Toward The Use Of The Time-Warping Principle With Discrete-Time Sequences

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An extension of the class of unitary time–warping operators to discrete–time sequences

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***Abstract*— This paper establishes a new coherent framework to extend the class of unitary warping operators [1] to the case of discrete–time sequences. Providing some *a priori* considerations on signals, we show that the class of discrete–time warping operators finds a natural description in linear shift–invariant spaces. On such spaces, any discrete–time warping operator can be seen as a non–uniform weighted resampling of the original signal. Then, gathering different results from the non–uniform sampling theory, we propose an efficient iterative algorithm to compute the inverse discrete–time warping operator and we give the conditions under which the warped sequence can be inverted. Numerical examples show that the inversion error is of the order of the numerical round–off limitations after few iterations.**

***Index Terms*— Time–frequency, Unitary equivalence, Implementation of time–warping operators, Non–stationary filtering.**

I. Introduction

Signal processing methods are often based on a change of the representation space. This change is generally performed by projecting the original space into another one, adapted to a particular class of signals. The underlying idea is that some spaces are better suited than others to highlight specific properties of signals. As a consequence, it is a natural feature to perform processing tasks on the projected space since the useful information is easily reachable. As a final step, a well–defined inverse projection allows to return back to the original domain. This projection–processing–inversion framework has been successfully used in various signal processing domains [2].

An interesting class of unitary projections is the class of time warping operators [1]. This class has been used in image processing [3] for non–linear coordinate transformations and morphing purposes. In signal processing, warping operators have been used to build time–frequency representations with reduced interference terms, the so–called VU–Cohen's class [1]. Despite some other applications, the reversibility property of the time warping operators has surprisingly not found signal–processing applications as is the case for other unitary transforms. Recently, we have shown that a projection–processing–inversion framework, in time–warped spaces, can be used for efficient non–stationary denoising purpose [4]. Still, because of our non–exact approach, cumulative errors led to inaccurate results in multi–stages processing.

As far as our knowledge, an extension of the class of time warping operators, while keeping in mind invertibility, has not been derived yet in the case of discrete–time signals. We believe that this lack may explain the small number of signal–processing methods based on this class of operators. As an attempt to fill this lack, this paper establishes a new coherent framework to extend the class of warping operators in the case of discrete–time sequences. Main difficulties in the inversion of the discrete–time operator is related to the inversion of the resampling operator.

This inversion deals with the recovery of a signal from its non–uniformly distributed samples which is a difficult problem. Using recent results from non–uniform sampling theory, we show that this problem can be solved by an iterative algorithm. We state that providing a dense enough warped sequence, any discrete–time warping operator can be numerically inverted. Then we gave a density relation between the derivative of the warping function and the shift–invariant space kernel to guarantee the existence of the inverse discrete–time warping operator. Performances of this procedure are illustrated on a toy example and show that the precision the inverse discrete–time warping operator reach the finite round–off precision in few iterations.

We believe that this new definition of the warping operators for discrete–time signals will give new insight in multi-components time–frequency signal processing and will leads to efficient signal–processing methods that are unknow so far.

The organization of this paper is as follows. Section II starts with the classical definition of the class of continuous time–warping operators and describes its mathematical properties. Then a discrete–time formulation is proposed in shift–invariant spaces. Section III states the equivalence between inversion of the discrete–

This work was supported by the French Military Center of Oceanography under the reaserch contract CA/2003/06/CMO.

time warping operator and the inversion of a resampling operator. Gathering different results from the non–uniform sampling theory, an efficient iterative implementation is proposed, and invertibility conditions on the resampling set are derived. Numerical and convergence results are given section IV-A, and concluding remarks are given section V.

II. Class of unitary time–warping operators

A. Continuous formulation

Unitary warping methods play an important role in large number of signal–processing applications. Such an application is the design of time–frequency distributions that match almost any one–to–one group delay or instantaneous frequency characteristics [5], [6], [7]. The principle of the time–warping concept has been introduced in [1] and is based on the transformation of the time axis by means of a non–linear mapping. This can be done by warping the original time t axis by an eventually non–linear warping function $w(t)$ designed to match the signal properties.

Let $x(t) \in L^2(\mathbb{R})$ be a square–integrable[1] signal and let

$$\left\{\mathcal{W},\quad w(t) \in \mathcal{C}^1,\ w'(t) > 0 : x(t) \to \mathcal{W}x(t)\right\}, \qquad (1)$$

be a the class of warping operators where $w'(t)$ is the derivative of $w(t)$ with respect to t and $\mathcal{C}^1$ is the class of derivable functions. Warping transformation is a linear transformation of the signal $x(t)$ into the warped domain, whose effect on $x(t)$ is defined in [1] by

$$\mathcal{W}x(t) = \left|\frac{dw(t)}{dt}\right|^{1/2} x\left(w(t)\right). \qquad (2)$$

In [1], the warping concept has been introduced in the context of unitary transformations. An operator $\mathcal{U}$ is said to be unitary if it preserves the energy of the transformed signal, that is $\|\mathcal{U}x\|_2^2 = \|x\|_2^2$ and if it preserves the inner product, that is $\langle \mathcal{U}x, \mathcal{U}y\rangle = \langle x, y\rangle$.

Thus, warping operators are unitary since the envelope $|dw(t)/dt|^{1/2}$ preserves the energy of the signal at the output of $\mathcal{W}$ that is for all $x, y \in L^2(\mathbb{R})$

$$\|x\|_2^2 = \int_{\mathbb{R}} \left|\frac{dw(t)}{dt}\right| \left|x\left(w(t)\right)\right|^2\ dt, \qquad (3)$$

$$= \int_{\mathbb{R}} |x(t)|^2\ dt, \qquad (4)$$

and preserves the inner product that is

$$\langle \mathcal{W}x, \mathcal{W}y\rangle = \int_{\mathbb{R}} \mathcal{W}\left(x(t)\ y^*(t)\right) dt \qquad (5)$$

$$= \int_{\mathbb{R}} x(t)\ y^*(t)\ dt. \qquad (6)$$

[1]The set of square–integrable functions denoted by $L^2(\mathbb{R})$ is a Hilbert space with inner product $\langle x, y\rangle = \int_{\mathbb{R}} x(t)y^*(t)dt$, where $y^*(t)$ denotes the conjugate, and with norm $\|x(t)\|_2 = \left(\int_{\mathbb{R}} x(t)^2 dt\right)^{1/2} < \infty$ for all $x(t),\ y(t) \in L^2(\mathbb{R})$.

B. Discrete formulation

For real–life applications, the continuous formulation of the class of warping operators defined in Sec. II-A has to be turned into a discrete formulation. Let $x[n] \in \mathbb{R}^N,\ n = 0, \ldots, N-1$ be the sequence obtained by uniform sampling of the continuous signal, $x[n] = \int_t x(t)\delta(t - nT)dt$ with T the sampling rate, and $x_{\mathcal{W}}[m] \in \mathbb{R}^M,\ m = 0, \ldots, M-1$ be the warped discrete sequence. Since we are now dealing with finite–length sequences, we shall restrict ourself to the class of warping functions defined in the interval $[0, (N-1)T]$, for which $w(0) = 0$, and $w(nT) = nT$. For the sake of notation simplicity, we denote by $\overline{m}$ the normalized sequence $m/(M-1),\ m = 1 \ldots M-1$. Then a straightforward definition for the sampled discrete–time warping operators is

$$(\mathcal{W}x)[m] = \left|\dot{w}_d\left(\overline{m}\right)\right|^{1/2} x\left(w_d\left(\overline{m}\right)(N-1)T\right), \qquad (7)$$

where the warping function $w_d(t)$ is defined by $\{w_d : [0,1] \to [0,1] \in \mathcal{C}^1|\ w_d(0) = 0,\ w_d(1) = 1,\ \dot{w}_d(t) \geq 0\}$. From this definition, the computation of the discrete–time warping operator requires samples $x\left(w_d\left(\overline{m}\right)(N-1)T\right)$. However, from the sequence $x[n]$, only samples $x(nT)$ are known and the recovery of the missing samples has to deal with this partial knowledge.

From the Shannon's theory [8], it is well known that any bandlimited signal can be exactly recovered from its uniform samples with the so–called *sinc interpolator* by

$$x(t) = \sum_n x[n]\mathrm{sinc}(t - nT), \qquad (8)$$

where $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$. However this method is generally not used because of the slow decay of the sinc function with order $\mathcal{O}(1/x)$ which is not well–suited for practical applications.

More powerful methods can be found in an interpolation perspective [9]. One of those is the general class of interpolators in *linear shift–invariant* spaces.

A linear shift–invariant space V_ϕ is uniquely determined by the *kernel* ϕ with $V_\phi = \mathrm{Span}(\{\phi(. - k),\ k \in \mathbb{Z}\})$. The general interpolation formula on V_ϕ is given by [10]

$$x = \sum_{k\in\mathbb{Z}} \underbrace{\Big[\sum_n x[n]\psi(k-n)\Big]}_{a_k} \phi(. - kT), \qquad (9)$$

where ψ is the impulse response of some projection filters, and the coefficients (a_k) are the result of the filtering. In the scope of this paper we shall restrict ourself to the case of *exact interpolation*, which is equivalent to $x[n] = x(t)|_{t=nT},\ n = 0..N-1$. The latter condition is met, in the shift–invariant space V_φ generated by the kernel $\varphi(.) = \sum_{k\in\mathbb{Z}} \psi(k-n)\phi(. - kT),\ k \in \mathbb{Z},\ t \in \mathbb{R}$, if and only if φ verifies the *exact interpolation condition*

$$\varphi(nT) = \sum_{k\in\mathbb{Z}} \psi(k-n)\phi(nT - kT) = \delta_{n,0},\ \forall n\mathbb{Z}, \qquad (10)$$

where $\delta_{n,m}$ denotes the Kronecker delta function. This interpolation method allows a degree of freedom on the

choice of the interpolation kernel $\varphi(t)$. In truth, this choice is a matter of *a priori* considerations on the signal $x(t)$. If one deals with bandlimited signals then $\varphi(t) = \mathrm{sinc}(t)$ has to be chosen to recover Equ. 8. On the other hand, if the signal $x(t)$ can be modelled by a spline, then the cardinal B–spline [10] is the optimal choice.

Let $\mathcal{S} : \{x[n]\} \rightarrow x_{\mathcal{S}}[m],\ n = 0..N-1,\ m = 0..M-1$ be the resampling operator on the shift–invariant space V_φ defined by

$$x_{\mathcal{S}}[m] = (\mathcal{S}x)[m] = \sum_n x[n]\varphi(f(m) - n), \qquad (11)$$

for some resampling mapping f. Defining $n_m = f(m) = (N-1)\ w(\overline{m})$, the set $\mathcal{X} = \{n_m\},\ m = 0..M-1$ is a non–uniform sampling set for the V_φ space. This gives the final expression for the class of discrete–time warping operators

$$x_{\mathcal{W}}[m] = (\mathcal{W}x)[m] = |\dot{w}_d(\overline{m})|^{1/2}\ x_{\mathcal{S}}[m], \qquad (12)$$

which can be seen as a weighted resampling in V_φ of the sequence $x[n]$.

III. DISCRETE-TIME INVERSE WARPING OPERATOR

A. Problem statement

Our starting point is the definition of the discrete-time inverse warping operator $\mathcal{W}^{-1}$

$$(\mathcal{W}^{-1}(\mathcal{W}x))[n] \triangleq x[n]. \qquad (13)$$

Then, defining $\mathcal{S}^{-1}$ the inverse sampling operator, and using Equ. 12 leads to

$$(\mathcal{W}^{-1}x_{\mathcal{W}})[n] = (\mathcal{S}^{-1}\,|\dot{w}_d(\overline{m})|^{-1/2}\,x_{\mathcal{W}}[m])[n]. \qquad (14)$$

Inversion of the discrete-time warping operator resumes to the inversion of the sampling operator which is a difficult task in shift–invariant spaces for any kernel function.

B. Equivalence in non–uniform sampling theory

The problem of recovering a signal $x \in V$ from a non–uniformly distributed set of samples is generally referred as a non–uniform sampling problem [11].

It can be shown that if the maximal gap between the samples $n_m,\ n_{m+1}$, is small enough, then any $x \in V_\varphi$ can be recovered from the set $\{x_{\mathcal{S}}[m]\}$, and one says that the sampling set $\mathcal{X}$ is stable in V_φ. Conditions on $\mathcal{X}$ to be stable in V_φ are discussed in Sec. III-C. Then, from [11] and [12] we derive the following iterative algorithm of the inverse sampling operator.

Alg. 1 (Inverse sampling operator): Let $\varphi(.)$ be a kernel for the shift–invariant space V_φ. For all $\varphi(t)$ verifying

$$\sum_n \sup_{t\in[0,1]} |\varphi(t-n)| < \infty,\ \forall\ n \in \mathbb{Z},\ t \in \mathbb{R}, \qquad (15)$$

$$\varphi(t)|_{t=nT} = \delta_{n,0},\ \forall n \in \mathbb{Z},\ t \in \mathbb{R}, \qquad (16)$$

and providing $\mathcal{X} = \{n_m\},\ m = 0..M-1$ a stable sampling set in V_φ, the uniform samples $x[n],\ n = 0..N-1$ for all $x \in V_\varphi$ can be recovered by the following iterative algorithm.

- Initialization

  $x^{(0)}[n] = x_{\mathcal{S}}[k],\ k = \underset{m}{\mathrm{argmin}}\,\{|n - n_m|\}$

  $x_{\mathcal{S}}^{(0)}[m] = \sum_{n=0}^{N-1} x^{(0)}[n]\varphi(n_m - n)$

- Until $\|x^{(p)} - x^{(p-1)}\|_2 < \varepsilon$ do

  $\Delta x^{(p)}[m] = x_{\mathcal{S}}[k] - x_{\mathcal{S}}^{(p-1)}[k],\ k = \underset{m}{\mathrm{argmin}}\,\{|n - n_m|\}$

  $x^{(p)}[n] = x^{(p-1)}[n] + \Delta x^{(p)}[n]$

  $x_{\mathcal{S}}^{(p)}[m] = \sum_{n=0}^{N-1} x^{(p)}[n]\varphi(n_m - n)$

- End

and $\lim_{p\rightarrow\infty} \left\|x[n] - x^{(p)}[n]\right\|_2 = 0$ with a geometric convergence.

Proof of convergence related to this algorithm are gathered in the appendix VI.

C. Maximal gap between samples

It is obvious that a signal $x \in V_\varphi$ is not always uniquely determined for all sampling set $\mathcal{X} = \{n_m\},\ m = 0..M-1$, especially if $\mathcal{X}$ contains large gaps. In the case of bandlimited function the Beurling–Landau's theorem [13] provides a condition on $\mathcal{X}$ to be stable. However, in the case of shift–invariant spaces, this result does not hold anymore and the exact conditions on $\mathcal{X}$ to be stable in V_φ are unknown so far. Recently, under–optimal stability conditions have been determined for shift–invariant spaces in [12].

Let B_m be the δ–*ball* defined by

$$B_m = \{x : |n_m - x| \leq \delta\},\ x \in [0, N-1]. \qquad (17)$$

We define the *maximal gap* the smallest δ such that

$$\bigcup_m B_m = [0, N-1]. \qquad (18)$$

Then it can be shown that the upper bound

$$\delta < \left\| \frac{\pi\ G_\varphi(\omega)}{T\ G_{\dot{\varphi}}(\omega)} \right\|_0, \qquad (19)$$

guarantees the sampling set $\mathcal{X}$ to be stable in V_φ. The functions $G_\varphi(\omega)$ and $G_{\dot{\varphi}}(\omega)$ are both related to the Fourier transform $\hat{\varphi}(\omega)$ of the kernel function $\varphi(t)$ by

$$G_\varphi(\omega) = \left(\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2 \right)^{1/2}, \qquad (20)$$

$$G_{\dot{\varphi}}(\omega) = \left(\sum_k |j\omega\ \hat{\varphi}(\omega + 2k\pi)|^2 \right)^{1/2}. \qquad (21)$$

Because $G_{\varphi}(\omega)$ and $G_{\dot{\varphi}}(\omega)$ are both 2π–periodic, the norm $\|.\|_0$ is given by $\|G_{.}(\omega)\|_0 = \inf_{\omega\in[0,2\pi]} G_{.}(\omega)$.

Since the maximal gap is equal to $\sup_m |n_{m+1} - n_m|/2$, it is easy to show that

$$\sup_m |n_{m+1} - n_m| \leq \sup_{t\in[0,1]}(\dot{w}_d(t)) \frac{2\,(N-1)}{M-1} \leq \delta, \quad (22)$$

and to establish the under–optimal stability condition such that

$$r_c = \frac{1}{2}\left\| \frac{G'_{\varphi_{\text{int}}}(f)}{G_{\varphi_{\text{int}}}(f)} \right\|_{\infty} \sup_{t\in[0,1]} \frac{dw_d(t)}{dt}, \quad (23)$$

and we denote this condition as the critical redundancy ratio $r_c = M/N$. This conditions implies that the time–warped sequence $x_{\mathcal{W}}[m]$ has always more samples than the original sequence $x[n]$ in order to guarantee a stable reconstruction of the signal from the inverse warping operator. Then, for any sequence $\mathcal{W}x[m]$, conditions under which the discrete–time warping operator can be inverted only depend on the kernel function and the maximum of the derivative of the warping function.

IV. Experimental results

In this section we demonstrate the performances of the proposed method. To do so, we illustrate our algorithm with two examples

A. Example 1: warping and unwarping of a discrete time–sequence

We illustrate, in this section, our method on a numerical example. We consider here the shift–invariant space V_{φ^a} generated by

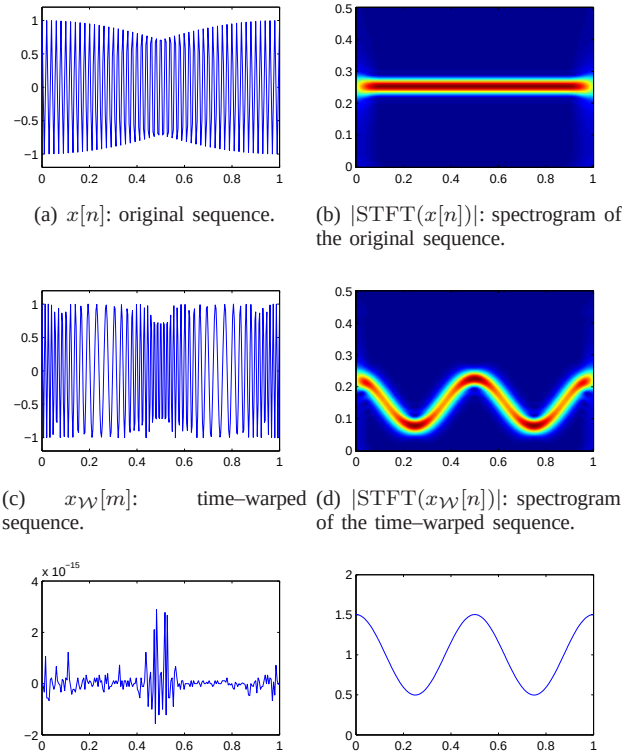
$$\varphi^a(t) = \text{sinc}(t)\cos\left(\frac{\pi t}{2a}\right)^2 \Pi_{[-a,a]}(t), \quad (24)$$

where the function $\Pi_{[-a,a]}(t) = 1,\ t < |a|,\ 0$ otherwise. $\varphi^a(t)$ has to be seen as an approximation of the sinc function in the sense that $\lim_{a\to\infty} \varphi^a(t) = \text{sinc}(t)$. This kernel belongs to the class of windowed–sinc interpolators [3] and is generally preferable to the sinc function since it has a compact support and leads to a reduction of the ringing artifacts.

For any $a < \infty$ it is obvious that $\varphi^a(t)$ verify Equ. 15 and Equ. 16, and the iterative algorithm always converges for a stable set $\mathcal{X}$.

The sequence $x[n] = \cos(2\pi 50\overline{n}),\ n = 0,\ldots,199$ is first generated. The discrete–time warping operator we use is defined by the warping function $w_d(t) = t + 0.04\ \sin(4\pi t)$. The warped sequence $x_{\mathcal{W}}[m],\ m = 0,\ldots,319$ is generated with $\varphi^5(t)$ by means of Equ. 12. Then we use Equ. 14 and Alg. 1 to recover the original sequence $x[n]$. Results of the numerical simulation are depicted in Fig. 1.

Fig. 1(a) shows the original discrete–time cosine sequence and Fig. 1(b) its time–frequency representation. Fig. 1(c) shows the warped sequence and Fig. 1(d) its time–frequency representation. The instantaneous frequency of the warped sequence is cosine modulated. This non–linear modulation effect comes from the derivative of the warping function represented Fig 1(f). Fig. 1(e) shows the difference between the original sequence and the sequence recovered with the inverse–warping operator after 45 iterations. As seen, the maximal reconstruction error is of the order of the round–off precision we used ($\varepsilon \approx 2.22\ 10^{-16}$ in our example).



(a) $x[n]$: original sequence. (b) $|\text{STFT}(x[n])|$: spectrogram of the original sequence.

(c) $x_{\mathcal{W}}[m]$: time–warped sequence. (d) $|\text{STFT}(x_{\mathcal{W}}[n])|$: spectrogram of the time–warped sequence.

(e) $x[n] - \left(\mathcal{W}^{-1}\left(\mathcal{W}x\right)\right)[n]$: reconstruction error. (f) $\dot{w}_d(t)$: Derivative of the warping function.

Fig. 1. Numerical example of discrete–time warping operators. In this example $N = 200$, $M = 320$ and $a = 5$. (a)(b) original cosine sequence. (c)(d) time–warped sequence. (e) reconstruction error between the original sequence and the sequence recovered from the warped sequence after 45 iterations. (f) derivative of the warping function.

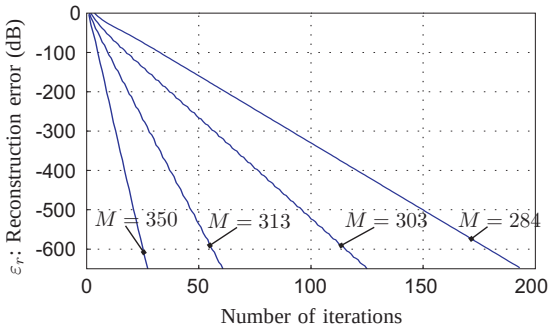


Fig. 2. Reconstruction error versus number of iterations for different sizes of resampling sets ($M = 284, 303, 313, 350$).

Fig. 2 and Fig. 3 show results of convergence. Fig 2 shows the reconstruction error $\varepsilon_r = 20\ \log\|x[n] - \mathcal{W}^{-1}(\mathcal{W}x)[n]\|_2$ as a function of the number of iterations, for different sizes of resampling sets. Clearly, the

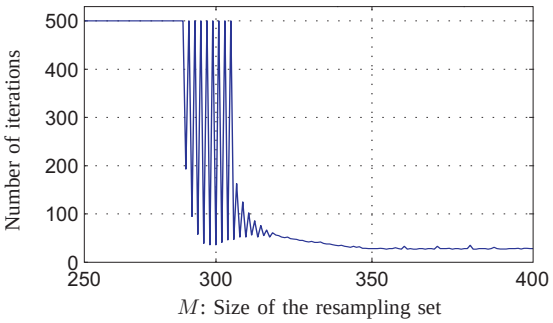


Fig. 3. Number of iterations necessary to reach $\varepsilon_r < -320$ dB, versus the size of the resampling set. A number of iterations equal to 500 signifies that the iterative algorithm does not converge for the current resampling set

reconstruction error is linearly decreasing on a dB scale as the iterations increase. This confirms the geometric convergence of the inverse sampling algorithm stated in Alg. 1.

As can be seen in Fig. 2, the size of the resampling set is critical as regards of the number of iterations needed to reach a fixed reconstruction error bound. As an example, one needs 10 times more iterations for a resampling set with size $M = 284$ than for a set with a size $M = 350$. This is an expected result since it is well-known that the repartition of the sampling set is related to the conditioning of the non–uniform sampling problem, and so to the convergence rate of the iterative reconstruction algorithm.

Fig 3 shows the number of iterations needed to reach the error bound $\varepsilon_r < -640$ dB, as a function of the size of the resampling set. In this example, a number of iterations equal 500 iterations signifies that the iterative algorithm does not converge for the current resampling set. Below $M = 280$ the resampling set is not stable and the iterative algorithm does not converge. Between $M = 280$ and $M = 318$ the resampling set is *critically stable* and a small perturbation of a stable set may give an unstable set. After $M = 318$, the resampling set is stable and the number of iterations needed to reach the fixed error bound is globally decreasing. This result speaks in favour of large values for M for practical applications. However, the size of the resampling set cannot be set as large as wanted for computation burden reasons and a trade–off has to be found between converging rate and computation cost.

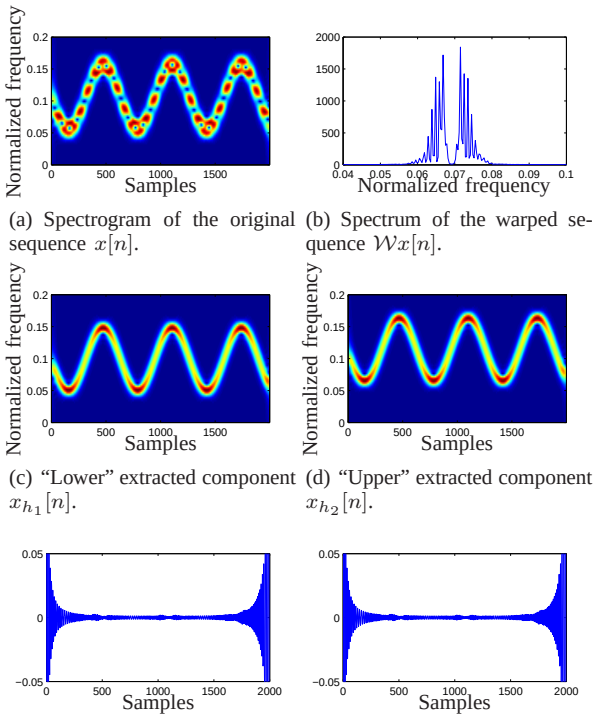
B. Example 2: separation of close time–frequency components

In this example we consider a signal made of two close time–frequency components. Let $x[n]$ be the sequence generated by the sum of two close cosine frequency modulated components $x_1[n]$ and $x_2[n]$ where

$$x_1[n] = \exp\left(2i\pi\left(5\cos(0.01\ n) + 0.10\ n\right)\right), \qquad (25)$$

$$x_2[n] = \exp\left(2i\pi\left(5\cos(0.01\ n) + 0.115\ n\right)\right), \qquad (26)$$

and where $n = 0, \ldots, 2000$. The spectrogram of the sequence $x[n]$ is given in Fig. 4(a). The two components



(a) Spectrogram of the original sequence $x[n]$.

(b) Spectrum of the warped sequence $\mathcal{W}x[n]$.

(c) "Lower" extracted component $x_{h_1}[n]$.

(d) "Upper" extracted component $x_{h_2}[n]$.

(e) Error of separation between the "lower" component and the extracted "lower" component.

(f) Error of separation between the "higher" component and the extracted "higher" component.

Fig. 4. Example of separation of two close frequency modulated signals based on the proposed time–varying filtering method.

are very close and have constructive and destructive interferences on the spectrogram which leads to a "poor" representation of the time–frequency content. The extraction task illustrated at Fig. 5 consists to separate the "upper" and the "lower" component into two signals. For this purpose, we choose the warping function $w(t) = \varphi^{-1}(t)$ such that

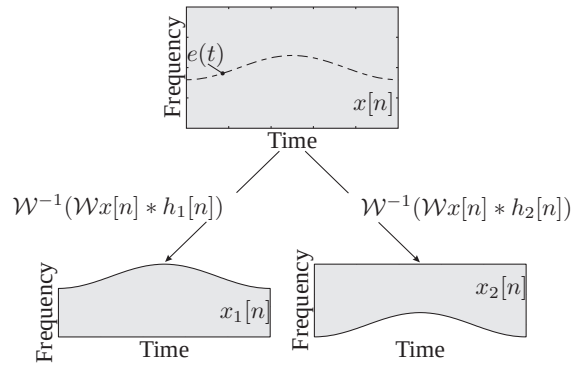


Fig. 5. Illustration of the extraction task based on time–warping operators. Top: Received signal $x[n]$. Bottom–left: "lowpass" filtered signal. Bottom–right: "highpass" filtered signal.

$$\varphi(t) = 5\cos(0.01\ n) + 0.11\ n, \qquad (27)$$

to generates the warping operator.

In Fig. 4(b), the spectrum of the sequence $\mathcal{W}x[n]$ is shown. In the warped domain, both components are well–separated as the energy of the spectrum falls to zeros around the normalized frequency 0.0707 Hz. Thus,

components can be separated by means of frequency filter. To do so, we first multiply the spectrum of the Fig. 4(b) by the frequency windows $\widehat{h_1}(f),\ \widehat{h_2}(f)$ given by

$$\widehat{h_1}(f) = \begin{cases} 1, & \text{if } f \in [0.0707, 0.5], \\ 0, & \text{else.} \end{cases} \tag{28}$$

$$\widehat{h_2}(f) = \begin{cases} 1, & \text{if } f \in [0, 0.0707], \\ 0, & \text{else.} \end{cases} \tag{29}$$

Then we recover signals $x_{h_1}[n]$ and $x_{h_2}[n]$, the estimates of respectively $x_1[n]$ and $x_2[n]$, by means of the inverse local harmonic Fourier with

$$x_{h_1}[n] = x[n] \overset{\varphi^{-1}(t)}{*} \left(\mathcal{F}^{-1}\widehat{h_1}(f)\right)[n], \tag{30}$$

$$x_{h_2}[n] = x[n] \overset{\varphi^{-1}(t)}{*} \left(\mathcal{F}^{-1}\widehat{h_2}(f)\right)[n]. \tag{31}$$

Then $x_{h_1}[n]$ contains the extracted "upper" component as $x_{h_2}[n]$ contains the extracted "lower" component. Spectrograms of the filtered sequences $x_{h_1}[n]$ and $x_{h_2}[n]$ are shown in Fig. 4(c) and Fig. 4(d). The error of the extraction procedure $\varepsilon_{x_i} = x_i[n] - x_{h_i}[n],\ i = 1, 2$ between the "true" component and the extracted component are shown Fig. 4(e) and Fig. 4(f).

As can be seen, the components are well–separated and the destructive interferences terms are suppressed. The SNR of the extracted components calculated by

$$SNR = 10\ \log_{10}\left(\frac{\|x_i[n]\|^2}{\|x_i[n] - x_{h_i}[n]\|^2}\right), \quad i = 1, 2, \tag{32}$$

is 31.41 dB for both which confirms a globally good performance of the extraction procedure. The error of separation is smaller in the middle of the signal than in the sides of the signal : as is also the case in linear time–invariant filtering this side–effect is due to the filter functions $\widehat{h_1}(f)$ and $\widehat{h_2}(f)$ that have a smoothing effect on the amplitude of the extracted sequences $x_{h_1}[n]$ and $x_{h_2}[n]$.

V. CONCLUSION

We have established a new coherent framework to extend the class of warping operators to the case of discrete–time sequences and defined conditions under which such operators are invertible.

We have first considered the original discrete signal as a sampling procedure in a shift–invariant space and shown that any discrete–time warping operator can be written as a weighted resampling of the original signal.

Before giving stability conditions on the resampling set, we have shown that any discrete–time warping operator can be inverted by an efficient iterative algorithm with geometric convergence.

Finally, we have illustrated, on a first example, the performances of the method on numerical examples and showed that the error of reconstruction of the inverse discrete–time warping operator is of the order of the round–off precision after few iterations. The second example showed that this class of transformation allows for the extraction of close time–frequency components. This has been done by means of a linear time–invariant filter, that has been applied in the warped domain.

We have already shown that time–warped spaces, can be used for efficient non–stationary denoising purpose [4]. We think that this new definition of the class of discrete–time warping operators can be useful for multi–stages signal denoising algorithms and separation of components with non–linear instantaneous frequency laws. Another issue is the definition of a new class of time–varying filter based on the demodulation properties of the time–warping operator.

VI. APPENDIX: CONVERGENCE PROOF OF THE ITERATIVE ALGORITHM

Theorem 1: [11] Let φ_i in $W_0(L^1_w)$ and let $\mathcal{P}$ be a bounded projection from L^p_ν onto $V^p_\nu(\varphi)$. Then there exist a density $\gamma > 0$ such that any $f \in V^p_\nu$ can be recovered from its samples $\{f(x_j) : x_j \in X\}$ on any σ–dense set $X = \{x_j, j \in J\}$ by the iterative algorithm

$$f^{(0)} = \mathcal{P}\mathcal{Q}f \tag{33}$$

$$f^{(n)} = \mathcal{P}\mathcal{Q}(f - f^{n-1}) + f^n \tag{34}$$

Then iterates f_n converges to f uniformly and in the $W(L^p_\nu)$ and L^p_ν norms. The convergence is geometric, that is,

$$\|f - f^{(n)}\|_{L^p_\nu} \leq C\|f - f^{(n)}\|_{W(L^p_\nu)} \leq C'\|f - f^{(n)}\|_{L^p_\nu}\alpha^n, \tag{35}$$

for some $\alpha = \alpha(\gamma) < 1$.

The task is now to define suitable operators $\mathcal{P}$ and $\mathcal{Q}$ in order to design the iterative algorithm. In Theo. 1, the quasi–interpolant $\mathcal{Q}$ have to be generated from a suitable partition of the unity [11]. An efficient possibility [12] is to define the *voronoi domain* V_i as

$$V_i = \{n :\ |n_k - n| < |n_i - n|,\ k \neq i\}. \tag{36}$$

Then, the quasi–interpolant operator $\mathcal{Q}$ can be defined by

$$\mathcal{Q}f = \sum_i f(n_i)\chi_{V_i}, \tag{37}$$

where χ_{V_i} is the characteristic function of V_i.

In Theo. 1, the projection operator $\mathcal{P}$ has to be bounded operator $\mathcal{P} : L^2(\mathbb{R}) \to V_{\varphi_i}$. In [11], gives a formulation, in L^p_ν, of a suitable class of operators:

$$\mathcal{P} : f \to \sum_k \langle f, \widetilde{\varphi_i}(. - k)\rangle \varphi_i(. - k), \tag{38}$$

where $\widetilde{\varphi_i}$ is the dual of φ_i. Since the interpolation kernel φ_i verify the exact interpolation condition defined in Equ. 10, then for any $f \in L^2(\mathbb{R})$ the operator $\mathcal{P}$ can be expressed by

$$\mathcal{P}f = \sum_k \left\langle \sum_n f(n)\varphi_i(. - n) + f_{\varphi_i^\perp}, \widetilde{\varphi_i}(. - k) \right\rangle \varphi_i(.-k), \tag{39}$$

where $f_{\varphi_i^\perp} \in \mathrm{Ker}(\mathcal{P})$. Simplifying and rewriting Equ. 39 leads to

$$\begin{aligned}\mathcal{P}f &= \sum_n f(n) \sum_k \langle \varphi_i(.-n), \widetilde{\varphi}_i(.-k) \rangle \, \varphi_i(.-k) \\ &= \sum_n f(n) \sum_k \delta[n-k] \varphi_i(.-k) \\ &= \sum_n f(n) \varphi_i(.-n) \end{aligned} \tag{40}$$

Finally, gathering Theo. 1, Equ. 37 and Equ. 40 leads to the reconstruction algorithm presented in Sec. III.

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