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Some convergence results on quadratic forms for random fields and application to empirical covariances

Frdric LAVANCIER Anne PHILIPPE

Universit de Nantes, Laboratoire de Mathmatiques Jean Leray

Abstract

Limit theorems are proved for quadratic forms of Gaussian random fields in presence of long memory. We obtain a non central limit theorem under a minimal integrability condition, which allows isotropic and anisotropic models. We apply our limit theorems and those of Ginovian (1999) to obtain the asymptotic behavior of the empirical covariances of Gaussian fields, which is a particular example of quadratic forms. We show that it is possible to obtain a Gaussian limit when the spectral density is not in L^2 . Therefore the dichotomy observed in dimension $d = 1$ between central and non central limit theorems cannot be stated so easily due to possible anisotropic strong dependence in $d > 1$.

Keywords: Gaussian Random field ; Long memory ; Non central limit theorems ; Quadratic mean convergence ; Stochastic integral ; Triangular array.

1 Introduction

Many statistical methods proposed for long memory processes are based on statistics, which can be expressed using quadratic forms (see Beran, 1994, for a review). The empirical covariance sequence (see Hosking, 1996; Hannan, 1976), the estimation of the long memory parameter using Whittle contrast (see Fox and Taqqu, 1986; Giraitis and Surgailis, 1990; Giraitis and Taqqu, 1999, for example) or using the integrated periodogram (see Lobato I. and Robinson P.M., 1996), some change point detection procedure (see Beran and Terrin, 1996) rely on quadratic forms.

It would be interesting to validate the same statistical tools for the study of random fields having long memory. As a first step, we study in this paper the convergence of the quadratic forms in the d -dimensional ($d > 1$) case.

Let us introduce our framework. Let $X = (X_n)_{n \in \mathbb{Z}^d}$ be a stationary L^2 random field having long memory, i.e. its covariance sequence σ is not a summable

series. We assume that X is a Gaussian process and admits the linear representation

$$X_n = \int_E a(x) e^{i\langle n, x \rangle} dW(x), \quad n \in \mathbb{Z}^d, \quad E = [-\pi, \pi]^d, \quad (1)$$

where W is the Gaussian white noise spectral measure and where the function a is in $L^2(E)$. $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^d . Hereafter, we denote by f the spectral density of X , f is proportional to $|a(x)|^2$, and we have

$$\sigma(n) = \hat{f}_n := \int_E e^{i\langle n, \lambda \rangle} f(\lambda) d\lambda. \quad (2)$$

The quadratic forms associated to X and $(g_i)_{i \in \mathbb{N}^d} \in \ell^1(\mathbb{N}^d)$ are defined as

$$Q_n = \frac{1}{n^d} \sum_{i \in A_n} \sum_{j \in A_n} g_{i-j} X_i X_j \quad (3)$$

where $A_n = \{1, \dots, n\}^d$. The statistics Q_n can be rewritten as a functional of the periodogram I_n of X . Indeed let

$$I_n(t) = n^{-d} \sum_{k, l \in A_n^2} X_k X_l e^{-i\langle k-l, t \rangle},$$

we have

$$Q_n = \int_E g(t) I_n(t) dt, \quad (4)$$

where

$$g(t) = (2\pi)^d \sum_{j \in \mathbb{Z}^d} g_j e^{i\langle j, t \rangle}.$$

In dimension one ($d = 1$), many papers deal with the asymptotic behavior of Q_n for long memory processes. When the intensity of the memory is not too strong or when specific conditions are imposed to g in order to kill the effect of the long memory involved by f , the convergence rate and the asymptotic normality obtained in short memory are preserved. The asymptotic normality of the quadratic forms is proved by Avram (1988), Fox and Taqqu (1987) and Ginovian and Sahakian (2005) in the Gaussian case and extended by Giraitis and Surgailis (1990) to include the non-Gaussian linear case. Avram's result is restricted to f and g in L^2 , but requires only integrability conditions. On the opposite, Fox and Taqqu (1987), Ginovian and Sahakian (2005) specify the behavior of f and g around zero. The extension to linear models obtained by Giraitis and Surgailis (1990) holds under the assumptions of Avram (1988) or Fox and Taqqu (1987).

Under other conditions on f and g fixing the behavior of both functions around zero, Rosenblatt (1961), Fox and Taqqu (1985), Terrin and Taqqu (1990) prove non central limit theorems for Gaussian processes. The normalization is not standard (i.e. it is not equal to $n^{-d/2}$) and the limiting process is not Gaussian. The extension to linear models is proved in Giraitis *et al* (1998) in the general context of bivariate Appell polynomials.

Fixing the behavior of f and g around zero, the dichotomy in $d = 1$ can be summarize as follows. Assume that

$$\begin{aligned} f(x) &\sim |x|^{2\alpha} \quad \text{when } x \rightarrow 0, \\ g(x) &\sim |x|^{2\beta} \quad \text{when } x \rightarrow 0. \end{aligned}$$

The limiting distribution depends on the value of $(\alpha + \beta)$.

- If $\alpha + \beta > -\frac{1}{4}$ then the limit is Gaussian and the normalization is $n^{-1/2}$ (Q_n satisfies a central limit theorem).
- If $\alpha + \beta < -\frac{1}{4}$ then the limiting distribution is not Gaussian and the normalization in this case is $n^{2(\alpha+\beta)}$, which is lower than $n^{-1/2}$ (Q_n satisfies a non central limit theorem).

The case $\alpha + \beta = \frac{1}{4}$ is investigated in Ginovian and Sahakian (2005) (up to a slowly varying function in f and g which vanishes at zero). In this case, the limit remains gaussian.

In the d -dimensional case ($d > 1$), a central limit theorem has been proved by Ginovian (1999), in the same spirit as Avram (1988) (i.e. f and g are in L^2 and satisfy an integrability condition). This result is recalled in Section 2.1. When f or g are not in L^2 , some central limit theorems have been proved under some specific conditions on these functions, see Theorem 3.1 in Doukhan et al. (1996) and Lemma 3.3 in Boissy et al. (2005).

On the other hand, few results are available on non-central limit theorems in dimension $d > 1$. In Doukhan et al. (1996), the asymptotic of Q_n is investigated when the memory of X is "isotropic" (see Lavancier (2005) Definition 1), i.e. the memory of X is due to $|x|^{2\alpha}$ in the expression of f where $|\cdot|$ denotes the Euclidian norm. The authors obtain a non-central limit theorem very similar to Fox and Taqqu (1985). However the investigation of "anisotropic" memory and more general functions g (including the particular case of the empirical covariance) is of great interest for statistical applications. In Lavancier (2005), different classes of random fields having long memory are described: The important role of anisotropic processes in terms of modeling is illustrated.

Our paper is organized as follows. In Section 2, after a reminder of the central limit theorem obtained by Ginovian (1999), we present a non-central limit theorem under a general integrability condition on f and g . We show that this condition is satisfied by a large class of isotropic and anisotropic models (including all previous results). Since random fields in $d > 1$ may exhibit much more different types of long memory than processes in $d = 1$, the dichotomy between central and non central limit theorems is not so simple to formulate. As already highlighted in Remark 4.2 of Dobrushin and Major (1979), a spectral singularity outside zero may lead to new limiting results. This fact is illustrated through examples in Section 2.3.

In Section 3, we apply limit theorems for quadratic forms to the convergence of the empirical covariance series. Contrary to the classical setting in dimension one, we show that it is possible to obtain a Gaussian limit in dimension $d > 1$ when f is not in L^2 .

2 Convergence of quadratic forms

2.1 Central limit theorem

We recall in this section the central limit theorem obtained by Avram (1988) in dimension $d = 1$ and extended by Ginovian (1999) to $d > 1$. This result will be useful in section 3, where the asymptotic behavior of the covariance series is investigated.

Theorem 1 (Avram (1988), Ginovian (1999)). *Let X be a random field as in (1) and denote f its spectral density. Let Q_n , defined in (3), be the quadratic form associated with g and X . If $f \in L^p$ and $g \in L^q$ with $p \geq 2$, $q \geq 2$ and $1/p + 1/q \leq 1/2$, then*

$$n^{d/2}(Q_n - E(Q_n)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^{3d} \int_E f^2(t)g^2(t)dt\right), \quad (5)$$

where $\xrightarrow{\mathcal{L}}$ denotes the convergence in law.

This result provides a large setting since no specific form for functions f and g is required, but only integrability conditions. In return, f and g are restricted to L^2 . A similar result is obtained in Doukhan et al. (1996), Th. 6.2.¹, but under semi-parametric assumptions on f and g .

When f or g are not in L^2 but a compensation between them occurs, a central limit theorem may still arise as in Fox and Taqqu (1985) for $d = 1$. To our knowledge, two situations in dimension $d > 1$ have been investigated so far. In dimension $d = 2$, assuming f and g have a tensorial product form around zero, Boissy et al. (2005) obtain a central limit theorem, which can be straightforwardly extended to $d > 2$. In Doukhan et al. (1996), Th. 6.1, the authors prove the same result when f has an isotropic form around zero and when g is regular enough and $g(0) = 0$.

It is known that the lonely condition $0 < \int_E f^2(t)g^2(t)dt < \infty$ is not sufficient to obtain a central limit theorem, as proved in Ginovian and Sahakian (2005), Prop. 2.2. A general setting leading to a central limit theorem (included f or g not in L^2) is still missing. This issue is not investigated in this paper.

2.2 Main result : non-central limit theorem

The following theorem gives the convergence of the quadratic forms under the general condition **(H)** below on f and g . Contrary to the previous studies in

¹Their result contains a misprint: $(2\pi)^d$ should be replaced by $(2\pi)^{3d}$ to agree with (5).

the one dimension case or in dimension $d > 1$, this condition does not require to specify the behavior of f around its singularities. Such a condition is satisfied by many examples as shown in Section 2.3.

Let ζ be a function defined on \mathbb{R}^d , hereafter we denote by $\zeta_{(p)}$ ($p \in \mathbb{N}$) the periodic function with period $2\pi p$ (with respect to each component), that coincides with ζ on the set pE^d ,

$$\zeta_{(p)}(x) = \zeta(x) \quad \forall x \in [-p\pi, p\pi]^d. \quad (6)$$

Theorem 2. *Let X be the linear field defined by (1) and Q_n the quadratic form defined by (3).*

Assume that for all $x \in E$ $a(-x) = \overline{a(x)}$, and $g(-x) = \overline{g(x)}$. Moreover, assume that

$$\begin{cases} a(x) = \tilde{a}_{(1)}(x)L_1(x), \\ g(x) = \tilde{g}_{(1)}(x)L_2(x), \end{cases}$$

where \tilde{a} is a homogeneous function of degree α (i.e. $\forall c > 0, \forall x \in E, \tilde{a}(cx) = c^\alpha \tilde{a}(x)$), where \tilde{g} is a homogeneous function of degree 2β and where L_1 and L_2 are bounded functions, continuous at zero and $L_i(0) \neq 0$ ($i = 1, 2$).

*If the following assumption **(H)** is satisfied*

$$\text{(H): } \iint_{\mathbb{R}^{2d}} \tilde{a}^2(x)\tilde{a}^2(y) \left[\int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1+|x_k+t_k|)(1+|y_k-t_k|)} dt \right]^2 dx dy < \infty,$$

then

$$n^{d+2\alpha+2\beta}(Q_n - E(Q_n)) \xrightarrow{L^2} L_1^2(0)L_2(0) \iint_{\mathbb{R}^{2d}} \tilde{a}(x)\tilde{a}(y) \int_{\mathbb{R}^d} \tilde{g}(t)H(x+t)H(y-t)dt dW(x)dW(y), \quad (7)$$

where $H(z) = \prod_{j=1}^d \frac{e^{iz_j}-1}{iz_j}$.

It seems that **(H)** is almost necessary for the convergence, since it guarantees that the integral in (7) exists.

The proof is relegated to Section 2.4. The main tool consists in rewriting Q_n as a double stochastic integral. Then the convergence in quadratic mean is deduced from a simple convergence in $L^2(\mathbb{R}^{2d})$. Note that contrary to the proofs for non central limit theorems in Terrin and Taqqu (1990) and Doukhan et al. (1996), we do not apply the scheme of convergence of Dobrushin and Major (1979), especially their Lemma 3 which involves a spectral measure convergence assumption.

2.3 Some class of models satisfying the hypothesis **(H)**

The setting of random fields allows many kind of dependencies (see Lavancier (2005)). Apart from its intensity, the dependence can be isotropic or can occur

all over several particular directions, depending on the form of the spectral density. Since we did not want to restrict ourselves to one particular case, Theorem 2 involves a general integrability condition, i.e. assumption **(H)**. We check in this section that this hypothesis is not too restrictive. Indeed, we prove that in all the preceding studies about quadratic forms of Gaussian fields, this condition is fulfilled. Moreover, it allows to extend to new cases, mainly when the dependence is not isotropic.

As a consequence of the following Lemmas and according to Theorem 2, a non-central theorem for Q_n arises for all the cases treated in this section.

Lemma 3. *Assume that there exists some positive constants c and c' such that*

$$\begin{cases} |\tilde{a}(x)| \leq c \prod_{i=1}^d |x_i|^{\alpha/d} \\ |\tilde{g}(x)| \leq c' \prod_{i=1}^d |x_i|^{2\beta/d}. \end{cases} \quad (8)$$

*If moreover $\alpha > -d/2$, $\beta > -d/2$ and $\alpha + \beta < -d/4$, then **(H)** is fulfilled.*

Remark 1. *The result of our Theorem 2 in the setting of Lemma 3 is the same as the convergence stated in dimension 1 by Terrin and Taqqu (1990) (Th. 1).*

The assumptions in Lemma 3 allow the filter $a(x)$ to be isotropic, i.e. equivalent at zero, up to a constant, to $|x|^\alpha$. This is the hypothesis done in Theorem 5.1 of Doukhan et al. (1996) where, moreover, it is assumed $\beta = 0$.

Proof. We have to check that the function defined on $(\mathbb{R}^d)^4$ by

$$(x, y, s, t) \mapsto \frac{\tilde{a}^2(x)\tilde{a}^2(y)|\tilde{g}(t)||\tilde{g}(s)|}{\prod_{k=1}^d (1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)}$$

is integrable on $(\mathbb{R}^d)^4$. From the hypothesis of Lemma 3, it is enough to prove that for all $k = 1, \dots, d$

$$(t_k, s_k, x_k, y_k) \mapsto \frac{|x_k|^{2\alpha/d}|y_k|^{2\alpha/d}|t_k|^{2\beta/d}|s_k|^{2\beta/d}}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)}$$

is integrable on \mathbb{R}^4 . Such integrals are studied in Lemma 1 of Terrin and Taqqu (1990): Under the assumptions of Lemma 3 on α and β , this integral is finite. \square

When the functions a and g involved in Theorem 2 do not satisfy (8), condition **(H)** may be investigated thanks to power counting theorems (cf. Theorem 2 in Terrin and Taqqu (1990)). The following lemma focus on a particular situation in dimension $d = 2$: When \tilde{a} admits one or two independent lines of singularities.

Lemma 4. *Assume that $d = 2$ and that \tilde{g} follows the same conditions as in (8).*

If, for $p \neq q$,

$$\tilde{a}(x_1, x_2) = |x_1 + px_2|^{\alpha_p}|x_1 + qx_2|^{\alpha_q},$$

*and if $\alpha_p > -1/2$, $\alpha_q > -1/2$, $\beta > -1$ and $\alpha_p + \alpha_q + \beta < -1/2$, then **(H)** is fulfilled.*

Proof. Checking **(H)** is equivalent to prove that

$$\int_{\mathbb{R}^8} \frac{|x_1 + px_2|^{2\alpha_p} |x_1 + qx_2|^{2\alpha_q} |y_1 + py_2|^{2\alpha_p} |y_1 + qy_2|^{2\alpha_q} |t_1|^\beta |t_2|^\beta |s_1|^\beta |s_2|^\beta}{\prod_{k=1}^2 (1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)} dt ds dx dy \quad (9)$$

is finite. We apply Theorem 2 of Terrin and Taqqu (1990). Let us introduce some notations. The integral above can be written :

$$\int_{\mathbb{R}^8} |L_1(u)|^{2\alpha_p} |L_2(u)|^{2\alpha_q} |L_3(u)|^{2\alpha_p} |L_4(u)|^{2\alpha_q} \times \\ |L_5(u)|^\beta |L_6(u)|^\beta |L_7(u)|^\beta |L_8(u)|^\beta \prod_{k=9}^{16} (1 + |L_k(u)|)^{-1} du, \quad (10)$$

where $u = (x_1, x_2, y_1, y_2, t_1, t_2, s_1, s_2)$ and the L_k 's are the linear functionals involved in (9). For instance $L_1(u) = x_1 + px_2$, $L_5(u) = t_1$, $L_9(u) = x_1 + t_1$.

Let $T = \{L_1, \dots, L_{16}\}$ and let γ_k be the exponent associated to L_k in (10). For instance, $\gamma_1 = 2\alpha_p$, $\gamma_5 = \beta$, $\gamma_9 = -1$. Consider now the subsets $W \subset T$ such that $\text{span}(W) \cap T = W$. A subset W is said padded if any $L_k \in W$ is a linear combination of the L_i 's in $W - \{L_k\}$.

The integrability of (9) near 0 is obvious since $2\alpha_p > -1$, $2\alpha_q > -1$ and $\beta > -1$. According to Theorem 2 in Terrin and Taqqu (1990), the integrability at infinity is achieved if for every padded W considered above but T , $d_\infty(W) := \text{rank}(T) - \text{rank}(W) + \sum_{T-W} \gamma_k < 0$.

The maximum value for $d_\infty(W)$ is obtained with $W = \{L_9, \dots, L_{16}\}$. In this case $d_\infty(W) = 8 - 6 + 4\alpha_p + 4\alpha_q + 4\beta$. This leads to $\alpha_p + \alpha_q + \beta < -1/2$. \square

2.4 Proof of Theorem 2

Proof. Let Z be a Gaussian random spectral measure associated to the measure μ . Major (1981) defines

$$\iint_{\mathbb{R}^{2d}} f(x, y) dZ(x) dZ(y)$$

for all $f \in H_\mu$ where H_μ denotes the space of functions $f: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that $f(-x, -y) = \overline{f(x, y)}$ and $\int |f(x, y)|^2 d\mu(x) d\mu(y) < \infty$. The second order moment of this integral satisfies, for all f in H_μ ,

$$E \left(\iint_{\mathbb{R}^{2d}} f(x, y) dZ(x) dZ(y) \right)^2 \leq 2 \int_{\mathbb{R}^{2d}} |f(x, y)|^2 d\mu(x) d\mu(y). \quad (11)$$

And the so-called Ito formula follows: For all f_1 and f_2 in H_μ ,

$$\iint_{\mathbb{R}^{2d}} f_1(x) f_2(y) dZ(x) dZ(y) = \int f_1(x) dZ(x) \int f_2(y) dZ(y) - \int f_1(x) \overline{f_2(x)} d\mu(x). \quad (12)$$

Now, let us rewrite Q_n as a double stochastic integral. According to (1),

$$\begin{aligned} Q_n - E(Q_n) &= \frac{1}{n^d} \sum_{k \in A_n} \sum_{l \in A_n} g_{k-l} (X_k X_l - r(l-k)) \\ &= \int_E g(t) \left[\frac{1}{n^d} \sum_{k \in A_n} \sum_{l \in A_n} e^{i\langle k-l, t \rangle} (X_k X_l - r(l-k)) \right] dt. \end{aligned} \quad (13)$$

From the Ito formula (12)

$$\begin{aligned} &X_k X_l - r(l-k) \\ &= \int_E a(x) e^{i\langle k, x \rangle} dW(x) \int_E a(y) e^{i\langle l, y \rangle} dW(y) - \int_E e^{i\langle l-k, x \rangle} a^2(x) d\mu(x) \\ &= \iint_{E^2} a(x) a(y) e^{i(\langle k, x \rangle + \langle l, y \rangle)} dW(x) dW(y), \end{aligned}$$

where μ , the spectral measure of ϵ , is proportional to the Lebesgue measure λ .

Assume that $g(-x) = \overline{g(x)}$ and $a(-x) = \overline{a(x)}$, then the function

$$(x, y) \mapsto a(x) a(y) \left[\frac{1}{n^d} \int_E g(t) \sum_{k \in A_n} e^{i\langle k, x+t \rangle} \sum_{l \in A_n} e^{i\langle l, y-t \rangle} dt \right] \quad (14)$$

belongs to H_λ . Therefore we can rewrite (13) as

$$Q_n - E(Q_n) = \iint_{E^2} a(x) a(y) \left[\frac{1}{n^d} \int_E g(t) H_n(x+t) H_n(y-t) dt \right] dW(x) dW(y), \quad (15)$$

where $H_n(t) = \sum_{k \in A_n} e^{i\langle k, t \rangle}$.

In (15), we make the change of variables $x \rightarrow x/n$, $y \rightarrow y/n$, $t \rightarrow t/n$. Since the Gaussian measure W satisfies, for all Borelian set A , $W(n^{-1}A) = n^{-1/2}W(A)$ and since \tilde{a} and \tilde{g} are homogeneous, we get

$$\begin{aligned} n^{d+2\alpha+2\beta}(Q_n - E(Q_n)) &= \\ &\iint_{nE^2} \tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) L_1\left(\frac{x}{n}\right) L_1\left(\frac{y}{n}\right) \psi_n(x, y) dW(x) dW(y), \end{aligned}$$

where

$$\Psi_n(x, y) = \int_{nE} \tilde{g}_{(n)}(t) L_2\left(\frac{t}{n}\right) \frac{1}{n^d} H_n\left(\frac{x+t}{n}\right) \frac{1}{n^d} H_n\left(\frac{y-t}{n}\right) dt.$$

As a consequence, according to (11), it suffices, for proving (7), to show that the following integral tends to zero as $n \rightarrow \infty$:

$$\iint_{nE^2} \left[\tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) L_1\left(\frac{x}{n}\right) L_1\left(\frac{y}{n}\right) \Psi_n(x, y) - \tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) L_1^2(0) \Psi(x, y) \right]^2 dx dy$$

where

$$\Psi(x, y) = L_2(0) \int_{\mathbb{R}^d} \tilde{g}(t) H(x+t) H(y-t) dt.$$

From the decomposition $A_n B - CD = (A_n - C)B + (B - D)C$, the L^2 -norm above is lower than the sum $2(I_1 + I_2)$ where

$$I_1 = \iint_{nE^2} \left[\tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) \Psi_n(x, y) - \tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) \Psi(x, y) \right]^2 \times \left[L_1\left(\frac{x}{n}\right) L_1\left(\frac{y}{n}\right) \right]^2 dx dy$$

and

$$I_2 = \iint_{nE^2} \left[L_1\left(\frac{x}{n}\right) L_1\left(\frac{y}{n}\right) - L_1^2(0) \right]^2 \left[\tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) \Psi(x, y) \right]^2 dx dy.$$

The following lemma will be useful.

Lemma 5.

$$(i) \quad \forall z = (z_1, \dots, z_d) \in nE, \quad \left| \frac{1}{n^d} H_n\left(\frac{z}{n}\right) \right| \leq \prod_{j=1}^d \pi \left(1 \wedge \frac{1}{|z_j|} \right).$$

$$(ii) \quad \forall z = (z_1, \dots, z_d) \in \mathbb{R}^d, \quad |H(z)| \leq \prod_{j=1}^d 2 \left(1 \wedge \frac{1}{|z_j|} \right).$$

$$(iii) \quad \text{For a.e. } z \in \mathbb{R}^d, \quad \lim_{n \rightarrow \infty} \left| \mathbb{1}_{nE}(z) \frac{1}{n^d} H_n\left(\frac{z}{n}\right) - H(z) \right| = 0.$$

Proof of Lemma 5. Since for all j , $|z_j| \leq n\pi$,

$$\begin{aligned} \left| \frac{1}{n^d} H_n\left(\frac{z}{n}\right) \right| &= \prod_{j=1}^d \left| \frac{1}{n} e^{i \frac{z_j}{n}} \frac{e^{iz_j} - 1}{e^{i \frac{z_j}{n}} - 1} \right| = \prod_{j=1}^d \left| \frac{\sin(z_j/2)}{z_j/2} \right| \left| \frac{z_j/2n}{\sin(z_j/2n)} \right| \\ &\leq \prod_{j=1}^d \pi \left(1 \wedge \frac{1}{|z_j|} \right). \end{aligned}$$

Similarly, for all $z \in \mathbb{R}^d$,

$$|H(z)| = \prod_{j=1}^d \left| \frac{e^{iz_j} - 1}{iz_j} \right| = \prod_{j=1}^d \left| \frac{\sin(z_j/2)}{z_j/2} \right| \leq \prod_{j=1}^d 2 \left(1 \wedge \frac{1}{|z_j|} \right).$$

Finally, for proving (iii), suppose first that $d = 1$. If $z \neq 0$,

$$\left| \frac{1}{n^d} H_n\left(\frac{z}{n}\right) - H(z) \right| = \left| \frac{1}{n} e^{i \frac{z}{n}} \frac{e^{iz} - 1}{e^{i \frac{z}{n}} - 1} - \frac{e^{iz} - 1}{iz} \right| \leq \frac{2}{|z|} \left| \frac{e^{i \frac{z}{n}} iz}{n(e^{i \frac{z}{n}} - 1)} - 1 \right|.$$

The norm in the right hand side term is equivalent to $|z|/n$ when n goes to infinity, hence it tends to 0. In dimension d , (iii) is proved by induction thanks to the decomposition $AB - CD = (A - C)B + (B - D)C$. \square

Let us first prove that I_1 asymptotically vanishes. Since L_1 is bounded, it suffices to prove the convergence to 0 of

$$I_{11} = \iint_{nE^2} \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) \left[\int_{nE} \Phi_n^{(11)}(t, x, y) dt \right]^2 dx dy, \quad (16)$$

with

$$\Phi_n^{(11)}(t, x, y) = \tilde{g}_{(n)}(t) \left(L_2 \left(\frac{t}{n} \right) - L_2(0) \right) \frac{1}{n^d} H_n \left(\frac{x+t}{n} \right) \frac{1}{n^d} H_n \left(\frac{y-t}{n} \right)$$

and

$$I_{12} = \iint_{nE^2} \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) L_2(0)^2 \left[\int_{nE} \Phi_n^{(12)}(t, x, y) dt \right]^2 dx dy, \quad (17)$$

with

$$\Phi_n^{(12)}(t, x, y) = \tilde{g}_{(n)}(t) \left[\frac{1}{n^d} H_n \left(\frac{x+t}{n} \right) \frac{1}{n^d} H_n \left(\frac{y-t}{n} \right) - H(x+t)H(y-t) \right]$$

In both I_{11} and I_{12} , the 2π -periodicity of g , H_n and H allows us to reduce the domain of integration nE (with respect to t) to

$$nD_{x,y} = \{|x-t| < n\pi\} \cap \{|y+t| < n\pi\} \cap nE.$$

Therefore, (i) of Lemma 5 can be applied and since L_2 is bounded,

$$\left| \int_{nE} \Phi_n^{(11)}(t, x, y) dt \right| \leq c \int_{nD_{x,y}} |\tilde{g}_{(n)}(t)| \prod_{k=1}^d \left(1 \wedge \frac{1}{|x_k + t_k|} \right) \left(1 \wedge \frac{1}{|y_k - t_k|} \right) dt,$$

where c is a positive constant. Hence,

$$I_{11} \leq c' \iint_{\mathbb{R}^{2d}} \tilde{a}^2(x) \tilde{a}^2(y) \left[\int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)} dt \right]^2 dx dy.$$

Besides, according to (i) of Lemma 5,

$$\begin{aligned} & \left| \mathbb{1}_{nE^2}(x, y) \mathbb{1}_{nD_{x,y}}(t_1) \mathbb{1}_{nD_{x,y}}(t_2) \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) \Phi_n^{(11)}(t_1, x, y) \Phi_n^{(11)}(t_2, x, y) \right| \\ & \leq c \tilde{a}^2(x) \tilde{a}^2(y) |\tilde{g}(t_1)| |\tilde{g}(t_2)| \left| L_2 \left(\frac{t_1}{n} \right) - L_2(0) \right| \left| L_2 \left(\frac{t_2}{n} \right) - L_2(0) \right|. \end{aligned}$$

From the continuity of L_2 at 0, this term tends to zero for any fixed $(x, y, t_1, t_2) \in \mathbb{R}^{4d}$. Therefore, thanks to assumption **(H)**, the Lebesgue's dominated convergence theorem applies and $\lim_{n \rightarrow \infty} I_{11} = 0$.

The convergence of I_{12} is proved similarly. From (i) and (ii) of Lemma 5, we have

$$\left| \int_{nE} \Phi_n^{(12)}(t, x, y) dt \right| \leq c \int_{nD_{x,y}} |\tilde{g}_{(n)}(t)| \prod_{j=1}^d \left(1 \wedge \frac{1}{|x_k + t_k|} \right) \left(1 \wedge \frac{1}{|y_k - t_k|} \right) dt$$

and

$$I_{12} \leq c' \iint_{\mathbb{R}^{2d}} \tilde{a}^2(x) \tilde{a}^2(y) \left[\int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)} dt \right]^2 dx dy,$$

where c and c' are positive constants.

Besides, for almost every $(x, y, t_1, t_2) \in \mathbb{R}^{4d}$, according to (iii) of Lemma 5,

$$\mathbb{I}_{nE^2}(x, y) \mathbb{I}_{nD_{x,y}}(t_1) \mathbb{I}_{nD_{x,y}}(t_2) \tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) \Phi_n^{(12)}(t_1, x, y) \Phi_n^{(12)}(t_2, x, y) \xrightarrow{n \rightarrow \infty} 0.$$

The Lebesgue's dominated convergence theorem applies thanks to **(H)** and I_{12} tends to zero when $n \rightarrow \infty$.

It is easy to see that I_2 tends similarly to zero. \square

3 Convergence of the empirical auto-covariance function

We present an application of the preceding theorems to the asymptotic law of the empirical covariance function in \mathbb{Z}^d . Indeed, for $h \in \mathbb{Z}^d$,

$$\hat{r}(h) = \frac{1}{n^d} \sum_{i \in A_n} X_i X_{i+h}$$

is a particular case of (3) with $g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle}$. We will consider further

$$\hat{\hat{r}}(h) = \frac{1}{n^d} \sum_{i \in A_n} (X_i - \bar{X}_n) (X_{i+h} - \bar{X}_n)$$

where $\bar{X}_n = n^{-d} \sum_{i \in A_n} X_i$ and we denote in the following $r(h) = E(X_i X_{i+h})$.

We are dealing with linear fields as (1). In dimension 1, this framework provides a dichotomy in the asymptotic behavior of $\hat{r}(h)$ depending on whether the spectral density of X belongs to L^2 or not (see Hosking (1996)). We prove later on the same kind of division in dimension d , although some kind of intermediate behavior can arise. Let us note that this dichotomy does no longer hold in $d = 1$ when X is not linear, see for instance Giraitis et al. (2000) in the case of a LARCH process or Giraitis and Taqqu (1999) for the consequences on the Whittle estimator.

When $f \in L^2$, we prove that $\hat{r}(h)$ and $\hat{\hat{r}}(h)$ follow the same central limit theorem. This is the object of Proposition 6.

When $f \notin L^2$, the asymptotic behavior of $\hat{r}(h)$ and $\hat{\hat{r}}(h)$ may differ and we focus on $\hat{r}(h)$ which is a proper quadratic form. In this setting, the asymptotic law comes from Theorem 2 where $\beta = 0$, provided condition **(H)** is satisfied. Therefore, the normalization and the limit in law depend on the filter a . We summarize in Proposition 7 and in the beginning of section 3.2 the two situations already studied in Lemma 3 and Lemma 4 before.

But Theorem 2 does not apply in the example of Lemma 4 when $\alpha_p \alpha_q = 0$ and $\beta = 0$. This corresponds to the particular situation when the long memory occurs only along one direction in dimension 2. In this case, we study the asymptotic behavior of $\hat{r}(h)$ in Proposition 9. It appears that a non-central limit theorem holds in the sense that the normalization is not $n^{d/2}$. Yet, contrary to the classical non central limit results for $\hat{r}(h)$ (see references therein), the limiting law is Gaussian. This shows that the covariance series of a linear field may be asymptotically gaussian even if its spectral density does not belong to L^2 .

3.1 General results in dimension d

Let us first present the central limit theorem for $\hat{r}(h)$ and $\hat{\hat{r}}(h)$ when $f \in L^2$.

Proposition 6. *Let X be the linear field defined by (1).*

If $f \in L^2$, then, for all $h \in \mathbb{Z}^d$, $n^{d/2}(\hat{r}(h) - r(h))$ and $n^{d/2}(\hat{\hat{r}}(h) - r(h))$ converge both in law to a Gaussian random variable with zero mean and variance $2(2\pi)^d \widehat{f_{2h}^2}$, where $\widehat{f_{2h}^2}$ stands for the $(2h)$ -th Fourier coefficients of f^2 as defined in (2).

Proof. The central limit theorem for $r(h)$ follows from Theorem 1 where $p = 2$, $q = +\infty$ and $g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle}$.

For $\hat{\hat{r}}(h)$, we prove that almost surely, $n^{d/2}(\hat{r}(h) - \hat{\hat{r}}(h)) = o(1)$. Indeed,

$$n^{d/2}(\hat{r}(h) - \hat{\hat{r}}(h)) = \frac{1}{n^{3d/2}} \sum_{k_1 \in A_n} \sum_{k_2 \in A_n} X_{k_1} X_{k_2+h}.$$

Using representation (1) of X , this term is equal to

$$\frac{1}{n^{3d/2}} \int_E f(x) e^{i\langle h, x \rangle} \left| \sum_{k \in A_n} e^{i\langle k, x \rangle} \right|^2 dx = \frac{1}{n^{d/2}} \int_E f(x) e^{i\langle h, x \rangle} \prod_{j=1}^d F_n(x_j) dx$$

where F_n denotes the Fejer kernel on $[-\pi, \pi]$. Now, from the Cauchy-Schwarz inequality, this last term is lower than

$$\frac{1}{n^{d/2}} \sqrt{\int_E f^2(x) \prod_{j=1}^d F_n(x_j) dx}.$$

Since $f^2 \in L^1$, the Lebesgue's Theorem implies that the integral above is a $o(n^d)$. Therefore, $n^{d/2}(\hat{r}(h) - \hat{\hat{r}}(h)) = o(1)$ and the central limit theorem for $\hat{\hat{r}}(h)$ is inherited from the one for $\hat{r}(h)$. □

Let us recall that the notation $\tilde{a}_{(1)}$ is defined at the beginning of Section 2.2. The following proposition provides central and non central limit for $\hat{r}(h)$, depending on the parameter α , when condition (18) below is fulfilled. This framework includes isotropic and anisotropic models. The result shows the same kind of dichotomy than in dimension $d = 1$. In section 4.2, some anisotropic models that do not follow this dichotomy are presented.

Proposition 7. *Let X be the linear field defined by (1).*

Assume that for all $x \in E$ $a(-x) = \overline{a(x)}$ and that $a(x) = \tilde{a}_{(1)}(x)L_1(x)$, where \tilde{a} is a homogeneous function of degree $\alpha > -d/2$ and L_1 is a bounded function, continuous at zero and non-null at 0. If

$$|\tilde{a}(x)| \leq c \prod_{i=1}^d |x_i|^{\alpha/d} \quad (18)$$

for some positive constant c , then for all $h \in \mathbb{Z}^d$,

- if $\alpha > -d/4$,

$$n^{d/2}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^d \widehat{f_{2h}^2}\right), \quad (19)$$

where $\widehat{f_{2h}^2}$ is the $(2h)$ -th Fourier coefficients of f^2 as defined in (2).

- if $\alpha < -d/4$,

$$n^{d+2\alpha}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} L_1^2(0) \iint_{\mathbb{R}^{2d}} \tilde{a}(x)\tilde{a}(y)H(x+y)dW(x)dW(y), \quad (20)$$

where $H(z) = \prod_{j=1}^d \frac{e^{iz_j} - 1}{iz_j}$.

Proof. In the case $\alpha > -d/4$, the convergence result is a consequence of Proposition 6.

For $\alpha < -d/4$, Lemma 3 applies since $\beta = 0$ and condition **(H)** in Theorem 2 is fulfilled. Let us justify the simplification of the limit in (20). Here

$$g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle},$$

so $\tilde{g}(t) = 1$ and $L_2(0) = (2\pi)^{-d}$. The simplification comes from the main term I_{12} in the proof of theorem 2, where we use the identity

$$\int_{nE} \frac{1}{n^d} H_n\left(\frac{x+t}{n}\right) \frac{1}{n^d} H_n\left(\frac{y-t}{n}\right) dt = (2\pi)^d \frac{1}{n^d} H_n\left(\frac{x+y}{n}\right).$$

The pointwise convergence of this last term relies on (iii) of Lemma 5 and an application of the Lebesgue's theorem concludes the proof of (20). \square

3.2 Some anisotropic examples in dimension $d = 2$

Starting with the anisotropic case studied in Lemma 4, we confirm that new limiting results can be obtained as suggested by Dobrushin and Major (1979) (Remark 4.2). If we suppose that $a(x) = \tilde{a}_{(1)}(x)L(x)$, where L is a bounded function, continuous and non-null at 0 and

$$\tilde{a}(x_1, x_2) = |x_1 + px_2|^{\alpha_p} |x_1 + qx_2|^{\alpha_q},$$

with $\alpha_p > -\frac{1}{2}$, $\alpha_q > -\frac{1}{2}$ and $(p, q) \in \mathbb{R}^2$, then, under the assumptions of Lemma 4

$$n^{2+2\alpha_p+2\alpha_q}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} Z$$

where Z is defined as the limit in (20).

This convergence is a simple application of Theorem 2. Condition **(H)** is fulfilled thanks to Lemma 4 and the simplification of the limit holds for the same reasons as for Proposition 7.

The assumptions of Lemma 4, in the case when $\beta = 0$, imply the existence of two lines where the spectral density is unbounded. When $\alpha_p = 0$ or $\alpha_q = 0$ (or $p = q$), that is when the long memory occurs along only one direction, Lemma 4 does not imply **(H)** when $\beta = 0$. This case provides a new limiting behavior for $\hat{r}(h)$ as stated in Proposition 9.

Before stating this result, we deduce in the following lemma the covariance structure of X when the spectral density is unbounded along a line that goes through the origin. We say that the process has a long memory along one direction if its covariance function is not summable along this direction.

Lemma 8. *Let $p \in \mathbb{R}$ and for all $(x_1, x_2) \in \mathbb{R}^2$,*

$$f(x_1, x_2) = \frac{1}{2\pi} \tilde{f}(x_1 + px_2),$$

where \tilde{f} is an even, non-negative, and 2π -periodic function on \mathbb{R} . Let us denote, for all $(h_1, h_2) \in \mathbb{Z}^2$,

$$\sigma(h_1, h_2) = \int_{[-\pi, \pi]^2} e^{i(h_1 x_1 + h_2 x_2)} f(x_1, x_2) dx_1 dx_2$$

and

$$\tilde{\sigma}(h_1) = \int_{-\pi}^{\pi} e^{ih_1 x} \tilde{f}(x) dx.$$

Then, for all $(h_1, h_2) \in \mathbb{Z}^2$, we have

$$\sigma(h_1, h_2) = \begin{cases} \tilde{\sigma}(h_1) & \text{if } h_2 = ph_1, \\ \frac{\sin((h_2 - ph_1)\pi)}{(h_2 - ph_1)\pi} \tilde{\sigma}(h_1) & \text{otherwise.} \end{cases}$$

Proof. The proof is given at the end of this section. □

Proposition 9. *Let X be a stationary Gaussian process in dimension $d = 2$. Let us suppose that its spectral density is*

$$f(x_1, x_2) = \frac{1}{2\pi} \tilde{f}(x_1 + px_2),$$

where $p \in \mathbb{Z}$ and where \tilde{f} , defined on $[-\pi, \pi]$, is a spectral density in dimension $d = 1$. Assume moreover that for $-1/2 < \alpha < 0$ and for all $x \in [-\pi, \pi]$, $\tilde{f}(x) = L(x)|x|^{2\alpha}$ where L is a bounded function, continuous at zero and non-null at 0. Then,

- if $\alpha > -1/4$,

$$n(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^2 \widehat{f_{2h}^2}\right), \quad (21)$$

where $\widehat{f_{2h}^2}$ is the $(2h)$ -th Fourier coefficients of f^2 as defined in (2).

- if $\alpha < -1/4$ and if $h_2 \neq ph_1$,

$$n^{(4\alpha+3)/2}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_{\alpha,p}^2\right), \quad (22)$$

where $\sigma_{\alpha,p}^2 = \lim_{n \rightarrow \infty} n^{4\alpha+3} \text{Var}(\hat{r}(h))$.

Proof. In the case $\alpha > -1/4$, the result is a consequence of Proposition 6. Let us focus on $\alpha < -1/4$. We restrict the proof to the case $p \geq 0$ since $p \leq 0$ can be treated in the same way.

We prove the result thanks to a central limit theorem for triangular arrays stated in Romano and Wolf (2000).

An alternative proof could be to start from the representation of $\hat{r}(h)$ in terms of a double stochastic integral as in (15), then to use the necessary and sufficient conditions for the convergence of such an integral given in Nualart and Peccati (2005) and Nualart and Ortiz-Latorre (2008). But checking these conditions leads to hard computations, which would not simplify or shorten the present proof.

So, let us take advantage of the natural triangular array structure that arises in our setting. Indeed, let $Y_{i_1, i_2} = X_{i_1, pi_1 + i_2}$ and let $i = i_1$, $j = i_2 - pi_1$, we have

$$\hat{r}(h_1, h_2) = \frac{1}{n^2} \sum_{i_1=1}^n \sum_{i_2=1}^n X_{i_1, i_2} X_{i_1+h_1, i_2+h_2} = \frac{1}{n^2} \sum_{j=-pn+1}^{n-p} \sum_{i \in B_j} Y_{i,j} Y_{i+h_1, j-ph_1+h_2},$$

where $B_j = \{i \mid 1 \wedge \frac{1-j}{p} \leq i \leq \frac{n-j}{p} \vee n\}$. Therefore $\hat{r}(h_1, h_2)$ is the triangular array

$$\hat{r}(h_1, h_2) = \sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j}$$

where

$$\tilde{Y}_{n,j} = n^{-2} \sum_{i \in B_{j-pn}} Y_{i, j-pn} Y_{i+h_1, j-pn-ph_1+h_2}.$$

Let us summarize the properties of $\tilde{Y}_{n,j}$.
From Lemma 8, we have

$$r(h_1, h_2) = \begin{cases} \tilde{r}(h_1) & \text{if } h_2 = ph_1, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

where \tilde{r} is the covariance function associated with \tilde{f} . Consequently $(Y_{i,j})$ is a zero mean Gaussian process such that

$$E(Y_{i,j}Y_{i+h_1,j+h_2}) = \begin{cases} \tilde{r}(h_1) & \text{if } h_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The process $\tilde{Y}_{n,j}$, viewed as a function of $((Y_{i,j-pn})_{i \in \mathbb{Z}}, (Y_{i+h,j-pn-ph_1+h_2})_{i \in \mathbb{Z}})$, is thus a $(h_2 - ph_1)$ -dependent process. Moreover, we can compute the moments of $\tilde{Y}_{n,j}$ thanks to the representation of the moments of Gaussian variables in terms of Wick's product. Since we have assumed $h_2 \neq ph_1$, we obtain

$$E(\tilde{Y}_{n,j}) = 0, \\ E(\tilde{Y}_{n,j_1}\tilde{Y}_{n,j_2}) = \begin{cases} n^{-4} \sum_{i_1, i_2 \in B_{j-pn}} \tilde{r}^2(i_2 - i_1) & \text{if } j_1 = j_2 = j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$E(\tilde{Y}_{n,j}^4) = 3n^{-8} \left(\sum_{i_1, i_2 \in B_{j-pn}} \tilde{r}^2(i_2 - i_1) \right)^2 \\ + 6n^{-8} \sum_{i_1, i_2, i_3, i_4 \in B_{j-pn}} \tilde{r}(i_2 - i_1)\tilde{r}(i_3 - i_2)\tilde{r}(i_4 - i_3)\tilde{r}(i_4 - i_1).$$

We are now in position to apply Theorem 2.1 in Romano and Wolf (2000) which gives sufficient condition for the convergence in law of a triangular array of m -dependent random variables to a normal distribution. Following the same notations as in this theorem, we choose $\delta = 2$ and $\gamma = 0$ and we look for Δ_n , K_n and L_n such that

- $E(\tilde{Y}_{n,j}^4) \leq \Delta_n$ for all j ,
- $Var\left(\sum_{j=a}^{a+k-1} \tilde{Y}_{n,j}\right) \leq k K_n$ for all a and for all $k \geq (h_2 - ph_1)$,
- $Var\left(\sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j}\right) \geq ((p+1)n - p) L_n$.

According to Romano and Wolf (2000), the convergence in law holds whenever $K_n/L_n = O(1)$ and $\Delta_n/L_n^2 = O(1)$.

From Theorem 2.24 of Zygmund (1959), we have $\tilde{r}(h) \sim c_\alpha h^{-2\alpha-1}$ when $h \rightarrow \infty$ where c_α is a constant depending on α and $L(0)$.

Since $(i \in B_j) \Rightarrow (1 \leq i \leq n)$, an integral test leads to $\Delta_n = O(n^{-8\alpha-8})$ and $K_n = O(n^{-4\alpha-4})$. For L_n , let us compute directly the equivalent of the variance. From the decomposition in terms of Wick's product, we obtain

$$\begin{aligned} \text{Var} \left(\sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) &= \text{Var} \left(n^{-2} \sum_{i=1}^n \sum_{j=1}^n X_{i,j} X_{i+h_1, j+h_2} \right) \\ &= n^{-4} \sum_{i_1, i_2, j_1, j_2=1}^n r(i_2-i_1, j_2-j_1) + r(i_2+h_1-i_1, j_2+h_2-j_1) r(i_2-i_1-h_1, j_2-j_1-h_2). \end{aligned}$$

In view of (23) and since $h_2 \neq ph_1$, most of the terms in the sum above vanish. We finally obtain

$$\text{Var} \left(\sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) = n^{-4} \sum_{(i_1, i_2, j_1) \in C} \tilde{r}^2(i_2 - i_1),$$

where $C = \{1 \leq i_1, i_2, j_1 \leq n, 1 \leq j_1 + p(i_2 - i_1) \leq n\}$. Let $k = i_2 - i_1$, we have

$$\begin{aligned} C &= \{1 \leq j \leq n, |k| \leq n-1, 1-pk \leq j \leq n-pk\} \\ &= \left\{ 0 \leq k \leq \left\lfloor \frac{n-1}{p} \right\rfloor, 1 \leq j \leq n-pk \right\} \cup \left\{ -\left\lfloor \frac{n-1}{p} \right\rfloor \leq k \leq -1, 1-pk \leq j \leq n \right\}. \end{aligned}$$

So,

$$\text{Var} \left(\sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) = n^{-3} \tilde{r}^2(0) + 2n^{-4} \sum_{k=1}^{\left\lfloor \frac{n-1}{p} \right\rfloor} \tilde{r}^2(k)(n-pk).$$

The latest sum involves positive terms which are equivalent to $c_\alpha^2 k^{-4\alpha-2}(n-pk)$ when $k \rightarrow \infty$. As a consequence, $\text{Var} \left(\sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) \sim \sigma_{\alpha,p}^2 n^{-4\alpha-3}$, where $\sigma_{\alpha,p}^2$ is a positive constant. This leads in particular to $L_n = O(n^{-4\alpha-4})$.

The conditions in Theorem 2.1 in Romano and Wolf (2000) are fulfilled and the convergence in law holds. \square

Proof of Lemma 8. When $p = 0$, the result is obvious. Let us assume, without loss of generality, that $p \geq 1$.

$$\sigma(h_1, h_2) = \frac{1}{2\pi} \int_{[-\pi, \pi]^2} e^{ih_1(x_1+px_2)} \tilde{f}(x_1+px_2) e^{i(h_2-ph_1)x_2} dx_1 dx_2.$$

Let the change of variables $u = x_1 + px_2$ and $v = x_2$:

$$\sigma(h_1, h_2) = \frac{1}{2\pi} \int_{-(p+1)\pi}^{(p+1)\pi} e^{ih_1 u} \tilde{f}(u) \left(\int_{-\pi \vee \frac{u-\pi}{p}}^{\pi \wedge \frac{u+\pi}{p}} e^{i(h_2-ph_1)v} dv \right) du.$$

Let us first suppose that $h_2 \neq ph_1$. When $p \geq 2$, the above domain of integration can be cut up as follows (the case $1 \leq p \leq 2$ is not detailed but can be treated similarly):

$$\sigma(h_1, h_2) = \frac{1}{2\pi(h_2 - ph_1)}(I_1 + I_2),$$

where

$$\begin{aligned} I_1 = & -i \int_{-(p+1)\pi}^{-(p-1)\pi} e^{ih_1 u} \tilde{f}(u) \left(e^{i(h_2 - ph_1)\frac{u+\pi}{p}} - e^{-i(h_2 - ph_1)\pi} \right) du \\ & + \int_{(p-1)\pi}^{(p+1)\pi} e^{ih_1 u} \tilde{f}(u) \left(e^{i(h_2 - ph_1)\pi} - e^{i(h_2 - ph_1)\frac{u+\pi}{p}} \right) du \end{aligned}$$

and

$$I_2 = -i \int_{-(p-1)\pi}^{(p-1)\pi} e^{ih_1 u} \tilde{f}(u) \left(e^{i(h_2 - ph_1)\frac{u+\pi}{p}} - e^{i(h_2 - ph_1)\frac{u-\pi}{p}} \right) du.$$

Some trigonometric computations lead to

$$I_1 = 2 \int_{(p-1)\pi}^{(p+1)\pi} \tilde{f}(u) \left(\sin(h_1 u + (h_2 - ph_1)\pi) - \sin\left(\frac{h_2 u}{p} - (h_2 - ph_1)\frac{\pi}{p}\right) \right) du,$$

$$I_2 = 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \int_0^{(p-1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du.$$

Now, let $s = u - [p]$ and $e = p - [p]$,

$$\begin{aligned} I_1 = & 2 \int_{-\pi+e\pi}^{\pi+e\pi} \tilde{f}(s + [p]\pi) \left(\sin(h_1 s + h_2 [p]\pi + (h_2 - ph_1)\pi) \right. \\ & \left. - \sin\left(\frac{h_2 s}{p} + h_2 \frac{[p]\pi}{p} - (h_2 - ph_1)\frac{\pi}{p}\right) \right) ds, \end{aligned}$$

$$\begin{aligned} I_2 = & 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \left(\int_0^{([p]-1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du + \right. \\ & \left. + \int_{-\pi}^{-\pi+e\pi} \tilde{f}(s + [p]\pi) \cos\left(\frac{h_2 s}{p} + h_2 \frac{[p]\pi}{p}\right) ds \right). \end{aligned}$$

The domain of integration in I_1 can be split into $-\pi < s < \pi$ and $s \in [-\pi, -\pi + e\pi] \cup [\pi, \pi + e\pi]$. From the 2π -periodicity of \tilde{f} , this is easy to check that, when summing up I_1 and I_2 , all the integrals involving e in their range of

integration sum up to zero. Hence $I_1 + I_2$ reduces to

$$\begin{aligned}
& 2 \int_{-\pi}^{\pi} \tilde{f}(s + \lfloor p \rfloor \pi) \left(\sin(h_1 s + h_2 \lfloor p \rfloor \pi + (h_2 - ph_1)\pi) \right. \\
& \quad \left. - \sin\left(\frac{h_2 s}{p} + h_2 \frac{\lfloor p \rfloor \pi}{p} - (h_2 - ph_1) \frac{\pi}{p}\right) \right) ds \\
& \quad + 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \int_0^{(\lfloor p \rfloor - 1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du. \quad (24)
\end{aligned}$$

The latest integral above is

$$\sum_{j=0}^{\lfloor p \rfloor - 2} \int_0^{\pi} \tilde{f}(u + j\pi) \cos\left(h_2 \frac{(u + j\pi)}{p}\right) du \quad (25)$$

and it is handled according to the parity of $\lfloor p \rfloor$ and j . Since \tilde{f} is a 2π -periodic function, $\tilde{f}(u + j\pi) = \tilde{f}(u)$ when j is even and $\tilde{f}(u + j\pi) = \tilde{f}(u - \pi)$ when j is odd.

When $\lfloor p \rfloor$ is even, the sum (25) above is then

$$\begin{aligned}
& \int_0^{\pi} \tilde{f}(u) \sum_{j=-\frac{\lfloor p \rfloor - 2}{2}}^{\frac{\lfloor p \rfloor - 2}{2}} \cos\left(h_2 \frac{(u + 2j\pi)}{p}\right) du \\
& \quad = \int_0^{\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) \frac{\sin\left(h_2 \frac{\lfloor p \rfloor \pi}{p} - h_2 \frac{\pi}{p}\right)}{\sin\left(\frac{h_2 \pi}{p}\right)} du.
\end{aligned}$$

When plugging in this latest result in (24), it simplifies and $I_1 + I_2$ becomes

$$2 \int_{-\pi}^{\pi} \tilde{f}(s) \sin(h_1 s + (h_2 - ph_1)\pi) ds = 2 \sin((h_2 - ph_1)\pi) \int_{-\pi}^{\pi} \tilde{f}(s) \cos(h_1 s) ds.$$

This proves the result of the lemma for $h_2 \neq ph_1$ in the case $\lfloor p \rfloor$ even.

When $\lfloor p \rfloor$ is odd, the sum (25) is

$$\begin{aligned}
& \int_0^{2\pi} \tilde{f}(u) \sum_{j=0}^{\frac{\lfloor p \rfloor - 3}{2}} \cos\left(h_2 \frac{(u + 2j\pi)}{p}\right) du = \\
& \quad \int_0^{2\pi} \tilde{f}(u) \sin\left(\frac{h_2 \pi \lfloor p \rfloor - 1}{p}\right) \frac{\cos\left(\frac{h_2 u}{p} + \frac{h_2 \pi \lfloor p \rfloor - 3}{p}\right)}{\sin\left(\frac{h_2 \pi}{p}\right)} du. \quad (26)
\end{aligned}$$

Now, in (24), we split the domain of the first integral into $-\pi < s < 0$ where $\tilde{f}(s + \lfloor p \rfloor \pi) = \tilde{f}(s + \pi)$ and $0 < s < \pi$ where $\tilde{f}(s + \lfloor p \rfloor \pi) = \tilde{f}(s - \pi)$. We apply respectively the change of variables $s = s + \pi$ and $s = s - \pi$. This allows to

exhibit the integral $2 \int_{-\pi}^{\pi} \tilde{f}(s) \sin(h_1 s + (h_2 - ph_1)\pi) ds$. With the help of (26), some trigonometric computations show that the remaining terms coming from this change of variables simplify with the remaining term in (24).

Therefore, when $h_2 \neq ph_1$, the result of the lemma is proved for all p .

The proof when $h_2 = ph_1$ is simpler and it can be conducted in the same way.

□

References

- Avram, F. (1988). On bilinear forms in Gaussian random variables and toeplitz matrices. *Probability Theory and Related Fields*, 79(1):37–45.
- Beran, J. (1994). *Statistics for long memory processes*. Chapman Hall.
- Beran, J. and Terrin, N. (1996). Testing for a change of the long-memory parameter. *Biometrika*, 83(3):627–638.
- Boissy, Y., Bhattacharyya, B. B., Li, X., and Richardson, G. D. (2005). Parameter estimates for fractional autoregressive spatial processes. *Annals of Statistics*, 33(6):2553–2567.
- Dobrushin, R. L. and Major, P. (1979). Non central limit theorems for non-linear functionals of gaussian fields. *Z. Warsch. verw. Geb.*, 50:27–52.
- Doukhan, P., Leon, J. R., and Soulier, P. (1996). Central and non central limit theorems for quadratic forms of a strongly dependent gaussian field. *Brazilian Journal of Prob. and. Stat.*, 10:205–223.
- Fox, R. and Taqqu, M. (1985). Non central limit theorem for quadratic forms in random variables having long-range dependence. *Annals of Probability*, 13:428–446.
- Fox, R. and Taqqu, M. (1987). Central limit theorem for quadratic forms in random variables having long-range dependence. *Probability Theory and Related Fields*, 74:213–240.
- Fox, R. and Taqqu, M. S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *The Annals of Statistics*, 14:517–532.
- Ginovian, M. S. (1999). Nonparametric estimation of the spectrum of homogeneous Gaussian Fields. *J. Contemp. Math. Anal.*, 34(2):1–15.
- Ginovian, M. S. and Sahakian, A. A. (2005). On the central limit theorem for Toeplitz quadratic forms of stationary sequences. *Theory Probability and its Applications*, 49(4):612–628.

- Giraitis, L., Robinson, P., and Surgailis, D. (2000). A model for long memory conditional heteroscedasticity. *The annals of Applied Probability*, 10(3):1002–1024.
- Giraitis, L. and Surgailis, D. (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotic normality of Whittles’s estimate. *Probab. Theory Related Fields*, 86:87–104.
- Giraitis, L. and Taqqu, M. S. (1999). Whittle estimator for finite-variance non-Gaussian time series with long memory. *Annals of Statistics*, 27(1):178–203.
- Giraitis, L., Taqqu, M. S. and Terrin, N. (1998). Limit theorems for bivariate Appell polynomials. Part II: Non-central limit theorems. *Probability Theory and Related Fields*, 110:333–367.
- Hannan, E. (1976). The asymptotic distribution of serial covariances. *Annals of Statistics*, 4(2):396–399.
- Hosking, J. R. M. (1996). Asmyptotic distributions of the sample mean, auto-covariances, and autocorrelations of long-memory time series. *J. of Econometrics*, 73:261–284.
- Katznelson, Y. (1968). *An introduction to harmonic analysis*. Dover Publications.
- Lavancier, F. (2005). Long memory random fields. in *Dependence in Probability and Statistics, Lecture Notes in Statistics, vol. 187. Bertail, P; Doukhan, P; Soulier, Ph (Eds.), Springer*.
- Lobato I. and Robinson P.M. (1996). Averaged periodogram estimation of long memory. *J. of Econometrics*, 73:303–324.
- Major, P. (1981). *Multiple Wiener-It Integrals*. Number 849 in Lecture Notes in Mathematics. Springer.
- Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Annals of Probability*, 33(1):177–193.
- Nualart, D. and Ortiz-Latorre, S. (2008). Central limit theorems ofr multiple stochastic integrals and Malliavin calculus. *Stochastic Processes and their Applications*, 118(4):614–628.
- Romano, J. and Wolf, M. (2000). A more general central limit theorem for m-dependent random variables with unbounded m. *Statistics and Probability Letters*, 47:115–124.
- Rosenblatt, M. (1961). Weak convergence to the rosenblatt process. In *Proceedings of the fifth Berkeley Symposium*, pages 67–74.
- Terrin, N. and Taqqu, M. (1990). A noncentral limit theorem for quadratic forms of Gaussian stationary sequences. *J. Theor. Probab.*, 3:449–475.
- Zygmund, A. (1959). *Trigonometric series*. Cambridge University Press.