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# Convergence of quadratic forms for random fields and its application to the convergence of empirical covariances

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## Abstract

Limit theorems are proved for quadratic forms of Gaussian random fields in presence of long memory. Similarly to the one dimensional case, we prove that the quadratic forms, appropriately normalized, may have Gaussian or non-Gaussian limits. However the dichotomy observed in  $d = 1$  cannot be stated so easily, due to the possible occurrence of anisotropic strong dependence in  $d > 1$ . We apply our theorems to obtain the asymptotic behavior of the empirical covariances, which is a particular example of quadratic forms.

**Keywords:** Gaussian Random field ; Long memory ; Central and non central limit theorems.

## 1 Introduction

Many statistical methods proposed for long memory processes are based on statistics, which can be expressed using quadratic forms (see Beran, 1994, for a review). The empirical covariance sequence (see Hosking, 1996; Hannan, 1976), the estimation of the long memory parameter using Whittle contrast (see Fox and Taqqu, 1986; Giraitis and Surgailis, 1990; Giraitis and Taqqu, 1999, for example) or using the integrated periodogram (see Lobato I. and Robinson P.M., 1996), some change point detection procedure (see Beran and Terrin, 1996) rely on quadratic forms.

It would be interesting to validate the same statistical tools for the study of random fields having long memory. As a first step, we study in this paper the convergence of the quadratic forms in the  $d$ -dimensional ( $d > 1$ ) case.

Let us introduce our framework. Let  $X = (X_n)_{n \in \mathbb{Z}^d}$  be a stationary  $L^2$  random field having long memory, i.e. its covariance sequence  $\sigma$  is not a summable series. We assume that  $X$  is a Gaussian process and admits the linear representation

$$X_n = \int_E a(x) e^{i \langle n, x \rangle} dW(x), \quad n \in \mathbb{Z}^d, \quad E = [-\pi, \pi]^d, \quad (1)$$

where  $W$  is the Gaussian white noise spectral measure and where the function  $a$  is in  $L^2(E)$ .  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^d$ . Hereafter, we denote by  $f$  the spectral density of  $X$ ,  $f$  is proportional to  $|a(x)|^2$ , and we have

$$\sigma(n) = \hat{f}_n = \int_E e^{i\langle n, \lambda \rangle} f(\lambda) d\lambda. \quad (2)$$

The quadratic forms associated to  $X$  and  $(g_i)_{i \in \mathbb{N}^d} \in \ell^1(\mathbb{N}^d)$  are defined as

$$Q_n = \frac{1}{n^d} \sum_{i \in A_n} \sum_{j \in A_n} g_{i-j} X_i X_j \quad (3)$$

where  $A_n = \{1, \dots, n\}^d$ . The statistics  $Q_n$  can be rewritten as a functional of the periodogram  $I_n$  of  $X$ . Indeed let

$$I_n(t) = n^{-d} \sum_{k, l \in A_n^2} X_k X_l e^{-i\langle k-l, t \rangle},$$

we have

$$Q_n = \int_E g(t) I_n(t) dt, \quad (4)$$

where

$$g(t) = (2\pi)^d \sum_{j \in \mathbb{Z}^d} g_j e^{i\langle j, t \rangle}.$$

In dimension one ( $d = 1$ ), many papers deal with the asymptotic behavior of  $Q_n$  for long memory processes. When the intensity of the memory is not too strong or when specific conditions are imposed to  $g$  in order to kill the effect of the long memory involved by  $f$ , the convergence rate and the asymptotic normality obtained in short memory are preserved. The asymptotic normality of the quadratic forms is proved by Avram (1988) and Fox and Taqqu (1987) in the Gaussian case and extended by Giraitis and Surgailis (1990) to include the non-Gaussian linear case. Avram's result is restricted to  $f$  and  $g$  in  $L^2$ , but requires only integrability conditions. On the opposite, Fox and Taqqu (1987) and Giraitis and Surgailis (1990) specify the behavior of  $f$  and  $g$  around zero.

Under other conditions on  $f$  and  $g$  fixing the behavior of both functions around zero, Rosenblatt (1961), Fox and Taqqu (1985), Terrin and Taqqu (1990) prove non central limit theorems. The normalization is not standard (i.e. it is not equal to  $n^{-d/2}$ ) and the limiting process is not Gaussian.

Fixing the behavior of  $f$  and  $g$  around zero, the dichotomy in  $d = 1$  can be summarize as follows. Assume that

$$\begin{aligned} f(x) &\sim |x|^{2\alpha} & \text{when } x \rightarrow 0, \\ g(x) &\sim |x|^{2\beta} & \text{when } x \rightarrow 0. \end{aligned}$$

The limiting distribution depends on the value of  $(\alpha + \beta)$ .

- If  $\alpha + \beta > -\frac{1}{4}$  then the limit is Gaussian and the normalization is  $n^{-1/2}$  ( $Q_n$  satisfies a central limit theorem).
- If  $\alpha + \beta < -\frac{1}{4}$  then the limiting distribution is not Gaussian and the normalization in this case is  $n^{2(\alpha+\beta)}$ , which is lower than  $n^{-1/2}$  ( $Q_n$  satisfies a non central limit theorem).

In the  $d$ -dimensional case ( $d > 1$ ), Doukhan et al. (1996) obtained first asymptotic results when the memory of  $X$  is "isotropic", i.e. the memory of  $X$  is due to  $|x|^\alpha$  in the expression of  $a$  where  $|\cdot|$  denotes the Euclidian norm (see Lavancier (2005) Definition 1). Assuming specific conditions on  $g$  and  $\alpha$ , they prove that the quadratic forms, appropriately normalized, may have Gaussian or non-Gaussian limits. Their results are very similar to Fox and Taqu (1987) and Fox and Taqu (1985). However the investigation of "anisotropic" memory and more general functions  $g$  (including the particular case of the empirical covariance) is of great interest for statistical applications. In Lavancier (2005), different classes of random fields having long memory are described: The important role of anisotropic processes in terms of modeling is illustrated.

Our paper is organized as follows. In Section 2, we generalize the central limit theorem proved in Avram (1988). In Section 3, a non central limit theorem is stated under an integrability condition on  $f$  and  $g$ . We show that this condition is satisfied by a large class of isotropic and anisotropic models (including the previous results in dimension  $d$ ). Since random fields in  $d > 1$  may exhibit much more different types of long memory than processes in  $d = 1$ , the dichotomy between central and non central limit theorems is not so simple to formulate. As already highlighted in Remark 4.2 of Dobrushin and Major (1979), a spectral singularity outside zero may lead to new limiting results. This fact is illustrated through examples in Section 3.2.

In Section 4, we apply our limit theorems to the convergence of the empirical covariance series. Contrary to the one-dimension case, we show that it is possible to obtain a Gaussian limit with a non standard normalization.

## 2 Central Limit Theorem

The following theorem extend the central limit theorem obtained by Avram (1988) to the dimension  $d > 1$ . The hypotheses do not require to specify the form of functions  $f$  and  $g$ , but only integrability conditions.

**Theorem 1.** *Let  $X$  be a random field as in (1) and denote  $f$  its spectral density. Let  $Q_n$ , defined in (4), be the quadratic form associated with  $g$  and  $X$ . If  $f \in L^p$  and  $g \in L^q$  with  $p \geq 2$ ,  $q \geq 2$  and  $1/p + 1/q \leq 1/2$ , then*

$$n^{d/2}(Q_n - E(Q_n)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^{3d} \int_E f^2(t)g^2(t)dt\right), \quad (5)$$

where  $\xrightarrow{\mathcal{L}}$  denotes the convergence in law.

**Remark 1.** This convergence result is comparable in terms of limit and normalization to that obtained by Doukhan et al. (1996), Th. 6.2., except that we do not fix semi-parametric forms on  $f$  and  $g$ . Note also that their result contains a misprint:  $(2\pi)^d$  should be replaced by  $(2\pi)^{3d}$  to suit (5).

*Proof.* We use the method of cumulants as in Avram (1988). We need a generalization of its results stated in Lemmas 2 and 3 below. Thanks to these Lemmas, the converge result of Theorem 1 is straightforward.  $\square$

Let  $T_n(f)$  and  $T_n(g)$  be the finite Toeplitz Operators respectively associated to  $f$  and  $g$  (see the Appendix).

**Lemma 2.** Denote  $cum_k$  the cumulants of order  $k$ ,

$$cum_k(n^{d/2}(Q_n - E(Q_n))) = \frac{2^{k-1}(k-1)!}{n^{kd/2}} Tr[(T_n(f)T_n(g))^k], \quad (6)$$

where  $Tr$  is the trace operator.

*Proof.* Let  $s : \{1, \dots, n^d\} \rightarrow A_n$  be a bijective function.

$$Q_n = \frac{1}{n^d} \sum_{i \in A_n} \sum_{j \in A_n} g_{i-j} X_i X_j = \frac{1}{n^d} \sum_{k=1}^{n^d} \sum_{l=1}^{n^d} g_{s_k - s_l} X_{s_k} X_{s_l} = \frac{1}{n^d} \tilde{X}' \tilde{G} \tilde{X},$$

where  $\tilde{X} = (X_{s_k})_{k=1, \dots, n^d}$ ,  $\tilde{G} = (g_{s_k - s_l})_{k, l=1, \dots, n^d}$ . From Grenander and Szeg (1958), we get

$$cum_k(\tilde{X}' \tilde{G} \tilde{X}) = 2^{k-1}(k-1)! Tr [(\tilde{\Gamma} \tilde{G})^k],$$

where  $\tilde{\Gamma} = (cov(X_{s_k}, X_{s_l}))_{k, l=1, \dots, n^d}$ . Now, we easily check that

$$Tr [(\tilde{\Gamma} \tilde{G})^k] = Tr [(T_n(f)T_n(g))^k]$$

and (6) is then straightforward.  $\square$

**Lemma 3.** If  $f \in L^p$  and  $g \in L^q$  with  $p \geq 2$ ,  $q \geq 2$  and  $1/p + 1/q \leq 1/2$ ,

$$(i) \quad \lim_{n \rightarrow \infty} \frac{1}{n^d} Tr[(T_n(f)T_n(g))^2] = (2\pi)^{3d} \int_E f^2(t)g^2(t)dt.$$

$$(ii) \quad \text{If } k \geq 3, \quad \lim_{n \rightarrow \infty} \frac{1}{n^{kd/2}} Tr[(T_n(f)T_n(g))^k] = 0.$$

*Proof.* The proof follows the same lines as the proof of Theorem 1 of Avram (1988).

The first point consists in checking (i) for polynomials functions in  $x \mapsto e^{ix}$ . By multilinearity it is enough to consider  $f(x) = e^{i\langle k, x \rangle}$  and  $g(x) = e^{i\langle l, x \rangle}$  where  $k \in \mathbb{Z}^d$  and  $l \in \mathbb{Z}^d$ . Easy computations lead to

$$Tr[(T_n(f)T_n(g))^2] = \begin{cases} (2\pi)^{4d} \prod_{j=1}^d (n - |k_j|) & \text{if } k + l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can rewrite

$$\text{Tr}[(T_n(f)T_n(g))^2] = (2\pi)^{3d} \prod_{j=1}^d (n - |k_j|) \int_E f^2(t)g^2(t)dt$$

and (i) is proved for polynomials functions.

The remaining of the proof of (i) is an inference on the number of non-polynomials functions. The arguments to achieve this inference and the proof of (ii) are exactly the same as in Avram (1988). We do not reproduce them, however we give below the two key results of the proof and check that they still hold in dimension  $d$ .

The first fundamental property is an inequality about the p-Schatten norms of finite Toeplitz operators recalled in Lemma 12 of the appendix.

The second key result is the  $L^p$ -convergence of the Fejer sum of a function  $f \in L^p$ . In dimension 1, see Theorem 2.11 in Katznelson (1968). In dimension  $d$ , the Fejer kernel is replaced by a tensor product and the Fejer sum of  $f$  is

$$\int_E f(x-t) \prod_{i=1}^d F_n(t_i) dt, \quad \forall x \in E, \quad (7)$$

where  $t = (t_1, \dots, t_d)$  and  $F$  denotes the Fejer kernel on  $[-\pi, \pi]$ .

The  $L^p$ -convergence of (7) is still true in dimension  $d$  since it relies on the kernel property

$$\lim_{n \rightarrow \infty} \int_{|t| > \delta} \prod_{i=1}^d F_n(t_i) dt = 0.$$

□

### 3 Non-central limit theorem

#### 3.1 Main result

The following theorem gives the convergence of the quadratic forms under the general condition (9) on the integrability of  $f$  and  $g$ . Contrary to the previous studies in the one dimension case or in dimension  $d > 1$ , this condition does not require to specify the behavior of  $f$  around its singularities. Such a condition is satisfied by many examples as shown in Section 3.2.

Let  $\zeta$  be a function defined on  $\mathbb{R}^d$ , hereafter we denote by  $\zeta_{(p)}$  ( $p \in \mathbb{N}$ ) the periodic function with period  $2\pi p$  (with respect to each component), that coincides with  $\zeta$  on the set  $pE^d$ ,

$$\zeta_{(p)}(x) = \zeta(x) \quad \forall x \in [-p\pi, p\pi]^d. \quad (8)$$

**Theorem 4.** *Let  $X$  be the linear field defined by (1) and  $Q_n$  the quadratic form defined by (3).*

Assume that for all  $x \in E$   $a(-x) = \overline{a(x)}$ , and  $g(-x) = \overline{g(x)}$ . Moreover, assume that

$$\begin{cases} a(x) = \tilde{a}_{(1)}(x)L_1(x), \\ g(x) = \tilde{g}_{(1)}(x)L_2(x), \end{cases}$$

where  $\tilde{a}$  is a homogeneous function of degree  $\alpha$  (i.e.  $\forall c > 0, \forall x \in E, \tilde{a}(cx) = c^\alpha \tilde{a}(x)$ ), where  $\tilde{g}$  is a homogeneous function of degree  $2\beta$  and where  $L_1$  and  $L_2$  are bounded functions, continuous at zero and  $L_i(0) \neq 0$  ( $i = 1, 2$ ).

If

$$\iint_{\mathbb{R}^{2d}} \tilde{a}^2(x)\tilde{a}^2(y) \left[ \int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1+|x_k+t_k|)(1+|y_k-t_k|)} dt \right]^2 dx dy < \infty, \quad (9)$$

then

$$n^{d+2\alpha+2\beta}(Q_n - E(Q_n)) \xrightarrow{\mathcal{L}} L_1^2(0)L_2(0) \iint_{\mathbb{R}^{2d}} \tilde{a}(x)\tilde{a}(y) \int_{\mathbb{R}^d} \tilde{g}(t)H(x+t)H(y-t)dt dW(x)dW(y), \quad (10)$$

where  $H(z) = \prod_{j=1}^d \frac{e^{iz_j} - 1}{iz_j}$ .

### 3.2 Some examples

The setting of random fields allows many kind of dependencies. Apart from its intensity, the dependence can be isotropic or can occur all over several particular directions, depending on the form of the spectral density. Since we did not want to restrict ourselves to one particular case, Theorem 4 involves a general condition, i.e. assumption (9). We check in this section that this hypothesis is not too restrictive. Indeed, we prove that in all the preceding studies about quadratic forms of Gaussian fields, this condition is fulfilled. Moreover, it allows to extend to new cases, mainly when the dependence is not isotropic.

**Corollary 5.** *Assume that it exists some positive constants  $c$  and  $c'$  such that*

$$\begin{cases} |\tilde{a}(x)| \leq c \prod_{i=1}^d |x_i|^{\alpha/d} \\ |\tilde{g}(x)| \leq c' \prod_{i=1}^d |x_i|^{2\beta/d}. \end{cases} \quad (11)$$

*If moreover  $\alpha > -d/2$ ,  $\beta > -d/2$  and  $\alpha + \beta < -d/4$ , then (9) is fulfilled.*

**Remark 2.** *When  $d = 1$ , the setting of Corollary 5 corresponds to the one of Terrin and Taqqu (1990). The result of our Theorem 4 is the same as the convergence stated in Theorem 1 in this article.*

*The assumptions in Corollary 5 allow the filter  $a(x)$  to be isotropic, i.e. equivalent at zero, up to a constant, to  $|x|^\alpha$ . This is the hypothesis done in Theorem 5.1 of Doukhan et al. (1996) where, moreover, it is assumed  $\beta = 0$ .*

*Proof.* We have to check that the function defined on  $(\mathbb{R}^d)^4$  by

$$(x, y, s, t) \mapsto \frac{\tilde{a}^2(x)\tilde{a}^2(y)|\tilde{g}(t)||\tilde{g}(s)|}{\prod_{k=1}^d (1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)}$$

is integrable on  $(\mathbb{R}^d)^4$ . From the hypothesis of Corollary 5, it is enough to prove that for all  $k = 1, \dots, d$

$$(t_k, s_k, x_k, y_k) \mapsto \frac{|x_k|^{2\alpha/d}|y_k|^{2\alpha/d}|t_k|^{2\beta/d}|s_k|^{2\beta/d}}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)}$$

is integrable on  $\mathbb{R}^4$ . Such integrals are studied in Lemma 1 of Terrin and Taquq (1990): Under the assumptions of Corollary 5 on  $\alpha$  and  $\beta$ , this integral is finite.  $\square$

When the functions  $a$  and  $g$  involved in Theorem 4 do not satisfy (11), condition (9) may be investigated thanks to power counting theorems (cf. Theorem 2 in Terrin and Taquq (1990)). The following corollary focus on a particular situation in dimension  $d = 2$ : When  $\tilde{a}$  admits two independent lines of singularities. Note that it also includes the case where  $\tilde{a}$  has only one line of singularity.

**Corollary 6.** *Assume that  $d = 2$  and that  $\tilde{g}$  follows the same conditions as in (11).*

*If, for  $p \neq q$ ,*

$$\tilde{a}(x_1, x_2) = |x_1 + px_2|^{\alpha_p}|x_1 + qx_2|^{\alpha_q},$$

*and if  $\alpha_p > -1/2$ ,  $\alpha_q > -1/2$ ,  $\beta > -1$  and  $\alpha_p + \alpha_q + \beta < -1/2$ , then (9) is fulfilled.*

*Proof.* Checking (9) is equivalent to prove that

$$\int_{\mathbb{R}^8} \frac{|x_1 + px_2|^{2\alpha_p}|x_1 + qx_2|^{2\alpha_q}|y_1 + py_2|^{2\alpha_p}|y_1 + qy_2|^{2\alpha_q}|t_1|^\beta|t_2|^\beta|s_1|^\beta|s_2|^\beta}{\prod_{k=1}^2 (1 + |x_k + t_k|)(1 + |y_k - t_k|)(1 + |x_k + s_k|)(1 + |y_k - s_k|)} dt ds dx dy \quad (12)$$

is finite. We apply Theorem 2 of Terrin and Taquq (1990). Let us introduce some notations. The integral above can be written :

$$\int_{\mathbb{R}^8} |L_1(u)|^{2\alpha_p}|L_2(u)|^{2\alpha_q}|L_3(u)|^{2\alpha_p}|L_4(u)|^{2\alpha_q} \times |L_5(u)|^\beta|L_6(u)|^\beta|L_7(u)|^\beta|L_8(u)|^\beta \prod_{k=9}^{16} (1 + |L_k(u)|)^{-1} du, \quad (13)$$

where  $u = (x_1, x_2, y_1, y_2, t_1, t_2, s_1, s_2)$  and the  $L_k$ 's are the linear functionals involved in (12). For instance  $L_1(u) = x_1 + px_2$ ,  $L_5(u) = t_1$ ,  $L_9(u) = x_1 + t_1$ .

Let  $T = \{L_1, \dots, L_{16}\}$  and let  $\gamma_k$  be the exponent associated to  $L_k$  in (13). For instance,  $\gamma_1 = 2\alpha_p$ ,  $\gamma_5 = \beta$ ,  $\gamma_9 = -1$

Consider now the subsets  $W \subset T$  such that  $\text{span}(W) \cap T = W$ . A subset  $W$  is said padded if any  $L_k \in W$  is a linear combination of the  $L_i$ 's in  $W - \{L_k\}$ .



The integrability of (12) near 0 is obvious since  $2\alpha_p > -1$ ,  $2\alpha_q > -1$  and  $\beta > -1$ . According to Theorem 2 in Terrin and Taqqu (1990), the integrability at infinity is achieved if for every padded  $W$  considered above but  $T$ ,  $d_\infty(W) := \text{rank}(T) - \text{rank}(W) + \sum_{T-W} \gamma_k < 0$ .

The maximum value for  $d_\infty(W)$  is obtained with  $W = \{L_9, \dots, L_{16}\}$ . In this case  $d_\infty(W) = 8 - 6 + 4\alpha_p + 4\alpha_q + 4\beta$ . This leads to  $\alpha_p + \alpha_q + \beta < -1/2$ .  $\square$

### 3.3 Proof of Theorem 4

*Proof.* The main tool we use consists in rewriting  $Q_n$  as a double stochastic integral. Then the convergence in law is deduced from a simple convergence in  $L^2(\mathbb{R}^{2d})$ . Note that contrary to the proofs for non central limit theorems in Terrin and Taqqu (1990) and Doukhan et al. (1996), we do not apply the scheme of convergence of Dobrushin and Major (1979), especially their Lemma 3 which involves a spectral measure convergence assumption.

Let  $Z$  be a Gaussian random spectral measure associated to the measure  $\mu$ . Major (1981) defines

$$\iint_{\mathbb{R}^{2d}} f(x, y) dZ(x) dZ(y)$$

for all  $f \in H_\mu$  where  $H_\mu$  denotes the space of functions  $f: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$  such that  $f(-x, -y) = \overline{f(x, y)}$  and  $\int |f(x, y)|^2 d\mu(x) d\mu(y) < \infty$ . The second order moment of this integral verifies, for all  $f$  in  $H_\mu$ ,

$$E \left( \iint f(x, y) dZ(x) dZ(y) \right)^2 \leq 2 \int_{\mathbb{R}^{2d}} |f(x, y)|^2 d\mu(x) d\mu(y). \quad (14)$$

And the so-called Ito formula follows: For all  $f_1$  and  $f_2$  in  $H_\mu$ ,

$$\iint f_1(x) f_2(y) dZ(x) dZ(y) = \int f_1(x) dZ(x) \int f_2(y) dZ(y) - \int f_1(x) \overline{f_2(x)} d\mu(x). \quad (15)$$

Now, let us rewrite  $Q_n$  as a double stochastic integral. According to (1),

$$\begin{aligned} Q_n - E(Q_n) &= \frac{1}{n^d} \sum_{k \in A_n} \sum_{l \in A_n} g_{k-l} (X_k X_l - r(l-k)) \\ &= \int_E g(t) \left[ \frac{1}{n^d} \sum_{k \in A_n} \sum_{l \in A_n} e^{i\langle k-l, t \rangle} (X_k X_l - r(l-k)) \right] dt. \end{aligned} \quad (16)$$

From the Ito formula (15)

$$\begin{aligned} &X_k X_l - r(l-k) \\ &= \int_E a(x) e^{i\langle k, x \rangle} dW(x) \int_E a(y) e^{i\langle l, y \rangle} dW(y) - \int_E e^{i\langle l-k, x \rangle} a^2(x) d\mu(x) \\ &= \iint_{E^2} a(x) a(y) e^{i(\langle k, x \rangle + \langle l, y \rangle)} dW(x) dW(y), \end{aligned}$$

where  $\mu$ , the spectral measure of  $\epsilon$ , is proportional to the Lebesgue measure  $\lambda$ .

Assume that  $g(-x) = \overline{g(x)}$  and  $a(-x) = \overline{a(x)}$ , then the function

$$(x, y) \mapsto a(x)a(y) \left[ \frac{1}{n^d} \int_E g(t) \sum_{k \in A_n} e^{i\langle k, x+t \rangle} \sum_{l \in A_n} e^{i\langle l, y-t \rangle} dt \right] \quad (17)$$

belongs to  $H_\lambda$ . Therefore we can rewrite (16) as

$$Q_n - E(Q_n) = \iint_{E^2} a(x)a(y) \left[ \frac{1}{n^d} \int_E g(t) H_n(x+t) H_n(y-t) dt \right] dW(x)dW(y), \quad (18)$$

where  $H_n(t) = \sum_{k \in A_n} e^{i\langle k, t \rangle}$ .

In (18), we make the change of variables  $x \rightarrow x/n$ ,  $y \rightarrow y/n$ ,  $t \rightarrow t/n$ . Since the Gaussian measure  $W$  verifies for all Borelian set  $A$ ,  $W(n^{-1}A) = n^{-1/2}W(A)$  and since  $\tilde{a}$  and  $\tilde{g}$  are homogeneous, we get

$$n^{d+2\alpha+2\beta}(Q_n - E(Q_n)) = \iint_{nE^2} \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)L_1\left(\frac{x}{n}\right)L_1\left(\frac{y}{n}\right)\psi_n(x,y)dW(x)dW(y),$$

where

$$\Psi_n(x, y) = \int_{nE} \tilde{g}_{(n)}(t)L_2\left(\frac{t}{n}\right)\frac{1}{n^d}H_n\left(\frac{x+t}{n}\right)\frac{1}{n^d}H_n\left(\frac{y-t}{n}\right)dt.$$

As a consequence, according to (14), it suffices, for proving (10), to show that the following integral tends to zero as  $n \rightarrow \infty$ :

$$\iint_{nE^2} \left[ \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)L_1\left(\frac{x}{n}\right)L_1\left(\frac{y}{n}\right)\Psi_n(x, y) - \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)L_1^2(0)\Psi(x, y) \right]^2 dx dy$$

where

$$\Psi(x, y) = L_2(0) \int_{\mathbb{R}^d} \tilde{g}(t)H(x+t)H(y-t)dt.$$

From the decomposition  $A_n B - CD = (A_n - C)B + (B - D)C$ , the  $L^2$ -norm above is lower than the sum  $2(I_1 + I_2)$  where

$$I_1 = \iint_{nE^2} \left[ \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)\Psi_n(x, y) - \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)\Psi(x, y) \right]^2 \times \left[ L_1\left(\frac{x}{n}\right)L_1\left(\frac{y}{n}\right) \right]^2 dx dy$$

and

$$I_2 = \iint_{nE^2} \left[ L_1\left(\frac{x}{n}\right)L_1\left(\frac{y}{n}\right) - L_1^2(0) \right]^2 \left[ \tilde{a}_{(n)}(x)\tilde{a}_{(n)}(y)\Psi(x, y) \right]^2 dx dy.$$

The following lemma will be useful.

**Lemma 7.**

$$(i) \quad \forall z = (z_1, \dots, z_d) \in nE, \quad \left| \frac{1}{n^d} H_n \left( \frac{z}{n} \right) \right| \leq \prod_{j=1}^d \pi \left( 1 \wedge \frac{1}{|z_j|} \right).$$

$$(ii) \quad \forall z = (z_1, \dots, z_d) \in \mathbb{R}^d, \quad |H(z)| \leq \prod_{j=1}^d 2 \left( 1 \wedge \frac{1}{|z_j|} \right).$$

$$(iii) \quad \text{For a.e. } z \in \mathbb{R}^d, \quad \lim_{n \rightarrow \infty} \left| \mathbb{1}_{nE}(z) \frac{1}{n^d} H_n \left( \frac{z}{n} \right) - H(z) \right| = 0.$$

*Proof of Lemma 7.* Since for all  $j$ ,  $|z_j| \leq n\pi$ ,

$$\begin{aligned} \left| \frac{1}{n^d} H_n \left( \frac{z}{n} \right) \right| &= \prod_{j=1}^d \left| \frac{1}{n} e^{i \frac{z_j}{n}} \frac{e^{i z_j} - 1}{e^{i \frac{z_j}{n}} - 1} \right| = \prod_{j=1}^d \left| \frac{\sin(z_j/2)}{z_j/2} \right| \left| \frac{z_j/2n}{\sin(z_j/2n)} \right| \\ &\leq \prod_{j=1}^d \pi \left( 1 \wedge \frac{1}{|z_j|} \right). \end{aligned}$$

Similarly, for all  $z \in \mathbb{R}^d$ ,

$$|H(z)| = \prod_{j=1}^d \left| \frac{e^{i z_j} - 1}{i z_j} \right| = \prod_{j=1}^d \left| \frac{\sin(z_j/2)}{z_j/2} \right| \leq \prod_{j=1}^d 2 \left( 1 \wedge \frac{1}{|z_j|} \right).$$

Finally, for proving (iii), suppose first that  $d = 1$ . If  $z \neq 0$ ,

$$\left| \frac{1}{n^d} H_n \left( \frac{z}{n} \right) - H(z) \right| = \left| \frac{1}{n} e^{i \frac{z}{n}} \frac{e^{i z} - 1}{e^{i \frac{z}{n}} - 1} - \frac{e^{i z} - 1}{i z} \right| \leq \frac{2}{|z|} \left| \frac{e^{i \frac{z}{n}} i z}{n(e^{i \frac{z}{n}} - 1)} - 1 \right|.$$

The norm in the right hand side term is equivalent to  $|z|/n$  when  $n$  goes to infinity, hence it tends to 0. In dimension  $d$ , (iii) is proved by induction thanks to the decomposition  $AB - CD = (A - C)B + (B - D)C$ .  $\square$

Let us first prove that  $I_1$  asymptotically vanishes. Since  $L_1$  is bounded, it suffices to prove the convergence to 0 of

$$I_{11} = \iint_{nE^2} \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) \left[ \int_{nE} \Phi_n^{(11)}(t, x, y) dt \right]^2 dx dy, \quad (19)$$

with

$$\Phi_n^{(11)}(t, x, y) = \tilde{g}_{(n)}(t) \left( L_2 \left( \frac{t}{n} \right) - L_2(0) \right) \frac{1}{n^d} H_n \left( \frac{x+t}{n} \right) \frac{1}{n^d} H_n \left( \frac{y-t}{n} \right)$$

and

$$I_{12} = \iint_{nE^2} \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) L_2(0)^2 \left[ \int_{nE} \Phi_n^{(12)}(t, x, y) dt \right]^2 dx dy, \quad (20)$$

with

$$\Phi_n^{(12)}(t, x, y) = \tilde{g}_{(n)}(t) \left[ \frac{1}{n^d} H_n \left( \frac{x+t}{n} \right) \frac{1}{n^d} H_n \left( \frac{y-t}{n} \right) - H(x+t)H(y-t) \right]$$

In both  $I_{11}$  and  $I_{12}$ , the  $2\pi$ -periodicity of  $g$ ,  $H_n$  and  $H$  allows us to reduce the domain of integration  $nE$  (with respect to  $t$ ) to

$$nD_{x,y} = \{|x-t| < n\pi\} \cap \{|y+t| < n\pi\} \cap nE.$$

Therefore, (i) of Lemma 7 can be applied and since  $L_2$  is bounded,

$$\left| \int_{nE} \Phi_n^{(11)}(t, x, y) dt \right| \leq c \int_{nD_{x,y}} |\tilde{g}_{(n)}(t)| \prod_{k=1}^d \left( 1 \wedge \frac{1}{|x_k + t_k|} \right) \left( 1 \wedge \frac{1}{|y_k - t_k|} \right) dt,$$

where  $c$  is a positive constant. Hence,

$$I_{11} \leq c' \iint_{\mathbb{R}^{2d}} \tilde{a}^2(x) \tilde{a}^2(y) \left[ \int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)} dt \right]^2 dx dy.$$

Besides, according to (i) of Lemma 7,

$$\begin{aligned} & \left| \mathbb{I}_{nE^2}(x, y) \mathbb{I}_{nD_{x,y}}(t_1) \mathbb{I}_{nD_{x,y}}(t_2) \tilde{a}_{(n)}^2(x) \tilde{a}_{(n)}^2(y) \Phi_n^{(11)}(t_1, x, y) \Phi_n^{(11)}(t_2, x, y) \right| \\ & \leq c \tilde{a}^2(x) \tilde{a}^2(y) |\tilde{g}(t_1)| |\tilde{g}(t_2)| \left| L_2 \left( \frac{t_1}{n} \right) - L_2(0) \right| \left| L_2 \left( \frac{t_2}{n} \right) - L_2(0) \right|. \end{aligned}$$

From the continuity of  $L_2$  at 0, this term tends to zero for any fixed  $(x, y, t_1, t_2) \in \mathbb{R}^{4d}$ . Therefore, thanks to assumption (9), the Lebesgue's dominated convergence theorem applies and  $\lim_{n \rightarrow \infty} I_{11} = 0$ .

The convergence of  $I_{12}$  is proved similarly. From (i) and (ii) of Lemma 7, we have

$$\left| \int_{nE} \Phi_n^{(12)}(t, x, y) dt \right| \leq c \int_{nD_{x,y}} |\tilde{g}_{(n)}(t)| \prod_{j=1}^d \left( 1 \wedge \frac{1}{|x_k + t_k|} \right) \left( 1 \wedge \frac{1}{|y_k - t_k|} \right) dt$$

and

$$I_{12} \leq c' \iint_{\mathbb{R}^{2d}} \tilde{a}^2(x) \tilde{a}^2(y) \left[ \int_{\mathbb{R}^d} |\tilde{g}(t)| \prod_{k=1}^d \frac{1}{(1 + |x_k + t_k|)(1 + |y_k - t_k|)} dt \right]^2 dx dy,$$

where  $c$  and  $c'$  are positive constants.

Besides, for almost every  $(x, y, t_1, t_2) \in \mathbb{R}^{4d}$ , according to (iii) of Lemma 7,

$$\mathbb{I}_{nE^2}(x, y) \mathbb{I}_{nD_{x,y}}(t_1) \mathbb{I}_{nD_{x,y}}(t_2) \tilde{a}_{(n)}(x) \tilde{a}_{(n)}(y) \Phi_n^{(12)}(t_1, x, y) \Phi_n^{(12)}(t_2, x, y) \xrightarrow{n \rightarrow \infty} 0.$$

The Lebesgue's dominated convergence theorem applies thanks to (9) and  $I_{12}$  tends to zero when  $n \rightarrow \infty$ .

It is easy to see that  $I_2$  tends similarly to zero.  $\square$

## 4 Convergence of the empirical auto-covariance function

We present an application of the preceding theorems to the asymptotic law of the empirical covariance function in  $\mathbb{Z}^d$ . Indeed, for  $h \in \mathbb{Z}^d$ ,

$$\hat{r}(h) = \frac{1}{n^d} \sum_{i \in A_n} X_i X_{i+h}$$

is a particular case of (3) with  $g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle}$ . We will consider further

$$\hat{\tilde{r}}(h) = \frac{1}{n^d} \sum_{i \in A_n} (X_i - \bar{X}_n) (X_{i+h} - \bar{X}_n)$$

where  $\bar{X}_n = n^{-d} \sum_{i \in A_n} X_i$  and we denote in the following  $r(h) = E(X_i X_{i+h})$ .

We are dealing with linear fields as (1). In dimension 1, this framework provides a dichotomy in the asymptotic behavior of  $\hat{r}(h)$  depending on whether the spectral density of  $X$  belongs to  $L^2$  or not (see Hosking (1996)). We prove later on the same kind of division in dimension  $d$ , although some kind of intermediate behavior can arise. Let us note that this dichotomy does no longer hold in  $d = 1$  when  $X$  is not linear, see for instance Giraitis et al. (2000) in the case of a LARCH process or Giraitis and Taqqu (1999) for the consequences on the Whittle estimator.

When  $f \in L^2$ , we prove that  $\hat{r}(h)$  and  $\hat{\tilde{r}}(h)$  follow the same central limit theorem. This is the object of Proposition 8.

When  $f \notin L^2$ , the asymptotic behavior of  $\hat{r}(h)$  and  $\hat{\tilde{r}}(h)$  may differ and we focus on  $\hat{r}(h)$  which is a proper quadratic form. In this setting, the asymptotic law comes from Theorem 4 where  $\beta = 0$ , provided condition (9) is satisfied. Therefore, the normalization and the limit in law depend on the filter  $a$ . We summarize in Proposition 9 and in the beginning of section 4.2 the two situations already studied in corollary 5 and corollary 6 before.

But Theorem 4 does not apply in the example of corollary 6 when  $\alpha_p \alpha_q = 0$  and  $\beta = 0$ . This corresponds to the particular situation when the long memory occurs only along one direction in dimension 2. In this case, we study the asymptotic behavior of  $\hat{r}(h)$  in Proposition 11. It appears that a non-central limit theorem holds in the sense that the normalization is not  $n^{d/2}$ . Yet, contrary to the classical non central limit results for  $\hat{r}(h)$  (see references therein), the limiting law is Gaussian.

### 4.1 General results in dimension $d$

Let us first present the central limit theorem for  $\hat{r}(h)$  and  $\hat{\tilde{r}}(h)$  when  $f \in L^2$ .

**Proposition 8.** *Let  $X$  be the linear field defined by (1).*

*If  $f \in L^2$ , then, for all  $h \in \mathbb{Z}^d$ ,  $n^{d/2}(\hat{r}(h) - r(h))$  and  $n^{d/2}(\hat{\tilde{r}}(h) - r(h))$  converge both in law to a Gaussian random variable with zero mean and variance*

$2(2\pi)^d \widehat{f_{2h}^2}$ , where  $\widehat{f_{2h}^2}$  stands for the  $(2h)$ -th Fourier coefficients of  $f^2$  as defined in (2).

*Proof.* The central limit theorem for  $r(h)$  follows from Theorem 1 where  $p = 2$ ,  $q = +\infty$  and  $g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle}$ .

For  $\tilde{r}(h)$ , we prove that almost surely,  $n^{d/2}(\hat{r}(h) - \tilde{r}(h)) = o(1)$ . Indeed,

$$n^{d/2}(\hat{r}(h) - \tilde{r}(h)) = \frac{1}{n^{3d/2}} \sum_{k_1 \in A_n} \sum_{k_2 \in A_n} X_{k_1} X_{k_2+h}.$$

Using representation (1) of  $X$ , this term is equal to

$$\frac{1}{n^{3d/2}} \int_E f(x) e^{i\langle h, x \rangle} \left| \sum_{k \in A_n} e^{i\langle k, x \rangle} \right|^2 dx = \frac{1}{n^{d/2}} \int_E f(x) e^{i\langle h, x \rangle} \prod_{j=1}^d F_n(x_j) dx$$

where  $F_n$  denotes the Fejer kernel on  $[-\pi, \pi]$ . Now, from the Cauchy-Schwarz inequality, this last term is lower than

$$\frac{1}{n^{d/2}} \sqrt{\int_E f^2(x) \prod_{j=1}^d F_n(x_j) dx}.$$

Since  $f^2 \in L^1$ , the Lebesgue's Theorem implies that the integral above is a  $o(n^d)$ . Therefore,  $n^{d/2}(\hat{r}(h) - \tilde{r}(h)) = o(1)$  and the central limit theorem for  $\tilde{r}(h)$  is inherited from the one for  $\hat{r}(h)$ .  $\square$

Recall that the notation  $\tilde{a}_{(1)}$  is defined at the beginning of Section 3.1. The following proposition provides central and non central limit for  $\hat{r}(h)$ , depending on the parameter  $\alpha$ , when condition (21) is fulfilled. This framework includes isotropic and anisotropic models. The result shows the same kind of dichotomy than in dimension  $d = 1$ . In section 4.2, some anisotropic models that does not follow this dichotomy are presented.

**Proposition 9.** *Let  $X$  be the linear field defined by (1).*

*Assume that for all  $x \in E$   $a(-x) = \overline{a(x)}$  and that  $a(x) = \tilde{a}_{(1)}(x)L_1(x)$ , where  $\tilde{a}$  is a homogeneous function of degree  $\alpha > -d/2$  and  $L_1$  is a bounded function, continuous at zero and non-null at 0. If*

$$|\tilde{a}(x)| \leq c \prod_{i=1}^d |x_i|^{\alpha/d} \quad (21)$$

*for some positive constant  $c$ , then for all  $h \in \mathbb{Z}^d$ ,*

- *if  $\alpha > -d/4$ ,*

$$n^{d/2}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^d \widehat{f_{2h}^2}\right), \quad (22)$$

*where  $\widehat{f_{2h}^2}$  is the  $(2h)$ -th Fourier coefficients of  $f^2$  as defined in (2).*

- if  $\alpha < -d/4$ ,

$$n^{d+2\alpha}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} L_1^2(0) \iint_{\mathbb{R}^{2d}} \tilde{a}(x)\tilde{a}(y)H(x+y)dW(x)dW(y), \quad (23)$$

$$\text{where } H(z) = \prod_{j=1}^d \frac{e^{iz_j} - 1}{iz_j}.$$

*Proof.* In the case  $\alpha > -d/4$ , the convergence result is a consequence of Proposition 8.

For  $\alpha < -d/4$ , corollary 5 applies since  $\beta = 0$  and condition (9) in Theorem 4 is fulfilled. Let us justify the simplification of the limit in (23). Here

$$g(t) = (2\pi)^{-d} e^{i\langle h, t \rangle},$$

so  $\tilde{g}(t) = 1$  and  $L_2(0) = (2\pi)^{-d}$ . The simplification comes from the main term  $I_{12}$  in the proof of theorem 4, where we use the obvious identity

$$\int_{nE} \frac{1}{n^d} H_n \left( \frac{x+t}{n} \right) \frac{1}{n^d} H_n \left( \frac{y-t}{n} \right) dt = (2\pi)^d \frac{1}{n^d} H_n \left( \frac{x+y}{n} \right).$$

The pointwise convergence of this last term relies on (iii) of Lemma 7 and an application of the Lebesgue's theorem concludes the proof of (23).  $\square$

## 4.2 Some anisotropic examples in dimension $d = 2$

Starting with the anisotropic case studied in Corollary 6, we confirm that new limiting results can be obtained as suggested by Dobrushin and Major (1979) (Remark 4.2). If we suppose that  $a(x) = \tilde{a}_{(1)}(x)L(x)$ , where  $L$  is a bounded function, continuous and non-null at 0 and

$$\tilde{a}(x_1, x_2) = |x_1 + px_2|^{\alpha_p} |x_1 + qx_2|^{\alpha_q},$$

with  $\alpha_p > -\frac{1}{2}$ ,  $\alpha_q > -\frac{1}{2}$  and  $(p, q) \in \mathbb{R}^2$ , then, under the assumptions of Corollary 6

$$n^{2+2\alpha_p+2\alpha_q}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} Z$$

where  $Z$  is defined as the limit in (23)

This convergence is a simple application of Theorem 4. Condition (9) is fulfilled thanks to corollary 6 and the simplification of the limit holds for the same reasons as for Proposition 9.

The assumptions of Corollary 6, in the case when  $\beta = 0$ , imply the existence of two lines where the spectral density is unbounded.

When  $\alpha_p = 0$  or  $\alpha_q = 0$  (or  $p = q$ ), that is when the long memory occurs along only one direction, Corollary 6 does not imply (9) when  $\beta = 0$ . This case provides a new limiting behavior for  $\hat{r}(h)$  as stated in Proposition 11. To prove this result, we need to obtain the covariance structure of  $X$  when the spectral density is unbounded along a line that goes through the origin. This is the object of the following lemma. We say that the process has a long memory along one direction if its covariance function is not summable along this direction.

**Lemma 10.** Let  $p \in \mathbb{R}$  and for all  $(x_1, x_2) \in \mathbb{R}^2$ ,

$$f(x_1, x_2) = \frac{1}{2\pi} \tilde{f}(x_1 + px_2),$$

where  $\tilde{f}$  is an even, non-negative, and  $2\pi$ -periodic function on  $\mathbb{R}$ . Let us denote, for all  $(h_1, h_2) \in \mathbb{Z}^2$ ,

$$\sigma(h_1, h_2) = \int_{[-\pi, \pi]^2} e^{i(h_1 x_1 + h_2 x_2)} f(x_1, x_2) dx_1 dx_2$$

and

$$\tilde{\sigma}(h_1) = \int_{-\pi}^{\pi} e^{ih_1 x} \tilde{f}(x) dx.$$

Then, for all  $(h_1, h_2) \in \mathbb{Z}^2$ , we have

$$\sigma(h_1, h_2) = \begin{cases} \tilde{\sigma}(h_1) & \text{if } h_2 = ph_1, \\ \frac{\sin((h_2 - ph_1)\pi)}{(h_2 - ph_1)\pi} \tilde{\sigma}(h_1) & \text{otherwise.} \end{cases}$$

*Proof.* The proof is given at the end of this section.  $\square$

**Proposition 11.** Let  $X$  be a stationary Gaussian process in dimension  $d = 2$ . Let us suppose that its spectral density is

$$f(x_1, x_2) = \frac{1}{2\pi} \tilde{f}(x_1 + px_2),$$

where  $p \in \mathbb{Z}$  and where  $\tilde{f}$ , defined on  $[-\pi, \pi]$ , is a spectral density in dimension  $d = 1$ . Assume moreover that for  $-1/2 < \alpha < 0$  and for all  $x \in [-\pi, \pi]$ ,  $\tilde{f}(x) = L(x)|x|^{2\alpha}$  where  $L$  is a bounded function, continuous at zero and non-null at 0. Then,

- if  $\alpha > -1/4$ ,

$$n(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(2\pi)^2 \widehat{f_{2h}^2}\right), \quad (24)$$

where  $\widehat{f_{2h}^2}$  is the  $(2h)$ -th Fourier coefficients of  $f^2$  as defined in (2).

- if  $\alpha < -1/4$  and if  $h_2 \neq ph_1$ ,

$$n^{2\alpha+3/2}(\hat{r}(h) - r(h)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_{\alpha,p}^2\right), \quad (25)$$

where  $\sigma_{\alpha,p}^2 = \lim_{n \rightarrow \infty} n^{-4\alpha-3} \text{Var}(\hat{r}(h))$ .

*Proof.* In the case  $\alpha > -1/4$ , the result is a consequence of Proposition 8. Let us focus on  $\alpha < -1/4$ . We restrict the proof to the case  $p \geq 0$  since  $p \leq 0$  can be treated in the same way.



We prove the result thanks to a central limit theorem for triangular arrays stated in Romano and Wolf (2000). Indeed, let  $Y_{i_1, i_2} = X_{i_1, pi_1 + i_2}$  and let  $i = i_1$ ,  $j = i_2 - pi_1$ , we have

$$\hat{r}(h_1, h_2) = \frac{1}{n^2} \sum_{i_1=1}^n \sum_{i_2=1}^n X_{i_1, i_2} X_{i_1+h_1, i_2+h_2} = \frac{1}{n^2} \sum_{j=-pn+1}^{n-p} \sum_{i \in B_j} Y_{i, j} Y_{i+h_1, j-ph_1+h_2},$$

where  $B_j = \{i \mid 1 \wedge \frac{1-j}{p} \leq i \leq \frac{n-j}{p} \vee n\}$ . So  $\hat{r}(h_1, h_2)$  is the triangular array

$$\hat{r}(h_1, h_2) = \sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n, j}$$

where  $\tilde{Y}_{n, j} = n^{-2} \sum_{i \in B_{j-pn}} Y_{i, j-pn} Y_{i+h_1, j-pn-ph_1+h_2}$ .

Let us summarize the properties of  $\tilde{Y}_{n, j}$ .

From Lemma 10, we have

$$r(h_1, h_2) = \begin{cases} \tilde{r}(h_1) & \text{if } h_2 = ph_1, \\ 0 & \text{otherwise.} \end{cases}$$

where  $\tilde{r}$  is the covariance function associated with  $\tilde{f}$ . Consequently  $(Y_{i, j})$  is a zero mean Gaussian process such that

$$E(Y_{i, j} Y_{i+h_1, j+h_2}) = \begin{cases} \tilde{r}(h_1) & \text{if } h_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore  $\tilde{Y}_{n, j}$ , viewed as a function of  $((Y_{i, j-pn})_{i \in \mathbb{Z}}, (Y_{i+h_1, j-pn-ph_1+h_2})_{i \in \mathbb{Z}})$ , is a  $(h_2 - ph_1)$ -dependent process. Moreover, we can compute the moments of  $\tilde{Y}_{n, j}$  thanks to the representation of the moments of Gaussian variables in terms of Wick's product. We obtain

$$E(\tilde{Y}_{n, j}) = 0, \\ E(\tilde{Y}_{n, j_1} \tilde{Y}_{n, j_2}) = \begin{cases} 0 & \text{if } j_1 \neq j_2, \\ \sum_{i_1, i_2 \in B_{j-pn}} \tilde{r}^2(i_2 - i_1) & \text{otherwise} \end{cases}$$

and

$$E(\tilde{Y}_{n, j}^4) = 3 \left( \sum_{i_1, i_2 \in B_{j-pn}} \tilde{r}^2(i_2 - i_1) \right)^2 + \\ + 6 \sum_{i_1, i_2, i_3, i_4 \in B_{j-pn}} \tilde{r}(i_2 - i_1) \tilde{r}(i_3 - i_2) \tilde{r}(i_4 - i_3) \tilde{r}(i_4 - i_1).$$

We are now in position to apply Theorem 2.1 in Romano and Wolf (2000) which gives sufficient condition for the convergence in law of a triangular array

of  $m$ -dependent random variables to a normal distribution. Following the same notations as in this theorem, we choose  $\delta = 2$  and  $\gamma = 0$  and we look for  $\Delta_n$ ,  $K_n$  and  $L_n$  such that:  $E(\tilde{Y}_{n,j}^2) \leq \Delta_n$  for all  $j$ ,

$$\text{Var} \left( \sum_{j=a}^{a+k-1} \tilde{Y}_{n,j} \right) \leq k K_n$$

for all  $a$  and for all  $k \geq (h_2 - ph_1)$ , and

$$\text{Var} \left( \sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) \geq ((p+1)n-p) L_n.$$

The convergence in law holds whenever  $K_n/L_n = O(1)$  and  $\Delta_n/L_n^2 = O(1)$ .

From Theorem 2.24 of Zygmund (1959), we have  $\tilde{r}(h) \sim c_\alpha h^{-2\alpha-1}$  when  $h \rightarrow \infty$  where  $c_\alpha$  is a constant depending on  $\alpha$  and  $L(0)$ . Therefore, since  $(i \in B_j) \Rightarrow (1 \leq i \leq n)$ , an integral test leads to  $\Delta_n = O(n^{-8\alpha})$  and  $K_n = O(n^{-4\alpha})$ . For  $L_n$ , note that when  $n \leq j \leq pn - p + 1$ , there are at least  $\lceil \frac{n-1}{p} \rceil - 1$  indexes  $i$  in  $B_{j-pn}$ , where  $\lceil x \rceil$  denotes the smallest integer not lower than  $x$ . Hence,

$$\begin{aligned} \text{Var} \left( \sum_{j=1}^{(p+1)n-p} \tilde{Y}_{n,j} \right) &\geq \sum_{j=n}^{pn-p+1} \sum_{i_1, i_2 \in B_{j-pn}} \tilde{r}^2(i_2 - i_1) \\ &\geq \sum_{j=n}^{pn-p+1} \sum_{|i|=0}^{\lceil \frac{n-1}{p} \rceil - 2} \tilde{r}^2(i) \left( \left\lceil \frac{n-1}{p} \right\rceil - 1 + |i| \right) \end{aligned}$$

and an integral test leads to  $L_n = O(n^{-4\alpha})$ .

The conditions in Theorem 2.1 in Romano and Wolf (2000) are fulfilled and the convergence in law holds.  $\square$

*Proof of Lemma 10.* When  $p = 0$ , the result is obvious. Let us assume, without loss of generality, that  $p \geq 1$ .

$$\sigma(h_1, h_2) = \frac{1}{2\pi} \int_{[-\pi, \pi]^2} e^{ih_1(x_1 + px_2)} \tilde{f}(x_1 + px_2) e^{i(h_2 - ph_1)x_2} dx_1 dx_2.$$

Let the change of variables  $u = x_1 + px_2$  and  $v = x_2$  :

$$\sigma(h_1, h_2) = \frac{1}{2\pi} \int_{-(p+1)\pi}^{(p+1)\pi} e^{ih_1 u} \tilde{f}(u) \left( \int_{-\pi \vee \frac{u-\pi}{p}}^{\pi \wedge \frac{u+\pi}{p}} e^{i(h_2 - ph_1)v} dv \right) du.$$

Let us first suppose that  $h_2 \neq ph_1$ . When  $p \geq 2$ , the above domain of integration can be cut up as follows (the case  $1 \leq p \leq 2$  is not detailed but can be treated similarly):

$$\sigma(h_1, h_2) = \frac{1}{2\pi(h_2 - ph_1)}(I_1 + I_2),$$

where

$$\begin{aligned} I_1 = & -i \int_{-(p+1)\pi}^{-(p-1)\pi} e^{ih_1 u} \tilde{f}(u) \left( e^{i(h_2 - ph_1)\frac{u+\pi}{p}} - e^{-i(h_2 - ph_1)\pi} \right) du \\ & + \int_{(p-1)\pi}^{(p+1)\pi} e^{ih_1 u} \tilde{f}(u) \left( e^{i(h_2 - ph_1)\pi} - e^{i(h_2 - ph_1)\frac{u+\pi}{p}} \right) du \end{aligned}$$

and

$$I_2 = -i \int_{-(p-1)\pi}^{(p-1)\pi} e^{ih_1 u} \tilde{f}(u) \left( e^{i(h_2 - ph_1)\frac{u+\pi}{p}} - e^{i(h_2 - ph_1)\frac{u-\pi}{p}} \right) du.$$

Some trigonometric computations lead to

$$I_1 = 2 \int_{(p-1)\pi}^{(p+1)\pi} \tilde{f}(u) \left( \sin(h_1 u + (h_2 - ph_1)\pi) - \sin\left(\frac{h_2 u}{p} - (h_2 - ph_1)\frac{\pi}{p}\right) \right) du,$$

$$I_2 = 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \int_0^{(p-1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du.$$

Now, let  $s = u - [p]$  and  $e = p - [p]$ ,

$$\begin{aligned} I_1 = & 2 \int_{-\pi+e\pi}^{\pi+e\pi} \tilde{f}(s + [p]\pi) \left( \sin(h_1 s + h_2 [p]\pi + (h_2 - ph_1)\pi) \right. \\ & \left. - \sin\left(\frac{h_2 s}{p} + h_2 \frac{[p]\pi}{p} - (h_2 - ph_1)\frac{\pi}{p}\right) \right) ds, \end{aligned}$$

$$\begin{aligned} I_2 = & 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \left( \int_0^{([p]-1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du + \right. \\ & \left. + \int_{-\pi}^{-\pi+e\pi} \tilde{f}(s + [p]\pi) \cos\left(\frac{h_2 s}{p} + h_2 \frac{[p]\pi}{p}\right) ds \right). \end{aligned}$$

The domain of integration in  $I_1$  can be splitted into  $-\pi < s < \pi$  and  $s \in [-\pi, -\pi + e\pi] \cup [\pi, \pi + e\pi]$ . From the  $2\pi$ -periodicity of  $\tilde{f}$ , this is easy to check that, when summing up  $I_1$  and  $I_2$ , all the integrals involving  $e$  in their

range of integration sum up to zero. Hence  $I_1 + I_2$  reduces to

$$\begin{aligned}
& 2 \int_{-\pi}^{\pi} \tilde{f}(s + \lfloor p \rfloor \pi) (\sin(h_1 s + h_2 \lfloor p \rfloor \pi + (h_2 - ph_1)\pi) \\
& \quad - \sin\left(\frac{h_2 s}{p} + h_2 \frac{\lfloor p \rfloor \pi}{p} - (h_2 - ph_1) \frac{\pi}{p}\right)) ds \\
& \quad + 4(-1)^{h_1} \sin\left(\frac{h_2 \pi}{p}\right) \int_0^{(\lfloor p \rfloor - 1)\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) du. \quad (26)
\end{aligned}$$

The latest integral above is

$$\sum_{j=0}^{\lfloor p \rfloor - 2} \int_0^{\pi} \tilde{f}(u + j\pi) \cos\left(h_2 \frac{(u + j\pi)}{p}\right) du \quad (27)$$

and it is handled according to the parity of  $\lfloor p \rfloor$  and  $j$ . Since  $\tilde{f}$  is a  $2\pi$ -periodic function,  $\tilde{f}(u + j\pi) = \tilde{f}(u)$  when  $j$  is even and  $\tilde{f}(u + j\pi) = \tilde{f}(u - \pi)$  when  $j$  is odd.

When  $\lfloor p \rfloor$  is even, the sum (27) above is then

$$\begin{aligned}
& \int_0^{\pi} \tilde{f}(u) \sum_{j=-\frac{\lfloor p \rfloor - 2}{2}}^{\frac{\lfloor p \rfloor - 2}{2}} \cos\left(h_2 \frac{(u + 2j\pi)}{p}\right) du \\
& \quad = \int_0^{\pi} \tilde{f}(u) \cos\left(\frac{h_2 u}{p}\right) \frac{\sin\left(h_2 \frac{\lfloor p \rfloor \pi}{p} - h_2 \frac{\pi}{p}\right)}{\sin\left(\frac{h_2 \pi}{p}\right)} du.
\end{aligned}$$

When plugging in this latest result in (26), it simplifies and  $I_1 + I_2$  becomes

$$2 \int_{-\pi}^{\pi} \tilde{f}(s) \sin(h_1 s + (h_2 - ph_1)\pi) ds = 2 \sin((h_2 - ph_1)\pi) \int_{-\pi}^{\pi} \tilde{f}(s) \cos(h_1 s) ds.$$

This proves the result of the lemma for  $h_2 \neq ph_1$  in the case  $\lfloor p \rfloor$  even.

When  $\lfloor p \rfloor$  is odd, the sum (27) is

$$\begin{aligned}
& \int_0^{2\pi} \tilde{f}(u) \sum_{j=0}^{\frac{\lfloor p \rfloor - 3}{2}} \cos\left(h_2 \frac{(u + 2j\pi)}{p}\right) du = \\
& \quad \int_0^{2\pi} \tilde{f}(u) \sin\left(\frac{h_2 \pi \lfloor p \rfloor - 1}{p}\right) \frac{\cos\left(\frac{h_2 u}{p} + \frac{h_2 \pi \lfloor p \rfloor - 3}{p}\right)}{\sin\left(\frac{h_2 \pi}{p}\right)} du. \quad (28)
\end{aligned}$$

Now, in (26), we split the domain of the first integral into  $-\pi < s < 0$  where  $\tilde{f}(s + \lfloor p \rfloor \pi) = \tilde{f}(s + \pi)$  and  $0 < s < \pi$  where  $\tilde{f}(s + \lfloor p \rfloor \pi) = \tilde{f}(s - \pi)$ . We apply respectively the change of variables  $s = s + \pi$  and  $s = s - \pi$ . This allows to

exhibit the integral  $2 \int_{-\pi}^{\pi} \tilde{f}(s) \sin(h_1 s + (h_2 - ph_1)\pi) ds$ . With the help of (28), some trigonometric computations show that the remaining terms coming from this change of variables simplify with the remaining term in (26).

Therefore, when  $h_2 \neq ph_1$ , the result of the lemma is proved for all  $p$ .

The proof when  $h_2 = ph_1$  is simpler and it can be conducted in the same way. □

## 5 Appendix on Toeplitz Operators

Let  $f \in L^p(E)$ ,  $1 \leq p \leq \infty$ . As explained below, for all  $1 \leq p \leq \infty$ , the finite Toeplitz Operator associated to  $f$ ,  $T_n(f)$ , is defined as the projection of a bounded Multiplication Operator. It can be identified to the “matrix representation”  $(\hat{f}_{p-q})_{p,q \in A_n^2}$ . We refer to Böttcher and Silbermann (1990) for further information about Toeplitz Operators.

When  $f \in L^\infty(E)$ , define  $T(f)$  the Multiplication Operator generated by  $f$

$$\forall h \in L^2(E), \quad T(f)(h) = fh.$$

Denote by  $P_n$  the projection acting on  $L^2(E)$  by

$$P_n \left( \sum_{k \in \mathbb{Z}^d} c_k e^{i\langle k, \cdot \rangle} \right) = \sum_{k \in A_n} c_k e^{i\langle k, \cdot \rangle}.$$

In this case the finite Toeplitz Operator associated to  $f$  is  $T_n(f) = P_n T(f) P_n$ . It can be identified to the “matrix representation”  $(\hat{f}_{p-q})_{p,q \in A_n^2}$  since

$$\int_E T_n(f)(e^{i\langle p, \lambda \rangle}) e^{-i\langle q, \lambda \rangle} d\lambda = \hat{f}_{p-q}.$$

When  $f \in L^p(E)$ ,  $1 \leq p < \infty$ , consider  $T(f)$  defined by

$$\langle T(f)(e^{i\langle p, \cdot \rangle}), e^{i\langle q, \cdot \rangle} \rangle = \begin{cases} \hat{f}_{q-p} & \text{if } (p, q) \in A_n^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\tilde{f} \equiv T(f)(1)$  belongs to  $L^\infty(E)$  and we can define  $T_n(f) \equiv T_n(\tilde{f})$  where  $T_n(\tilde{f})$  is the finite Toeplitz Operator defined as above.  $T_n(f)$  has obviously the same “matrix representation”  $(\hat{f}_{p-q})_{p,q \in A_n^2}$ .

The  $p$ -Schatten norm of a symmetric bounded linear operator  $A$  with eigenvalues  $\{\lambda_n\}_{n \geq 1}$  is  $\|A\|_p^p = \sum_{n \geq 1} \lambda_n^p$ ,  $1 \leq p < \infty$ , and  $\|A\|_\infty = \sup_n \{\lambda_n\}$ . We recall the following inequality for the  $p$ -Schatten norms of finite Toeplitz operators:

**Lemma 12.** *Let  $f \in L^p(E)$ ,  $1 \leq p \leq \infty$ , then*

$$\|T_n(f)\|_p \leq n^{d/p} \|f\|_p.$$

This result is stated in dimension 1 in Lemma 1 of Avram (1988) and generalized in dimension  $d$  in Proposition A.1 of Doukhan et al. (1996).

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