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► **To cite this version:**

Xiaonan Ma, Weiping Zhang. Geometric quantization for proper moment maps. 2008. hal-00348681v1

HAL Id: hal-00348681

<https://hal.science/hal-00348681v1>

Preprint submitted on 19 Dec 2008 (v1), last revised 19 Sep 2012 (v2)

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GEOMETRIC QUANTIZATION FOR PROPER MOMENT MAPS

XIAONAN MA AND WEIPING ZHANG

Abstract. We establish a geometric quantization formula for Hamiltonian actions of a compact Lie group acting on a non-compact symplectic manifold such that the associated moment map is proper. In particular, we resolve the conjecture of Vergne in this non-compact setting.

0. Introduction

The famous geometric quantization conjecture of Guillemin and Stenzel [9] states that for a compact pre-quantizable symplectic manifold admitting a Hamiltonian action of a compact Lie group, the principle of "quantization commutes with reduction" holds. This conjecture was first proved independently by Meinrenken [14] and Vergne [23] for the case where the Lie group is abelian, and then by Meinrenken [15] in the general case. The singular reduction case was proved by Meinrenken-Sjamaar in [16]. There are also an analytic approach to the original conjecture developed by Tian and Zhang [20] as well as a proof developed by Paradan [17] by making use of the theory of transversally elliptic operators, see also [24] for an excellent survey.

It is natural to consider the generalizations of the above results to actions on non-compact spaces. One of the aspects of this issue has been considered by Weitsman in [26], where the properness of the associated moment map is assumed. In her ICM 2006 Plenary lecture [25], Vergne made a quantization conjecture (under the assumption that the zero point set of the vector field generated by the moment map is compact), which generalizes the original Guillemin-Stenzel conjecture to this non-compact setting. A special case of this conjecture had indeed been verified already by Paradan in [18] where he proved a quantization formula valid for the case where a maximal compact subgroup of a non-compact real semi-simple Lie group acts on the co-adjoint orbits of the real semi-simple Lie group itself.

The purpose of this paper is to establish a general quantization formula in this framework of a compact group acting on a non-compact space with proper moment map. As we will see, our result could be viewed as an extended version of the conjecture of Vergne, in the sense that we do not make any extra assumptions beside the properness of the moment map.

To be more precise, let $(M; \omega)$ be a non-compact symplectic manifold with symplectic form ω . We assume that $(M; \omega)$ is prequantizable, that is, there exists a complex line bundle L (called a prequantized line bundle) carrying a Hermitian metric h^L and a

Hermitian connection r^L such that

$$(0.1) \quad \frac{p-1}{2} r^L{}^2 = ! :$$

We also assume that there exists an almost complex structure J on TM such that

$$(0.2) \quad g^{TM}(u;v) = ! (u;Jv); \quad u; v \in TM$$

defines a Riemannian metric on TM .

Let G be a compact connected Lie group with Lie algebra denoted by \mathfrak{g} . We assume that G acts on the left on M and this action can be lifted to an action on L^*M . Moreover, we assume that G preserves g^{TM} , J , h^L and r^L .

For any $K \in \mathfrak{g}$, let K^M be the vector field generated by K over M .

Let $\mu : M \rightarrow \mathfrak{g}^*$ be defined by the Kostant formula [10]

$$(0.3) \quad \frac{p-1}{2} \mu(K) := \langle K, \mu \rangle = r^L_{K^M} \quad L_K; \quad K \in \mathfrak{g} :$$

Then μ is the corresponding moment map, i.e. for any $K \in \mathfrak{g}$,

$$(0.4) \quad d \langle K, \mu \rangle = \langle K^M, ! \rangle :$$

We call the G -action with a moment map $\mu : M \rightarrow \mathfrak{g}^*$ verifying (0.4) a Hamiltonian action.

From now on, we assume that the following fundamental assumption holds.

Fundamental Assumption. The moment map $\mu : M \rightarrow \mathfrak{g}^*$ is proper, in the sense that the inverse image of a compact subset is compact.

Let T be a maximal torus of G , $C_G \subset \mathfrak{g}$ be a Weyl chamber associated to T , Λ be the weight lattice, and $\Phi = \Lambda \setminus C_G$ be the set of dominant weights. Then the ring of characters $R(G)$ of G has a \mathbb{Z} -basis V^μ , $\mu \in \Phi$: V^μ is the irreducible G -representation with highest weight μ .

Take any $\mu \in \Phi$. If μ is a regular value of the moment map, then one can construct the Marsden-Winstein symplectic reduction $(M_\mu; !)$, where $M_\mu = \mu^{-1}(\mu) = G \cdot \mu$ is a compact (as μ is proper) orbifold. Moreover, the line bundle L (resp. the almost complex structure J) induces a prequantized line bundle L_μ (resp. an almost complex structure J_μ) over $(M_\mu; !)$. One can then construct the associated Spin^c -Dirac operator (twisted by L_μ), $D_+^L : \Gamma^{0,\text{even}}(M_\mu; L_\mu) \rightarrow \Gamma^{0,\text{odd}}(M_\mu; L_\mu)$ (cf. Section 1.6, (1.5)) on M_μ whose index

$$(0.5) \quad Q(L_\mu) := \dim \text{Ker } D_+^L - \dim \text{Coker } D_+^L \in 2\mathbb{Z};$$

is well-defined. If $\mu \in \Phi$ is not a regular value of μ , then by proceeding as in [16] (cf. [17, x7.4]), one still gets a well-defined quantization number $Q(L_\mu)$ extending the above definition¹.

On the other hand, let \mathfrak{g} be equipped with an Ad_G -invariant metric. Set $H = \mu^{-1}(\mu)$. Then since μ is proper, for any $c > 0$, $U_c := \mu^{-1}([0; c]) = \{x \in M : H(x) \leq c\}$ is a compact subset of M .

¹See also Section 2.1 for a standard perturbative definition.

Recall that by Sard's theorem, the set of critical values of the function $H : M \rightarrow \mathbb{R}$ is of measure zero.

Let $X^H = -J(dH)$ be the Hamiltonian vector field associated to H .

For any regular value $c > 0$ of H , one knows X^H is nowhere zero on $@U_c = H^{-1}(c)$. Thus, according to Atiyah [1, x1, x3] and Paradan [17, x3] (cf. also Vergne [23]), the triple $(U_c; X^H; L)$ defines a transversally elliptic symbol (corresponding to the Spivak-Dirac operator (twisted by L) on M) associated to the G -action on U_c . And according to Atiyah [1, x1], it admits a well-defined transversal index whose character is a distribution on G .

For any \mathfrak{g} , let $Q(L)_c \in \mathbb{Z}$ denote the \mathfrak{g} -component of this transversal index.

Theorem 0.1. For any \mathfrak{g} , there exists $c > 0$ such that $Q(L)_c \in \mathbb{Z}$ does not depend on $c > c$, with c a regular value of H . And $Q(L)_{c=0} \in \mathbb{Z}$ does not depend on $c > 0$, with c a regular value of H .

According to Theorem 0.1, for any \mathfrak{g} , we have a well-defined integer $Q(L)_{\mathfrak{g}}$ not depending on the regular value $c > 0$. From now on we denote it by $Q(L)_{\mathfrak{g}}$.

We can now state our main result as follows.

Theorem 0.2. For any \mathfrak{g} , the following identity holds,

$$(0.6) \quad Q(L)_{\mathfrak{g}} = Q(L)_{\mathfrak{g}};$$

Remark 0.3. If the zero set of X^H is compact, then Theorem 0.1 was already known in [17] and [25], while Theorem 0.2 was conjectured by Vergne in [25, x4.3]. Thus Theorem 0.2 can be thought of as an extended version of the Vergne conjecture.

If we set

$$(0.7) \quad Q_G(L)^1 = \sum_{\mathfrak{g} \in \widehat{G}} Q(L)_{\mathfrak{g}} V^{\mathfrak{g}};$$

then by Theorem 0.2, $Q_G(L)^1$ equals to the formal geometric quantization in the sense of [26, Definition 4.1] and [19, Definition 1.2]. In particular, it verifies the functorial quantization property described as follows.

It is clear that if M is compact, then Theorem 0.1 holds tautologically and Theorem 0.2 is the Guillemin-Stenzel conjecture proved in [16]. Let $(N; !^N)$ be such a pair with N being compact, F the notation for the prequantized line bundle over N , etc. Combining Theorem 0.2 with the result [26, Theorem 1] (cf. also [19, Theorem 1.5]) one gets the following functorial quantization result.

Let $L \otimes F$ be the prequantized line bundle over $M \times N$ obtained by the tensor product of the natural liftings of L and F to $M \times N$.

Theorem 0.4. For the induced action of G on $(M \times N; !^M \otimes !^N)$ and $L \otimes F$, the following identity holds,

$$(0.8) \quad Q(L \otimes F)_{\mathfrak{g}} = \sum_{\mathfrak{g} \in \widehat{G}} Q(L)_{\mathfrak{g}} Q(F)_{\mathfrak{g}};$$

By taking N to be the orbits of the co-adjoint action of G on \mathfrak{g}^* , one recovers Theorem 0.2 from Theorem 0.4 immediately. Thus Theorem 0.2 and 0.4 are actually equivalent. In this paper we will write out our proof for Theorem 0.4 directly. In this way we also get a new proof of the following functorial quantization result due to Weinstein [26, Theorem 1] and Paradan [19, Theorem 1.5], without using the symplectic cut techniques there.

Corollary 0.5. Under the assumptions in Theorem 0.4, the following identity holds,

$$(0.9) \quad Q_{(L, F)} = \int_{2\widehat{G}}^X Q(L) Q(F) :$$

Now let K be a compact subgroup of G such that the moment map of the induced Hamiltonian action of K on M also verifies the fundamental assumption of being proper. Then by combining Theorem 1.2, (0.7) with [19, Theorem 1.3], one gets the following consequence on the relation between $Q_G(L)$ and $Q_K(L)$.

Corollary 0.6. Any irreducible representation of K has a finite multiplicity in $Q_G(L)$. Moreover, the following identity holds, when both sides are viewed as virtual representation spaces of K ,

$$(0.10) \quad Q_G(L) = Q_K(L) :$$

Our proof of Theorems 0.1 and 0.4 is analytic. It makes use of Brawerman's analytic interpretation [5, x14] of the transversal index of a wide class of transversally elliptic operators covering the ones mentioned above. The main idea is that through the analytic interpretation of Brawerman, one can further express this transversal index by using the Atiyah-Patodi-Singer type index² for Dirac type operators on manifolds with boundary [2]. One can then apply the analytic methods developed in [20] and [22] to study the corresponding quantization problem.

Indeed, after we interpret the transversal index by the APS type index, it is almost direct to prove Theorem 0.1 by applying the analytic techniques in [20] and [22].

The proof of Theorem 0.4 needs more effort. Tautologically, one would adapt the idea of two steps deformations appeared in [18, x3] to the current situation. The main point comes from the fact that, after choosing the suitable (already non-trivial) first step deformation, in order to pass from the boundaries of submanifolds of $M \times N$ to boundaries of specific forms obtained by the product of a boundary in M with N , one may encounter a lot of zero points of the vector fields used in the deformation. It is then necessary to eliminate the potential contributions caused by these possible zero points.³

Recall that in the analytic proof of the original Guillemin-Stenzel conjecture developed in [20], one already encounters the local pointwise estimates around the zero points of the Hamiltonian vector fields used in the deformation there. It is remarkable

²We will briefly call it as APS index in what follows.

³The situation considered in [18] is much simpler as there is no zero point of the vector fields used in the corresponding deformation, he can then apply directly the homotopy invariance of the transversal index.

that such kind of local estimates still hold in the current (apparently more sophisticated) non-compact situation.

We would like to point out that one may indeed formulate Theorems 0.1, 0.2 and 0.4 directly using the APS type index which is more intrinsic (as the definition of the transversal index involved depends on certain extra structures which go beyond the symplectic structures involved).

It would also be interesting to give a direct analytic proof of Corollary 0.6 without using the symplectic cut techniques in [19].

The rest of this paper is organized as follows. In Section 1, we first apply the results of Braverman [5] to interpret certain transversal index as a kind of APS type index. We then apply the analytic approach of the quantization formulas developed in [20], [22] and prove Theorem 0.1. In Section 2, we present our proof of Theorem 0.4 modulo a vanishing result, Theorem 2.4, whose proof will be carried out in Section 3.

Some results of this paper have been announced in [13].

Acknowledgements. We would like to thank Professor Jean-Michel Bismut for many helpful discussions. The work of the second author was partially supported by MOEC and NNSFC. Part of the paper was written while the author was visiting the School of Mathematics of Fudan University during November and December of 2008. He would like to thank Professor Jiaying Hong and other members of the School for hospitality.

1. Transversal index and quantization for proper moment maps

In this section, we prove Theorem 0.1. In doing so, we first express the transversal index appearing in the context as certain APS type index, with the help of a result by Braverman [5], then we apply the analytic methods developed in [20] and [22] to complete the proof of Theorem 0.1.

This section is organized as follows. In Section 1.1, we recall the definition certain transversal index in the sense of Atiyah [1] and Paradan [17] for group actions on manifolds with boundary. In Section 1.2 we consider certain APS type index in the same framework as in Section 1.1. In Section 1.3 we prove an invertibility result for some induced boundary operator. In Section 1.4 we introduce the idea of spectral flow in our context and use it to complete the proof of a result stated in Section 1.2. In Section 1.5, we give an APS index interpretation of the transversal index in Section 1.1 by applying a result of Braverman [5]. In Section 1.6, we prove Theorem 0.1 by applying the analytic method developed in [20] and [22].

1.1. Transversal index. Let M be an even dimensional compact oriented Spin^c -manifold with non-empty boundary ∂M . Let E be a complex vector bundle over M .

Let G be a compact connected Lie group with Lie algebra denoted by \mathfrak{g} . We assume that G acts on the left on M and that this action can be lifted to an action of G on E .

Let the tangent vector bundle $\pi: TM \rightarrow M$ carry a G -invariant metric g^{TM} . Then one identifies TM and TM via g^{TM} .

For $K \geq 2$, we denote by $K_x^M = \frac{\partial}{\partial t} e^{tK} \Big|_{t=0}$ the corresponding vector field on M .

Following [1, p. 7] (cf. [17, x3]), set

$$(1.1) \quad T_G M = \{v \in TM : v(x); K^M(x) = 0 \text{ for all } K \in \mathfrak{g}\} :$$

Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the bundle of spinors associated to the spin^c -structure on TM and g^{TM} . For any $V \in TM$, the Clifford action $c(V)$ exchanges $S_+(TM)$, the Z_2 -grading on $S(TM)$.

Let $\rho : M \rightarrow \mathfrak{g}$ be a G -equivariant map. Let ρ^M denote the vector field over M such that $\rho^M(x)$ equals to the value at x of the vector field generated by $\rho(x) \in \mathfrak{g}$ over M .

We make the assumption that ρ^M is nowhere zero on ∂M .

$$(1.2) \quad \rho^M_E : \text{Hom}(S_+(TM) \otimes E; S_-(TM) \otimes E) \text{ denote the symbol defined by}$$

$$(1.2) \quad \rho^M_E(x; v(x)) = \frac{P}{|c(v + \rho^M)|} \text{Id}_{E_{(x, v(x))}} \text{ for } (x; v(x)) \in T_x M :$$

Since ρ^M is nowhere zero on ∂M , by (1.1) and (1.2) one sees that the zero set of the restriction of ρ^M_E on $T_G M$ is contained in a compact subset of $\mathring{M} \times T_G \mathring{M}$ (where $\mathring{M} = M \setminus \partial M$ is the interior of M). Thus, it defines a G -transversally elliptic symbol on $T_G \mathring{M}$ in the sense of Atiyah [1, x1, x3] and Paradan [17, x3] (see also Vergne [23]), which in turn determines a transversal index⁴

$$(1.3) \quad \text{Ind} \rho^M_E = \sum_{2\hat{G}} \text{Ind} \rho^M_E \hat{V};$$

with each $\text{Ind}(\rho^M_E) \in \mathbb{Z}$.

1.2. The Atiyah-Patodi-Singer (APS) index. We continue the setting in Section 1.1.

Let h^E be a G -invariant Hermitian metric on E , r^E a G -invariant Hermitian connection with respect to h^E . Let $h^{S(TM) \otimes E}$ be the metric on $S(TM) \otimes E$ induced by the metrics on $S(TM)$ and on E .

Let $r^{S(TM)}$ be the Clifford connection on $S(TM)$ induced by the Levi-Civita connection r^{TM} of g^{TM} and a Hermitian connection on the line bundle defining the spin structure. Let $r^{S(TM) \otimes E}$ be the Hermitian connection on $S(TM) \otimes E$ obtained by the tensor product of the connections $r^{S(TM)}$ and r^E .

Let dv_M be the Riemannian volume form on $(M; g^{TM})$. For $s \in \mathcal{C}^1(M; S(TM) \otimes E)$, its L_2 -norm $\|s\|_0$ is defined by

$$(1.4) \quad \|s\|_0^2 = \int_M \langle s, s \rangle_{S(TM) \otimes E}(x) dv_M(x) :$$

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{C}^1(M; S(TM) \otimes E)$ corresponding to $\| \cdot \|_0^2$.

One can construct now the Spin^c -Dirac operator (twisted by E) (cf. [11, Appendix D]) by

$$(1.5) \quad D^E = \sum_{i=1}^{\dim M} c(e_i) r_{e_i}^{S(TM) \otimes E} : \mathcal{C}^1(M; S(TM) \otimes E) \rightarrow \mathcal{C}^1(M; S(TM) \otimes E);$$

where $\{e_i\}$ is an orthonormal basis of TM . Then D^E is G -equivariant and formally self-adjoint. Denote by D^E the restrictions of D^E on $\mathcal{C}^1(M; S_+(TM) \otimes E)$ respectively.

⁴We will not exploit the distribution nature of this index here.

Following [8, p. 139], let $\epsilon > 0$ be less than the injectivity radius of g^{T^*M} . We use the inward geodesic flow to identify a neighborhood of the boundary ∂M with the collar $\partial M \times [0; \epsilon]$. Let $e_{\dim M}$ be the inward unit normal vector field perpendicular to ∂M . Let $e_1; \dots; e_{\dim M - 1}$ be an oriented orthonormal basis of $T\partial M$ so that $e_1; \dots; e_{\dim M - 1}; e_{\dim M}$ is an oriented orthonormal basis of $TM|_{\partial M}$. By using parallel transport with respect to r^{T^*M} along the unit speed geodesics perpendicular to ∂M , $e_1; \dots; e_{\dim M}$ forms an oriented orthonormal basis of TM over $\partial M \times [0; \epsilon]$.

Let $D_{\partial M}^E : \mathcal{C}^1(\partial M; (S(T^*M)|_{\partial M})^E) \rightarrow \mathcal{C}^1(\partial M; (S(T^*M)|_{\partial M})^E)$ be the induced (by D^E) Dirac operator on ∂M defined by (cf. [8, p. 142])

$$(1.6) \quad D_{\partial M}^E = \sum_{i=1}^{\dim X - 1} c(e_{\dim M}) c(e_i) r_{e_i}^{S(T^*M)|_{\partial M}} + \frac{1}{2} \sum_{i=1}^{\dim X - 1} \epsilon_i \epsilon_i$$

where

$$(1.7) \quad \epsilon_{ij} = r_{e_i}^{T^*M} e_j; e_{\dim M}|_{\partial M}; \quad 1 \leq i, j \leq \dim M - 1;$$

is the second fundamental form of the isometric embedding $\{e_M : \partial M \hookrightarrow M\}$. Let $D_{\partial M}^E$ be the restriction of $D_{\partial M}^E$ to $\mathcal{C}^1(\partial M; (S(T^*M)|_{\partial M})^E)$.

As in (1.4), we define the inner product $\langle \cdot, \cdot \rangle_{\partial M}$ and the L_2 -norm $\|\cdot\|_{\partial M}$ on $\mathcal{C}^1(\partial M; (S(T^*M)|_{\partial M})^E)$.

By [8, Lemma 2.2], $D_{\partial M}^E$ is a formally self-adjoint first order elliptic operator intrinsically defined on ∂M ⁵.

On the other hand, since G acts on M and thus preserves ∂M , one has

$$(1.12) \quad \int_{\partial M} \langle D_{\partial M}^E \mathcal{C}^1(\partial M; T\partial M) \rangle$$

⁵In fact, one has for any $s \in \mathcal{C}^1(M; S(T^*M)|_{\partial M})^E$, the following identities hold on ∂M ,

$$(1.8) \quad D^E s = c(e_{\dim M}) r_{e_{\dim M}}^{S(T^*M)|_{\partial M}} s + c(e_{\dim M}) D_{\partial M}^E \left(\frac{1}{2} \sum_{i=1}^{\dim X - 1} \epsilon_i (s_{j_M}^i) \right);$$

$$c(e_{\dim M}) D_{\partial M}^E (s_{j_M}^i) = D_{\partial M}^E c(e_{\dim M}) (s_{j_M}^i);$$

In particular, let r^{T^*M} be the Levi-Civita connection on $(T\partial M; g^{T^*M})$, with g^{T^*M} the metric on $T\partial M$ induced by g^{T^*M} , we define $e(X) = c(e_{\dim M}) c(X)$, for any $X \in T\partial M$, then $e(\cdot)$ induces a Clifford action of the Clifford algebra $C(T\partial M)$ on $(S(T^*M)|_{\partial M})^E$, and the Hermitian connection

$$(1.9) \quad r_X^{(S(T^*M)|_{\partial M})^E} = r_X^{S(T^*M)|_{\partial M}} + \frac{1}{2} \sum_{i,j=1}^{\dim X - 1} \epsilon_{ij} c(e_{\dim M}) c(e_j) hX; e_i \epsilon_i$$

is the $e(\cdot)$ -Clifford connection on $(S(T^*M)|_{\partial M})^E$, verifying for example that for any $X; Y \in \mathcal{C}^1(\partial M; T\partial M)$,

$$(1.10) \quad h r_X^{(S(T^*M)|_{\partial M})^E} e(Y); e(Y) = e r_X^{T^*M} Y$$

Thus $D_{\partial M}^E$ is the Dirac operator on $(S(T^*M)|_{\partial M})^E$ associated to $r^{(S(T^*M)|_{\partial M})^E}$, i.e.

$$(1.11) \quad D_{\partial M}^E = \sum_{j=1}^{\dim X - 1} e(e_j) r_{e_j}^{(S(T^*M)|_{\partial M})^E}$$

From (1.11), we understand that $D_{\partial M}^E$ is formally self-adjoint and intrinsically defined on ∂M .

By (1.12) and following [8, Lemma 2.2] and [22, x1c)] (see also [5, x2.8]), set for $T \geq R$,

$$(1.13) \quad \begin{aligned} D_T^E &= D^E + \frac{p-1}{2} T C^M : \mathcal{C}^1(M; S(TM) \otimes E) \rightarrow \mathcal{C}^1(M; S(TM) \otimes E); \\ D_{\neq T}^E &= D^E + \frac{p-1}{2} T C^M : \mathcal{C}^1(M; S(TM) \otimes E) \rightarrow \mathcal{C}^1(M; S(TM) \otimes E); \end{aligned}$$

and

$$(1.14) \quad \begin{aligned} D_{\partial M, T}^E &= D_{\partial M}^E + \frac{p-1}{2} T C^{(\dim M)} C^M \\ &: \mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M}) \rightarrow \mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M}); \\ D_{\partial M, \neq T}^E &= D_{\partial M, \neq T}^E : \mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M}) \rightarrow \mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M}). \end{aligned}$$

Then $D_{\partial M, T}^E$ is also formally self-adjoint and preserves $\mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M})$. For any $\lambda \in \text{Spec} D_{\partial M, T}^E$, the spectrum of $D_{\partial M, T}^E$, let $E_{\lambda, T}$ be the corresponding eigenspace. Let $P_{>0, T}$ (resp. $P_{>0, \neq T}$) be the orthogonal projection from the L^2 -completions of $\mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M})$ onto ${}_{>0}E_{\lambda, T}$ (resp. ${}_{>0}E_{\lambda, \neq T}$).

For any $T \geq R$, let $(D_{+, T}^E; P_{>0, \neq T})$ (resp. $(D_{\neq T}^E; P_{>0, T})$) denote the corresponding Atiyah-Patodi-Singer boundary value problem [2]. More precisely, the boundary condition of $D_{+, T}^E$ is $P_{>0, \neq T}(s|_{\partial M}) = 0$ for $s \in \mathcal{C}^1(M; S_+(TM) \otimes E)$ (resp. of $D_{\neq T}^E$ is $P_{>0, T}(s|_{\partial M}) = 0$ for $s \in \mathcal{C}^1(M; S(TM) \otimes E)$).

Both $(D_{+, T}^E; P_{>0, \neq T})$ and $(D_{\neq T}^E; P_{>0, T})$ are elliptic, and $(D_{\neq T}^E; P_{>0, T})$ is the adjoint of $(D_{+, T}^E; P_{>0, \neq T})$ (cf. (1.8), [8, Theorem 2.3]). Moreover, it is clear that both $(D_{+, T}^E; P_{>0, \neq T})$ and $(D_{\neq T}^E; P_{>0, T})$ are G -equivariant.

$$(1.15) \quad \begin{aligned} \text{Let } Q_{\text{APS}, T}^M(E; \cdot) \in \mathbb{Z}, \quad \hat{\mathcal{C}} \text{ be defined by} \\ Q_{\text{APS}, T}^M(E; \cdot) \in \hat{\mathcal{C}} = \text{Ind } D_{+, T}^E; P_{>0, \neq T} \\ \in \text{Ker } D_{+, T}^E; P_{>0, \neq T} \oplus \text{Ker } D_{\neq T}^E; P_{>0, T} : \end{aligned}$$

The following important result will be proved in the next two subsections.

Proposition 1.1. For any $\lambda \in \hat{\mathcal{C}}$, there exists $T > 0$ such that $Q_{\text{APS}, T}^M(E; \lambda)$ does not depend on $T > T_0$.

1.3. An estimate on the boundary. The proof of Proposition 1.1 consists two steps. In the first step, we show that the boundary operator $D_{\partial M, T}^E$, when restricted to the λ -component, is invertible when $T > 0$ is large. Then in the second step, we apply the spectral flow idea to complete the proof of Proposition 1.1.

In this subsection, we carry out the first step.

For any $\lambda \in \hat{\mathcal{C}}$, let $\mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M})$ denote the λ -component of $\mathcal{C}^1(\partial M; S(TM) \otimes E)|_{\partial M}$. We denote the restriction of $D_{\partial M, T}^E$ on $\mathcal{C}^1(\partial M; (S(TM) \otimes E)|_{\partial M})$ by $D_{\partial M, T}^E(\lambda)$.⁶

Proposition 1.2. For any $\lambda \in \hat{\mathcal{C}}$, there exists $T > 0$ such that for any $T > T_0$, $D_{\partial M, T}^E(\lambda)$ is invertible.

⁶Since $D_{\partial M, T}^E$ is G -equivariant, $D_{\partial M, T}^E(\lambda)$ is well-defined.

Proof. Let \mathfrak{g} (thus \mathfrak{g}^*) be equipped with an Ad_G -invariant metric. Let $h_1; \dots; h_{\dim \mathfrak{g}}$ be an orthonormal basis of \mathfrak{g} with its dual basis V_i of \mathfrak{g}^* . Then one has

$$(1.16) \quad \psi(x) = \sum_{i=1}^{\dim M} \psi_i(x) V_i;$$

where $\psi_i, 1 \leq i \leq \dim M$, are smooth functions on M . From (1.16), one gets

$$(1.17) \quad \Delta_M \psi(x) = \sum_{i=1}^{\dim M} \psi_i(x) \Delta_M V_i(x);$$

Recall that from (1.12), $\Delta_M \in \mathcal{C}^1(\mathcal{O}_M; T^*\mathcal{O}_M)$. From (1.6), (1.14) and (1.17), one finds,

$$(1.18) \quad \begin{aligned} D_{\mathcal{O}_M; T}^E \psi^2 &= D_{\mathcal{O}_M}^E \psi^2 - \frac{1}{2} \sum_{i=1}^{\dim M} \Delta_M \psi_i^2 \\ &+ \frac{1}{2} \sum_{i=1}^{\dim M} \Delta_M \psi_i^2 - \sum_{i=1}^{\dim M} \psi_i \Delta_M \psi_i \\ &= D_{\mathcal{O}_M}^E \psi^2 - \frac{1}{2} \sum_{i=1}^{\dim M} \Delta_M \psi_i^2 + \sum_{i=1}^{\dim M} \psi_i \Delta_M \psi_i : \end{aligned}$$

For any $1 \leq i \leq \dim M$, let L_{V_i} denote the Lie derivative of V_i acting on $\mathcal{C}^1(\mathcal{O}_M; S^*(TM) \otimes E)$ and thus also on $\mathcal{C}^1(\mathcal{O}_M; (S^*(TM) \otimes E)|_{\mathcal{O}_M})$. Then

$$r_{V_i}^{S(TM) \otimes E} \in \mathcal{C}^1(\mathcal{O}_M; \text{End}(S(TM) \otimes E))$$

is a bounded operator.

By (1.17), we write

$$(1.19) \quad r_M^{S(TM) \otimes E} = \sum_{i=1}^{\dim M} L_{V_i} + \sum_{i=1}^{\dim M} r_{V_i}^{S(TM) \otimes E} \psi_i;$$

It is clear that when restricted on $\mathcal{C}^1(\mathcal{O}_M; (S(TM) \otimes E)|_{\mathcal{O}_M})$, each L_{V_i} is bounded. On the other hand, since ψ^2 is nowhere zero on \mathcal{O}_M , there exists $C > 0$ such that

$$(1.20) \quad \psi^2 > 4C \quad \text{on } \mathcal{O}_M;$$

From (1.18)–(1.20), there exists a positive constant $C > 0$ such that for any $s \in \mathcal{C}^1(\mathcal{O}_M; (S(TM) \otimes E)|_{\mathcal{O}_M})$, one has

$$(1.21) \quad D_{\mathcal{O}_M; T}^E s^2 \geq D_{\mathcal{O}_M}^E s^2 - TC \|s\|_{\mathcal{O}_M}^2 + T^2 C \|s\|_{\mathcal{O}_M}^2;$$

From (1.21), one sees easily that Proposition 1.2 holds for $T = 2C = C$.

1.4. Spectral flow and a proof of Proposition 1.1. It is an easy matter to extend the idea of spectral flow ([3, x7]) to the current component situation. Recall that such an extension to the G -invariant case has already been considered in [22, x4a)].

Let $\text{FD}_t; 0 \leq t \leq 1$ be a one parameter smooth family of self-adjoint G -equivariant Dirac type operators acting on $\mathcal{C}^1(\mathcal{O}_M; (S_+(TM) \otimes E)|_{\mathcal{O}_M})$. We define the component spectral flow of $\text{FD}_t; 0 \leq t \leq 1$, denoted by $\text{Sf FD}_t; 0 \leq t \leq 1$, to be the spectral flow of the family of self-adjoint Fredholm operators $\text{FD}_t(\cdot); 0 \leq t \leq 1$ in the sense of

Atiyah-Patodi-Singer [3, x7], where for any $t \in [0; 1]$, $D_t(\cdot)$ is the restriction of D_t on $\mathcal{C}^1(\partial M; (S_+(TM) \oplus E)|_{\partial M})$.

In view of (1.15), one can then proceed as in [22, Theorem 4.2] (cf. also [6, Theorem 1.1]) to get that for any $T > T_0$,

$$(1.22) \quad Q_{APS;T}^M(E; \cdot) - Q_{APS;T}^M(E; \cdot) = \int_{\partial M} D_{\partial M;T}^E; T_0 \leq t \leq T :$$

By (1.22) and Proposition 1.2, we get Proposition 1.1.

Remark 1.3. Indeed, since the induced boundary operators are invertible for $T > T_0$, the APS boundary problems involved form a continuous family of Fredholm operators, which implies the constancy of the associated index. By this one avoids the consideration of spectral flow in the proof of Proposition 1.1.

1.5. Transversal index and the APS index. Denote by $Q_{APS}^M(E; \cdot)$ the quantization number $Q_{APS;T}^M(E; \cdot)$ for $T > T_0$ appearing in Proposition 1.1.

The following main result of this subsection identifies the transversal index in Section 1.1 with the above APS type index.

Theorem 1.4. The following identity holds for \mathbb{D}^m ,

$$(1.23) \quad \text{Ind } D_{\mathbb{D}^m}^E = Q_{APS}^M(E; \cdot) :$$

Proof. We can enlarge the manifold M a little bit to get a compact Spin^c -manifold U with boundary such that $M \cup \partial U = \mathbb{D}^m$, the interior of U , and that everything extends in a G -invariant way to U from M , moreover, the corresponding vector field ξ^U is nowhere zero on $U \setminus \mathring{M}$, with \mathring{M} the interior part of M . The existence of U is clear.

Then by proceeding as in [5, x14], one can construct a complete metric $g^{\widehat{U}}$ on \widehat{U} such that $g^{\widehat{U}}|_{\widehat{M}} = g^{\widehat{M}}$, as well as an admissible (G -invariant) function f on \widehat{U} in the sense of [5, Definition 2.6] such that $f|_{\widehat{M}} = 1$, and $f > \frac{1}{2}$ on $\widehat{U} \setminus \mathring{M}$, $f \rightarrow +1$ near ∂U .

Let

$$(1.24) \quad D_{\widehat{U};T}^E = D^E + \frac{p-1}{2} f c(\xi^U)$$

be the corresponding (G -equivariant) Dirac type operator on \widehat{U} . Then it is shown in [5, x14] that when restricted to a fixed \mathbb{R} -component, for any $T > 0$, the restricted operator $D_{\widehat{U};T}^E(\cdot)$ is a Fredholm operator, moreover, one has

$$(1.25) \quad \text{Ind } D_{\widehat{U};T}^E(\cdot) = \text{Ind } D_{\mathbb{R};T}^U(\cdot);$$

where $D_{\widehat{U};T}^E(\cdot)$ is the restriction of $D_{\widehat{U};T}^E(\cdot)$ on the \mathbb{R} -component of $\mathcal{C}^1(\widehat{U}; (S_+(T\widehat{U}) \oplus E))$ as in (1.13).

As ξ^U is nowhere zero on $U \setminus \mathring{M}$, by the excision formula of the transversal index [1, x3],

$$(1.26) \quad \text{Ind } D_{\mathbb{D}^m}^E = \text{Ind } D_{\mathbb{R};T}^U(\cdot) :$$

Now, the boundary ∂M cuts \widehat{U} into two manifolds with boundary $\partial M : M$ and $\widehat{U} \setminus \mathring{M}$. Let $D_{\widehat{U} \setminus \mathring{M};T;APS}^E$ (resp. $D_{M;T;APS}^E$) be the corresponding APS type operators (i.e. the

APS type boundary value problem s) on $\widehat{\mathcal{U}nM}$ (resp. M) respectively as in Section 1.2. Let $D_{\widehat{\mathcal{U}nM}; T}^E(\cdot)$ (resp. $D_M; T_{APS}^E(\cdot)$) be the restriction onto the corresponding $\widehat{\mathcal{U}nM}$ -components.

By proceeding as in [7, x3], which applies to the Fredholm operators here (one may deform the metrics and operators involved in a G -invariant way to the situation of product structure near $\mathcal{Q}M$, which does not alter the restriction of $D_{\widehat{\mathcal{U}nM}; T}^E$ on $\mathcal{Q}M$ (thus it does not alter the indices considered), if necessary), one deduces that if $T > T_0$ so that Proposition 1.2 holds, then one has the following splitting formula, in which each index does not depend on $T > T_0$,

$$(1.27) \quad \text{Ind } D_{\widehat{\mathcal{U}nM}; T}^E(\cdot) = \text{Ind } D_{\widehat{\mathcal{U}nM}; T_{APS}}^E(\cdot) + \text{Ind } D_M; T_{APS}^E(\cdot) :$$

Lemma 1.5. The following identity holds for $T > T_0$,

$$(1.28) \quad \text{Ind } D_{\widehat{\mathcal{U}nM}; T_{APS}}^E(\cdot) = 0 :$$

Proof. Let U^0 be a G -invariant open subset of $\widehat{\mathcal{U}nM}$ such that $U \cap U^0$ is compact in $\widehat{\mathcal{U}nM}$. Let $U^{\overline{0}}$ be a G -invariant open subset of $(U \cap M) \setminus \mathcal{Q}M$ such that $U^0 \cup U^{\overline{0}} = U \cap \widehat{\mathcal{U}nM}$ and $\overline{U^{\overline{0}}}$ is compact in $\widehat{\mathcal{U}nM}$. Let f_1, f_2 be two G -invariant functions on U with $\text{supp}(f_1) \subset U^0$, $\text{supp}(f_2) \subset U^{\overline{0}}$ such that $f_1; f_2$ forms a partition of unity associated with the open covering $U^0; U^{\overline{0}}$ of $U \cap \widehat{\mathcal{U}nM}$. The existence of $f_1; f_2$ is clear.

As the vector field U is nowhere zero on $U \cap \widehat{\mathcal{U}nM}$, and $f > 1/2$ on $U \cap \widehat{\mathcal{U}nM}$, by proceeding as in [5, x9.3] and [22, x2] respectively, one knows that for any smooth sections s lying in the $\widehat{\mathcal{U}nM}$ -component of $\mathcal{C}^1(\widehat{\mathcal{U}nM}; S(T\widehat{\mathcal{U}nM}) \otimes E_{\widehat{\mathcal{U}nM}})$ with compact support and verifying the APS boundary condition on $\mathcal{Q}M$ as in Section 1.2, one has

$$(1.29) \quad D_{\widehat{\mathcal{U}nM}; T}^E(f_1 s)_0^2 > C \left(D_{\widehat{\mathcal{U}nM}; T}^E(f_1 s)_0^2 + (T - b)k f_1 s k_0^2 \right) ;$$

$$(1.30) \quad D_{\widehat{\mathcal{U}nM}; T}^E(f_2 s)_0^2 > C^0 \left(D_{\widehat{\mathcal{U}nM}; T}^E(f_2 s)_0^2 + (T - b^0)k f_2 s k_0^2 \right)$$

respectively, for some positive constants $C; C^0; b^0$, and b .

From (1.29) and (1.30) and by taking $T > T_0$ large enough and proceeding the gluing arguments as in [4, pp. 115-116], one gets Lemma 1.5.

On the other hand, since $f_j \leq 1$, one sees directly, in view of Proposition 1.2, that when $T > T_0$,

$$(1.31) \quad \text{Ind } D_M; T_{APS}^E(\cdot) = Q_{APS}^M(E; \cdot) :$$

From (1.25)-(1.31), we get (1.23).

Remark 1.6. Theorem 1.4 allows us to use analytic method to deal with the transversal index problems. This is the viewpoint we adopt in this paper.

1.6. A proof of Theorem 0.1. We now apply Theorem 1.4 to the setting considered in Introduction. We have the canonical splitting $TM|_R \subset T^{(1;0)}M \oplus T^{(0;1)}M$, for the

complexification of TM , with

$$(1.32) \quad \begin{aligned} T^{(1,0)}M &= \text{fu } 2 \text{ } TM \quad \mathbb{R} \subset \mathbb{C}; Ju = \overline{p} \text{ } \text{lug}; \\ T^{(0,1)}M &= \text{fu } 2 \text{ } TM \quad \mathbb{R} \subset \mathbb{C}; Ju = p \text{ } \text{lug}; \end{aligned}$$

Let $T^{(0,1)}M$ be the dual of $T^{(1,0)}M$. For any ω a 1-form on M , we denote by ω^c its metric dual.

The almost complex structure J on M determines a canonical Spin^c -structure on TM with the associated Hermitian line bundle $\det(T^{(1,0)}M)$ and we have

$$(1.33) \quad S(TM) = T^{(0,1)}M; \quad S(TM) = \begin{matrix} \text{even} \\ \text{odd} \end{matrix} T^{(0,1)}M;$$

For any $W \in TM$, we write its complexification as $W = w + \overline{w} \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\overline{w} \in T^{(0,1)}M$ be the metric dual of w (cf. [4, x5]). Then

$$(1.34) \quad c(W) = \frac{p}{2} (\overline{w} \wedge \frac{1}{\overline{w}})$$

defines the canonical Clifford action of W on $(T^{(0,1)}M)$. It exchanges $\text{even}(T^{(0,1)}M)$ and $\text{odd}(T^{(0,1)}M)$.

The Levi-Civita connection r^{TM} together with the almost complex structure J induces via projection a canonical Hermitian connection $r^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This induces a Hermitian connection r^{\det} on $\det(T^{(1,0)}M)$. The Clifford connection $r^{(T^{(0,1)}M)}$ on $T^{(0,1)}M$ is induced by the Levi-Civita connection r^{TM} and the connection r^{\det} (cf. [20, x1a], [12, x1.3], [11, Appendix D]).

We take $E = L$ and denote ${}^0(M; L) = \mathcal{C}^1(M; (T^{(0,1)}M) \otimes L)$. We denote by $L_2({}^0(M; L))$ the L_2 -completion of the elements in ${}^0(M; L)$ with compact support.

Recall that we assume that the moment map $\mu: M \rightarrow \mathfrak{g}$ is proper. For a regular value $c > 0$ of $H = \int \dot{f}$, denote by M_c the G -invariant manifold with boundary

$$(1.35) \quad M_c = \{x \in M : H(x) = c\};$$

Also recall that X^H is the Hamiltonian vector field of H , i.e. $\mathbb{L}_{X^H} \mu = dH$. As in (1.16), we can write

$$(1.36) \quad \begin{aligned} \mathbb{L}_{X^H} \mu &= \sum_{i=1}^n \mu_i h_i; \end{aligned}$$

under the identification of \mathfrak{g} with \mathfrak{g} , as

$$(1.37) \quad \begin{aligned} \mathbb{L}_{X^H} \mu &= \sum_{i=1}^n \mu_i V_i; \end{aligned}$$

and from (0.4), one has (cf. [20, (1.19)]),

$$(1.38) \quad X^H = J(dH) = 2 \sum_{i=1}^n \mu_i (d \mu_i) = 2 \sum_{i=1}^n \mu_i V_i^M = 2^M;$$

Since $c > 0$ is a regular value of H , by (1.38) one knows that $X^H = 2^M$ is nowhere zero on $\partial M_c = H^{-1}(c)$.

We can now restate Theorem 0.1 as follows.

Theorem 1.7. For any $\phi \in \mathcal{P}$, there exists $c > 0$ such that for any regular values $c^0; c$ of H with $c^0; c > c$, the following identity for transversal indices holds,

$$(1.39) \quad \text{Ind}_{L; }^{M_c} = \text{Ind}_{L; }^{M_{c^0}} :$$

If $\phi = 0$, we can take $c = 0$.

Proof. Let $c^0 > c > 0$ be two regular values of H . Let M_{c^0} denote the manifold with boundary $M_{c^0} = \overline{M_{c^0} \setminus M_c}$.

By the additivity of the transversal index (cf. [1, Theorem 3.7, x6] and [17, Prop. 4.1]), one has

$$(1.40) \quad \text{Ind}_{L; }^{M_{c^0}} - \text{Ind}_{L; }^{M_c} = \text{Ind}_{L; }^{M_{c^0} \setminus M_c} :$$

Thus (1.39) is equivalent to say that when $c > 0$ is large enough, one has $\text{Ind}_{L; }^{M_{c^0} \setminus M_c} = 0$; which by Theorem 1.4 is equivalent to say that

$$(1.41) \quad Q_{\text{APS}}^{M_{c^0} \setminus M_c}(L;) = 0 :$$

If $\phi = 0$, by [22, Theorem 4.3], we get (1.41) for $c^0 > c > 0$. Thus we get (1.39) when $\phi = 0$.

From (1.13), we see that in the current situation, one has

$$(1.42) \quad D_T^L = D^L + \frac{p-1}{2} T c X^H : \mathcal{O}^0(M_{c^0}; L) \rightarrow \mathcal{O}^0(M_{c^0}; L) :$$

Let $e_j; \dots; e_{\dim M}$ be an orthonormal basis of TM_{c^0} . By [20, Theorem 1.6], one has the following Bochner type formula,

$$(1.43) \quad \begin{aligned} D_T^L{}^2 &= D^L{}^2 + \frac{p-1}{4} T^2 \sum_{j=1}^{\dim M} c(e_j) c_r{}_{e_j}^{TM} X^H \\ &\quad - \frac{p-1}{2} T \text{Tr} r^{T(1;0)M} X^H \sum_{i=1}^{\dim M} \mathcal{L}_{V_i} + \frac{T}{2} \sum_{i=1}^{\dim G} p-1 c J V_i^M c V_i^M + V_i^M{}^2 \\ &\quad + 4 T H \sum_{i=1}^{\dim G} \frac{p-1}{2} T \mathcal{L}_{V_i} + \frac{T^2}{4} X^H{}^2 : \end{aligned}$$

Let $\mathcal{O}^0(M_{c^0}; L)$ denote the $\mathcal{O}^0(M_{c^0}; L)$ -component of $\mathcal{O}^0(M_{c^0}; L)$. Then each $\mathcal{L}_{V_i}, 1 \leq i \leq \dim G$, acts on $\mathcal{O}^0(M_{c^0}; L)$ as a linear bounded operator $A(V_i)$ and we denote its operator norm by $\|A(V_i)\|$. Set

$$(1.44) \quad F_T^L = D_T^L{}^2 + 2 \sum_{i=1}^{\dim G} \mathcal{L}_{V_i} - 2 \sum_{i=1}^{\dim G} A(V_i) :$$

Lemma 1.8. There exists $c > 0$ such that for any $c > c$, and any $s \in L_2(\mathcal{O}^0(M; L))$, $\text{supp}(s) \subset M \setminus M_c$, one has

$$(1.45) \quad \sum_{i=1}^{\dim G} \|A(V_i) s\| \leq 6 \|H s\| :$$

Proof. By (1.37), one verifies that

$$(1.46) \quad \begin{aligned} & \sum_{i=1}^{\dim G} \left(\sum_{j=1}^{\dim G} |A_{ij}(V_i)s_j|^2 \right) \leq \frac{1}{2} \sum_{i=1}^{\dim G} \left(k_i s_{i0}^2 + kA_{ii}(V_i) s_{i0}^2 \right) \\ & \leq \frac{1}{2} \sum_{i=1}^{\dim G} \left(k_i s_{i0}^2 + \frac{1}{2} \sum_{j=1}^{\dim G} kA_{ij}(V_i) k^2 s_{j0}^2 \right) : \end{aligned}$$

From (1.46), one then verifies easily that Lemma 1.8 holds for $c = \sum_{i=1}^{\dim G} kA_{ii}(V_i) k^2$. Note that c is bigger than the absolute value of the eigenvalue of the Casimir operator of G acting on the irreducible G -representation with highest weight λ , and $c = 0$ if $\lambda = 0$.

By (1.43)–(1.45) and proceeding in exactly the same way as in the proof of [20, Proposition 2.2], one verifies that the following result holds.

Proposition 1.9. Let $c^0 > c > c$ be two regular values of H . Then for any $x \in M_{c^0} \cap (\mathcal{M}_c \setminus \mathcal{M}_{c^0})$, there exists an open neighborhood U_x of x in $M_{c^0} \cap (\mathcal{M}_c \setminus \mathcal{M}_{c^0})$ such that there exist $C_x > 0$, $b_x > 0$ such that for any $T > 1$ and any $s \in \mathcal{S}^0(M; L)$ with $\text{supp}(s) \subset U_x$, one has

$$(1.47) \quad \text{Re } F_T^L s; s > C_x \left(D_T^L s \right)_0^2 + (T - b_x) k s k_0^2 :$$

On the other hand, since X^H is nowhere zero on \mathcal{M}_{c^0} , the estimates described in [22, Proposition 2.4] holds near \mathcal{M}_{c^0} for the $-$ component here as well. By this and by Proposition 1.9, one can then proceed the gluing argument as in [22, Theorem 2.6], which goes back to [4, pp. 115–117], to the the similar estimates for the $-$ component (instead of the G -invariant component there), to see that the similar estimates in [22, Theorem 2.6] still holds here on M_{c^0} for the $-$ component.

To be more precise, by Proposition 1.2, one knows that there exists $T > 0$ such that for any $T > T$, $D_{\mathcal{M}_{c^0}; T}^L$ is invertible.

From (1.21) for $D_{\mathcal{M}_{c^0}; T}^L$ and (1.43), and by the same argument as in the proof of [22, Proposition 2.4], we get : there exists an open neighborhood U of \mathcal{M}_{c^0} and constants $T_0 > 0$, $C, C_1 > 0$ such that for any $T > T_0$ and $s \in \mathcal{S}^0(M_{c^0}; L)$, with $\text{supp}(s) \subset U$ and $P_{>0; T} s|_{\mathcal{M}_{c^0}} = 0$, the following inequality holds,

$$(1.48) \quad k D_T^L s k_0^2 > C \left(D_T^L s \right)_0^2 + (T - C_1) k s k_0^2 :$$

On the other hand, by (1.44) one knows that for any $s \in \mathcal{S}^0(M_{c^0}; L)$, one has

$$(1.49) \quad D_T^L s = F_T^L s :$$

By these and by the above discussions one sees that there exist $C^0 > 0$, $b > 0$ such that for any $T > T$ and any $s \in \mathcal{S}^0(M_{c^0}; L)$ such that $P_{>0; T} (s|_{\mathcal{M}_{c^0}}) = 0$, one has

$$(1.50) \quad D_T^L s \left(\right)_0^2 > C^0 \left(D_T^L s \right)_0^2 + (T - b) k s k_0^2 :$$

By taking $T > T$ sufficiently large in (1.50), one gets (1.41), which completes the proof of Theorem 1.7 and thus Theorem 0.1.

Remark 1.10. If the zero set of $X^H = 2^M$ is compact, then Theorem 0.1 is known in [17] and [25] already.

Remark 1.11. Our proof of Theorem 0.4 in the following two sections follows the similar line of the arguments as in the above proof of Theorem 0.1. However, we have to deal with more subtle deformations and estimates, some of which are highly non-trivial. See the next two sections for more details.

2. Quantization for proper moment maps: a proof of Theorem 0.4

In this section, we prove Theorem 0.4 modulo a vanishing result Theorem 2.4 which will be proved in Section 3.

This section is organized as follows: In Section 2.1, we give a reformulation of Theorem 0.4 as Theorem 2.3. In Section 2.2, we state a vanishing result Theorem 2.4 away from ${}^1(0)$. In Section 2.3, we give a proof of Theorem 2.3 by using Theorem 2.4.

2.1. A reformulation of Theorem 0.4. For convenience, we recall the basic setting.

Let $(M; \omega)$, $(N; \omega^N)$ be two symplectic manifolds with symplectic forms $\omega; \omega^N$ respectively. We assume that N is compact.

Let $(L; h^L)$ be a Hermitian line bundle on M with Hermitian connection r^L , and $(F; h^F)$ a Hermitian line bundle on N with Hermitian connection r^F . Let $R^L = (r^L)^2$, $R^F = (r^F)^2$ be the associated curvatures.

We suppose that

$$(2.1) \quad \omega = \frac{p}{2} \omega^L; \quad \omega^N = \frac{p}{2} \omega^F :$$

Let $J^M; J^N$ be almost complex structures on $TM; TN$ respectively such that $\omega(\cdot; \cdot)$ defines a metric g^M on TM , and $\omega^N(\cdot; \cdot)$ defines a metric g^N on TN .

Let G be a compact connected Lie group with Lie algebra \mathfrak{g} , and \mathfrak{g} admits an Ad_G -invariant metric. Suppose that G acts (by left) on $M; N$ and its actions on $M; N$ lift on L and F respectively. Moreover, we assume that the G -action preserves the above metrics and connections on $TM; TN; L; F$ and $J^M; J^N$.

For any $K \in \mathfrak{g}$, let $K^M \in \mathcal{C}^1(M; TM)$ denote the vector field generated by K on M . Recall that the moment map $\mu : M \rightarrow \mathfrak{g}$ has been defined in (0.3). Let $\mu^N : N \rightarrow \mathfrak{g}$ be the moment map defined in the same way for $(N; \omega^N)$ and $(F; h^F; r^F)$.

We will use the same notation for the natural extension of the objects on M, N to $M \times N$. In particular, $L \otimes F$ is the Hermitian line bundle on $M \times N$ induced by L and F with the Hermitian connection induced by $r^L; r^F$.

The G -action on $M \times N$ is defined by $g \cdot (x; y) = (gx; gy)$ for $(x; y) \in M \times N$. We define the symplectic form $\omega_{(x; y)}$ and the almost complex structure J on $M \times N$ by

$$(2.2) \quad \omega_{(x; y)} = \omega_x + \omega_y^N; \quad J = (J^M; J^N) :$$

The induced moment map $\mu : M \times N \rightarrow \mathfrak{g}$ is given by

$$(2.3) \quad \mu(x; y) = \mu(x) + \mu(y) :$$

Recall that we have assumed that the moment map $\mu : M \rightarrow \mathfrak{g}$ is proper. Since N is compact, one sees that the moment map μ is also proper.

Recall that for any $2 \in \mathcal{P}$, the index $Q(L \rightarrow F)$ has been defined by applying Theorem 0.1 on $M \rightarrow N$, while $Q((L \rightarrow F))$ is the quantization number defined in [16] which by [17, x7.4], can be defined as follows.

If 2 is a regular value of the moment map μ , then one can construct the Marsden–Weinstein symplectic reduction $((M \rightarrow N); \mu^{-1}(2))$, where $(M \rightarrow N) = \mu^{-1}(2)/G$ is a compact orbifold. Moreover, $L \rightarrow F$ (resp. J) induces a prequantized line bundle $(L \rightarrow F)$ (resp. an almost complex structure J) over $((M \rightarrow N); \mu^{-1}(2))$. One then constructs the associated Spin^c -Dirac operator (twisted by $(L \rightarrow F)$) on $(M \rightarrow N)$ whose index $Q((L \rightarrow F))$ as in (0.5) is well-defined.

If $2 \in \mathcal{P}$ is not a regular value of μ , then take a $2 \in \mathcal{P}$ sufficiently close to 2 such that a is a regular value of μ , and by replacing 2 by a , we get the index $Q((L \rightarrow F_a))$. For a regular value of μ and close enough to 2 , $Q((L \rightarrow F_a))$ does not depend on a and we denote it as $Q((L \rightarrow F))$.

The following result can be viewed as a quantization formula for the $\mu^{-1}(0)$ (or G -invariant) component.

Theorem 2.1. The following identity holds,

$$(2.4) \quad Q(L \rightarrow F)^{=0} = Q((L \rightarrow F))_{=0} :$$

Proof. If M is compact, this is the Guillemin–Stenzel conjecture proved in [15] and [16].

In the general case, for any two regular values $c^0 > c > 0$ of j^2 , let $Q_{\text{APS}}^{(M \rightarrow N)_{c;c^0}}(L \rightarrow F; \mu)^{=0}$ be the APS type index defined in Section 1.5 for the current situation, where $(M \rightarrow N)_{c;c^0} = \{x \in M \rightarrow N : c \leq j^2(x) \leq c^0\}$. Then by [22, Theorem 4.3], one finds

$$(2.5) \quad Q_{\text{APS}}^{(M \rightarrow N)_{c;c^0}}(L \rightarrow F; \mu)^{=0} = 0 :$$

By Theorems 0.1, 1.4, (1.40) and (2.5), and by taking $c^0 > 0$ large enough, one sees that for any regular value $c > 0$ of j^2 , one has

$$(2.6) \quad \text{Ind}_{=0}^{(M \rightarrow N)_c} = \text{Ind}_{=0}^{(M \rightarrow N)_{c^0}} = Q(L \rightarrow F)^{=0} ;$$

where $(M \rightarrow N)_c = \{x \in M \rightarrow N : j^2(x) \leq c\}$, and similarly for $(M \rightarrow N)_{c^0}$.

If 0 is a regular value of μ , then from [22, Theorem 4.3], $Q_{\text{APS}}^{(M \rightarrow N)_c}(L \rightarrow F; \mu)^{=0} = Q((L \rightarrow F))_{=0}$. From Theorem 1.4, (2.6), we get (2.4).

On the other hand, if we take $c > 0$ small enough so that there is no critical point of j^2 besides $\mu^{-1}(0)$, then by [17, Prop. 7.10] and [18] one knows that even if 0 is a singular value of μ ,

$$(2.7) \quad \text{Ind}_{=0}^{(M \rightarrow N)_c} = Q((L \rightarrow F))_{=0} :$$

From (2.6) and (2.7), one gets (2.4) which completes the proof of Theorem 2.1.

Remark 2.2. See Remark 2.8 for an outline of an analytic proof of (2.7).

By Theorem 2.1, one can reformulate Theorem 0.4 as follows.

Theorem 2.3. The following identity holds,

$$(2.8) \quad Q(L - F)^0 = \int_{2\hat{G}} Q(L) Q(F) :$$

In the rest of this section, we will present a proof of Theorem 2.3 modulo a vanishing result away from $\mu^{-1}(0)$, which will be stated in the next subsection and proved in the next section.

2.2. A vanishing result away from $\mu^{-1}(0)$. In this section, we state a vanishing result which can be thought of the first step in the two steps deformation proof of Theorem 2.3.

We use the Ad_G -invariant metric on \mathfrak{g} to identify \mathfrak{g} and \mathfrak{g}^* . Let $V_i, 1 \leq i \leq \dim G$, be an orthonormal basis of \mathfrak{g} , then we can write the moment maps μ^M and μ^N as

$$(2.9) \quad \mu^M = \sum_{i=1}^{\dim G} \langle V_i, \cdot \rangle V_i; \quad \mu^N = \sum_{i=1}^{\dim G} \langle V_i, \cdot \rangle V_i.$$

For any $1 \leq i \leq \dim G$, denote by $V_i^M; V_i^N; V_i^{M \times N}$ the Killing vector fields on $M \times N; M \times N; M \times N$ induced by V_i respectively. Then one verifies easily from (0.4) that

$$(2.10) \quad V_i^{M \times N} = V_i^M + V_i^N; \\ d^M_{V_i} = J^M V_i^M; \quad d^N_{V_i} = J^N V_i^N.$$

Let $D^{L, F} : \mu^{-1}(0) \rightarrow \mu^{-1}(0); (M \times N; L - F) \rightarrow (M \times N; L - F)$ be the Spin Dirac operator on $M \times N$ (cf. Section 1.6, (1.5)).

By Sard's theorem, the set of critical values of the functions j^2 and $\frac{1}{2}j^2$ on $M \times N$ has measure zero in \mathbb{R} . Especially, for any $C > 0$, there exists $C^0 > C$ such that C^0 is a regular value for the functions j^2 and $\frac{1}{2}j^2$ on $M \times N$.

As N is a compact manifold, there exists $C_0 > 0$ such that $j \geq C_0$ on N .

For $A > 6C_0^2$ large enough which is a regular value of the functions j^2 and $\frac{1}{2}j^2$ on $M \times N$, we define

$$(2.11) \quad M = \{ (x; y) \in M \times N; j(x; y)^2 > A; j(x; y)^2 \leq 2A \}; \\ M_A = \{ (x; y) \in M \times N; j(x; y)^2 \leq A \}; \quad @M_A = \{ (x; y) \in M \times N; j(x; y)^2 = A \}; \\ M_1 = \{ (x; y) \in M \times N; j(x; y)^2 = A \}; \\ M_2 = \{ (x; y) \in M \times N; j(x; y)^2 = 2A \};$$

By our choice of A , we know that M_1, M_2 are smooth sub-manifolds of $M \times N$. Moreover, as

$$(2.12) \quad j^2 = j^2 + j^2 + 2h; \quad i$$

and $A > 6C_0^2$, we know that $M_1 \setminus M_2 = \emptyset$.⁷ Thus M is a smooth manifold with boundary $@M$ and

$$(2.13) \quad @M = M_1 \cup M_2; \quad M_1 = @M_A \cup N;$$

⁷Indeed, by (2.12), one gets $j > j \geq C_0$. Thus, if $j^2 = 2A$, then $j > A^{1/2} + ((\frac{p}{2} - 1)A^{1/2} - C_0) > A^{1/2}$ when $A > 6C_0^2$.

We will introduce a deformation of the vector field $\sum_{j=1}^{\dim G} \rho_j (V_j^M + V_j^N)$ on M_1 to the vector field $\sum_{j=1}^{\dim G} \rho_j (V_j^M + V_j^N)$ on M_2 , where $\rho_j = \rho_j + \rho_j, j = 1, \dots, \dim G$, are the coordinates of ρ associated to the basis $V_j, j = 1, \dots, \dim G$, of \mathfrak{g} .

Let $\rho_j, j = 1, \dots, \dim G$, be smooth functions on M such that there exists a strictly positive function $\rho \in C^1(\partial M)$ such that

$$(2.14) \quad \rho_j(x; y) = \begin{cases} \rho_j(x; y) + \rho_j(x; y) & \text{on } M_1; \\ \rho_j(x; y) + \rho_j(x; y) & \text{on } M_2; \end{cases}$$

Let X_ρ be the vector field on M defined by

$$(2.15) \quad X_\rho = \sum_{j=1}^{\dim G} \rho_j (V_j^M + V_j^N) :$$

As ρ_j and ρ_j are G -invariant functions on $M \setminus N, M_1$ and M_2 are G -submanifolds with boundary of $M \setminus N$, in particular,

$$(2.16) \quad \rho_j^M(x; y) \in T\partial M; \quad X_\rho(x; y) \in T\partial M; \quad \text{for } (x; y) \in \partial M :$$

As A is a regular value of the functions ρ_j and $\frac{1}{2}\rho_j$ on $M \setminus N$, we know that

$$(2.17) \quad X_\rho = \begin{cases} \sum_{j=1}^{\dim G} \rho_j (V_j^M + V_j^N) \notin 0 & \text{pointwise over } M_1; \\ \sum_{j=1}^{\dim G} \rho_j (V_j^M + V_j^N) \notin 0 & \text{pointwise over } M_2; \end{cases}$$

Thus X_ρ is nowhere zero on ∂M .

Clearly, X_ρ is induced on M by the G -equivariant map $\mathcal{X}^\rho : M \rightarrow \mathfrak{g}$ defined by

$$(2.18) \quad \mathcal{X}^\rho = \sum_{j=1}^{\dim G} \rho_j V_j :$$

We can now state the main result of this subsection as follows.

Theorem 2.4. When $A > 0$ is large enough, there exist functions $\rho_j, 1 \leq j \leq \dim G$, verifying the above properties, such that the following identity for the APS type index holds,

$$(2.19) \quad \text{Ind}_{\text{APS}}^M(L, F; \mathcal{X}^\rho) = 0 :$$

The proof of Theorem 2.4 will be given in the next section.

2.3. A proof of Theorem 2.3. We continue the discussion in the previous subsection.

Denote by $\mathcal{M}_1 = \{ (x; y) \in M \setminus N : \rho_j(x; y) \geq Ag, j = 1, \dots, \dim G \}$, $\mathcal{M}_2 = \{ (x; y) \in M \setminus N : \rho_j(x; y) \geq 2Ag \}$. Then $\mathcal{M}_1, \mathcal{M}_2$ are submanifolds with boundary of $M \setminus N$ such that $\partial \mathcal{M}_i = M_i, i = 1, 2$. Recall also $\mathcal{M}_2 = \mathcal{M}_1 \cup M$.

Let $\mathcal{M}_2 \rightarrow \mathfrak{g}$ be a G -equivariant map such that the induced vector field $X_{\mathcal{M}_2}$ on \mathcal{M}_2 verifies $X_{\mathcal{M}_2}|_{\partial \mathcal{M}_2} = X_\rho$. The existence of \mathcal{M}_2 is clear.

From (2.17), the positivity of ρ on ∂M and the additivity of the transversal index (cf. [L, Theorem 3.7, x6] and [L7, Prop. 4.1]), one has,

$$(2.20) \quad \text{Ind}_{L, F; \tilde{X}}^{\mathcal{M}_2} = \text{Ind}_{L, F; \tilde{X}}^M + \text{Ind}_{L, F}^{\mathcal{M}_1} :$$

From Theorems 1.4, 2.4 and (2.20), one gets

$$(2.21) \quad \text{Ind}_{=0} \mathcal{M}_2 \Big|_{L, F}; = \text{Ind}_{=0} \mathcal{M}_1 \Big|_{L, F}; :$$

Lemma 2.5. The following identities hold

$$(2.22) \quad \begin{aligned} \text{Ind}_{=0} \mathcal{M}_1 \Big|_{L, F}; &= \text{Ind}_{=0} \mathcal{M}_1 \Big|_{L, F}; ; \\ \text{Ind}_{=0} \mathcal{M}_2 \Big|_{L, F}; &= \text{Ind}_{=0} \mathcal{M}_2 \Big|_{L, F}; : \end{aligned}$$

Proof. For any $t \in [0; 1]$, set $\mathcal{M}_t = (1-t)\mathcal{M}_1 + t\mathcal{M}_2$, $\mathcal{M}'_t = (1-t)\mathcal{M}'_1 + t\mathcal{M}'_2$. Let $\mathcal{M}_t^1, \mathcal{M}_t^2$ be the induced vector fields on $\mathcal{M}_1, \mathcal{M}_2$ respectively. Then by (2.17) and the positivity of ν , one sees that \mathcal{M}_t^1 (resp. \mathcal{M}_t^2) is nowhere zero on $\mathcal{C}\mathcal{M}_1 = M_1$ (resp. $\mathcal{C}\mathcal{M}_2 = M_2$). Formula (2.22) then follows from the homotopy invariance of the transversal index (cf. [1, Theorems 2.6, 3.7] and [17, x3]).

Recall that by our assumption, the induced vector field \mathcal{M}^M of $\nu: M \rightarrow \mathfrak{g}$ is nowhere zero on $\mathcal{C}M_A$.

Proposition 2.6. The following identity holds,

$$(2.23) \quad \text{Ind}_{=0} \mathcal{M}_1 \Big|_{L, F}; = \sum_{2\widehat{G}} \text{Ind}_{L; Q(F)} \mathcal{M}^A \Big|_{L, F}; :$$

Proof. Proposition 2.6 can be proved by using the homotopy invariance of the transversal index. Here, we will develop an analytic proof.

Recall that $\mathcal{M}^{M, N}$ denotes the vector field generated by ν on $M \times N$. One has

$$(2.24) \quad \mathcal{M}^{M, N} = \sum_{j=1}^{\dim G} V_j^M + V_j^N :$$

For any $t \in [0; 1]$, we define the deformation

$$(2.25) \quad \mathcal{M}_t = \sum_{j=1}^{\dim G} V_j^M + (1-t)V_j^N :$$

For any $t \in [0; 1]$, $T > 0$, we define the following Dirac type operators as in (1.13),

$$(2.26) \quad \begin{aligned} D_T^{L, F}(t) &= D^{L, F} + \frac{1}{2} \mathcal{C} \mathcal{M}_t \\ &: 0; (\mathcal{M}_1; L, F) ! 0; (\mathcal{M}_1; L, F); \\ D_{\neq T}^{L, F}(t) &= D_T^{L, F}(t) j_{0; \text{odd}}^{\text{even}} (\mathcal{M}_1; L, F) : \end{aligned}$$

Let $(D_{+; T}^{L, F}(t); P_{> 0; +; T}(t))$ denote the canonically associated (elliptic) APS boundary value problem defined similarly as in Section 1.2. Let $\text{Ind}_{=0} (D_{+; T}^{L, F}(t); P_{> 0; +; T}(t))$ denote the G -invariant part of the corresponding index.

Lemma 2.7. There exists $T_0 > 0$ such that for any $T > T_0$, $\text{Ind}_{=0} (D_{+; T}^{L, F}(t); P_{> 0; +; T}(t))$ does not depend on $t \in [0; 1]$.

Proof. For any $T > 0, t \in [0; 1]$, let $D_{\mathcal{M}_1; T}^{L, F}(t)$ be the Dirac type operator on the boundary $\partial\mathcal{M}_1$ induced by $D_{\mathcal{M}_1}^{L, F}(t)$ (cf. (1.14)).

In view of Section 1.4, in order to prove Lemma 2.7, we need only to show that there exists $T_0 > 0$ such that when restricted to the subspace of G -invariant sections of ${}^0(\mathcal{M}_1; L^{L, F})_{\partial\mathcal{M}_1}$, $D_{\mathcal{M}_1; T}^{L, F}(t)$ is invertible for any $T > T_0, t \in [0; 1]$.

On M_1 , set

$$(2.27) \quad X(t) = \frac{h_{\mathcal{M}_1}(t); \mathbb{M}^N}{j^{\mathbb{M}^N}}; \quad Y(t) = \mathcal{M}_1(t) \cdot X(t);$$

Then $X(t), Y(t)$ are perpendicular to each other. Moreover, one verifies that the following identity holds on M_1 ,

$$(2.28) \quad \mathfrak{X}(t)j = \frac{\sum_{i=1}^{\dim G} |V_i^M|^2 + (1-t) \sum_{i=1}^{\dim G} |V_i^N|^2}{\sum_{i=1}^{\dim G} |V_i^M|^2 + \sum_{i=1}^{\dim G} |V_i^N|^2} > c_A > 0$$

for some positive constant c_A which might depend on A .

Now we write $D_{\mathcal{M}_1; T}^{L, F}(t)$ explicitly as, in view of (1.14),

$$(2.29) \quad \begin{aligned} D_{\mathcal{M}_1; T}^{L, F}(t) &= D_{\mathcal{M}_1}^{L, F} \cdot \frac{1-T}{2} c(e_{\dim M}) c(\mathcal{M}_1(t)) \\ &= D_{\mathcal{M}_1}^{L, F} \cdot \frac{1-T}{2} c(e_{\dim M}) c(Y(t)) \cdot \frac{1-T}{2} c(e_{\dim M}) c(X(t)); \end{aligned}$$

Then as $X(t), Y(t) \in T\mathcal{M}_1$ are perpendicular to each other, one has

$$(2.30) \quad \begin{aligned} |D_{\mathcal{M}_1; T}^{L, F}(t)|^2 &= \frac{1-T}{2} |D_{\mathcal{M}_1}^{L, F}; c(e_{\dim M}) c(X(t))|^2 \\ &\quad + \frac{T^2}{4} \mathfrak{X}(t)j^2 + |D_{\mathcal{M}_1}^{L, F} \cdot \frac{1-T}{2} c(e_{\dim M}) c(Y(t))|^2; \end{aligned}$$

By (2.28) and proceeding similarly as in the proof of Proposition 1.2, especially (1.18), (1.19), one gets that there exist $T_0 > 0$ and $C > 0$ such that for any $T > T_0, t \in [0; 1]$ and any G -invariant sections s of ${}^0(\mathcal{M}_1; L^{L, F})_{\partial\mathcal{M}_1}$, one has

$$(2.31) \quad \frac{1-T}{2} |D_{\mathcal{M}_1}^{L, F}; c(e_{\dim M}) c(X(t))|^2 + \frac{T}{4} \mathfrak{X}(t)j^2 |s|^2 > C |hs; s|^2;$$

From (2.31), one sees that for any $T > T_0, t \in [0; 1]$, when restricted to the subspace of G -invariant sections of ${}^0(\mathcal{M}_1; L^{L, F})_{\partial\mathcal{M}_1}$, $D_{\mathcal{M}_1; T}^{L, F}(t)$ is invertible, thus $(D_{\mathcal{M}_1; T}^{L, F}(t); P_{>0; T})$ form a continuous family of Fredholm operators, which implies the constancy of the associated index. The proof of Lemma 2.7 is thus completed.

Coming back to the proof of Proposition 2.6. Let $M_A = \sum_{i=1}^{\dim G} |V_i^{M_A}|^2$ be the vector field on M_A induced by \mathbb{M}^A . By (2.25), one has

$$(2.32) \quad \mathcal{M}_1(1) = \sum_{i=1}^{\dim G} |V_i^M|^2 = M_A; \quad \text{on } \mathcal{M}_1;$$

Now we deform inside M_A (leaving the data on $@M_A$ unchanged) in G -invariant manner the metrics and connections as well as M_A to the situation that everything is of product nature near $@M_A$. We denote also the spinor $S(TM_A)$ of M_A (associated to the product metric near the boundary now) obtained from $(T^{(0,1)}M)$. Then when taking product with N , we also deform things on \mathcal{M}_1 to a situation which is of product nature near $@\mathcal{M}_1 = M_1 = @M_A \times N$.

We then attach an infinite cylinder $@M_A \times [0; +1)$ along the boundary $@M_A$ and extend everything in M_A to the now complete manifold \widetilde{M}_A with cylindrical end, and \widetilde{M}_A is constant along the direction $[0; +1)$. By taking product with N , we get similar construction $\widetilde{\mathcal{M}}_1$ for \mathcal{M}_1 . We denote the extended subjects on the obtained manifolds with cylindrical end by a "\(\tilde{\)" modifying notation. Then $S(T\widetilde{\mathcal{M}}_1) = S(T\widetilde{M}_A) \otimes (T^{(0,1)}N)$.

Now since for $T > T_0$, $D_{@M_1; T}^{L, F}(1)$ is invertible on the subspace of G -invariant subspace, the standard arguments in [2, Prop. 3.11] show that

$$(2.33) \quad \mathcal{D}_{+; T}^{\widetilde{L}, F} = \mathcal{D}_+^{\widetilde{L}, F} + \frac{P-1}{2} T C_{\widetilde{M}_A} : L^2(\widetilde{\mathcal{M}}_1; S_+(T\widetilde{\mathcal{M}}_1)) \otimes F^G \rightarrow L^2(\widetilde{\mathcal{M}}_1; S(T\widetilde{\mathcal{M}}_1)) \otimes F^G;$$

where we use the superscript G to denote the subspace of G -invariant sections, is a Fredholm operator. Moreover, its index equals to $\text{Ind}_{=0}(D_{+; T}^{L, F}(1); P_{>0; T}(1))$.

Let $e_i; \dim M$ (resp. $f_j; \dim N$) denote the orthonormal basis of TM_A (resp. TN). Then we can write

$$(2.34) \quad \mathcal{D}_T^{\widetilde{L}, F} = \sum_{i=1}^{\dim M} c(e_i) r_{e_i}^{S(T\widetilde{\mathcal{M}}_1) \widetilde{L}, F} + \frac{P-1}{2} T C_{\widetilde{M}_A} + \sum_{j=1}^{\dim N} c(f_j) r_{f_j}^{S(T\widetilde{\mathcal{M}}_1) \widetilde{L}, F};$$

where $\sum_{i=1}^{\dim M} c(e_i) r_{e_i}^{S(T\widetilde{\mathcal{M}}_1) \widetilde{L}, F} + \frac{P-1}{2} T C_{\widetilde{M}_A}$ (resp. $\sum_{j=1}^{\dim N} c(f_j) r_{f_j}^{S(T\widetilde{\mathcal{M}}_1) \widetilde{L}, F}$) is lifted from a corresponding operator on \widetilde{M}_A (resp. N) denoted by $\mathcal{D}_T^{\widetilde{L}}$ (resp. D^F).

By (2.34), one has

$$(2.35) \quad \|\mathcal{D}_T^{\widetilde{L}, F}\|^2 = \|\mathcal{D}_T^{\widetilde{L}}\|^2 + \|D^F\|^2;$$

from which one gets

$$(2.36) \quad \text{Ker}_{L^2} \mathcal{D}_T^{\widetilde{L}, F} = \text{Ker}_{L^2} \mathcal{D}_T^{\widetilde{L}} \oplus \text{Ker } D^F;$$

Since N is compact, $\dim(\text{Ker } D^F)$ is of finite dimension. From (2.33), (2.36) and the standard arguments in [2, Prop. 3.11], one then deduces that when $T > 0$ is large enough,

$$(2.37) \quad \text{Ind}_{=0} D_{+; T}^{L, F}(1); P_{>0; T}(1) = \sum_{2\widehat{G}} Q_{APS}^{M_A}(L; \cdot) \cdot Q(F);$$

By taking $A > 0$ large enough, from Theorem 1.4, Lemma 2.7 and (2.37), one gets (2.23), which completes the proof of Proposition 2.6.

From (2.21)–(2.23) and by taking $A > 0$ large enough, in view of Theorems 1.4 and 1.7, one gets (2.8) which completes the proof of Theorem 2.3.

The proof of Theorem 2.4 is thus also completed.

Remark 2.8. By combining the argument in this subsection with those of [21] and [22], one is able to get a new proof of (2.7). Indeed, if $c > 0$ is a regular value of j_j^2 , for a $2g$ close enough to 0 $2g$ which is a regular value for \quad , let O_a be the co-adjoint orbit of a and consider the moment map $\mu_a : (M \times N) \times O_a \rightarrow \mathfrak{g}$ defined by $\mu_a(x; y) = \langle x \rangle \cdot y$ for $x \in (M \times N) \times O_a$ and $y \in O_a$. Then as j_j is small enough, $dj_{t_a} j_j^2, t \in [0; 1]$, are nowhere zero on $\partial(M \times N) \times O_a$. One can then use the argument in this subsection to relate the G -invariant APS type index on $(M \times N) \times O_a$ to that on $(M \times N) \times O_a$, which in turn leads to the contribution on $\mu_a^{-1}(0) = G \times \mu_a^{-1}(G a) = G$, by combining the arguments in [22] and [21].

3. Vanishing result: a proof of Theorem 2.4

In this section, we will establish our vanishing result, Theorem 2.4. Assume that X is a suitable vector field on M (depend on the parameter A) in (2.15) deforming $M \times N$ on M_1 to $M \times N$ on M_2 , we need to prove that when $T > 0$ is large enough, the G -invariant component of the APS type index of the deformed Dirac operator D_T^M in (3.9) on M is zero. In fact, we will prove that after we fix A large enough, the restriction of D_T^M with APS boundary condition on the G -invariant sections is invertible for T large enough.

As X is nowhere zero on ∂M , by the argument of (1.48) (cf. [22, Prop. 2.4]), we have an estimate (3.117) for D_T^M on an open neighborhood of ∂M similar to (1.48). By the gluing argument as in [4, p. 115-116], if we establish the local estimate (3.94) inside M , then we can conclude Theorem 2.4.

As we are interested in the restriction of D_T^M to the G -invariant subspace of $L^2(M; L(F))$, we will take the term $\int_M L_{V_j}$ in (3.11) as zero. If $z \notin \text{zero}(X) = \{z \in M; X(z) = 0\}$, then from the Bochner type formula (3.11), the term $\frac{T^2}{4} \langle X \rangle^2$ is the leading term and we get easily (3.94). If $z \in \text{zero}(X)$, then we hope to adapt the argument in [20, x2b] to get the local estimate around z . Basically, we hope $\int_M \langle X \rangle^2$ will be a positive term, and it will control all the tensors in (3.11). As there is no assumption on the vector fields V_j^M and their covariant derivatives on M , we can not expect to get our estimate by a simple argument. To control locally the covariant derivatives of V_j^M in (3.11) near $z \in \text{zero}(X)$, we will use the harmonic oscillator argument from [20, x2b]), thus we like to impose that the vector field $\langle X \rangle V_j^M$ is a Hamiltonian vector field along the direction M .

As N is compact, the tensor V_j^N and their derivatives are bounded. Thus if on $\text{zero}(X)$, we can control $\int_M \langle X \rangle^2 (I_1 + I_2)$ which involve V_j^M and $(d^M \langle X \rangle)$ in (3.10) by $\int_M \langle X \rangle^2$, then basically, we can achieve our local estimate. What we prove in Proposition 3.11 (whose proof occupies from Section 3.3 until Section 3.7) is that for our choice $\langle X \rangle$ in (3.4), $\int_M \langle X \rangle^2 (I_1 + I_2)$ is bounded from below uniformly on $A > A_0$, any regular value of j_j^2 and $\frac{1}{2} j_j^2$, and on $\text{zero}(X) \subset M$. Moreover $\int_M \langle X \rangle^2 > A$ on M , thus when $A > 0$ is large enough, the term $\int_M \langle X \rangle^2$ will really control the tensors which do not involve the derivative of V_j^M in (3.11). In this way, we can obtain the local estimate (3.94) around each $z \in \text{zero}(X)$. Now from the gluing argument of [4, p. 115-116], (3.94) and (3.117), we get (3.118), which imply the invertibility of the restriction of the operator D_T^M with APS boundary condition to the G -invariant part. Especially we establish Theorem 2.4.

This section is organized as follows. In Section 3.1, we propose a construction of the deformed vector field X in (2.15) which depends on two functions $e; e_0$. In Section 3.2, we establish a Bochner type formula (3.11) for the Dirac operator deformed by X . In Section 3.3, we study the relation on the zero set of X in M of the vector fields generated by the group action, and in Section 3.4, we study the asymptotics of the functions appeared in the above relations when the parameter $A \rightarrow +\infty$. In Section 3.5, we compute precisely the tensors involving the vector field V_j^M and $(d^M)_j$ on the zero set of X in the Bochner formula (3.11). In Section 3.6, we study the coefficients appeared in the computation in Section 3.5. In Section 3.7, we prove in Proposition 3.11 that the sum of the tensors involving the vector field V_j^M and $(d^M)_j$ on the zero set of X in the Bochner formula is uniformly bounded from below for our choice $e; e_0$. In Section 3.8, under the help of Prop. 3.11, for A large enough fixed, we establish the local estimate around each point inside M . Finally in Section 3.9, we prove Theorem 2.4.

We use the notation as in Section 2, and when a subscript index appears two times in a formula, we sum up with this index.

3.1. The construction of the deformed vector field. We first specify the deformation vector field used in Section 2.2.

Let $e; e_0 \in C^1(\mathbb{R})$ be such that

$$(3.1) \quad \begin{aligned} e(t) &= \begin{cases} t^2; & \text{for } t \leq \frac{1}{3}; \\ 1; & \text{for } t > \frac{2}{3}; \end{cases} & e_0(t) &= \begin{cases} 1 - t^2; & \text{for } t \leq \frac{1}{3}; \\ 2(1 - t); & \text{for } t > \frac{2}{3}; \end{cases} \\ e(t) + e_0(t) &> \frac{29}{27}; & e_0(t) &< 0; & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}; \end{aligned}$$

The existence of $e; e_0$ is easy to see. For example, one may set $e_0(t) = 1 - t^2$ on $t \leq \frac{3}{8}$; $e_0(t) = 1$, $e_0(t) = 2(1 - t)$ on $t > \frac{5}{8}$; and e_0, e_0 are linear on $\frac{3}{8} \leq t \leq \frac{5}{8}$. By an approximation argument from e_0, e_0 , one gets $e; e_0$ verifying (3.1).

For $A > 0$, set

$$(3.2) \quad e(t) = e^{-\frac{t}{A}} - 1; \quad e_0(t) = e^{-\frac{t}{A}} - 1;$$

Let $\{V_j\} \in C^1(M \rightarrow N)$, $\{g_j\} : M \rightarrow N \rightarrow g_j' g_j$ be defined by

$$(3.3) \quad g_j = j^2 + \dots; \quad g_j = \dots;$$

Recall that $V_1; \dots; V_{\dim V}$ is an orthonormal basis of $g; V_i^M; V_i^N$ are the Killing vector fields on $M; N$ induced by V_i . For any function Q with values in g , we will denote Q_i its i -component with respect to the base $\{V_i\}$. Set⁸

$$(3.4) \quad \begin{aligned} X &= 2 \sum_j (1 + \dots) j^2 + \dots; \\ X &= 2 \sum_j \dots \end{aligned}$$

Let X be the vector field on M in (2.11) defined by

$$(3.5) \quad X = \sum_j V_j^M + V_j^N;$$

⁸See Lemma 3.7 for a possible motivation of introducing the j 's here.

Then one verifies,

$$(3.6) \quad \begin{aligned} X &= 2 \sum_j V_j^M + V_j^N \quad \text{on } M_1; \\ X &= 2 + \frac{8}{A} \sum_i \sum_j V_j^M + V_j^N \quad \text{on } M_2; \end{aligned}$$

We can now state the main result of this section, which precise Theorem 2.4, as follows.

Theorem 3.1. When $A > 0$ large enough is a regular value for both $\sum_j \hat{f}_j^2$ and $\frac{1}{2} \sum_j \hat{f}_j^2$, X is nowhere zero on M_1 and M_2 . Moreover, we have

$$(3.7) \quad Q_{APS}^M(L, F; X)^0 = 0:$$

We devote the rest of this section to the proof of Theorem 3.1.

3.2. A Bochner type formula for the deformation by X . We will fix the symplectic form ω^M and the almost complex structure J^M on M induced from $M \rightarrow N$. Then the associated Riemannian metric g^{TM} is also induced by g^{TM} and g^{TN} . Let r^{TM} be the Levi-Civita connection on $(M; g^{TM})$.

For any function ϕ on $M \rightarrow N$, we denote by $d^M \phi, d^N \phi$ its differential along M, N respectively.

We denote also V_j^M the Killing vector field on M induced by $V_j \in \mathfrak{g}$, then

$$(3.8) \quad V_j^M = V_j^{M \rightarrow N} = V_j^M + V_j^N:$$

Let $r^{T^{(1;0)M}}, r^{T^{(1;0)N}}$ be the connections on $T^{(1;0)M}, T^{(1;0)N}$ induced by the Levi-Civita connections r^{TM}, r^{TN} as explained in Section 1.6. Let $r^{(0;1)}$ be the Hermitian connection on $(T^{(0;1)}(M \rightarrow N)) \rightarrow L \rightarrow F$ induced by the Clifford connections on $(T^{(0;1)M})$ and on $(T^{(0;1)N})$ and the connections r^L, r^F .

Definition 3.2. For $T \in \mathbb{R}$, let D_T^M be the operator defined by

$$(3.9) \quad D_T^M = D^{L \rightarrow F} + \frac{p-1}{2} c(X) : \mathbb{R}^0; (M; L \rightarrow F) \rightarrow \mathbb{R}^0; (M; L \rightarrow F):$$

Recall that when a subscript index appears two times in a formula, we sum up with this index. Set

$$(3.10) \quad \begin{aligned} I_1 &= \frac{1}{4} c(d^M \sum_j c(V_j^M + 2V_j^N) + \frac{1}{2} c(d^N \sum_j c(V_j^M)); \\ I_2 &= \frac{1}{4} (1 + \frac{J^M}{1}) \sum_j V_j^M; d^M \sum_j; \\ I_3 &= \frac{1}{2} c(d^N \sum_j c(V_j^N)); \end{aligned}$$

Let $\{e_k g_{k=1}^{dim M}\}$ (resp. $\{f_i g_{i=1}^{dim N}\}$) be an orthonormal frame of TM (resp. TN), then $\{e_a g_{a=1}^{dim M}\} = \{e_k g\}$ [$\{f_i g\}$ is an orthonormal frame of TM].

The following Bochner type formula holds for $(D_T^M)^2$.

Theorem 3.3. The following Bochner type formula holds

$$\begin{aligned}
 (3.11) \quad D_T^M \|\cdot\|^2 &= D^{L_F} \|\cdot\|^2 + \frac{\int_X \dim M}{4} \sum_{k=1}^p c(e_k) c(r_{e_k}^{TM}) \sum_j V_j^M \\
 &+ \frac{\int_X \dim N}{4} \sum_{i=1}^p c(f_i) c(r_{f_i}^{TN}) \sum_j V_j^N \sum_i \frac{\int_X h}{2} \text{Tr } r^{T(1;0)M} \sum_j V_j^M \sum_i \\
 &\quad \frac{\int_X h}{2} \text{Tr } r^{T(1;0)N} \sum_j V_j^N \sum_i \\
 &+ 2 \int_X \sum_j L_{V_j} + \frac{\int_X}{4} \|\cdot\|^2 + \int_X (\mathbb{I} + \mathbb{I}_2 + \mathbb{I}_3) :
 \end{aligned}$$

Proof. For any $1 \leq a \leq \dim M$, it is clear that

$$(3.12) \quad r_{e_a}^{0; \cdot} c(X) = c(X) r_{e_a}^{0; \cdot} + c(r_{e_a}^{TM} X) :$$

By (3.9), (3.12), we have

$$\begin{aligned}
 (3.13) \quad D_T^M \|\cdot\|^2 &= D^{L_F} \|\cdot\|^2 + \frac{\int_X \dim M}{2} \sum_{a=1}^p c(e_a) r_{e_a}^{0; \cdot} c(X) + c(X) c(e_a) r_{e_a}^{0; \cdot} + \frac{\int_X}{4} \|\cdot\|^2 \\
 &= D^{L_F} \|\cdot\|^2 + \frac{\int_X \dim M}{2} \sum_{a=1}^p c(e_a) c(r_{e_a}^{TM} X) \int_X r_X^{0; \cdot} + \frac{\int_X}{4} \|\cdot\|^2 :
 \end{aligned}$$

By [20, Lemma 1.5],

$$\begin{aligned}
 (3.14) \quad r_{V_j^M}^{0; \cdot} &= L_{V_j} + 2 \int_X \mathbb{I}(\cdot; V_j) \\
 &+ \frac{\int_X \dim M}{4} \sum_{a=1}^p c(e_a) c(r_{e_a}^{TM} V_j^M) + \frac{\int_X h}{2} \text{Tr } r^{T(1;0)M} \sum_j V_j^M \sum_i :
 \end{aligned}$$

Thus by (3.5), (3.14),

$$\begin{aligned}
 (3.15) \quad r_X^{0; \cdot} &= \int_X L_{V_j} + 2 \int_X \mathbb{I}(\cdot; V_j) \\
 &+ \frac{\int_X \dim M}{4} \sum_{k=1}^p c(e_k) c(r_{e_k}^{TM} V_j^M) \sum_j + \frac{\int_X \dim N}{4} \sum_{i=1}^p c(f_i) c(r_{f_i}^{TN} V_j^N) \sum_j \\
 &+ \frac{\int_X h}{2} \sum_j \text{Tr } r^{T(1;0)M} \sum_j V_j^M \sum_i + \frac{\int_X h}{2} \sum_j \text{Tr } r^{T(1;0)N} \sum_j V_j^N \sum_i :
 \end{aligned}$$

Thus from (3.5), (3.10), (3.14) and (3.15), we get

$$\begin{aligned}
 (3.16) \quad r_X^{0; \cdot} &= \int_X L_{V_j} + 2 \int_X \mathbb{I}(\cdot; V_j) + \frac{\int_X \dim M}{4} \sum_{k=1}^p c(e_k) c(r_{e_k}^{TM}) \sum_j V_j^M \\
 &+ \frac{\int_X \dim N}{4} \sum_{i=1}^p c(f_i) c(r_{f_i}^{TN}) \sum_j V_j^N + \frac{\int_X}{4} c(d^M) \sum_j c(V_j^M) \\
 &+ \frac{\int_X h}{2} \sum_j \text{Tr } r^{T(1;0)M} \sum_j V_j^M \sum_i + \frac{\int_X h}{2} \sum_j \text{Tr } r^{T(1;0)N} \sum_j V_j^N \sum_i \mathbb{I} :
 \end{aligned}$$

Also by (3.5),

(3.17)

$$\begin{aligned} \frac{1}{2} \sum_{a=1}^{\dim M} c(e_a) c r_{e_a}^{TM} X &= \frac{1}{2} \sum_{k=1}^{\dim M} c(e_k) c r_{e_k}^{TM} V_j^M + \frac{1}{2} c d^M_j c V_j^N \\ &+ \frac{1}{2} \sum_{i=1}^{\dim N} c(f_i) c r_{f_i}^{TN} V_j^N + \frac{1}{2} c d^N_j c V_j^M + V_j^N : \end{aligned}$$

By (3.10), (3.13), (3.16) and (3.17), the proof of Theorem 3.3 is completed.

3.3. Relations between terms in (3.11) near zero(X): Part 1. In this subsection, as well as the next a few subsections, we establish certain formulas concerning the relationships between the terms appearing in (3.11), on the zero set of X in M . These relations are crucial for the proof of Theorem 3.1.

Set

$$\begin{aligned} 1 &= 1 + {}^0(j^2)(j^2 - j^2); \\ 2 &= 1 - 2 {}^0(\)_{i i 1}; \\ (3.18) \quad 4 &= 1 - (\) - 2 {}^0(\) (j^2)_{i i}; \\ 5 &= (1 - (\))_1 - (j^2) \\ &= 1 - (\) - (j^2) + (1 - (\)) {}^0(j^2)(j^2 - j^2); \end{aligned}$$

These functions appear naturally on zero(X), as is clear from the following (3.19) and Lemma 3.4.

From (3.4), (3.18), one gets

$$(3.19) \quad j = 2 {}_1 j + 2 (j^2) j;$$

From (3.4), (3.18) and (3.19), one sees

$$\begin{aligned} (3.20) \quad j &= 2 j + 2(1 - (\))_j - 2 {}^0(\)_{i i} 2 {}_1 j + 2 (j^2) j \\ &= 2 {}_2 j + 2 {}_4 j; \end{aligned}$$

Lemma 3.4. On zero(X) = $\{x \in M : X(x) = 0\}$, we have

$$(3.21) \quad 2 {}_j V_j^M = 4 {}_j V_j^M; \quad 2 {}_j V_j^M = 2 {}_5 j V_j^M;$$

and

$$(3.22) \quad 2 {}_j V_j^N = 4 {}_j V_j^N; \quad 2 {}_j V_j^N = 2 {}_5 j V_j^N; \quad 2 {}_j V_j^N = (2 - 4) {}_j V_j^N;$$

Proof. By (2.15), one sees that

$$(3.23) \quad X = 0 \text{ if and only if } \sum_j X {}_j V_j^M = 0 \text{ and } \sum_j X {}_j V_j^N = 0;$$

From (3.20), the equation $\sum_j {}_j V_j^M = 0$ in (3.23) is equivalent to the first equation of (3.21).

By (3.18), (3.19) and the first equation of (3.21), we have on $\text{zero}(X)$,

$$\begin{aligned}
 (3.24) \quad \Delta_j V_j^M &= \Delta_{12} V_j^M + 2(j^2) \Delta_j V_j^M \\
 &= \Delta_{12} (1 - \Delta_j^2) (j^2) \Delta_j V_j^M + 2(j^2) (1 - \Delta_j^2) \Delta_j V_j^M \\
 &= 2(1 - \Delta_j^2) \Delta_j V_j^M = 2\Delta_{5j} V_j^M :
 \end{aligned}$$

In the same way, by (3.20), $\Delta_j V_j^N = 0$ in (3.23) is equivalent to the first equation of (3.22).

By (2.3) and the first equation of (3.22), as in (3.24), we have

$$\begin{aligned}
 (3.25) \quad \Delta_j V_j^N &= \Delta_{12} V_j^N + 2(j^2) \Delta_j V_j^N = 2\Delta_{5j} V_j^N ; \\
 \Delta_j V_j^N &= \Delta_j V_j^N + \Delta_j V_j^N = (2 - 4) \Delta_j V_j^N :
 \end{aligned}$$

The proof of Lemma 3.4 is completed.

3.4. Estimates of Δ_i ; $i = 1; 2; 4; 5$, when $A > 0$ is large. Recall that Δ_j , Δ_{5j} have been defined in (3.2), (3.3).

Lemma 3.5. There exists $A_0 > 0$ such that for $A > A_0$, we have

$$(3.26) \quad A < \Delta_j < 2A ; \quad \text{on } M \setminus n @ M :$$

Thus

$$(3.27) \quad 0 < (1 - \Delta_j) < 1 \quad \text{on } M \setminus n @ M :$$

Moreover, uniformly on M and the parameter A , we have

$$(3.28) \quad \Delta_1 = 1 + O(A^{-1/2}); \quad \Delta_2 = 1 + O(A^{-1/2});$$

$$(3.29) \quad \Delta_4 = (1 - \Delta_j)(1 + O(A^{-1/2}));$$

$$(3.30) \quad \Delta_5 = (1 - \Delta_j^2) (1 + O(A^{-1/2}));$$

Finally, for any $A > A_0$, we have

$$\begin{aligned}
 (3.31) \quad \Delta_1 (1 - \Delta_j^2) &< 0 \quad \text{if } (x; y) \in M \setminus n @ M ; \\
 &= 0 \quad \text{if } (x; y) \in @ M :
 \end{aligned}$$

Proof. As N is compact, there exists $C_0 > 0$ such that

$$(3.32) \quad |j| \leq C_0 :$$

From (2.11), (2.12) and (3.32), one sees that on M , one has

$$\begin{aligned}
 (3.33) \quad 2A &> j^2 + j^2 + 2h ; \quad i \\
 &> j^2 + j^2 - \frac{1}{3}j^2 + 3j^2 > \frac{2}{3}j^2 - 2C_0^2 :
 \end{aligned}$$

From (3.33), one knows that when $A > 0$ is large enough, one has on M that

$$(3.34) \quad A^{-1/2} \leq |j| < \frac{1}{3}A^{-1/2} :$$

From (2.3), (2.11), (3.1)–(3.3) and (3.34), one deduces that for $A > 0$ large enough, one also has

$$(3.35) \quad |j| < 2A^{-1/2}; \quad |j^2 - j^2| = O(A^{-1/2}); \quad \Delta_j = j^2 + O(A^{-1/2}) \text{ on } M :$$

If $j \leq 1:3A$, then by (3.35), when $A > 0$ is large enough, one has $\dots < \frac{4A}{3}$. Then by (3.1)–(3.3), one has for $\dots < \frac{4A}{3}$,

$$(3.36) \quad \begin{aligned} \frac{1}{A} \dots &= \frac{j \dots}{A} \dots + \frac{1}{A} \frac{j \dots}{A} \dots (j \dots - j \dots) \\ &= \frac{j \dots}{A} \dots (1 + O(A^{-1/2})): \end{aligned}$$

If $j > 1:3A$, then by (3.35), one has

$$(3.37) \quad \dots > A + O(3A) + O(A^{-1/2}) > 1:2A;$$

when $A > 0$ is large enough.

From (3.1), (3.2), (3.3), (3.35), (3.36) and (3.37), one gets (3.26) and (3.27).

From (2.11), (3.2), (3.3) and (3.32), one gets

$$(3.38) \quad j \dots < 2A^{-1/2} \quad \text{on } M :$$

From (3.2), (3.18), (3.32), (3.35) and (3.38), we get (3.28).

On the other hand, if $j \leq 1:3A$ and $A > 0$ is large enough so that $\dots < \frac{4A}{3}$, then by (3.1) and (3.2), one has

$$(3.39) \quad \begin{aligned} 1 \dots &= \frac{1}{A} \dots^3; \\ \dots &= \frac{3}{A} \frac{1}{A} \dots^2; \quad \dots = \frac{6}{A^2} \frac{1}{A} \dots \end{aligned}$$

From (3.1), (3.2), (3.32), (3.38) and (3.39), one finds

$$(3.40) \quad \begin{aligned} \dots(j \dots) &= (1 \dots)^{4-3} O(A^{-1}); \\ \dots(j \dots)_{i \dots} &= (1 \dots) O(A^{-3/2}): \end{aligned}$$

By (3.32), (3.38) and (3.40), we see that (3.29) holds for $A > 0$ large enough and $j \leq 1:3A$.

While if $j > 1:3A$, by (3.37), when $A > 0$ is large enough, $\dots > 1:1A$, and thus by (3.1) and (3.2), one has $1 \dots > 1 \dots (1:1A) > 0$, from which (3.29) holds tautologically.

The proof of (3.29) is thus completed.

If $j > \frac{5A}{3}$, then from (3.1),

$$(3.41) \quad (j \dots) = 1; \quad \dots(j \dots) = \dots(j \dots) = 0;$$

from which, one gets

$$(3.42) \quad 1 \dots (j \dots) = \dots; \quad \dots = \dots:$$

Thus from (3.18) and (3.42), we get (3.30) and (3.31).

Now by (3.1), (3.2), (3.3) and (3.35), it is clear that when $A > 0$ is large enough,

$$(3.43) \quad \dots = j \dots + \dots j \dots O(A^{-1/2}):$$

Let $0 < \epsilon_0 < \frac{1}{9}$ be such that

$$(3.44) \quad e(t) + e(t) > \frac{28:5}{27}; \quad \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}:$$

By (3.1), the existence of ρ_0 is clear.

If $\frac{(4-\rho_0)A}{3} \leq j \leq \frac{5A}{3}$, then from (3.1), (3.43) and (3.44), one sees that for A large enough,

$$(3.45) \quad 1 - \rho_0(j) \leq \frac{1}{27}$$

thus we get again (3.30) and (3.31) from (3.18).

If $j \leq \frac{(4-\rho_0)A}{3}$, then by (3.35), one sees that when $A > 0$ is large enough, one has $\rho_0(j) < \frac{4A}{3}$. Thus, by (3.1), (3.2), (3.36) and (3.39), one has

$$(3.46) \quad \begin{aligned} \rho_0(j) - \rho_0(j^2) &= \frac{j^2}{A} - \frac{j^2}{A} (1 + O(A^{-1/2})) - O(A^{-1}) \\ &= \frac{j^2}{A} - O(A^{-1}); \\ \rho_0(j) - \rho_0(j^2) &= \frac{j^2}{A} - \frac{j^2}{A} (1 + O(A^{-1/2})) : \end{aligned}$$

from which and (3.18) we get again (3.30) and (3.31).

The proof of (3.30) and (3.31) is now complete, as well as Lemma 3.5.

The following Lemma will also be used in the proof of Proposition 3.11.

Lemma 3.6. There exists $A_0 > 0$ such that for any $A > A_0$,

$$(3.47) \quad 1 < \frac{\rho_0(j) - \rho_0(j^2)}{\rho_0(j^2)} < 12 \quad \text{on } M_n \otimes M :$$

Proof. In fact, by (3.27) and (3.31),

$$(3.48) \quad \rho_0(j) - \rho_0(j^2) < \rho_0(j^2) < 0 \quad \text{on } M_n \otimes M :$$

Thus we need to prove that

$$(3.49) \quad \frac{\rho_0(j) - \rho_0(j^2)}{\rho_0(j^2)} < 0 \quad \text{if } (x, y) \in M_n \otimes M :$$

If $j > \frac{5A}{3}$, then $\rho_0(j) = 1$, thus (3.49) holds.

Let $\rho_0 > 0$ be defined as in (3.44).

If $A < j \leq \frac{(4-\rho_0)A}{3}$, then by (3.2), (3.36) and (3.39)

$$(3.50) \quad \begin{aligned} \frac{\rho_0(j) - \rho_0(j^2)}{\rho_0(j^2)} &= \frac{j^2}{A} - \frac{j^2}{A} (1 + O(A^{-1/2})) - O(A^{-1}) \\ &= \frac{j^2}{A} - O(A^{-1}) - \frac{j^2}{A} (1 + O(A^{-1/2})) : \end{aligned}$$

From (3.50), we know (3.49) holds for A large enough.

If $\frac{(4-\rho_0)A}{3} \leq j \leq \frac{5A}{3}$, then by (3.45), as $0 < \rho_0(j) < 1$, we get, when $A > 0$ is large enough, that

$$(3.51) \quad \frac{\rho_0(j) - \rho_0(j^2)}{\rho_0(j^2)} \leq \frac{11}{27} + \rho_0(j) < \frac{4}{27} :$$

Thus (3.47) holds on $M_n \otimes M$.

3.5. Relations between terms in (3.11) near zero (X): Part 2. The first of two relations we would include in this subsection is

Lemma 3.7. We have

$$(3.52) \quad jV_j^M = J^M (d^M j j) :$$

Proof. By (2.3), (3.3), we have

$$(3.53) \quad \begin{aligned} d_j &= 2^{1+\ell(j)} (j j) (j j - j j) - (j j) - j d^{M-j+2} (j j) - j d_j \\ &= j d^{M-j+2} (j j) - j d^N j; \\ d_j &= d_j - \ell(j) j d - \ell(j) d^N j; \end{aligned}$$

From (2.10) and (3.53), we get

$$(3.54) \quad \begin{aligned} (d_j) &= J^M V_j^M + J^N V_j^N; \\ (d_j) &= \sum_k J^M V_k^M + 2 \ell(j) \sum_k J^N V_k^N; \\ (d_j) &= J^M V_j^M - \ell(j) \sum_k J^M V_k^M \\ &\quad + (1 - \ell(j)) J^N V_j^N - 2 \ell(j) \sum_k J^N V_k^N; \end{aligned}$$

From (3.4), (3.54), we get

$$(3.55) \quad jV_j^M = 2 \sum_j V_j^M - \ell(j) \sum_k V_k^M = J^M (d^M j j) :$$

Thus we get (3.52).

On the other hand, by Lemma 3.5, we see that when $A > 0$ is large enough, $\ell_2 > 1=2$:

To state the second relation we would include in this subsection, we introduce the following two functions,

$$(3.56) \quad \begin{aligned} I_4 &= 2^{-\ell(j)} \ell(j) \ell(j) \ell(j) \ell(j) \frac{4}{2} + 4^{-\ell(j)} \ell(j) \ell(j) \ell(j) \ell(j) \frac{4}{2} \\ &\quad + 2^{-\ell(j)} \ell(j) \ell(j) + (\ell(j))^2 \frac{5}{2} + 2^{-\ell(j)} \frac{5}{2}; \\ I_5 &= 2^{-\ell(j)} \ell(j) \ell(j) \ell(j) \ell(j) \frac{\binom{2}{2} \binom{4}{4}}{2} \\ &\quad + 2^{-\ell(j)} \ell(j) \ell(j) \ell(j) \ell(j) + 2 (\ell(j))^2 \ell(j) \ell(j) \frac{\binom{2}{2} \binom{4}{5}}{2} \\ &\quad + \ell(j) \frac{\binom{2}{2} \binom{4}{5}}{2} + 1 - 2 \ell(j) \frac{5}{2} - \ell(j) \frac{4}{2}; \end{aligned}$$

which appear naturally in the following lemma.

Lemma 3.8. On $\text{zero}(X) \rightarrow M$, we have

$$(3.57) \quad \frac{1}{4}c((d^M_j))c(V_j^M) + \frac{1}{4} \left(1 + \frac{J^M}{1} V_j^M \right); (d^M_j) \\ = \frac{1}{2} \left(c(J^M V_j^M)c(V_j^M) + \frac{1}{1} J_j^M \right) \\ + I_4 \left(c(J^M V_k^M)c(V_j^M) + \frac{1}{1} X_{k,j} V_k^M \right)!$$

and

$$(3.58) \quad \frac{1}{4}c((d^M_j))c(V_j^N) = \frac{1}{2} \left(c(J^M V_j^M)c(V_j^N) + I_4 \left(c(J^M V_k^M)c(V_j^N) \right); \right. \\ \left. \frac{1}{4}c((d^N_j))c(V_j^M) = \frac{1}{2} \left(c(J^N V_j^N)c(V_j^M) + I_5 \left(c(J^N V_k^N)c(V_j^M) \right); \right. \right.$$

Proof. By (2.3), (3.4), we have

$$(3.59) \quad d_j = 2^0(j^2)(j^2 - j^2)_{j,k} - 2^0(j^2)_{j,k} + 2^0(j^2)_{j,k} - 2^0(j^2)_{j,k} d^M_k \\ + 4^0(j^2)_{j,k} d_k + 2 + 2^0(j^2)(j^2 - j^2) d^M_j + 2(j^2) d^N_j!$$

From (2.10), (3.59), we get

$$(3.60) \quad (d_j) = 4^0(j^2)(j^2 - j^2)_{j,k} J^M V_k^M \\ + 4^0(j^2)(j_k + k_j) J^M V_k^M \\ + 2 + 2^0(j^2)(j^2 - j^2) J^M V_j^M \\ + 4^0(j^2)_{j,k} J^N V_k^N + 2(j^2) J^N V_j^N!$$

Now from (3.4), (3.54) and (3.60), we get

$$(3.61) \quad (d^M_j) = 2^0(\)_{i,i,j} (d^M_j) - 2^0(\)_{i,i} (d^M_j) + j (d^M_i) \\ + 2(d^M_j) \\ = 2^0(\)_{i,i,j,k} J^M V_k^M - 2^0(\)_{i,i} 4^0(j^2)(j^2 - j^2)_{j,k} J^M V_k^M \\ + 4^0(j^2)(j_k + k_j) J^M V_k^M + 2 + 2^0(j^2)(j^2 - j^2) J^M V_j^M \\ 2^0(\)_{i,j} J^M V_i^M - 0(\)_{i,k} J^M V_k^M + 2 J^M V_j^M - 0(\)_{j,k} J^M V_k^M!$$

Thus from (3.18) and (3.61),

$$(3.62) \quad (d^M_j) = 2 J^M V_j^M - 8^0(\)^0(j^2)_{i,i}(j^2 - j^2)_{j,k} J^M V_k^M \\ 8^0(\)^0(j^2)_{i,i}(j_k + k_j) J^M V_k^M \\ + 2 \left(0(\)_{i,i} + (0(\))^2_{i,i} \right)_{j,k} J^M V_k^M - 2^0(\)_{(k,j)+(j,k)} J^M V_k^M!$$

By (3.62),

$$\begin{aligned}
(3.63) \quad c((d^M_j)^i) c(V_j^M) &= 2 \sum_{i,j} c(J^M V_j^M) c(V_j^M) \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} (j^2 - j^2) c(\sum_k J^M V_k^M) c(\sum_j V_j^M) \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} c(\sum_k J^M V_k^M) c(\sum_j V_j^M) + c(\sum_j J^M V_j^M) c(\sum_k V_k^M) \\
&\quad + 2 \sum_{i,j} \omega(\cdot)_{i,i} + (\sum_{i,j} \omega(\cdot)_{i,i})^2 \sum_{i,j} c(\sum_k J^M V_k^M) c(\sum_j V_j^M) \\
&\quad \quad \quad 2^0(\cdot) c(\sum_j J^M V_j^M) c(\sum_k V_k^M) + c(\sum_k J^M V_k^M) c(\sum_j V_j^M)
\end{aligned}$$

and by using the fact that J^M is anti-symmetric, $\sum_{i,j} \omega(j^2)_{i,i}$ on TM is \mathbb{C} -bilinear, we get

$$\begin{aligned}
(3.64) \quad 1 + \sum_{i,j} J^M V_j^M ; (d^M_j)^i &= 2 \sum_{i,j} J^M V_j^M \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} (j^2 - j^2) \sum_{i,j} J^M V_j^M \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} \sum_{i,j} J^M V_j^M ; \sum_k V_k^M \\
&\quad + 2 \sum_{i,j} \omega(\cdot)_{i,i} + (\sum_{i,j} \omega(\cdot)_{i,i})^2 \sum_{i,j} \sum_k V_k^M \\
&\quad \quad \quad 2^0(\cdot) \sum_{i,j} J^M V_j^M ; \sum_k V_k^M :
\end{aligned}$$

For the term $c((d^M_j)^i) c(V_j^N)$, from (3.18), (3.62), we have

$$\begin{aligned}
(3.65) \quad c((d^M_j)^i) c(V_j^N) &= 2 \sum_{i,j} c(J^M V_j^M) c(V_j^N) \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} (j^2 - j^2) c(\sum_k J^M V_k^M) c(\sum_j V_j^N) \\
&\quad 8^0(\cdot) \sum_{i,j} \omega(j^2)_{i,i} c(\sum_k J^M V_k^M) c(\sum_j V_j^N) + c(\sum_j J^M V_j^M) c(\sum_k V_k^N) \\
&\quad + 2 \sum_{i,j} \omega(\cdot)_{i,i} + (\sum_{i,j} \omega(\cdot)_{i,i})^2 \sum_{i,j} c(\sum_k J^M V_k^M) c(\sum_j V_j^N) \\
&\quad \quad \quad 2^0(\cdot) c(\sum_j J^M V_j^M) c(\sum_k V_k^N) + c(\sum_k J^M V_k^M) c(\sum_j V_j^N) :
\end{aligned}$$

Moreover, from (2.10), (3.54), (3.60) and $\sum_{i,j} \omega(j^2)_{i,i} = \sum_{i,j} \omega(j^2)_{i,i}$, we get

$$\begin{aligned}
(3.66) \quad (d^N_j)^i &= 2^0(\cdot) \sum_{i,j} (d^N_j)^i - 2^0(\cdot) \sum_{i,j} (d^N_i)^j \\
&\quad 2 \sum_{i,j} \omega(j^2)_{i,i} (d^N_j)^i - 2^0(\cdot) \sum_{i,j} (d^N_i)^j + 2 (d^N_j)^i \\
&= 2^0(\cdot) \sum_{i,j} 4 \sum_{i,j} \omega(j^2)_{i,i} \sum_{i,j} J^N V_k^N + 2 \sum_{i,j} (j^2) J^N V_j^N \\
&\quad 2^0(\cdot) \sum_{i,j} (1 - \sum_{i,j} \omega(j^2)_{i,i}) J^N V_i^N - 2^0(\cdot) \sum_{i,j} (j^2)_{i,i} J^N V_k^N \\
&\quad 4 \sum_{i,j} \omega(j^2)_{i,i} \sum_{i,j} J^N V_k^N - 2^0(\cdot) \sum_{i,j} (i - \sum_{i,j} \omega(j^2)_{i,i}) J^N V_i^N \\
&\quad \quad \quad + 2 (1 - \sum_{i,j} \omega(j^2)_{i,i}) J^N V_j^N - 2^0(\cdot) \sum_{i,j} (j^2)_{i,i} J^N V_k^N :
\end{aligned}$$

From (3.18) and (3.66),

$$(3.67) \quad (d^N_j) = 2_4 J^N V_j^N \quad 8^0(\)^0(j \hat{f})_{i \ i \ j \ k} J^N V_k^N \\ + \quad 4^{\omega(\)} (j \hat{f})_{i \ i} + 4(\ ^0(\))^2 (j \hat{f})_{i \ i}^2 \quad 2^0(\)_{j \ k} J^N V_k^N \\ 2^0(\)(1 \quad 2(\))_{j \ i} J^N V_i^N \quad 4^0(\) (j \hat{f})_{j \ k} J^N V_k^N :$$

From (3.67), one has

$$(3.68) \quad c((d^N_j))c(V_j^M) = 2_4 \quad c(J^N V_j^N)c(V_j^M) \\ 8^0(\)^0(j \hat{f})_{i \ i} c(k J^N V_k^N)c(j V_j^M) \\ + \quad 4^{\omega(\)} (j \hat{f})_{i \ i} + 4(\ ^0(\))^2 (j \hat{f})_{i \ i}^2 \quad 2^0(\) c(k J^N V_k^N)c(j V_j^M) \\ 2^0(\)(1 \quad 2(\))c(i J^N V_i^N)c(j V_j^M) \\ 4^0(\) (j \hat{f})_{j \ k} c(k J^N V_k^N)c(j V_j^M) :$$

On zero(X), from (3.21), (3.63) and (3.64), we then get

$$(3.69) \quad \frac{1}{4}c((d^M_j))c(V_j^M) + \frac{1}{4} \quad 1 + \frac{J^M}{\mathbb{P} \equiv 1} V_j^M ; (d^M_j) \\ = \frac{1}{2} \quad 2 \quad c(J^M V_j^M)c(V_j^M) + \frac{1}{\mathbb{P} \equiv 1} J_j^M \hat{f} \\ + \frac{1}{4} \quad 8^0(\)^{\omega(j \hat{f})_{i \ i}} (j \hat{f})_{i \ i} (j \hat{f})_{j \ j} \quad \frac{4}{2}^2 \\ + 8^0(\)^0(j \hat{f})_{i \ i} \frac{2}{2}^4 + 2 \quad \omega(\)_{i \ i} + (\ ^0(\))^2 \frac{2}{i} \quad \frac{2}{2}^5 \quad 2 \\ + 2^0(\) \frac{4}{2}^5 \quad c(k J^M V_k^M)c(j V_j^M) + \frac{1}{\mathbb{P} \equiv 1} \quad X \quad k V_k^M \quad 2 \quad ! :$$

From (3.69) we get (3.57).

On zero(X), from (3.21), (3.22) and (3.65), as in (3.69), we get the first equation of (3.58).

On zero(X), from (3.21), (3.22) and (3.68), we get

$$(3.70) \quad \frac{1}{4}c((d^N_j))c(V_j^M) = \frac{1}{2} \quad 4 \quad c(J^N V_j^N)c(V_j^M) \\ + \frac{1}{4} \quad 8^0(\)^0(j \hat{f})_{i \ i} \frac{(2 \quad 4)}{2}^4 \\ 4^{\omega(\)} (j \hat{f})_{i \ i} + 4(\ ^0(\))^2 (j \hat{f})_{i \ i}^2 \quad 2^0(\) \quad \frac{2(2 \quad 4)}{2}^5 \\ + 2^0(\)(1 \quad 2(\)) \frac{2}{2}^5 \quad 4^0(\) (j \hat{f})_{j \ k} \frac{2}{2}^4 \quad c(k J^N V_k^N)c(j V_j^M) :$$

From (3.70), we get the second equation of (3.58).

3.6. Estimates of I_4 and I_5 . In this section, we establish the following estimate result for the terms I_4, I_5 appeared in Subsection 3.5.

Lemma 3.9. When $A > 0$ is large enough, we have

$$(3.71) \quad \begin{aligned} I_4 &= 2^{-\alpha} \left(1 - \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right) \right); \\ I_5 &= -\alpha \left(1 - \left(\frac{j^2}{j^2} \right) \right)^2 \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right); \end{aligned}$$

Moreover

$$(3.72) \quad \begin{aligned} I_4 &> 0 \quad \text{if } (x; y) \in M_n \setminus M; \\ I_4 &= 0 \quad \text{if } (x; y) \in M = M_1 \cup M_2; \end{aligned}$$

Proof. If $j^2 \leq \frac{5A}{3}$, by Lemma 3.5, (3.2), (3.35), (3.36) and (3.38), we get

$$(3.73) \quad \begin{aligned} I_4 &= -\alpha \left(\frac{j^2}{j^2} \right) O(A^{-1}) \left(1 - \left(\frac{j^2}{j^2} \right) \right)^2 - \alpha \left(\frac{j^2}{j^2} \right) O(A^{-1-2}) \left(1 - \left(\frac{j^2}{j^2} \right) \right) \\ &\quad + 2^{-\alpha} \left(\frac{j^2}{j^2} \right) \left(1 - \left(\frac{j^2}{j^2} \right) \right)^2 \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right) \\ &\quad + 2^{-\alpha} \left(\frac{j^2}{j^2} \right) \left(1 - \left(\frac{j^2}{j^2} \right) \right) \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right); \end{aligned}$$

By (3.1), (3.36), (3.39), (3.45) and (3.46), there exist $C > 0$ such that for any $A > A_0$, if $j^2 \leq \frac{5A}{3}$, then

$$(3.74) \quad \begin{aligned} 1 - \left(\frac{j^2}{j^2} \right) &\geq C \frac{j^2}{j^2} \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right); \quad \left(\frac{j^2}{j^2} \right) \geq C \frac{j^2}{j^2} \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right); \\ A^{-\alpha} \left(\frac{j^2}{j^2} \right) &\geq C \frac{j^2}{j^2} \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right); \end{aligned}$$

By (3.39), (3.73) and (3.74), we get (3.71) for I_4 .

If $j^2 > \frac{5A}{3}$, then by (3.1), (3.41), (3.42) and (3.56),

$$(3.75) \quad \begin{aligned} I_4 &= O(A^{-3-2}) \left(\frac{j^2}{j^2} \right)^2 - 2^{-\alpha} \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right) \\ &= -2^{-\alpha} \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right) \left(1 + O(A^{-1-2}) \right); \end{aligned}$$

Thus we get the first equation of (3.71).

As $e^0(t) < 0$ for $t > 0$, thus $-\alpha < 0$ on $j^2 > A$, from (3.31), (3.71), we get (3.72).

If $j^2 \leq \frac{5A}{3}$, by (3.1), (3.43), for A large enough, $\left(\frac{j^2}{j^2} \right) > \left(\frac{5A}{3} \right) + O(A^{-1-2}) > \frac{1}{3}$, and so by (3.28) and (3.29),

$$(3.76) \quad \frac{j^2}{j^2} - \frac{j^2}{j^2} = \left(\frac{j^2}{j^2} \right) + O(A^{-1-2});$$

Thus if $j^2 \leq \frac{5A}{3}$, by Lemma 3.5, (3.1), (3.38), (3.40) and (3.76), we have

$$(3.77) \quad \begin{aligned} I_5 &= -\alpha \left(\frac{j^2}{j^2} \right) \left(1 - \left(\frac{j^2}{j^2} \right) \right) O(A^{-1-2}) + \left(1 - \left(\frac{j^2}{j^2} \right) \right) \left(\frac{j^2}{j^2} \right) O(A^{-3-2}) \\ &\quad + -\alpha \left(\frac{j^2}{j^2} \right) \left(1 - \left(\frac{j^2}{j^2} \right) \right) + O(A^{-1-2}) \left(1 - \left(\frac{j^2}{j^2} \right) \right) \left(\frac{j^2}{j^2} \right) \\ &\quad - \alpha \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right) \left(\frac{j^2}{j^2} \right) + O(A^{-1-2}); \end{aligned}$$

Now by (3.39), (3.48), (3.74) and (3.77), we get (3.71) for I_5 .

If $j^2 > \frac{5A}{3}$, then by (3.18) and (3.41), we have $\frac{j^2}{j^2} = 1$ and

$$(3.78) \quad \frac{j^2}{j^2} - \frac{j^2}{j^2} = \left(\frac{j^2}{j^2} \right);$$

Thus if $j \hbar > \frac{5A}{3}$, by (3.1), (3.2), (3.38), (3.41), (3.42), (3.56) and (3.78), we get

$$(3.79) \quad I_5 = \left(\int_M \langle \cdot, \cdot \rangle \right)^2 \left(\int_M \langle \cdot, \cdot \rangle \right)^2 O(A^{-1/2}) + \int_M \langle \cdot, \cdot \rangle \left(\int_M \langle \cdot, \cdot \rangle (1 + O(A^{-1/2})) \right) \\ (1 - 2 \int_M \langle \cdot, \cdot \rangle) \left(\int_M \langle \cdot, \cdot \rangle (1 + O(A^{-1/2})) \right) \left(\int_M \langle \cdot, \cdot \rangle (1 + O(A^{-1/2})) \right) :$$

As $0 < \int_M \langle \cdot, \cdot \rangle < 1$ on $M \cap \text{supp} \mu$, from (3.2) and (3.79), we get if $j \hbar > \frac{5A}{3}$,

$$(3.80) \quad I_5 = \int_M \langle \cdot, \cdot \rangle \left(\int_M \langle \cdot, \cdot \rangle (1 + O(A^{-1/2})) \right) :$$

From (3.41), (3.80), we get again (3.71) for I_5 .

The proof of Lemma 3.9 is completed.

3.7. Estimates of $I_1 + I_2$. In this subsection, we establish an estimate result on zero (X). Before doing this, we first prove the following lemma.

For any $x \in M; y \in N, W \in T_x M$, let $B(W) \in \text{End}((T^{(0,1)}(M \times N))_{(x,y)})$ be defined by

$$(3.81) \quad B(W) = \int_M \langle \cdot, \cdot \rangle c(J^* W) c(W) + \int_N \langle \cdot, \cdot \rangle :$$

Lemma 3.10. For any $W \in T_x M, V \in T_y N$, the endomorphisms $B(W), \int_M \langle \cdot, \cdot \rangle c(W) c(V)$ of $(T^{(0,1)}(M \times N))_{(x,y)}$ are self-adjoint and for any $k > 0$,

$$(3.82) \quad B(W) > 0; \\ \int_M \langle \cdot, \cdot \rangle c(W) c(V) > \frac{1}{2k} B(W) - k \int_N \langle \cdot, \cdot \rangle :$$

Proof. As $W; V$ are orthogonal to each other, $B(W), \int_M \langle \cdot, \cdot \rangle c(W) c(V)$ are self-adjoint.

For $W \in T_x M, V \in T_y N$, we write their complexification as $W = w + \bar{w}, V = v + \bar{v}$, with $w \in T^{(1,0)} M, \bar{w} \in T^{(0,1)} M, v \in T^{(1,0)} N, \bar{v} \in T^{(0,1)} N$. Let $\bar{w} \in T^{(0,1)} M, \bar{v} \in T^{(0,1)} N$ be the metric duals of $w; v$. Then we deduce from (1.34) that

$$(3.83) \quad B(W) = c(w + \bar{w}) c(w + \bar{w}) + \int_N \langle \cdot, \cdot \rangle \\ = 2(\bar{w} \wedge + \dot{\bar{w}})(\bar{w} \wedge - \dot{\bar{w}}) + 2 \int_N \langle \cdot, \cdot \rangle = 4\bar{w} \wedge \dot{\bar{w}} :$$

Thus we get the first equation of (3.82) which has been obtained in [20, (2.9), (2.13)].

Moreover,

$$(3.84) \quad \int_M \langle \cdot, \cdot \rangle c(W) c(V) = \int_M \langle \cdot, \cdot \rangle (\bar{w} \wedge \bar{v} \wedge - \bar{w} \wedge \dot{\bar{v}} - \dot{\bar{w}} \bar{v} \wedge + \dot{\bar{w}} \dot{\bar{v}}) :$$

If $\mathcal{I} \in (T^{(0,1)}(M \times N))_{(x,y)}$, then we can write

$$(3.85) \quad \mathcal{I} = \bar{w} \wedge \bar{v} \wedge - \bar{w} \wedge \dot{\bar{v}} - \dot{\bar{w}} \bar{v} \wedge + \dot{\bar{w}} \dot{\bar{v}} ;$$

and $\mathcal{I}_i (i = 1; 2; 3; 4)$ do not contain the terms \bar{w} or \bar{v} . Then we get

$$(3.86) \quad \int_M \langle \cdot, \cdot \rangle c(W) c(V) = \int_M \langle \cdot, \cdot \rangle (\bar{w} \wedge \bar{v} \wedge - \bar{w} \wedge \dot{\bar{v}} - \dot{\bar{w}} \bar{v} \wedge + \dot{\bar{w}} \dot{\bar{v}}) \\ = \int_M \langle \cdot, \cdot \rangle (\bar{w} \wedge \bar{v} \wedge - \bar{w} \wedge \dot{\bar{v}} - \dot{\bar{w}} \bar{v} \wedge + \dot{\bar{w}} \dot{\bar{v}}) \\ = \int_M \langle \cdot, \cdot \rangle (\bar{w} \wedge \bar{v} \wedge - \bar{w} \wedge \dot{\bar{v}} - \dot{\bar{w}} \bar{v} \wedge + \dot{\bar{w}} \dot{\bar{v}}) :$$

From (3.86),

$$(3.87) \quad \text{hb}(W); \mathcal{I}_i = 4 \int_M \langle \cdot, \cdot \rangle \bar{w} \wedge \bar{v} \wedge - 4 \int_M \langle \cdot, \cdot \rangle \bar{w} \wedge \dot{\bar{v}} - 4 \int_M \langle \cdot, \cdot \rangle \dot{\bar{w}} \bar{v} \wedge + 4 \int_M \langle \cdot, \cdot \rangle \dot{\bar{w}} \dot{\bar{v}} :$$

From (3.86), (3.87), we get for any $k > 0$,

$$(3.88) \quad \begin{aligned} h^{\mathbb{P}} \text{---} 1c(W) c(V) ; i = 4jw^2jjj^2 \text{Im} h_{1; 4i} \quad 4jw^2jjj^2 \text{Im} h_{2; 3i} \\ > \frac{2}{k} jw^2jj^2 \quad 2kj^2jj^2 : \end{aligned}$$

From (3.87), (3.88), we get the second equation of (3.82).

Recall that the terms I_1, I_2 have been defined in (3.10). We now state the crucial estimate for $I_1 + I_2$ as follows.

Proposition 3.11. There exist $C > 0, A_0 > 0$ such that for any $A > A_0$ and $(x; y) \in \text{zero}(X) \cap M$, we have

$$(3.89) \quad h^{\mathbb{P}} \text{---} 1(I_1 + I_2) > C \text{Id} :$$

Proof. By (3.10), and apply Lemma 3.10 to (3.57) and (3.58) with $k = 8$, we get

$$(3.90) \quad \begin{aligned} h^{\mathbb{P}} \text{---} 1(I_1 + I_2) > \frac{1}{2} \quad \frac{1}{8} \quad \frac{1}{8} \quad \prod_{j=1}^{\dim G} B(V_j^M) \quad (8_2 + 8_4) \quad \prod_{j=1}^{\dim G} jV_j^N \\ + I_4 \quad \frac{1}{8} I_4 \quad \frac{1}{8} I_5 \quad \prod_{j=1}^{\dim G} jV_j^M \\ (16I_4 + 16I_5) \quad \prod_{j=1}^{\dim G} jV_j^N : \end{aligned}$$

By Lemma 3.5, $0 < (\) < 1$ on $M \cap \text{supp}(\)$, we get for A large enough

$$(3.91) \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{3}{4} + \frac{1}{8} (\) + O(A^{-1/2}) > 0 :$$

From Lemmas 3.6, 3.9, we know that when $A > 0$ is large enough,

$$(3.92) \quad I_4 \quad \frac{1}{8} I_4 \quad \frac{1}{8} I_5 > \frac{1}{8} I_4 > 0 \quad \text{if } (x; y) \in M \cap (M_1 \cup M_2) :$$

As jV_j^N have upper bound not depending on $x \in M$, thus V_j^N , $\prod_{j=1}^{\dim G} jV_j^N$ are uniformly bounded on M .

As $\text{zero}(X) \cap M \cap \text{supp}(\)$, from (3.82), (3.90), (3.91) and (3.92), there exist $A_0 > 0, C > 0$ such that for any $A > A_0$, and $(x; y) \in \text{zero}(X) \cap M$, (3.89) holds.

3.8. A local point estimate around $\text{zero}(X)$. Recall that D_T^M is the deformed Dirac operator on M defined by (3.9) according to the functions defined in (3.1), (3.2) and (3.3). For any $T > 0$, let $F_T^M : \mathcal{O}(\mathbb{R}^n; L \otimes F) \rightarrow \mathcal{O}(\mathbb{R}^n; L \otimes F)$ be defined by

$$(3.93) \quad F_T^M = D_T^M + h^{\mathbb{P}} \text{---} 1T \quad jL_j V_j :$$

The purpose of this subsection is to prove the following local pointwise estimate around each $x \in \text{zero}(X)$.

Proposition 3.12. There exists $A_0 > 0$ such that for any $A > A_0$ a regular value of both j^f and $\frac{1}{2}j^f$, and that if M is constructed as in (2.11), then for any $z \in M \setminus n @ M$, there exist an open neighborhood U_z of z and $C_z > 0; b_z > 0$ such that for any $s \in \mathbb{R}^0; (M; L \rightarrow F)$ with $\text{supp}(s) \subset U_z$ and $T > 1$, one has

$$(3.94) \quad \text{Re } F_T^M(s); s > C_z D^{L, F} s_0^2 + (T - b_z) \text{ksk}_0^2 :$$

Proof. Recall that $\{f_i\}$ is an orthonormal frame of TN . Set

$$(3.95) \quad R_j = \frac{1}{4} \sum_{i=1}^p c(f_i) c(r_{f_i}^{TN} V_j^N) - \frac{1}{2} \text{Tr } r^{T(1;0)N} V_j^N \mathbb{1}_{T(1;0)N}^i :$$

By (3.11), (3.93) and (3.95),

$$(3.96) \quad F_T^M = D^{L, F; 2} + \frac{1}{4} \sum_{k=1}^p c(e_k) c(r_{e_k}^{TM} (j V_j^M)) - \frac{1}{2} \text{Tr } r^{T(1;0)M} (j V_j^M) \mathbb{1}_{T(1;0)M}^i + T^2 j_j + j R_j + \frac{1}{4} (\mathbb{1} + I_2 + I_3) + \frac{T^2}{4} j^f :$$

By (3.10), (3.18), (3.19), (3.35), (3.67), $\mathbb{1} = \mathbb{1} + \mathbb{1}$ and $\mathbb{1} = \mathbb{1}(\cdot)$, we know that there exists $C > 0$ such that for any A large enough, on M

$$(3.97) \quad (d^N j) \leq C; \quad \text{thus } |j|_3 \leq C :$$

Recall that by (3.20), $j = 2 j_2 + 2 j_4$. Thus from Lemma 3.5, (2.3), (2.12), (3.32) and (3.34)

$$(3.98) \quad \begin{aligned} j_j &= O(A^{-1/2}); \\ j_j &= 2 j_2 j^f + 2 j_4 j^f + 2(\mathbb{1} + \mathbb{1})h; i \\ &= 2 j^f (1 + O(A^{-1/2})): \end{aligned}$$

By Proposition 3.11, (3.2), (3.38), (3.97) and (3.98), there exists $A_0 > 0$, such that for any $A > A_0$ which is a regular values of the functions j^f and $\frac{1}{2}j^f$ on $M \setminus N$, and $(x; y) \in \text{fix} = 0g \subset M$, we have

$$(3.99) \quad \begin{aligned} 2 j_j + \frac{1}{2} j (R_j + R_j) + \frac{1}{4} (\mathbb{1} + I_2 + I_3) &> A; \\ j^0(\cdot)_{i, j} &< \frac{1}{4}: \end{aligned}$$

Let's $(x; y) \in \text{fix} \subset M \setminus n @ M$.

If $(x_0; y_0) \neq 0$, then (3.94) follows trivially from (3.96).

If $(x_0; y_0) = 0$, we will adapt the argument in [20, x2b)].

Let $e_1; \dots; e_{\dim M}$ be an orthogonal frame of TM . Set

$$(3.100) \quad \begin{aligned} M &= \sum_{j=1}^{\dim M} (r_{e_j}^{0;})^2 r_{r_{e_j}^{TM} e_j}^{0;}; \\ N &= \sum_{j=1}^{\dim N} (r_{f_j}^{0;})^2 r_{r_{f_j}^{TN} f_j}^{0;}; \quad M \cdot N = M + N; \end{aligned}$$

be the Bochner Laplacians acting on $\mathcal{O}^0(M; L)$, $\mathcal{O}^0(N; F)$, $\mathcal{O}^0(M \cdot N; L \otimes F)$. From the Lichnerowicz formula for $D^{L \otimes F/2}$ [12, Theorem 1.3.5], [11, Appendix D], one finds on M ,

$$(3.101) \quad D^{L \otimes F/2} = M \cdot N + O(1);$$

Let $e_1; \dots; e_{\dim M}$ be an orthogonal base of $T_{x_0}M$. Let $(x_1; \dots; x_{\dim M}; x)$ be the normal coordinate system with respect to $\{e_j\}_{j=1}^{\dim M}$ near x_0 . Certainly, we can choose $e_1; \dots; e_{\dim M}$ such that the function $j^2(x; y_0)$ has the following expression near x_0 ,

$$(3.102) \quad j^2(x; y_0) = j^2(x_0; y_0) + \sum_{j=1}^{\dim M} a_j x_j^2 + O(|x|^3);$$

where the a_j 's may possibly be zero.

The following Lemma is an analogue of [20, Lemma 2.3].

Lemma 3.13. The following inequality holds at the point $(x_0; y_0)$,

$$(3.103) \quad \frac{1}{4} \sum_{k=1}^{\dim M} c(e_k) c(r_{e_k}^{TM} (j V_j^M)) - \frac{1}{2} \text{Tr} r^{T^{(1;0)M}} (j V_j^M) \Big|_{T^{(1;0)M}} > \sum_{j=1}^{\dim M} a_j j;$$

where the inequality is strict if at least one of the a_j 's is negative.

Proof. Let $e_1; \dots; e_{\dim M}$ be an orthonormal frame of TM near x_0 so that at x_0 , $e_j = e_j$ for $1 \leq j \leq \dim M$. From Lemma 3.7 and (3.102), we find that

$$(3.104) \quad (j V_j^M)_{(x; y_0)} = 2 \sum_{j=1}^{\dim M} t_j(x; y_0) J^M e_j; \quad \text{and } t_j(x; y_0) = a_j x_j + O(|x|^2);$$

From (3.81), (3.83), (3.104), we deduce that at the point $(x_0; y_0)$,

$$(3.105) \quad \begin{aligned} & \frac{1}{4} \sum_{k=1}^{\dim M} c(e_k) c(r_{e_k}^{TM} (j V_j^M)) - \frac{1}{2} \text{Tr} r^{T^{(1;0)M}} (j V_j^M) \Big|_{T^{(1;0)M}} \\ &= \frac{1}{2} \sum_{j=1}^{\dim M} a_j c(e_j) c(J^M e_j) - \frac{1}{2} \sum_{j=1}^{\dim M} (1 + \sum_{j=1}^{\dim M} a_j J^M e_j) \cdot e_j \\ &= \frac{1}{2} \sum_{j=1}^{\dim M} a_j (B(e_j) - 2) > \sum_{j=1}^{\dim M} a_j j; \end{aligned}$$

where the last inequality is strict if at least one of the a_j 's is negative.

Let $(y_1; \dots; y_{\dim X})$ be a normal coordinate system near y_0 on $U_{y_0} \subset N$. Let F_T^M be formal adjoint of F_T^M . From (3.96), (3.99), (3.101), (3.103) and (3.104), we find that near $(x_0; y_0)$,

$$(3.106) \quad \frac{1}{2} (F_T^M + F_T^M) > \sum_{j=1}^{\dim M} T \dot{p}_j^2 + T^2 \sum_{j=1}^{\dim M} t_j(x; y_0)^2 + \frac{T^2}{4} \sum_{j=1}^{\dim G} V_j^N + TA + O(1 + T|x|):$$

Following [20, x2b)], let $\epsilon > 0$, which will be further fixed later, be sufficiently small so that the orthonormal frame $e_j, g_{j=1}^{\dim M}$ is well defined over

$$(3.107) \quad B_\epsilon(x_0) = \{x \in M; d(x; x_0) < \epsilon\}:$$

Let (r_{e_j}) be the formal adjoint of $r_{e_j}^{0;}$ on $B_\epsilon(x_0)$. Set

$$(3.108) \quad \frac{M}{T} = \sum_{j=1}^{\dim M} (r_{e_j}) + T \sum_{j=1}^{\dim M} (\text{sgn } a_j) t_j(x; y_0) r_{e_j}^{0;} + T \sum_{j=1}^{\dim M} (\text{sgn } a_j) t_j(x; y_0) :$$

Clearly, $\frac{M}{T}$ is nonnegative when acting on compactly supported sections over $B_\epsilon(x_0)$. Moreover, we verify from (3.104) that

$$(3.109) \quad \frac{M}{T} = \sum_{j=1}^{\dim M} T \dot{p}_j^2 + T^2 \sum_{j=1}^{\dim M} t_j(x; y_0)^2 + O(\epsilon_x + 1 + T|x|);$$

where by $O(\epsilon_x + 1 + T|x|)$ we mean a first order differential operator

$$(3.110) \quad \sum_{j=1}^{\dim M} b_j(x) \frac{\partial}{\partial x_j} + d(x) \quad \text{with } b_j(x) = O(1); \quad d(x) = O(1 + T|x|):$$

We will also use similar notation for other operators.

From (3.106), (3.108), (3.109), we get, when acting on sections with compact support in $B_\epsilon(x_0) \cap U_{y_0}$, for any $k > 1$,

$$(3.111) \quad \begin{aligned} \frac{1}{2} (F_T^M + F_T^M) &> \sum_{j=1}^{\dim M} \frac{M}{T} + TA + O(\epsilon_x + 1 + T|x|) \\ &> \frac{1}{k} \sum_{j=1}^{\dim M} \frac{1}{k} \sum_{j=1}^{\dim M} T \dot{p}_j^2 + TA + O(\epsilon_x + 1 + T|x|): \end{aligned}$$

For any $s \in L^2(M; L^2(F))$ with $\text{supp}(s) \subset B_\epsilon(x_0) \cap U_{y_0}$, by (3.101), it is standard that there exist $C_1; C_2; C_3 > 0$ such that

$$(3.112) \quad \begin{aligned} \langle s; s \rangle &> C_1 k D^{L^2(F)} s k_0^2 - C_2 k s k_0^2; \\ \langle (1 + T|x|) s; s \rangle &\leq C_3 (1 + T\epsilon) k s k_0^2: \end{aligned}$$

Also by the elliptic estimate and Cauchy inequality, there exist $C_4; C_5 > 0$ such that

$$(3.113) \quad \langle \partial_x s; s \rangle \leq C_4 \epsilon k D^{L^2(F)} s k_0^2 + \frac{C_5}{\epsilon} k s k_0^2:$$

From (3.111)–(3.113), we get

$$(3.114) \quad \operatorname{Re} \operatorname{tr} \int_T^M s; s_i > \frac{C_1}{k} C_4 \epsilon^2 k D^{L, F} s k_0^2 + T A C_3 \epsilon^2 \frac{1}{k} \sum_{j=1}^{\dim M} |j_j|^2 k s k_0^2 \\ \frac{C_5}{\epsilon^2} + \frac{C_2}{k} + C_3 k s k_0^2:$$

Now, we take k large enough such that $A \frac{1}{k} \sum_{j=1}^{\dim M} |j_j|^2 > A=2$, then we choose ϵ small enough such that

$$(3.115) \quad \frac{C_1}{k} C_4 \epsilon^2 > 0; \quad A=2 \quad C_3 \epsilon^2 > A=4:$$

With this choice of ϵ , by (3.114)–(3.115), we get Proposition 3.12 at $(x_0; y_0)$.

3.9. Proof of Theorem 3.1. We now assume that $A > A_0$ is a regular value for both j_j^2 and $\frac{1}{2} j_j^2$, with A_0 verifying the conditions in Proposition 3.12. In particular, by (3.6), X is nowhere zero on $\partial M = M_1 \cup M_2$.

Let e_n be the inward pointing unit normal at any boundary point of M . Set

$$(3.116) \quad D_{\partial M; T} = D_{\partial M}^{L, F} - \frac{1}{2} \operatorname{tr} c(e_n) c(X) \\ : \mathcal{O}^0(M; L, F)_{\partial M} \rightarrow \mathcal{O}^0(M; L, F)_{\partial M}$$

be the Dirac type operator on ∂M induced by D_T^M .

Since X is nowhere zero on ∂M , in view of (3.5), by Proposition 1.2, we know that when restricted to the G -invariant subspace of $\mathcal{O}^0(M; L, F)_{\partial M}$, there exists $T_1 > 0$ such that for any $T > T_1$, $D_{\partial M; T}$ is invertible. Moreover, a similar formula like (1.21) holds.

One can then proceed as in the proof of (1.48), to see that there exists an open neighborhood $U_{\partial M}$ of ∂M and positive constants $C_1 > 0, b_1 > 0$ such that for $T > T_1$ and any G -invariant element of $\mathcal{O}^0(M; L, F)$ such that $\operatorname{supp}(s) \subset U_{\partial M}$ and $P_{>0; T}(s|_{\partial M}) = 0$, where $P_{>0; T}$ are the APS projections associated to $D_{\partial M; T}$ (cf. Section 1.2), one has

$$(3.117) \quad D_T^M s_0^2 > C_1 D^{L, F} s_0^2 + (T - b_1) k s k_0^2 :$$

From (3.93), (3.117) and Proposition 3.12, one can then proceed as in the proof of (1.50) to see that there exist $C > 0, b > 0$ such that for any $T > T_1$ and G -invariant element of $\mathcal{O}^0(M; L, F)$ such that $P_{>0; T}(s|_{\partial M}) = 0$, one has

$$(3.118) \quad D_T^M s_0^2 > C D^{L, F} s_0^2 + (T - b) k s k_0^2 :$$

By taking $T > T_1$ large enough, we get Theorem 3.1.

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