  by replacing  $(l_0; i]$  with  $(l_0 - 1; i - 1]$ .

In the sequel, we identify  $B_e(\infty)$  with the crystal structure obtained on  $\Psi_e$  by considering the operators  $\widetilde{f}_i$ ,  $i \in \mathbb{Z}/e\mathbb{Z}$  (see also Remark 2.7). Then  $\rho$  induces an involution on  $B_e(\infty)$  and the crystal operators  $\widetilde{f}_i$  and  $\widehat{f}_i$  are related by (2). We denote by  $\text{wt}(\psi)$  the weight of the aperiodic multisegment  $\psi$  considered as a vertex of the crystal  $B_e(\infty)$ . Set

$$(3) \quad \text{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \text{wt}_i(\psi) \Lambda_i.$$

For any  $i \in \mathbb{Z}/e\mathbb{Z}$ , define  $\varepsilon_i(\psi) = \max\{k \in \mathbb{N} \mid \widetilde{e}_i^k(\psi) \neq 0\}$  and  $\varphi_i(\psi) = \text{wt}_i(\psi) + \varepsilon_i(\psi)$ .

**2.3. The Kashiwara involution.** The Kashiwara involution  $*$  is the  $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -antiautomorphism such that  $q^* = q$  and defined on the generators as follows:

$$(4) \quad e_i^* = e_i, \quad f_i^* = f_i, \quad t_i^* = t_i^{-1}.$$

Since  $*$  stabilizes  $\mathcal{U}_v^-$ , it induces an involution (also denoted  $*$ ) on  $B_e(\infty)$  the crystal graph of  $\mathcal{U}_v^-$ . By setting for any vertex  $b \in B_e(\infty)$  and any  $i \in \mathbb{Z}/e\mathbb{Z}$

$$(5) \quad \widetilde{e}_i^*(b) = \widetilde{e}_i(b^*), \quad \widetilde{f}_i^*(b) = \widetilde{f}_i(b^*), \quad \varepsilon_i^*(b) = \varepsilon_i(b^*) \quad \text{and} \quad \varphi_i^*(b) = \varphi_i(b^*)$$

we obtain another crystal structure on  $B_e(\infty)$  (see [18]).

Let  $i \in \mathbb{Z}/e\mathbb{Z}$  and write  $B_i$  for the crystal with set of vertices  $\{b_i(k) \mid k \in \mathbb{Z}\}$  and such that

$$\begin{aligned} \text{wt}(b_i(k)) = k\alpha_i, \quad \varepsilon_j(b_i(k)) = \begin{cases} -k & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases}, \quad \varphi_j(b_i(k)) = \begin{cases} k & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases} \\ \tilde{e}_j b_i(k) = \begin{cases} b_i(k+1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{and} \quad \tilde{f}_j b_i(k) = \begin{cases} b_i(k-1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \end{aligned}$$

Set  $b_i = b_i(0)$ .

Recall the action of  $\tilde{e}_i$  and  $\tilde{f}_i$  on the tensor product  $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$  of the crystals  $B$  and  $B'$  :

$$(6) \quad \tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v), \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v), \end{cases}$$

and

$$(7) \quad \tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v), \\ \tilde{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v). \end{cases}$$

The embedding of crystals  $\theta_i : B_e(\infty) \hookrightarrow B_e(\infty) \otimes B_i$  which sends the highest weight vertex  $b_{\emptyset}$  of  $B_e(\infty)$  on  $b_{\emptyset} \otimes b_i$  permits to compute the action of the operators  $\tilde{e}_i^*$  and  $\tilde{f}_i^*$  at least on a theoretical point of view.

**Proposition 2.6.** (see Proposition 8.1 in [18]) Consider  $b \in B_e(\infty)$  and set  $\varepsilon_i^*(b) = m$ . Then we have

- (1)  $\theta_i(b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^m b_i$ ,
- (2)  $\theta_i(\tilde{f}_i^* b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^{m+1} b_i$  and
- (3)  $\theta_i(\tilde{e}_i^* b) = (\tilde{e}_i^*)^m b \otimes \tilde{f}_i^{m-1} b_i$  if  $m > 0$  and  $\theta_i(\tilde{e}_i^* b) = 0$  if  $m = 0$ .

*Remark 2.7.*

- (1) By 2.2,  $\Psi_e$  is equipped with two crystal structures. One is obtained from the action of the crystal operators  $\hat{f}_i, i \in \mathbb{Z}/e\mathbb{Z}$  and the other one is related to the operators  $\tilde{f}_i, i \in \mathbb{Z}/e\mathbb{Z}$  and yields the Kashiwara crystal graph structure  $B_e(\infty)$  on  $\Psi_e$ . We shall see in Section 4, that the actions of the operators  $\hat{f}_i$  and  $\tilde{f}_i^*$  with  $i \in \mathbb{Z}/e\mathbb{Z}$  coincide.
- (2) Observe that Proposition 2.6 does not provide an efficient procedure for computing the involution  $*$ . Indeed, in order to obtain  $\theta_i(b)$ , we have first to determine a path from  $b$  to the highest weight vertex of  $B_e(\infty)$ . Moreover, computing a section of the embedding  $\theta_i$  is difficult in general.

**2.4. Crystals of highest weight  $\mathcal{U}_v$ -modules.** Let  $l \in \mathbb{N}$  and consider  $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l$ .  $\mathbf{v}$  is called a multicharge and  $l$  is by definition the level of  $\mathbf{v}$ . One can then associate to  $\mathbf{v}$  the abstract  $\mathcal{U}_v$ -irreducible module  $V_e(\Lambda_{\mathbf{v}})$  with highest weight  $\Lambda_{\mathbf{v}} = \Lambda_{v_0(\text{mode})} + \dots + \Lambda_{v_{l-1}(\text{mode})}$ . There exist distinct realizations of  $V_e(\Lambda_{\mathbf{v}})$  as an irreducible component of a Fock space

$\mathfrak{F}_e^{\mathbf{v}}$  whose structure depends on  $\mathbf{v}$ . As a  $\mathbb{C}(v)$ -vector space, the Fock space  $\mathfrak{F}_e^{\mathbf{v}}$  of level  $l$  admits the set of all  $l$ -partitions as a natural basis. Namely the underlying vector space is

$$\mathfrak{F}_e = \bigoplus_{n \geq 0} \bigoplus_{\lambda \in \Pi_{l,n}} \mathbb{C}(v)\lambda$$

where  $\Pi_{l,n}$  is the set of  $l$ -partitions with rank  $n$ . Consider  $\mathbf{v} = (v_0, \dots, v_{l-1}) \in (\mathbb{Z}/e\mathbb{Z})^l$ . We write  $\mathbf{v} \in \mathfrak{v}$  when  $v_c \in \mathfrak{v}_c$  for any  $c = 0, \dots, l-1$ . As  $\mathcal{U}_v$ -modules, the Fock spaces  $\mathfrak{F}_e^{\mathbf{v}}$ ,  $\mathbf{v} \in \mathfrak{v}$  are all isomorphic but with distinct actions for  $\mathcal{U}_v$ . For each of these actions, the empty  $l$ -partition  $\emptyset = (\emptyset, \dots, \emptyset)$  is a highest weight vector of highest weight  $\Lambda_{\mathbf{v}}$ . We denote by  $V_e(\mathbf{v})$  the irreducible component with highest weight vector  $\emptyset$  in  $\mathfrak{F}_e^{\mathbf{v}}$ . The modules  $V_e(\mathbf{v})$  when  $\mathbf{v}$  runs over  $\mathfrak{v}$  are all isomorphic to the abstract module  $V_e(\Lambda_{\mathbf{v}})$ . However, the actions of the Chevalley operators on these modules do not coincide in general.

The module  $\mathfrak{F}_e^{\mathbf{v}}$  admits a crystal graph  $B_e^{\mathbf{v}}$  labelled by  $l$ -partitions. Let us now recall the crystal structures on  $B_e^{\mathbf{v}}$  and  $B_e(\mathbf{v})$  the crystal associated to  $V_e(\mathbf{v})$ . We will omit the description of the  $\mathcal{U}_v$ -module structures on  $\mathfrak{F}_e^{\mathbf{v}}$  and  $V_e(\mathbf{v})$  which are not needed in our proofs (see [16] for a complete exposition). Let  $\lambda = (\lambda^0, \dots, \lambda^{l-1})$  be an  $l$ -partition (identified with its Young diagram). Then, the nodes of  $\lambda$  are the triplets  $\gamma = (a, b, c)$  where  $c \in \{0, \dots, l-1\}$  and  $a, b$  are respectively the row and column indices of the node  $\gamma$  in  $\lambda^c$ . The content of  $\gamma$  is the integer  $c(\gamma) = b - a + v_c$  and the residue  $\text{res}(\gamma)$  of  $\gamma$  is the element of  $\mathbb{Z}/e\mathbb{Z}$  such that

$$(8) \quad \text{res}(\gamma) \equiv c(\gamma) \pmod{e}.$$

We say that  $\gamma$  is an  $i$ -node of  $\lambda$  when  $\text{res}(\gamma) \equiv i \pmod{e}$ . This node is removable when  $\gamma = (a, b, c) \in \lambda$  and  $\lambda \setminus \{\gamma\}$  is an  $l$ -partition. Similarly  $\gamma$  is addable when  $\gamma = (a, b, c) \notin \lambda$  and  $\lambda \cup \{\gamma\}$  is an  $l$ -partition.

The structure of crystal on  $B_e^{\mathbf{v}}$  (and in fact, the structure of  $\mathcal{U}_v$ -module on  $\mathfrak{F}_e^{\mathbf{v}}$  itself) is conditioned by the total order  $\prec_{\mathbf{v}}$  on the set of addable and removable  $i$ -nodes of the multipartitions. Consider  $\gamma_1 = (a_1, b_1, c_1)$  and  $\gamma_2 = (a_2, b_2, c_2)$  two  $i$ -nodes in  $\lambda$ . We define the order  $\prec_{\mathbf{v}}$  by setting

$$\gamma_1 \prec_{\mathbf{v}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2. \end{cases}$$

Starting from any  $l$ -partition  $\lambda$ , consider its set of addable and removable  $i$ -nodes. Let  $w_i$  be the word obtained first by writing the addable and removable  $i$ -nodes of  $\lambda$  in increasing order with respect to  $\prec_{\mathbf{v}}$  next by encoding each addable  $i$ -node by the letter  $A$  and each removable  $i$ -node by the letter  $R$ . Write  $\tilde{w}_i = A^r R^s$  for the word derived from  $w_i$  by deleting as many of the factors  $RA$  as possible. If  $r > 0$ , let  $\gamma$  be the rightmost addable  $i$ -node in  $\tilde{w}_i$ . When  $\tilde{w}_i \neq \emptyset$ , the node  $\gamma$  is called the good  $i$ -node.

**Proposition 2.8.** *The crystal graph  $B_e^{\mathbf{v}}$  of  $\mathfrak{F}_e^{\mathbf{v}}$  is the graph with*



- (1) *vertices: the  $l$ -partitions,*
- (2) *edges:  $\lambda \xrightarrow{i} \mu$  if and only if  $\mu$  is obtained by adding to  $\lambda$  its good  $i$ -node.*
- (3) *for any  $i \in \mathbb{Z}/e\mathbb{Z}$ ,  $\varepsilon_i(\lambda) = s$  and  $\varphi_i(\lambda) = r$ .*

Since  $V_e(\mathbf{v})$  is the irreducible module with highest weight vector  $\emptyset$  in  $\mathfrak{F}_e^{\mathbf{v}}$ , its crystal graph  $B_e(\mathbf{v})$  can be realized as the connected component of highest weight vertex  $\emptyset$  in  $B_e^s$ . The vertices of  $B_e(\mathbf{v})$  are labelled by  $l$ -partitions called Uglov  $l$ -partitions associated to  $\mathbf{v}$ .

Set

$$(9) \quad \mathcal{V}_l = \{\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l \mid v_0 \leq \dots \leq v_{l-1} \text{ and } v_{l-1} - v_0 < e\}.$$

**Definition 2.9.** Assume that  $\mathbf{v} \in \mathcal{V}_l$ . The  $l$ -partition  $\lambda = (\lambda^0, \dots, \lambda^{l-1})$  is a FLOTW  $l$ -partition associated to  $\mathbf{v}$  if it satisfies the two following conditions :

- (1) for all  $i = 1, 2, \dots$ , we have :
 
$$\lambda_i^j \geq \lambda_{i+v_{j+1}-v_j}^{j+1} \text{ for all } j = 0, \dots, l-2 \text{ and } \lambda_i^{l-1} \geq \lambda_{i+e+v_0-v_{l-1}}^0.$$
- (2) for all  $k > 0$ , among the residues appearing in  $\lambda$  at the right ends of the length  $k$  rows, at least one element of  $\{0, 1, \dots, e-1\}$  does not occur.

The following result has been obtained by Jimbo, Misra, Miwa and Okado [16] but the presentation we adopt here comes from [8].

**Proposition 2.10.** *When  $\mathbf{v} \in \mathcal{V}_l$ , the set of vertices of  $B_e(\mathbf{v})$  coincides with the set of FLOTW  $l$ -partitions associated to  $\mathbf{v}$ .*

Let us denote by  $\Phi_e(\mathbf{v})$  the set of FLOTW  $l$ -partitions associated to  $\mathbf{v}$ .

Consider  $\mathbf{v} \in \mathcal{V}_l$  and  $\lambda \in \Phi_e(\mathbf{v})$ . We associate to each non zero part  $\lambda_i^c$  of  $\lambda$  the segment

$$(10) \quad [(1-i+v_c)(\text{mode}), (2-i+v_c)(\text{mode}), \dots, (\lambda_i^c - i + v_c)(\text{mode})].$$

The multisegment  $f_{\mathbf{v}}(\lambda)$  is then the formal sum of all the segments associated to the parts  $\lambda_i^c$  of  $\lambda$ . Since  $f_{\mathbf{v}}(\lambda)$  is aperiodic by (2) of Definition 2.9, the map:

$$f_{\mathbf{v}} : B_e(\mathbf{v}) \rightarrow \Psi_e$$

is well-defined.

**Example 2.11.** Let  $e = 4$ , we consider the FLOTW bipartition (2.1, 1) associated to  $\mathbf{v} = (0, 1)$  then

$$f_{\mathbf{v}}(2.1, 1) = [0, 1] + [3] + [1].$$

Let  $\mathbf{v} = (0, 1, 3)$  and consider the FLOTW 3-partition (2, 1, 1). We have :

$$f_{\mathbf{v}}(2, 1, 1) = [0, 1] + [1] + [3].$$

Let  $T_\Lambda = \{t_\Lambda\}$  be the crystal defined by  $\text{wt}(t_{\Lambda_{\mathbf{v}}}) = \Lambda$ ,  $\epsilon_i(t_{\Lambda_{\mathbf{v}}}) = \varphi_i(t_{\Lambda_{\mathbf{v}}}) = -\infty$  and  $\tilde{e}_i t_{\Lambda_{\mathbf{v}}} = \tilde{f}_i t_{\Lambda_{\mathbf{v}}} = 0$ . We have a unique crystal embedding  $B_e(\mathbf{v}) \hookrightarrow B_e(\infty) \otimes T_\Lambda$ . The following theorem has been established in [3].

**Theorem 2.12.** *For any  $\mathbf{v} \in \mathcal{V}_l$ , the map  $f_{\mathbf{v}}$  coincides with the unique crystal embedding  $B_e(\mathbf{v}) \hookrightarrow B_e(\infty) \otimes T_\Lambda$ .*

According to Proposition 8.2 in [18], we have

$$f_{\mathbf{v}}(\Phi_e(\mathbf{v})) = \{\psi \in \Psi_e \mid \epsilon_i(\psi^*) \leq r_i \text{ for any } i \in \mathbb{Z}/e\mathbb{Z}\}$$

where  $r_i$  is the number of coordinates in  $\mathbf{v}$  equal to  $i$  and  $\psi^*$  is the image of  $\psi$  under the Kashiwara involution of the crystal  $B_e(\infty)$ .

Given any  $\psi \in \Psi_e$ , write  $\mathbf{v}(\psi)$  for the element of  $\mathcal{V}_l$  defined by the conditions

$$(11) \quad r_i = \epsilon_i(\psi^*) = \epsilon_i^*(\psi) \text{ for any } i \in \mathbb{Z}/e\mathbb{Z}.$$

Then, by the previous considerations, there exists a unique  $l$ -partition  $\underline{\lambda}(\psi) = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e(\mathbf{v}(\psi))$  such that  $f_{\mathbf{v}(\psi)}(\underline{\lambda}(\psi)) = \psi$ .

### 3. THE ZELEVINSKY INVOLUTION OF $\mathcal{H}_n^a(q)$

**3.1. Three natural involutions on  $\mathcal{H}_n^a(q)$ .** Denote by  $\mathcal{H}_n(q)$  the Hecke algebra of type  $A$  with parameter  $q$  over the field  $\mathbb{F}$ . This is the unital associative  $\mathbb{F}$ -algebra generated by  $T_1, \dots, T_{n-1}$  and the relations:

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (i = 1, \dots, n-2), \\ T_i T_j &= T_j T_i \quad (|j - i| > 1), \\ (T_i - q)(T_i + 1) &= 0 \quad (i = 1, \dots, n-1). \end{aligned}$$

The affine Hecke algebra  $\mathcal{H}_n^a(q)$  is the  $\mathbb{F}$ -algebra which as an  $\mathbb{F}$ -module is isomorphic to

$$\mathcal{H}_n(q) \otimes_{\mathbb{R}} \mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The algebra structure is obtained by requiring that  $\mathcal{H}_n(q)$  and  $\mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  are both subalgebras and for any  $i = 1, \dots, n$

$$T_i X_i T_i = q X_{i+1}, \quad T_i X_j = X_j T_i \text{ if } i \neq j.$$

In the sequel, we assume that  $q = \xi$  is a primitive  $e$ -th root of the unity and write  $\mathcal{H}_n^a(\xi)$  for the affine Hecke algebra with parameter  $\xi$ . We have three involutive automorphisms  $\tau$ ,  $\flat$  and  $\sharp$  on  $\mathcal{H}_n^a(\xi)$ . There are defined on the generators as follows :

$$(12) \quad \begin{aligned} T_i^\tau &= -\xi T_{n-i}^{-1}, & X_j^\tau &= X_{n+1-j}, \\ T_i^\flat &= T_{n-i}, & X_j^\flat &= X_{n+1-j}^{-1}, \\ T_i^\sharp &= -\xi T_i^{-1}, & X_j^\sharp &= X_j^{-1}. \end{aligned}$$

The involution  $\sharp$  has been considered by Iwahori and Matsumoto [12] and the involution  $\tau$ , which is called the Zelevinsky involution, by Moeglin and Waldspurger [22]. One can easily check that they are connected as follows:

$$(13) \quad \forall x \in \mathcal{H}_n^a(\xi), \quad x^\tau = (x^\flat)^\sharp = (x^\sharp)^\flat.$$

**3.2. The involutions  $\tau$ ,  $\sharp$  and  $\flat$  on  $B_e(\infty)$ .** We denote by  $\text{Mod}_n^a$  the category of finite-dimensional  $\mathcal{H}_n^a(\xi)$ -modules such that for  $j = 1, \dots, n$  the eigenvalues of  $X_j$  are power of  $\xi$ .

For any multisegment  $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i)}(l; i]$ , we write

$$|\psi| = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} lm_{(l;i)}.$$

The geometric realization of  $\mathcal{H}_n^a(\xi)$  due to Ginzburg permits to label the simple  $\mathcal{H}_n^a(\xi)$ -modules in  $\text{Mod}_n^a$  by the aperiodic multisegments such that  $|\psi| = n$ . We do not use Ginzburg's construction in the sequel and just refer to [6] (see also [3] and [20]) for a complete exposition or a short review. Let  $L_\psi$  be the simple  $\mathcal{H}_n^a(\xi)$ -module corresponding to  $\psi$  under this parametrization. The three involutions  $\tau, \flat, \sharp$  on  $\mathcal{H}_n^a(\xi)$  induce involutions on the set of simple  $\mathcal{H}_n^a(\xi)$ -modules that we will denote in the same way. This yields involutions on the set of aperiodic multisegments (also denoted by  $\tau, \flat$  and  $\sharp$ ) satisfying

$$L_\psi^\tau = L_{\psi^\tau}, \quad L_\psi^\flat = L_{\psi^\flat}, \quad L_\psi^\sharp = L_{\psi^\sharp}$$

for each aperiodic multisegment  $\psi$ . Thus we have three involutions on the vertices of  $B_e(\infty)$ . By (13), they verify the relation

$$(14) \quad \tau = \sharp \circ \flat = \flat \circ \sharp.$$

By [20, §2.4], for each aperiodic multisegment  $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i)}[i; l)$ , we have  $\psi^\flat = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i)}(l; -i]$ . By comparing with the action (1) of  $\rho$  on  $B_e(\infty)$ , this immediately gives the following lemma :

**Lemma 3.1.** *The involutions  $\rho$  and  $\flat$  coincide on  $B_e(\infty)$ . For any  $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i)}[i; l)$ , we have*

$$\psi^\flat = \psi^\rho = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i)}(l; -i].$$

Since the action of  $\rho = \flat$  on  $B_e(\infty)$  is immediate, it is equivalent to describe  $\tau$  or  $\sharp$  on  $B_e(\infty)$ . The following proposition makes explicit the involution  $\sharp$  on  $B_e(\infty)$ .

**Theorem 3.2.** ([20]) *Let  $\psi$  be an aperiodic multisegment. Then  $\psi^\sharp$  is the aperiodic multisegment obtained from  $\psi$  by the 2-fold symmetry  $i \leftrightarrow -i$  in the graph  $B_e(\infty)$ .*

#### 4. THE EQUALITY $* = \tau$ ON $B_e(\infty)$

The aim of this section is to prove that  $*$  and  $\tau$  coincide on  $B_e(\infty)$ . To do this, it suffices to establish the following theorem.

**Theorem 4.1.** *For any multisegment  $\psi \in \Psi_e$  and any  $j \in \mathbb{Z}/e\mathbb{Z}$ , we have  $\tilde{f}_j^*(\psi) = \widehat{f}_j(\psi)$ .*

Indeed, assuming this theorem holds, we easily derive the equality  $* = \tau$ .

**Corollary 4.2.** *The involutions  $*$  and  $\tau$  coincide on  $B_e(\infty)$ .*

*Proof.* Let  $\psi \in \Psi_e$ . Then, there exists  $i_1, \dots, i_n$  in  $\mathbb{Z}/e\mathbb{Z}$  such that

$$\psi = \tilde{f}_{i_1} \dots \tilde{f}_{i_n} \cdot \emptyset$$

Hence, we obtain

$$\begin{aligned} \psi^* &= \tilde{f}_{i_1}^* \dots \tilde{f}_{i_n}^* \cdot \emptyset \\ &= \tilde{f}_{i_1} \dots \tilde{f}_{i_n} \cdot \emptyset \end{aligned}$$

Using (2), this gives

$$\begin{aligned} \psi^* &= \rho(\tilde{f}_{-i_1} \dots \tilde{f}_{-i_n} \cdot \emptyset) \\ &= (\psi^\sharp)^\rho = (\psi^\rho)^\sharp = \psi^\tau \end{aligned}$$

where the two last equalities follow from Lemma 3.1 and (14).  $\square$

To prove Theorem 4.1, we are going to proceed by induction on  $|\psi|$ . By using Proposition 2.6, we shall see that it suffices in fact to establish the equivalence

$$\tilde{f}_i \widehat{f}_j(\psi) = \widehat{f}_j \tilde{f}_i(\psi) \iff \tilde{f}_i \tilde{f}_j^*(\psi) = \tilde{f}_j^* \tilde{f}_i(\psi)$$

for any  $i, j \in \mathbb{Z}/e\mathbb{Z}$ . Now, by definition, the operator  $\tilde{f}_i$  (resp.  $\widehat{f}_j$ ) adds an entry  $i$  on the right end (resp. an entry  $j$  at the left end) of one of the segments of  $\psi$ . This will imply that

- $\tilde{f}_i \widehat{f}_j(\psi) = \widehat{f}_j \tilde{f}_i(\psi)$  except possibly when  $i = j$  and  $\tilde{f}_i(\psi) = \psi + [i]$  or  $\widehat{f}_j(\psi) = \psi + [j]$ .

On the other hand, it is easy to derive from Proposition 2.6 that

- $\tilde{f}_i \tilde{f}_j^*(\psi) = \tilde{f}_j^* \tilde{f}_i(\psi)$  except possibly when  $i = j$  and  $\tilde{f}_i(\psi) = \psi + [i]$  or  $\tilde{f}_i^*(\psi) = \psi + [i]$ .

The case where the operators do not commute being easily tractable, this will imply theorem 4.1.

**4.1. More on the crystal operators  $\tilde{f}_i$  and  $\widehat{f}_i$ .** We begin with refinements of the actions of the operators  $\tilde{f}_i$  and  $\widehat{f}_i$ . In [3], we have obtained an alternative description of the action of the crystal operators on  $\Psi_e$ . Consider  $\psi \in \Psi_e$  and  $i \in \mathbb{Z}/e\mathbb{Z}$ . We encode the segments in  $\psi$  with *tail*  $i$  (resp.  $i-1$ ) by the symbol  $R$  (resp. by the symbol  $A$ ). For any nonnegative integer  $l$ , write  $w_{i,l} = R^{m(l;i)} A^{m(l;i-1)}$  where  $m(l;i)$  and  $m(l;i-1)$  are respectively the number of segments  $(l;i]$  and  $(l;i-1]$  in  $\psi$ . Set  $w_i = \prod_{l \geq 1} w_{i,l}$ . Write  $\tilde{w}_i = A^{a_i(\psi)} R^{r_i(\psi)}$  for the word derived from  $w_i$  by deleting as many of the factors  $RA$  as possible. If  $a_i(\psi) > 0$ , we denote by  $l_{0,i}(\psi) > 0$  the length of the rightmost segment  $A$  in  $\tilde{w}_i$ . If  $a_i(\psi) = 0$ , set  $l_{0,i}(\psi) = 0$ . When there is no risk of confusion, we simply write  $l_0$  instead of  $l_{0,i}(\psi)$ .

**Lemma 4.3.** ([3]) *With the above notation we have*

$$(1) \quad \varepsilon_i(\psi) = r_i(\psi)$$

(2)

$$\tilde{f}_i \psi = \begin{cases} \psi + (l_0; i] - (l_0 - 1; i - 1] & \text{if } a_i(\psi) > 0, \\ \psi + (1; i] & \text{if } a_i(\psi) = 0. \end{cases}$$

We can compute similarly the action of the crystal operators  $\widehat{f}_i$  (with  $i \in \mathbb{Z}/e\mathbb{Z}$ ) on  $\psi$ . We encode the segments in  $\psi$  with *head*  $i$  (resp.  $i + 1$ ) by the symbol  $\widehat{R}$  (resp. by the symbol  $\widehat{A}$ ). For any nonnegative integer  $l$ , write  $\widehat{w}_{i,l} = \widehat{R}^{m_{[i;l]}} \widehat{A}^{m_{[i+1;l]}}$  where  $m_{[i;l]}$  and  $m_{[i+1;l]}$  are respectively the number of segments  $[i; l]$  and  $[i + 1; l]$  in  $\psi$ . Set  $\widehat{w}_i = \prod_{l \geq 1} \widehat{w}_{i,l}$ . Write  $\overline{w}_i = \widehat{A}^{\widehat{a}_i(\psi)} \widehat{R}^{\widehat{r}_i(\psi)}$  for the word derived from  $\widehat{w}_i$  by deleting as many of the factors  $\widehat{R}\widehat{A}$  as possible. If  $\widehat{a}_i(\psi) > 0$ , let  $\widehat{l}_{0,i}(\psi) > 0$  be length of the rightmost segment  $\widehat{A}$  in  $\overline{w}_i$ . If  $\widehat{a}_i(\psi) = 0$ , set  $\widehat{l}_{0,i}(\psi) = 0$ . When there is no risk of confusion, we also simply write  $\widehat{l}_0$  instead of  $\widehat{l}_{0,i}(\psi)$ .

**Lemma 4.4.** *With the above notation, we have*

- (1)  $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$  where  $\widehat{\varepsilon}_i(\psi) = \max\{p \mid \widehat{e}_i^p(\psi) \neq 0\}$ .  
(2)

$$(15) \quad \widehat{f}_i \psi = \begin{cases} \psi + [i; \widehat{l}_0] - [i + 1; \widehat{l}_0 - 1] & \text{if } \widehat{a}_i(\psi) > 0, \\ \psi + [i; 1] & \text{if } \widehat{a}_i(\psi) = 0. \end{cases}$$

*Remark 4.5.* By Theorem 9.13 of [10], for any  $i \in \mathbb{Z}/e\mathbb{Z}$ , the integer  $\varepsilon_i(\psi) = r_i(\psi)$  (resp.  $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$ ) gives the maximal size of a Jordan block with eigenvalue  $\xi^i$  corresponding to the action of the generator  $X_n$  (resp.  $X_1$ ) on the simple  $\mathcal{H}_n^a(\xi)$ -module  $L_\psi$ .

**4.2. Equality of the crystal operators  $\tilde{f}_i^*$  and  $\widehat{f}_i$ .** Our purpose is now to establish the equality

$$(16) \quad \tilde{f}_i^*(\psi) = \widehat{f}_i(\psi) \text{ for any } \psi \in \Psi_e.$$

This is achieved by showing that the relations  $\tilde{f}_i^* \tilde{f}_j \psi = \tilde{f}_j \tilde{f}_i^* \psi$  and  $\widehat{f}_i \widehat{f}_j \psi = \widehat{f}_j \widehat{f}_i \psi$  are both equivalent to a very simple condition on  $\psi$ .

**Lemma 4.6.** *Put  $i \in \mathbb{Z}/e\mathbb{Z}$ .*

- (1) *Consider  $\psi, \chi \in \Psi_e$  such that  $\psi = \widehat{f}_i \chi$  and put  $j \in \mathbb{Z}/e\mathbb{Z}$ . Then we have:*

$$l_{0,j}(\chi) \neq l_{0,j}(\psi) \iff i = j, \widehat{a}_i(\chi) = 0 \text{ and } a_i(\chi) = 1.$$

- (2) *Consider  $\psi, \chi \in \Psi_e$  such that  $\psi = \tilde{f}_i \chi$  and put  $j \in \mathbb{Z}/e\mathbb{Z}$ . Then we have:*

$$\widehat{l}_{0,j}(\chi) \neq \widehat{l}_{0,j}(\psi) \iff i = j, a_i(\chi) = 0 \text{ and } \widehat{a}_i(\chi) = 1.$$

*Proof.* 1: Assume first  $\widehat{l}_{0,i}(\chi) = \widehat{l}_0 > 1$ . Hence  $\widehat{a}_i(\chi) > 0$  and  $\psi = \chi - [i + 1, \dots, i + \widehat{l}_0 - 1] + [i, \dots, i + \widehat{l}_0 - 1]$ . If  $j \notin \{(i + \widehat{l}_0 - 1)(\text{mode}), (i + \widehat{l}_0)(\text{mode})\}$ , neither  $[i + 1, \dots, i + \widehat{l}_0 - 1]$  or  $[i, \dots, i + \widehat{l}_0 - 1]$  are segments  $A$  or  $R$  for

$j$ . We have  $w_j(\psi) = w_j(\chi)$  and then  $l_{0,j}(\chi) = l_{0,j}(\psi)$ . Thus we can restrict ourselves to the cases  $j \in \{(i + \widehat{l}_0 - 1)(\text{mode}), (i + \widehat{l}_0)(\text{mode})\}$ . We write

$$\widehat{w}_i(\chi) = \cdots [i, \dots, i + \widehat{l}_0 - 2]^{m_{[i, \dots, i + \widehat{l}_0 - 2]}} [i + 1, \dots, i + \widehat{l}_0 - 1]^{m_{[i+1, \dots, i + \widehat{l}_0 - 1]}} \\ [i, \dots, i + \widehat{l}_0 - 1]^{m_{[i, \dots, i + \widehat{l}_0 - 1]}} [i + 1, \dots, i + \widehat{l}_0]^{m_{[i+1, \dots, i + \widehat{l}_0]}} \cdots$$

where we have only pictured the segments of length  $\widehat{l}_0 - 1$  and  $\widehat{l}_0$  of  $\widehat{w}_i(\chi)$ . Since  $\psi = \widehat{f}_i \chi$ , we have

$$\widehat{w}_i(\psi) = \cdots [i, \dots, i + \widehat{l}_0 - 2]^{m_{[i, \dots, i + \widehat{l}_0 - 2]}} [i + 1, \dots, i + \widehat{l}_0 - 1]^{m_{[i+1, \dots, i + \widehat{l}_0 - 1]}}^{-1} \\ [i, \dots, i + \widehat{l}_0 - 1]^{m_{[i, \dots, i + \widehat{l}_0 - 1]}} + 1 [i + 1, \dots, i + \widehat{l}_0]^{m_{[i+1, \dots, i + \widehat{l}_0]}} \cdots$$

In particular, by (15), we must have  $m_{[i, \dots, i + \widehat{l}_0 - 2]} < m_{[i+1, \dots, i + \widehat{l}_0 - 1]}$  and  $m_{[i, \dots, i + \widehat{l}_0 - 1]} \geq m_{[i+1, \dots, i + \widehat{l}_0]}$ .

When  $j = (i + \widehat{l}_0 - 1)(\text{mode})$ ,  $[i + 1, \dots, i + \widehat{l}_0 - 1]$  and  $[i, \dots, i + \widehat{l}_0 - 1]$  are of type  $R$  for  $j$ . Hence, by considering only the segments of lengths  $\widehat{l}_0 - 1$  and  $\widehat{l}_0$ , we can write

$$w_j(\chi) = \cdots [i + 1, \dots, i + \widehat{l}_0 - 1]^{m_{[i+1, \dots, i + \widehat{l}_0 - 1]}} [i, \dots, i + \widehat{l}_0 - 2]^{m_{[i, \dots, i + \widehat{l}_0 - 2]}} \\ [i, \dots, i + \widehat{l}_0 - 1]^{m_{[i, \dots, i + \widehat{l}_0 - 1]}} [i - 1, \dots, i + \widehat{l}_0 - 2]^{m_{[i-1, \dots, i + \widehat{l}_0 - 2]}} \cdots$$

and

$$w_j(\psi) = \cdots [i + 1, \dots, i + \widehat{l}_0 - 1]^{m_{[i+1, \dots, i + \widehat{l}_0 - 1]}}^{-1} [i, \dots, i + \widehat{l}_0 - 2]^{m_{[i, \dots, i + \widehat{l}_0 - 2]}} \\ [i, \dots, i + \widehat{l}_0 - 1]^{m_{[i, \dots, i + \widehat{l}_0 - 1]}} + 1 [i - 1, \dots, i + \widehat{l}_0 - 2]^{m_{[i-1, \dots, i + \widehat{l}_0 - 2]}} \cdots$$

Since  $m_{[i, \dots, i + \widehat{l}_0 - 2]} < m_{[i+1, \dots, i + \widehat{l}_0 - 1]}$ , the cancellation procedures of the factors  $RA$  in  $w_j(\chi)$  and  $w_j(\psi)$  yield the same final word. Hence  $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$  and we have also  $l_{0,j}(\chi) = l_{0,j}(\psi) = 1$ .

When  $j = (i + \widehat{l}_0)(\text{mode})$ ,  $[i + 1, \dots, i + \widehat{l}_0 - 1]$  and  $[i, \dots, i + \widehat{l}_0 - 1]$  are of type  $A$  for  $j$ . We obtain also  $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$  by considering the segments of lengths  $\widehat{l}_0 - 1$  and  $\widehat{l}_0$ . Thus  $l_{0,j}(\chi) = l_{0,j}(\psi)$ .

Observe that we have always  $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$  for any  $j \in \mathbb{Z}/e\mathbb{Z}$  when  $\widehat{l}_0 > 1$ . In particular

$$(17) \quad \widehat{a}_i(\chi) > 0 \implies a_j(\chi) = a_j(\widehat{f}_i \chi) \text{ for any } j \in \mathbb{Z}/e\mathbb{Z}.$$

Now assume  $\widehat{l}_0 = 1$ , that is  $\psi = \chi + [i]$ . Write

$$\widehat{w}_i(\chi) = [i]^{m_{[i]}} [i + 1]^{m_{[i+1]}} \cdots \text{ and } \widehat{w}_i(\psi) = [i]^{m_{[i]} + 1} [i + 1]^{m_{[i+1]}} \cdots$$

with  $m_{[i]} \geq m_{[i+1]}$ .

When  $j = i + 1(\text{mode})$ ,  $[i]$  is of type  $A$  for  $j$  and  $[i + 1]$  is of type  $R$ . Thus we can write

$$w_j(\chi) = [i + 1]^{m_{[i+1]}} [i]^{m_{[i]}} \cdots \text{ and } w_j(\psi) = [i + 1]^{m_{[i+1]}} [i]^{m_{[i]} + 1} \cdots$$

Since  $m_{[i]} \geq m_{[i+1]}$ , the rightmost segments  $A$  in  $\tilde{w}_j(\chi)$  and  $\tilde{w}_j(\psi)$  are the same and we have yet  $l_{0,j}(\chi) = l_{0,j}(\psi)$ .

When  $j = i(\text{mode})$ ,  $[i]$  is of type  $R$  for  $j$ . Observe that  $\hat{a}_i(\chi) = 0$ . Set  $\tilde{w}_i(\chi) = A^{a_i(\chi)} R^{r_i(\chi)}$ . Then  $\tilde{w}_i(\psi)$  is obtained by applying the cancellation procedure of the factors  $RA$  to the word  $w = RA^{a_i(\chi)} R^{r_i(\chi)}$ . Clearly,  $l_{0,j}(\chi) \neq l_{0,j}(\psi)$  if and only if  $a_i(\chi) = 1$  for in this case we have  $l_{0,j}(\chi) > 1$  and  $l_{0,j}(\psi) = 1$ . This proves assertion 1.

2: The arguments are similar to those used in the proof of 1.  $\square$

**Proposition 4.7.** *For any  $\chi \in \Psi_e$  and  $i, j \in \mathbb{Z}/e\mathbb{Z}$ , we have  $\tilde{f}_i \tilde{f}_j \chi \neq \tilde{f}_j \tilde{f}_i \chi \iff i = j$  and  $a_i(\chi) + \hat{a}_i(\chi) = 1$ .*

*Proof.* Assume  $i \neq j$  or,  $i = j$  and  $a_i(\chi) + \hat{a}_i(\chi) > 1$ . Then by assertions 1 and 2 of the previous lemma, we have  $\hat{l}_{0,j}(\chi) = \hat{l}_{0,j}(\tilde{f}_i \chi) = \hat{l}_0$  and  $l_{0,i}(\chi) = l_{0,i}(\tilde{f}_j \chi) = l_0$ . Hence

$$\tilde{f}_i \tilde{f}_j \chi = \chi + [j; \hat{l}_0] + (l_0; i] - [j + 1; \hat{l}_0 - 1] - (l_0 - 1; i - 1] = \tilde{f}_j \tilde{f}_i \chi$$

with  $[j + 1; \hat{l}_0 - 1] = \emptyset$  if  $\hat{l}_0 = 1$  and  $(l_0 - 1; i - 1] = \emptyset$  if  $l_0 = 1$ .

Now, assume  $i = j$ ,  $a_i(\chi) = 1$  and  $\hat{a}_i(\chi) = 0$ . We have

$$\tilde{f}_i \tilde{f}_i \chi = \chi + 2[i] \text{ and } \tilde{f}_i \tilde{f}_i \chi = \chi + [i] + [i - l_0 + 1, \dots, i] - [i - l_0 + 1, \dots, i - 1]$$

with  $l_0 = l_{0,i}(\chi) > 1$ . Similarly, if we assume  $i = j$ ,  $a_i(\chi) = 1$  and  $\hat{a}_i(\chi) = 0$ , we obtain

$$\tilde{f}_i \tilde{f}_i \chi = \chi + 2[i] \text{ and } \tilde{f}_i \tilde{f}_i \chi = \chi + [i] + [i + 1, \dots, i + \hat{l}_0 - 1] - [i, \dots, i + \hat{l}_0 - 1]$$

with  $\hat{l}_0 = \hat{l}_{0,i}(\chi) > 1$ . In both cases,  $\tilde{f}_i \tilde{f}_j \chi \neq \tilde{f}_i \tilde{f}_i \chi$  which completes the proof. Observe that we then have

$$(18) \quad \tilde{f}_i \tilde{f}_i \chi = (\tilde{f}_i)^2 \chi \text{ and } \tilde{f}_i \tilde{f}_i \chi = (\tilde{f}_i)^2 \chi.$$

$\square$

**Proposition 4.8.** *Consider  $\psi \in \Psi_e$  and  $i, j \in \mathbb{Z}/e\mathbb{Z}$ .*

(1) *If  $i \neq j$ , we have  $\tilde{f}_i \tilde{f}_j^* \psi = \tilde{f}_j^* \tilde{f}_i \psi$ .*

(2) *If  $i = j$ , set  $m = \varepsilon_i^*(\psi)$ . Then  $\tilde{f}_i \tilde{f}_i^* \psi \neq \tilde{f}_i^* \tilde{f}_i \psi \iff \varphi_i((\tilde{e}_i^*)^m \psi) = \varepsilon_i^*(\psi) + 1$ .*

*Proof.* 1: This is a classical property of crystals. Write  $\theta_j(\psi) = (\tilde{e}_j^*)^m \psi \otimes \tilde{f}_j^m b_j$  where  $m = \varepsilon_j^*(\psi)$ . Then by (6), we have  $\theta_j(\tilde{f}_i \psi) = \tilde{f}_i (\tilde{e}_j^*)^m \psi \otimes \tilde{f}_j^m b_j$  for  $i \neq j$ . By Proposition 2.6, we obtain  $\theta_j(\tilde{f}_j \tilde{f}_i \psi) = \tilde{f}_i (\tilde{e}_j^*)^m \psi \otimes \tilde{f}_j^{m+1} b_j$ . We have also  $\theta_j(\tilde{f}_j^* \psi) = (\tilde{e}_j^*)^m \psi \otimes \tilde{f}_j^{m+1} b_j$  and since  $i \neq j$ , this yields  $\theta_j(\tilde{f}_i \tilde{f}_j^* \psi) = \tilde{f}_i (\tilde{e}_j^*)^m \psi \otimes \tilde{f}_j^{m+1} b_j$ . Hence  $\theta_j(\tilde{f}_i \tilde{f}_j^* \psi) = \theta_j(\tilde{f}_j^* \tilde{f}_i \psi)$  and we have  $\tilde{f}_i \tilde{f}_j^* \psi = \tilde{f}_j^* \tilde{f}_i \psi$  because  $\theta_j$  is an embedding of crystals.

2: We derive by using the same arguments

$$\theta_i(\tilde{f}_i \tilde{f}_i^* \psi) = \begin{cases} \tilde{f}_i (\tilde{e}_i^*)^m \psi \otimes \tilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\tilde{e}_i^*)^m \psi) > m + 1, \\ (\tilde{e}_i^*)^m \psi \otimes \tilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\tilde{e}_i^*)^m \psi) \leq m + 1. \end{cases}$$

We have also

$$\theta_i(\tilde{f}_i^* \tilde{f}_i \psi) = \begin{cases} \tilde{f}_i(\tilde{e}_i^*)^m \psi \otimes \tilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\tilde{e}_i^*)^m \psi) > m, \\ (\tilde{e}_i^*)^m \psi \otimes \tilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\tilde{e}_i^*)^m \psi) \leq m. \end{cases}$$

Thus we obtain  $\theta_i(\tilde{f}_i \tilde{f}_i^* \psi) = \theta_i(\tilde{f}_i^* \tilde{f}_i \psi)$  except when  $\varphi_i((\tilde{e}_i^*)^m \psi) = m + 1$ . Observe that we have in this case

$$(19) \quad \tilde{f}_i \tilde{f}_i^* \psi = (\tilde{f}_i^*)^2 \psi \neq (\tilde{f}_i)^2 \psi = \tilde{f}_i^* \tilde{f}_i \psi.$$

□

**Lemma 4.9.** Consider  $\psi \in \Psi_e$  and  $i \in \mathbb{Z}/e\mathbb{Z}$ . Set  $\text{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \text{wt}_i(\psi) \Lambda_i$ . Then we have

- (1)  $\text{wt}_i(\psi) = a_i(\psi) - r_i(\psi) + \hat{a}_i(\psi) - \hat{r}_i(\psi)$ ,
- (2)  $\varphi_i(\psi) = a_i(\psi) + \hat{a}_i(\psi) - \hat{r}_i(\psi)$ .

*Proof.* 1: Set

$$\psi = \sum_{l \geq 1} m_{(l; i]}(l; i] = \sum_{l \geq 1} m_{[i; l]}[i; l).$$

During the cancellation procedure described in § 4.1, pairs of segments  $(R, A)$  or  $(\hat{R}, \hat{A})$  are deleted. Thus assertion 1 is equivalent to the equality  $\text{wt}_i(\psi) = \Delta_i(\psi)$  where

$$(20) \quad \Delta_i(\psi) = \sum_{l \geq 1} m_{(l; i-1]} - m_{(l; i]} + \sum_{l \geq 1} m_{[i+1; l]} - m_{[i; l]}.$$

We proceed by induction on  $|\psi|$ . For  $\psi = \emptyset$ , (20) is satisfied. Now assume the equalities (20) hold for any  $i \in \mathbb{Z}/e\mathbb{Z}$  with  $|\psi| = n$ . Set  $\psi' = \tilde{f}_j \psi$ . We have  $\text{wt}(\psi') = \text{wt}(\psi) - \alpha_j$ . Since  $\alpha_j = 2\Lambda_j - \Lambda_{j+1} - \Lambda_{j-1}$ , this gives

$$(21) \quad \text{wt}_i(\psi') = \begin{cases} \text{wt}_i(\psi) & \text{if } i \notin \{j-1, j, j+1\}, \\ \text{wt}_i(\psi) - 2 & \text{if } i = j, \\ \text{wt}_i(\psi) + 1 & \text{if } i \in \{j-1, j+1\}. \end{cases}$$

The multisegment  $\psi'$  is obtained by adding the segments  $[j]$  to  $\psi$  or by replacing a segment  $(l-1; j-1]$  in  $\psi$  by the segment  $(l, j]$ . This shows that the relations (21) are also satisfied by the  $\Delta_i(\psi')$ 's. Hence  $\Delta_i(\psi') = \text{wt}_i(\psi')$  for any  $i \in \mathbb{Z}/e\mathbb{Z}$ .

2: By (3), we have  $\text{wt}_i(\psi) = \varphi_i(\psi) - \varepsilon_i(\psi)$ . Lemma 4.3 then gives  $\text{wt}_i(\psi) = \varphi_i(\psi) - r_i(\psi)$ . Thus  $\varphi_i(\psi) = a_i(\psi) + \hat{a}_i(\psi) - \hat{r}_i(\psi)$  by 1. □

To prove (16), we are going to proceed by induction on  $n = |\psi|$ . We easily check that  $\tilde{f}_i^*(\emptyset) = \hat{f}_i(\emptyset) = [i]$  for the empty multisegment  $\emptyset$ . Now assume that  $\tilde{f}_i^*(\psi) = \hat{f}_i(\psi)$  holds for any multisegment  $\psi \in \Psi_e$  such that  $|\psi| \leq n$ .

**Proposition 4.10.** Under the previous induction hypothesis we have for any  $\chi \in \Psi_e$  such that  $|\chi| \leq n$

$$(22) \quad \tilde{f}_i \hat{f}_j \chi \neq \hat{f}_j \tilde{f}_i \chi \iff \tilde{f}_i \tilde{f}_j^* \chi \neq \tilde{f}_j^* \tilde{f}_i \chi \iff i = j \text{ and } a_i(\chi) + \hat{a}_i(\chi) = 1.$$



*Proof.* Note first that the proposition does not directly follows from the induction hypothesis for  $|\tilde{f}_j\chi| = n + 1$ . By this induction hypothesis, we have  $(\widehat{e}_i^*)^{\varepsilon_i^*(\chi)}\chi = (\widehat{e}_i)^{\widehat{r}_i(\chi)}\chi$ . Set  $\chi' = (\widehat{e}_i)^{\widehat{r}_i(\chi)}\chi$ . Assertion 2 of Lemma 4.9 gives

$$\varphi_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi') - \widehat{r}_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi) + \widehat{r}_i(\chi)$$

for we have  $\widehat{r}_i(\chi') = 0$  and  $\widehat{a}_i(\chi') = \widehat{a}_i(\chi) + \widehat{r}_i(\chi)$ . Observe that  $\varepsilon_i^*(\chi) = \widehat{r}_i(\chi)$  be the induction hypothesis. Moreover, we have  $a_i(\chi) = a_i(\chi')$  by (17) since  $\widehat{a}_i(\varphi) > 0$  for any  $\varphi = (\widehat{e}_i)^a\chi$  with  $a \in \{1, \dots, \widehat{r}_i(\chi)\}$ . This gives the equivalences

$$\varphi_i(\chi') = \varepsilon_i^*(\chi) + 1 \iff a_i(\chi') + \widehat{a}_i(\chi) = 1 \iff a_i(\chi) + \widehat{a}_i(\chi) = 1.$$

Now Propositions 4.7 and 4.8 yields ( 22).  $\square$

We immediately derive by using (2) : We are now able to prove the main result of this section.

**Theorem 4.11.** *For any multisegment  $\psi \in \Psi_e$  and any  $j \in \mathbb{Z}/e\mathbb{Z}$ , we have  $\widetilde{f}_j^*(\psi) = \widehat{f}_j(\psi)$ .*

*Proof.* We argue by induction on  $n = |\psi|$ . We already know that for all  $j \in \mathbb{Z}/e\mathbb{Z}$ , we have  $\widetilde{f}_j^*(\emptyset) = \widehat{f}_j(\emptyset) = [j]$ . Now assume  $\widetilde{f}_j^*\chi = \widehat{f}_j\chi$  for any  $j \in \mathbb{Z}/e\mathbb{Z}$  and any  $\chi \in \Psi_e$  such that  $|\chi| \leq n$ . Consider  $\psi \in \Psi_e$  such that  $|\psi| = n + 1$ . There exists  $i \in \mathbb{Z}/e\mathbb{Z}$  and  $\chi \in \Psi_e$  such that  $\psi = \widetilde{f}_i\chi$  and  $|\chi| = n$ .

When  $i \neq j$  or  $a_i(\chi) + \widehat{a}_i(\chi) > 1$ , we have by Proposition 4.10  $\widetilde{f}_j^*\psi = \widetilde{f}_j^*\widetilde{f}_i\chi = \widetilde{f}_i\widetilde{f}_j^*\chi$ . By our induction hypothesis, we can thus write  $\widetilde{f}_j^*\psi = \widetilde{f}_i\widehat{f}_j\chi$ . Since  $a_i(\chi) + \widehat{a}_i(\chi) > 1$ , this finally gives  $\widetilde{f}_j^*\psi = \widehat{f}_j\widetilde{f}_i\chi = \widehat{f}_j\psi$ .

When  $i = j$  and  $a_i(\chi) + \widehat{a}_i(\chi) = 1$ , we obtain  $\widetilde{f}_i^*\psi = \widetilde{f}_i^*\widetilde{f}_i\chi = \widetilde{f}_i^2\psi$  by (19). Similarly, we have  $\widehat{f}_i\psi = \widehat{f}_i\widetilde{f}_i\chi = \widetilde{f}_i^2\psi$  by (18). Thus  $\widetilde{f}_i^*\psi = \widehat{f}_i\psi$  which completes the proof.  $\square$

*Remark 4.12.* Theorem 4.1 and Proposition 4.10 notably imply the equivalence

$$\widetilde{f}_i\widetilde{f}_j^*\psi = \widetilde{f}_j^*\widetilde{f}_i\psi \iff i \neq j \text{ or } a_i(\psi) + \widehat{a}_i(\psi) > 1$$

for any  $\psi \in \Psi_e$ .

## 5. AFFINE HECKE ALGEBRA OF TYPE A AND ARIKI-KOIKE ALGEBRAS

**5.1. Identification of simple modules.** Let  $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathcal{V}_l$ . The Ariki-Koike algebra  $\mathcal{H}_n^{\mathbf{v}}(\xi)$  is the quotient  $\mathcal{H}_n^a(\xi)/I_{\mathbf{v}}$  where  $I_{\mathbf{v}} = \langle P_{\mathbf{v}} = \prod_{i=0}^{l-1} (X_1 - \xi^{v_i}) \rangle$ . Then each simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module is isomorphic to a simple  $\mathcal{H}_n^a(\xi)$ -module of  $\text{Mod}_n^a$ . By the Specht module theory developed by Dipper, James and Mathas [7], the simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules are parametrized by certain  $l$ -partitions of  $n$  called Kleshchev multipartitions. Let  $\Phi_e^K(\mathbf{v})$  be

the set of Kleshchev  $l$ -partitions. Given  $\mu$  in  $\Phi_e^K(\mathbf{v})$ , write  $D^\mu$  for the simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module associated to  $\mu$  under this parametrization. In fact, we shall need in the sequel the parametrization of the simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules by FLOTW  $l$ -partitions. The correspondence between the parametrizations by Kleshchev and FLOTW  $l$ -partitions has been detailed in [14]. In particular, the bijection  $\Gamma : \Phi_e(\mathbf{v}) \rightarrow \Phi_e^K(\mathbf{v})$  is an isomorphism of  $\mathcal{U}_v$ -crystals which can easily be made explicit. This means that, given any  $\lambda$  in  $\Phi_e(\mathbf{v})$ , we can compute  $\Gamma(\lambda)$  directly from  $\lambda$  without using the crystal structures on  $\Phi_e(\mathbf{v})$  and  $\Phi_e^K(\mathbf{v})$ . We then set  $\tilde{D}^\lambda = D^{\Gamma(\lambda)}$ . This gives the natural labelling

$$\text{Irr}(\mathcal{H}_n^{\mathbf{v}}(\xi)) = \{\tilde{D}^\lambda \mid \lambda \in \Phi_e(\mathbf{v})\}$$

which coincides with the parametrization of the simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -modules in terms of Geck-Rouquier canonical basic set obtained in [13].

The simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module  $L_\psi$  with  $\psi \in \Psi_e$  isomorphic to  $\tilde{D}^\lambda$  is given by the following theorem (see [3, Thm 6.2]).

**Theorem 5.1.** *Let  $\lambda \in \Phi_e(\mathbf{v})$  then*

$$\tilde{D}^\lambda \simeq L_{f_{\mathbf{v}}(\lambda)}$$

where  $f_{\mathbf{v}}$  is the crystal embedding of Theorem 2.12.

Conversely, given any simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module  $L_\psi$ , it is natural to search for the Ariki-Koike algebras  $\mathcal{H}_n^{\mathbf{v}}(\xi)$  with  $\mathbf{v}$  in  $\mathcal{V}_l$  and the simple  $\mathcal{H}_n^{\mathbf{v}}(\xi)$ -module  $\tilde{D}^\lambda$  such that  $\tilde{D}^\lambda \simeq L_\psi$ . This problem turns out to be more complicated. Indeed we have first to determine all the multicharges  $\mathbf{v}$  such that  $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$  and next we need to compute the  $l$ -partition  $\lambda$  satisfying  $f_{\mathbf{v}}(\lambda) = \psi$ . Note that  $\lambda$  is necessarily unique for a given  $\mathbf{v}$  since  $f_{\mathbf{v}}$  is injective. We will then say that  $\mathbf{v}$  is an *admissible multicharge* with respect to  $\psi$  when  $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$ . Then  $\lambda = f_{\mathbf{v}}^{-1}(\psi)$  is its corresponding *admissible multipartition*. In the next paragraphs, we shall completely solve the problem of determining all the admissible multicharges and FLOTW multipartitions associated to an aperiodic multisegment  $\psi$ . To obtain the corresponding Kleshchev multipartition, it then suffices to apply  $\Gamma$ .

**5.2. Admissible multicharges.** Let  $\psi \in \Psi_e$ . To find a multicharge  $\mathbf{v}$  such that  $f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset$ , we compute  $\varepsilon_i^*(\psi)$  for all  $i \in \mathbb{Z}/e\mathbb{Z}$  by using the equality  $\varepsilon_i^*(\psi) = \hat{r}_i(\psi)$  established in Theorem 4.1. For a multicharge  $\mathbf{v}$  in  $\mathcal{V}_l$  and  $i \in \mathbb{Z}/e\mathbb{Z}$ , let  $\kappa_i(\mathbf{v})$  be the nonnegative integers such that

$$\mathbf{v} = (\underbrace{0, \dots, 0}_{\kappa_0(\mathbf{v})}, \underbrace{1, \dots, 1}_{\kappa_1(\mathbf{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\kappa_{e-1}(\mathbf{v})}).$$

Then we have

$$f_{\mathbf{v}}^{-1}(\psi) \neq \emptyset \iff \forall i \in \mathbb{Z}/e\mathbb{Z}, \kappa_i(\mathbf{v}) \geq \varepsilon_i^*(\psi).$$

Observe that the multicharge  $\mathbf{v}(\psi)$  with  $\kappa_i(\mathbf{v}(\psi)) = \varepsilon_i^*(\psi)$  (defined at the end of §2.4) is the multicharge of minimal level among all the admissible

multicharges. It is of particular interest for the computation of the involution  $\sharp$  as we shall see in §5.2.

**5.3. Admissible multipartitions.** Consider  $\psi \in \Psi_e$ ,  $l \in \mathbb{N}$  and an admissible multicharge  $\mathbf{v} \in \mathcal{V}_l$  with respect to  $\psi$ . The aim of this section is to give a simple procedure for computing the admissible  $l$ -partition  $\boldsymbol{\lambda} \in \Phi_e(\mathbf{v})$  associated to  $\psi$  (i.e. such that  $f_{\mathbf{v}}(\boldsymbol{\lambda}) = \psi$ ).

We begin with a general lemma on FLOTW  $l$ -partitions. Consider  $\mathbf{v} \in \mathcal{V}_l$  and  $\boldsymbol{\lambda} \in \Phi_e(\mathbf{v})$  a non-empty  $l$ -partition. Let  $m$  be the length of the minimal non zero part of  $\boldsymbol{\lambda}$ . Let  $\boldsymbol{\mu}$  be the  $l$ -partition obtained by deleting in  $\boldsymbol{\lambda}$  the parts of length  $m$ .

**Lemma 5.2.** *The  $l$ -partition  $\boldsymbol{\mu}$  belongs to  $\Phi_e(\mathbf{v})$ .*

*Proof.* Assume that  $\boldsymbol{\mu} \notin \Phi_e(\mathbf{v})$ . Then one of the following situations happens.

- (i) There exists  $c \in \{0, 1, \dots, l-1\}$  and  $i \in \mathbb{N}$  such that  $\mu_i^c < \mu_{i+v_{c+1}-v_c}^{c+1}$ . This implies in particular that  $\mu_{i+v_{c+1}-v_c}^{c+1} \neq 0$ . Since  $\boldsymbol{\lambda}$  belongs to  $\Phi_e(\mathbf{v})$ , we have  $\lambda_i^c \geq \lambda_{i+v_{c+1}-v_c}^{c+1}$ . Thus  $\lambda_{i+v_{c+1}-v_c}^{c+1} = \mu_{i+v_{c+1}-v_c}^{c+1}$ ,  $\lambda_i^c = m$  and  $\mu_i^c = 0$ . We have  $\lambda_{i+v_{c+1}-v_c}^{c+1} \neq 0$  and  $\lambda_{i+v_{c+1}-v_c}^{c+1} \leq m$ . This contradicts the fact that  $\boldsymbol{\mu}$  is obtained from  $\boldsymbol{\lambda}$  by deleting the minimal nonzero parts.
- (ii) There exists  $i \in \mathbb{N}$  such that  $\mu_i^{l-1} < \mu_{i+v_0-v_{l-1}+e}^0$ . We obtain a contradiction similarly. □

For  $\psi \in \Psi_e$ , define  $l_1 > \dots > l_r > 0$  as the decreasing sequence of (distinct) lengths of the segments appearing in  $\psi$ . For any  $t = 1, \dots, r$ , write  $a_t$  for the number of segments in  $\psi$  with length  $l_t$ . Set  $\psi_0 = \emptyset$  the empty multisegment and  $\psi_r = \psi$ . For any  $1 \leq t \leq r-1$  let  $\psi_t$  be the multisegment obtained from  $\psi$  by deleting successively the segments of length  $l_{t+1}, \dots, l_r$ . Clearly  $\psi_t$  is aperiodic.

Assume  $\boldsymbol{\lambda} \in \Phi_e(\mathbf{v})$  is associated to  $\psi$ . Since  $f_{\mathbf{v}}(\boldsymbol{\lambda}) = \psi$ , the sequence  $l_1 > \dots > l_r$  is also the decreasing sequence of distinct parts appearing in  $\boldsymbol{\lambda}$ . Moreover, for any  $t = 1, \dots, r$ ,  $\boldsymbol{\lambda}$  contains  $a_t$  parts equal to  $l_t$ . Set  $\boldsymbol{\lambda}[r] = \boldsymbol{\lambda}$ . Let  $\boldsymbol{\lambda}[t]$ ,  $t = 0, \dots, r-1$  be the  $l$ -partitions obtained by deleting successively the parts of lengths  $l_r, \dots, l_{t+1}$  in  $\boldsymbol{\lambda}$ . By Lemma 5.2, the  $l$ -partitions  $\boldsymbol{\lambda}[t]$ ,  $t = 0, \dots, r-1$  all belong to  $\Phi_e(\mathbf{v})$ . Since  $f_{\mathbf{v}}(\boldsymbol{\lambda}) = \psi$  we must also have

$$f_{\mathbf{v}}(\boldsymbol{\lambda}[t]) = \psi_t \text{ for any } t = 0, \dots, r$$

by definition of the map  $f_{\mathbf{v}}$  (see (10)).

We are going to compute  $\boldsymbol{\lambda}$  from  $\psi$  by induction on the lengths  $l_1 > \dots > l_r > 0$  of the segments of  $\psi$ . To do this, we have to determine the sequence of  $l$ -partitions  $\boldsymbol{\lambda}[t]$ ,  $t = 0, \dots, r$  associated to the segments  $\psi_t$ ,  $t = 0, \dots, r$ .

We have  $\lambda[0] = \emptyset$ . Thus, it suffices to explain how  $\lambda[t+1] \in \Phi_e(\mathbf{v})$  can be obtained from  $\lambda[t] \in \Phi_e(\mathbf{v})$ .

The  $l$ -partition  $\lambda[t+1]$  is constructed by adding  $a_{t+1}$  parts of lengths  $l_{t+1}$  to  $\lambda[t]$  such that the parts added give segments  $[k_{t+1}; l_{t+1})$  in the correspondence (10). Since the nonzero parts  $\lambda[t]$  are greater to  $l_{t+1}$ , these new parts can only appear on the bottom of the partitions composing  $\lambda[t]$ . The procedure for computing  $\lambda[t+1]$  from  $\lambda[t]$  can be decomposed in the following three steps.

- (1) For  $c = 0, \dots, l-1$ , consider the integers

$$i_c = \min\{a \in \mathbb{N} \mid \lambda[t]_a^c = 0\},$$

that is the sequence of depths of the partitions appearing in  $\lambda$ .

- (2) Let  $c_1, c_2, \dots, c_p \in \{0, \dots, l-1\}$  be such that

$$k_{t+1} \equiv 1 - i_{c_1} + v_{c_1} \equiv \dots \equiv 1 - i_{c_p} + v_{c_p} \pmod{e},$$

with  $p \geq a_{t+1}$ . These integers must exist by (10) for  $f_{\mathbf{v}(\psi)}^{-1}(\psi_{t+1}) \neq \emptyset$ .

Without loss of generality, we can assume  $c_1 \triangleleft \dots \triangleleft c_p$  where  $\triangleleft$  is the total order on  $\{0, \dots, l-1\}$  such that

$$(23) \quad c \triangleleft c' \iff \begin{cases} \text{(i)} : v_c - i_c < v_{c'} - i_{c'} \text{ or} \\ \text{(ii)} : v_c - i_c = v_{c'} - i_{c'} \text{ and } c < c' \text{ as integers.} \end{cases}$$

- (3) The problem reduces to determine  $a_{t+1}$  partitions among the partitions  $\lambda[t]^{c_f}$ ,  $f = 1, \dots, p$  which, once completed with a part  $l_{t+1}$ , yield an  $l$ -partition of  $\Phi_e(\mathbf{v})$ . Set  $S[t+1] = \{(c_1, i_{c_1}), \dots, (c_{a_{t+1}}, i_{c_{a_{t+1}}})\}$ .

Let  $\widehat{\lambda}[t+1]$  be the  $l$ -partition defined by

$$\widehat{\lambda}[t+1]_i^c = \begin{cases} \lambda[t]_i^c & \text{if } (c, i) \notin S[t+1], \\ l_{t+1} & \text{if } (c, i) \in S[t+1]. \end{cases}$$

So  $\widehat{\lambda}[t+1]$  is obtained by adding  $a_{t+1}$  parts  $l_{t+1}$  on the bottom of the partitions  $\lambda[t]^c$  with  $c \in \{c_1, \dots, c_{a_{t+1}}\}$ . This means that the new parts are added on the bottom of the  $a_{t+1}$  first partitions considered following (23).

**Lemma 5.3.** *With the above notation,  $\widehat{\lambda}[t+1] = \lambda[t+1]$ .*

*Proof.* It suffices to prove that  $\widehat{\lambda}[t+1]$  belongs to  $\Phi_e(\mathbf{v})$ . Indeed, this will give  $f_{\mathbf{v}}(\widehat{\lambda}[t+1]) = f_{\mathbf{v}}(\lambda[t+1]) = \psi_{t+1}$  and thus,  $\widehat{\lambda}[t+1] = \lambda[t+1]$  since  $f_{\mathbf{v}}$  is an embedding. The second condition to be a FLOTW  $l$ -partition is clearly satisfied for the multisegment  $\psi_{t+1}$  is aperiodic. We have to check that the first condition also holds.

Assume that  $\widehat{\lambda}[t+1]$  does not satisfy condition 1 of Definition 2.9.

Suppose first we have  $\widehat{\lambda}[t+1]_i^s < \widehat{\lambda}[t+1]_{i+v_{s+1}-v_s}^{s+1}$  where  $s \in \{1, \dots, l-1\}$  and  $i$  is a nonnegative integer. Since  $\lambda[t] \in \Phi_e(\mathbf{v})$ , we have  $\widehat{\lambda}[t+1]_i^s = \lambda[t]_i^s = 0$  and  $\lambda[t]_{i+v_{s+1}-v_s}^{s+1} = 0$ ,  $\widehat{\lambda}[t+1]_{i+v_{s+1}-v_s}^{s+1} = l_{t+1}$ . Thus  $(s+1, i+v_{s+1}-v_s) \in S[t+1]$ . We have two cases to consider.

- Assume  $i + v_{s+1} - v_s > 1$  and  $\widehat{\lambda}[t+1]_{i+v_{s+1}-v_s-1}^{s+1} = \lambda[t]_{i+v_{s+1}-v_s-1}^{s+1} > 0$ . Then  $i = 1$  or  $\widehat{\lambda}[t+1]_{i-1}^s \neq 0$ . Indeed we must have  $\widehat{\lambda}[t+1]_{i-1}^s = \widehat{\lambda}[t]_{i-1}^s \geq \lambda[t]_{i+v_{s+1}-v_s-1}^{s+1}$  because  $\lambda[t]$  belongs to  $\Phi_e(\mathbf{v})$ . We have  $(s+1, i+v_{s+1}-v_s) \in S[t+1]$ . In particular

$$k_{t+1} \equiv v_{s+1} - (i + v_{s+1} - v_s) + 1 \equiv v_s - i + 1 \pmod{e}.$$

Since  $\lambda[t]_i^s = 0$ , this means that  $(s, i) \in S[t+1]$ . But this is a contradiction. Indeed by condition (ii) of (23), we should have  $\widehat{\lambda}[t+1]_i^s = l_{t+1} \neq 0$ .

- Assume  $i + v_{s+1} - v_s = 1$  then  $i = 1$  and we have  $v_{s+1} = v_s$ . Thus  $(s, i) \in S[t+1]$  and we derive a contradiction similarly.

Now suppose we have  $\widehat{\lambda}[t+1]_i^{l-1} > \widehat{\lambda}[t+1]_{i+v_0-v_{l-1}+e}^0$ . The proof is analogue. We obtain that  $(l-1, i) \in S[t+1]$  and  $(0, i+v_0-v_{l-1}+e) \in S[t+1]$ . This contradicts condition (i) of (23).  $\square$

By using the above procedure, we are now able to compute the  $l$ -partitions  $\lambda[t]$ ,  $t = 1, \dots, r$  from  $\psi$  and from its associated admissible multicharge  $\mathbf{v}$ . This thus gives a recursive algorithm for computing the admissible  $l$ -partition  $\lambda$  from  $\psi$ .

**5.4. Example.** Let  $e = 4$ . We consider the following aperiodic multisegment

$$\psi = [0; 6] + [0; 5] + [3; 5] + [1; 4] + 2[3; 3] + [0; 3] + [2; 2] + [2; 1].$$

We have  $\widehat{w}_0(\psi) = \widehat{R}\widehat{A}\widehat{R}\widehat{R}$ ,  $\widehat{w}_1(\psi) = \widehat{A}\widehat{A}\widehat{R}$ ,  $\widehat{w}_2(\psi) = \widehat{R}\widehat{R}\widehat{A}\widehat{A}\widehat{A}$  and  $\widehat{w}_3(\psi) = \widehat{R}\widehat{R}\widehat{A}\widehat{R}\widehat{R}\widehat{A}$ . This gives

$$\varepsilon_0^*(\psi) = 2, \quad \varepsilon_1^*(\psi) = 1, \quad \varepsilon_2^*(\psi) = 0, \quad \varepsilon_3^*(\psi) = 0.$$

Thus the multicharge  $(0, 0, 1)$  is an admissible multicharge. Actually this is the one with minimal level. We now use the above algorithm to compute the associated admissible  $l$ -partition  $\lambda$ . Using the same notation as above, we successively obtain

$$\begin{aligned} \lambda[0] &= (\emptyset, \emptyset), & \lambda[1] &= (6, \emptyset, \emptyset), & \lambda[2] &= (6.5, 5, \emptyset), & \lambda[3] &= (6.5, 5, 4), \\ \lambda[4] &= (6.5, 5.3, 4.3.3), & \lambda[5] &= (6.5.2, 5.3, 4.3.3), & \lambda[6] &= (6.5.2, 5.3.1, 4.3.3) \\ \lambda &= (6.5.2, 5.3.1, 4.3.3) \text{ is the admissible 3-partition associated to the multicharge } (0, 0, 1) \text{ and } \psi. \end{aligned}$$

$$f_{(0,0,1)}(6.5.2, 5.3.1, 4.3.3) = \psi.$$

This means that the modules  $L_\psi$  and  $\widetilde{D}^\lambda$  are isomorphic.

The multicharge  $(0, 0, 1, 2, 3)$  is another example of an admissible multicharge (with level 5) and its associated admissible multipartition is  $\lambda = (6.3, 5.3, 4.3, 2, 5.1)$ .

6. COMPUTATION OF THE INVOLUTION  $\sharp$

**6.1. The generalized Mullineux involution.** The two fold symmetry  $i \longleftrightarrow -i$  defines a skew crystal isomorphism from  $B_e(\mathbf{v})$  to  $B_e(\mathbf{v}^\sharp)$  where  $\mathbf{v} = (v_0, \dots, v_{l-1})$  and  $\mathbf{v}^\sharp = (-v_{l-1}, \dots, -v_0)$  belong to  $\mathcal{V}_l$  (see (9)). Given  $\lambda \in \Phi_e(\mathbf{v})$ , write  $m_l^{\mathbf{v}}(\lambda) \in \Phi_e(\mathbf{v}^\sharp)$  for the image of  $\lambda$  under this skew isomorphism. In [9], Ford and Kleshchev proved that for  $l = 1$ , the map  $m_l^{\mathbf{v}}$  reduces to the Mullineux involution  $m_1$  on  $e$ -restricted partitions. Thus we call  $m_l^{\mathbf{v}}$  the generalized Mullineux involution.

By § 5.1, the set  $\Phi_e^K(\mathbf{v})$  of Kleshchev  $l$ -partitions has also the structure of an affine crystal isomorphic to  $B_e(\mathbf{v})$ . In particular the two fold symmetry  $i \longleftrightarrow -i$  also defines a bijection  $m_l^{\mathbf{v},K}$  from  $\Phi_e^K(\mathbf{v})$  to  $\Phi_e^K(\mathbf{v}^\sharp)$ . In [15], we gave an explicit procedure yielding  $m_l^{\mathbf{v},K}$ . Given  $\lambda = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e^K(\mathbf{v})$ , the  $l$ -partition  $\mu = m_l^{\mathbf{v},K}(\lambda)$  is obtained by computing first

$$\nu = (m_1(\lambda^0), \dots, m_1(\lambda^{l-1}))$$

i.e. the  $l$ -partition obtained by applying the Mullineux map to each partition of  $\lambda$ . The  $l$ -partition  $\nu$  does not belong to  $\Phi_e^K(\mathbf{v}^\sharp)$  in general and we have then to apply a straightening algorithm (detailed in [15] §4.3) to obtain  $\mu$ . As already noted in §5.1, we have a bijection (in fact a crystal isomorphism)

$$\Gamma : \Phi_e(\mathbf{v}) \rightarrow \Phi_e^K(\mathbf{v})$$

which can be made explicit by using the results of [14]. This permits to compute the map  $m_l^{\mathbf{v}}$  since

$$(24) \quad m_l^{\mathbf{v}} = \Gamma^{-1} \circ m_l^{\mathbf{v},K} \circ \Gamma.$$

*Remark 6.1.*

- (1) The previous procedure yielding the generalized Mullineux map  $m_l^{\mathbf{v}}$  can be optimized. In particular the conjugation by the map  $\Gamma$  can be avoided. Nevertheless, the pattern of the computation remains essentially the same : it uses the original Mullineux map  $m_1$  and the results of [14] on affine crystal isomorphisms. Since it requires some technical combinatorial developments which are not essential for our purposes, we have chosen to omit it here.
- (2) Note also that in the case  $e = \infty$ , the map  $\Gamma$  is the identity and  $m_1$  is simply the conjugation operation on the partitions. As observed in [15, §4.4], the algorithm for computing  $m_l^{\mathbf{v}} = m_l^{\mathbf{v},K}$  then considerably simplifies.

**6.2. The algorithm.** Let  $\psi \in \Psi_e$  then, to compute  $\psi^\sharp$ , we first determine an admissible multicharge  $\mathbf{v}$  with respect to  $\psi$  and the associated admissible multipartition  $\lambda$ . Then we apply the above algorithm to compute  $m_l^{\mathbf{v}}(\lambda)$ . It turns out that the complexity of this algorithm considerably increases with the level of  $\mathbf{v}$ . Hence, the use of the admissible multicharge  $\mathbf{v}(\psi)$  with minimal level is preferable. Let us summarize the different steps of the procedure we have to apply to compute  $\psi^\sharp$  :

- (1) For  $i \in \mathbb{Z}/e\mathbb{Z}$ , we compute  $\varepsilon_i^*(\psi)$ . To do this, we use Theorem 4.1 which gives the equalities  $\varepsilon_i^*(\psi) = \widehat{r}_i(\psi)$  for all  $i \in \mathbb{Z}/e\mathbb{Z}$ . We then put

$$\mathbf{v}(\psi) = (\underbrace{0, \dots, 0}_{\varepsilon_0^*(\mathbf{v})}, \underbrace{1, \dots, 1}_{\varepsilon_1^*(\mathbf{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\varepsilon_{e-1}^*(\mathbf{v})}).$$

By Theorem 4.1,  $\mathbf{v}(\psi)$  is an admissible multicharge in  $\mathcal{V}_l$ .

- (2) Using §5.3, we compute the admissible FLOTW multipartition  $\underline{\lambda}(\psi)$  with respect to  $\mathbf{v}(\lambda)$  and  $\psi$ .
- (3) Using §6.1, we compute the image  $m_l^{\mathbf{Y}}(\underline{\lambda}(\psi))$  of  $\underline{\lambda}(\psi)$  under the generalized Mullineux involution,
- (4) We finally obtain the aperiodic multisegment  $\psi^\sharp = f_{\mathbf{v}(\psi)}(m_l^{\mathbf{Y}}(\underline{\lambda}(\psi)))$  using §2.4.

*Remark 6.2.* In the case where  $e = \infty$ , our algorithm for computing the Zelevinsky involution is essentially equivalent to that described by Moeglin and Waldspurger in [22] except we use multipartitions rather than multisegments.

## 7. FURTHER REMARKS

**7.1. Computation of the Kashiwara involution.** We have established in Section 4, that the crystal operators  $\widetilde{f}_i^*$  and  $\widehat{f}_i$  coincide for any  $i \in \mathbb{Z}/e\mathbb{Z}$ . Given  $\psi \in \Psi_e$ , we can thus compute  $\psi^*$  by determining a path  $\psi = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \emptyset$  in  $B_e(\infty)$  from the empty multisegment to  $\psi$ . We have then  $\psi^* = \widehat{f}_{i_1} \cdots \widehat{f}_{i_n} \emptyset$ .

By combining the algorithm described in Section 6 for computing the involution  $\sharp$  with the relation  $* = \tau = \rho \circ \sharp$ , we obtain another procedure computing  $*$  on  $B_e(\infty)$ . This procedure does not required the determination of a path in the crystal  $B_e(\infty)$ .

**Example 7.1.** Assume  $e = 2$ , the involution  $\sharp$  is nothing but the identity and, thus the Kashiwara involution coincides with  $\rho$  on  $B_e(\infty)$ .

**7.2. Crystal commutor for  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ .** In [17], Kamnitzer and Tingley introduced a crystal commutor for any symmetrizable Kac-Moody algebra. Recall that a crystal commutor for  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$  is a crystal isomorphism

$$\sigma_{\mathbf{v}, \mathbf{v}'} : B_e(\mathbf{v}) \otimes B_e(\mathbf{v}') \rightarrow B_e(\mathbf{v}') \otimes B_e(\mathbf{v}).$$

This isomorphism is unique if and only if  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$  does not contain two isomorphic connected components that is, if the decomposition of the corresponding tensor product is without multiplicity. Such a crystal commutor is defined by specifying the images of the highest weight vertices of  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ . It is easy to verify by using (7) that the highest weight vertices of  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$  are precisely the vertices of the form  $\emptyset \otimes \lambda$  with

$\lambda \in B_e(\mathbf{v}')$  such that  $\varepsilon_i(\lambda) \leq r_i$  for any  $i \in \mathbb{Z}/e\mathbb{Z}$  ( $r_i$  is the number of coordinates in  $\mathbf{v}$  equal to  $i$ ). Denote by  $\mathcal{H}_{\mathbf{v},\mathbf{v}'}$  the set of highest weight vertices in  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ .

For any  $\lambda \in B_e(\mathbf{v})$ , write for short  $\lambda^* = \underline{\lambda}(f_{\mathbf{v}'}(\lambda))^*$  (see the definition of  $\underline{\lambda}$  below (11)). Since  $*$  is an involution, we have  $\lambda^* \in B_e(\mathbf{w})$  where  $\mathbf{w} \in \mathcal{V}_l$  is the multicharge having  $\varepsilon_i(\lambda)$  coordinates equal to  $i$  and level  $l = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \varepsilon_i(\lambda)$ . The condition  $\varepsilon_i(\lambda) \leq r_i$  for any  $i \in \mathbb{Z}/e\mathbb{Z}$  then implies that  $\lambda^* \in B_e(\mathbf{v}')$ . We have the following theorem which is the main result of [17].

**Theorem 7.2.**

- (1) Assume  $\emptyset \otimes \lambda \in \mathcal{H}_{\mathbf{v},\mathbf{v}'}$ . Then  $\emptyset \otimes \lambda^* \in \mathcal{H}_{\mathbf{v}',\mathbf{v}}$ .
- (2) The map

$$\sigma_{\mathbf{v},\mathbf{v}'} : \begin{cases} \mathcal{H}_{\mathbf{v},\mathbf{v}'} \rightarrow \mathcal{H}_{\mathbf{v}',\mathbf{v}} \\ \emptyset \otimes \lambda \mapsto \emptyset \otimes \lambda^* \end{cases}$$

defines a crystal commutor for  $B_e(\mathbf{v}) \otimes B_e(\mathbf{v}')$ .

The results established in Sections 4 and 6 then permit to compute the crystal commutor of Kamnitzer and Tingley for affine type  $A$  crystals.

**Example 7.3.** Assume  $e = 2$ . Then the crystal commutor  $\sigma_{\mathbf{v},\mathbf{v}'}$  satisfies  $\sigma_{\mathbf{v},\mathbf{v}'}(\emptyset \otimes \lambda) = (\emptyset \otimes \underline{\lambda}(\rho \circ f_{\mathbf{v}'}(\lambda)))$  for any  $\emptyset \otimes \lambda \in \mathcal{H}_{\mathbf{v},\mathbf{v}'}$ .

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