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ON THE TOTAL ORDER OF REDUCIBILITY OF A PENCIL OF ALGEBRAIC PLANE CURVES

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Abstract. In this paper, the problem of bounding the number of reducible curves in a pencil of algebraic plane curves is addressed. Unlike most of the previous related works, each reducible curve of the pencil is here counted with its appropriate multiplicity. It is proved that this number of reducible curves, counted with multiplicity, is bounded by d^2-1 where d is the degree of the pencil. Then, a sharper bound is given by taking into account the Newton's polygon of the pencil.

Introduction

Given a pencil of algebraic plane curves such that a general element is irreducible, the purpose of this paper is to give a sharp upper bound for the number of reducible curves in this pencil. This question has been widely studied in the literature, but never, as far as we know, by counting the reducible factors with their multiplicities.

Let $r(X,Y)=f(X,Y)/g(X,Y)$ be a rational function in $\mathbb{K}(X,Y)$, where $\mathbb{K}$ is an algebraically closed field. It is commonly said to be non-composite if it cannot be written $r=u\circ h$ where $h(X,Y)\in\mathbb{K}(X,Y)$ and $u\in\mathbb{K}(T)$ such that $\deg(u)\geq 2$ (recall that the degree of a rational function is the maximum of the degrees of its numerator and denominator after reduction). If $d=\max(\deg(f),\deg(g))$, we define

$$f^\sharp(X,Y,Z)=Z^d f\left(\frac{X}{Z},\frac{Y}{Z}\right),\quad g^\sharp(X,Y,Z)=Z^d g\left(\frac{X}{Z},\frac{Y}{Z}\right)$$

that are two homogeneous polynomials of the same degree d in $\mathbb{K}[X,Y,Z]$. The set

$$\sigma(f,g)=\{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}\mid \mu f^\sharp+\lambda g^\sharp \text{ is reducible in } \mathbb{K}[X,Y,Z]\}\subset\mathbb{P}^1_{\mathbb{K}}$$

is the spectrum of r and a classical theorem of Bertini and Krull implies that it is finite if r is non-composite. Actually, $\sigma(f,g)$ is finite if and only if r is non-composite and if and only if the pencil of projective algebraic plane curves $\mu f^\sharp+\lambda g^\sharp=0$, $(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}$, has an irreducible general element (see for instance [Jou79, Chapitre 2, Théorème 3.4.6] and [Bod08, Theorem 2.2] for detailed proofs). Notice that the study of $\sigma(f,g)$ is trivial if $d=1$. Therefore, throughout this paper we will always assume that $d\geq 2$.

Given $(\mu:\lambda)\in\sigma(f,g)$, a complete factorization of the polynomial $\mu f^\sharp+\lambda g^\sharp$ is of the form

$$(\star)\qquad \mu f^\sharp+\lambda g^\sharp=\prod_{i=1}^{n(\mu:\lambda)} P_{(\mu:\lambda),i}^{e_{(\mu:\lambda),i}}$$

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where each polynomial $P_{(\mu:\lambda),i}$ is irreducible and homogeneous in $\mathbb{K}[X,Y,Z]$. If $\sigma(f,g)$ is finite the *total order of reducibility*[1] $\rho(f,g)$ of r is then defined by

$$\rho(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \big(n(\mu:\lambda)-1\big).$$

Observe that this sum is finite because $n(\mu:\lambda)\neq 1$ implies that $(\mu:\lambda)\in\sigma(f,g)$.

It is known that $\rho(f,g)$ is bounded above by d^2-1 where d stands for the degree of r. As far as we know, the first related result has been given by Poincaré [Poi91]. He showed that

$$|\sigma(f,g)| \leq (2d-1)^2+2d+2$$

This bound was improved only very recently by Ruppert [Rup86] who proves that $|\sigma(f,g)|$ is bounded by d^2-1. This result was obtained as a byproduct of a very interesting technique developed by the author to decide the reducibility of an algebraic plane curve. Later on, Stein studied a less general question but gave a stronger result: he proves that if $g=1$ then $\rho(f,1)\leq d-1$. Its approach, based on the study of the multiplicative group of all the divisors of the reducible curves in the pencil, is entirely different from that of Ruppert. Then, Stein's bound was improved in [Kal92] and after that several papers [Lor93, Vis93, AHS03, Bod08] developed techniques with similar flavors to deal with the general case $\rho(f,g)$. All of them obtained the bound $\rho(f,g)\leq d^2-1$ but also provide some various extensions: In [Lor93] the bound is proved in arbitrary characteristic, in [Bod08] it is shown that a direct generalization of Stein's result yields the bound $\rho(f,g)\leq d^2+d-1$, in [Vis93] the result is generalized to a very general ground variety and finally, in [AHS03] the authors were interested in a total reducibility order over a field $\mathbb{K}$ that is not necessarily algebraically closed.

The aim of this paper is to study the total order of reducibility by counting the multiplicities. More precisely, for each $(\mu:\lambda)\in\sigma(f,g)$ define

$$m(\mu:\lambda) := \sum_{i=1}^{n(\mu:\lambda)} e_{(\mu:\lambda),i}$$

from the factorization $(\star)$. This number is the number of factors of $\mu f^\sharp+\lambda g^\sharp$ where the multiplicities of the factors are counted. In particular, it is clear that $n(\mu:\lambda)\leq m(\mu:\lambda)$. We define the total order of reducibility with multiplicities of the rational function r as the integer

$$m(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \big(m(\mu:\lambda)-1\big)$$

Obviously, it always holds that $0\leq\rho(f,g)\leq m(f,g)$. Moreover, notice that unlike $\rho(f,g)$, $m(f,g)$ takes into account those curves in the pencil that are geometrically irreducible but scheme-theoretically non-reduced. However, it is proved in [AHS03, General Mixed Primset Theorem, p 74] that the number of such curves is at most 4 in our context; we will come back to this point in Section 2.

The first main result of this paper is that the upper bound d^2-1 for $\rho(f,g)$ is also valid for $m(f,g)$. This is the content of Section 2 where it is assumed that the characteristic of $\mathbb{K}$ is zero. Our method, which is inspired by [Rup86], is elementary

[1]This terminology is taken from [Ste89].

compared to the previously mentioned papers. Roughly speaking, we will transform the pencil of curves into a pencil of matrices and obtain in this way the claimed bound as a consequence of rank computations of some matrices that we will study in Section 1. In this way, the known inequality $\rho(f,g) \leq d^2 - 1$ is easily obtained. Moreover, we will actually not only bound $m(f,g)$ by d^2-1, but a bigger quantity that takes into account the multiple factors of the reducible elements in the pencil. Notice that we will also show the same bound holds in the case where $r = f/g$ is a rational function in an arbitrary number of variables via a classical use of Bertini's Theorem at the end of Section 2.

The second main result of this paper, given in Section 3, is a refined upper bound for $m(f,g)$ which is obtained by considering the Newton's polygons of the polynomials f and g. This result also gives a bound for the total order of reducibility $\rho(f,g)$ which is new and sharper. Notice that in this section the characteristic of $\mathbb{K}$ will be assumed to be 0 or $> d(d-1)$ where d denotes the degree of $r = f/g$.

Notations. Throughout this paper, $\mathbb{K}$ stands for an algebraically closed field of characteristic p. Given a polynomial f, $\deg(f)$ denotes its total degree and $\partial_X f$ (resp. $\partial_Y f$) denotes the partial derivative of f with respect to the variable X (resp. to Y). Also, for any integer n the notation $\mathbb{K}[X,Y]_{\leq n}$ stands for the set of all the polynomials in $\mathbb{K}[X,Y]$ with total degree less or equal to n; the notation $\mathbb{K}[X,Y,Z]_n$ stands for the set of all homogeneous polynomials of degree n in $\mathbb{K}[X,Y,Z]$.

1. Ruppert's linear map

In the paper [Rup86], Ruppert introduced an original technique to decide whether a plane algebraic curve is reducible. Its formulation relies on the computation of the first de Rham's cohomology group of the complementary of the plane curve by means of linear algebra methods. Later, Gao followed this approach to obtain an algorithm for the factorization of a bivariate polynomial [Gao03].

From now on, we will always assume in this section that the characteristic of the algebraically closed field $\mathbb{K}$ is $p = 0$.

For ν a positive integer and $f(X,Y) \in \mathbb{K}[X,Y]$ a polynomial of degree $d \leq \nu$, define the $\mathbb{K}$-linear map

$$\begin{array}{rcl} \mathcal{G}_\nu(f) : \mathbb{K}[X,Y]_{\leq \nu-1} \times \mathbb{K}[X,Y]_{\leq \nu-1} & \longrightarrow & \mathbb{K}[X,Y]_{\leq \nu+d-2} \\ (G,H) & \mapsto & f^2\left(\partial_Y\left(\frac{G}{f}\right) - \partial_X\left(\frac{H}{f}\right)\right) \\ & & = \begin{vmatrix} f & \partial_Y f \\ G & \partial_Y G \end{vmatrix} - \begin{vmatrix} f & \partial_X f \\ H & \partial_X H \end{vmatrix} \end{array}$$

Let $f_1, \ldots, f_r$ be the irreducible factors of f. If $\gcd(f, \partial_X f)$ is a nonzero constant in $\mathbb{K}$ then it is proved in [Gao03] that $\ker \mathcal{G}_d(f)$ is a $\mathbb{K}$-vector space of dimension r and that the set

$$\left\{ \left(\frac{f}{f_i}\partial_X f_i, \frac{f}{f_i}\partial_Y f_i \right) \mid i = 1, \ldots, r \right\}$$

is a basis of this kernel. This result provides an explicit description of the kernel of the linear map $\mathcal{G}_d(f)$ if the polynomial $f(X,Y)$ does not have any square factor. In order to investigate this kernel in the general case, that is to say for an arbitrary

polynomial $f \in \mathbb{K}[X,Y]$ and for an arbitrary integer $\nu \geq \deg(f)$, we interpret it in terms of algebraic de Rham cohomology.

Let ν be a positive integer and $0 \neq f(X,Y) \in \mathbb{K}[X,Y]$ be a polynomial of degree $d \leq \nu$. Assume that $f = f_1^{e_1} \cdots f_r^{e_r}$ is a factorization of f where each polynomial f_i is irreducible and denote by $\mathcal{C}$ the algebraic curve defined by the equation $f = 0$. The first algebraic de Rham cohomology $H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C})$ is the quotient of the closed 1-differential forms $w \in \Omega_{\mathbb{K}[X,Y]_f/\mathbb{K}}$ of $\mathbb{K}[X,Y]_f$ over $\mathbb{K}$ by the exact 1-forms.

By definition of $\mathcal{G}_\nu(f)$, a couple $(G,H) \in \mathbb{K}[X,Y]_{\leq \nu-1} \times \mathbb{K}[X,Y]_{\leq \nu-1}$ belongs to the kernel of $\mathcal{G}_\nu(f)$ if and only if the 1-form $\frac{1}{f}(G\mathbf{d}X + H\mathbf{d}Y)$ is closed. Therefore, the kernel of $\mathcal{G}_\nu(f)$ is in correspondence with the closed 1-differential forms $w \in \Omega_{\mathbb{K}[X,Y]_f/\mathbb{K}}$ that can be written $w = \frac{1}{f}(G\mathbf{d}X + H\mathbf{d}Y)$ for some polynomials G and H of degree less or equal to $\nu - 1$. As a consequence of Ruppert's results in [Rup86] (see also [Sch07, Theorem 8.3]), these particular closed 1-forms are sufficient to give a represent for any element in $H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C})$, that is to say that the canonical map

$$\ker \mathcal{G}_\nu(f) \to H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C})$$

is surjective. Actually, the closed 1-forms $\frac{\mathbf{d}f_1}{f_1}, \ldots, \frac{\mathbf{d}f_r}{f_r}$ are known to form a basis of $H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C})$ (see loc. cit. or for instance [Dim92, Chapter 6]). It follows that

$$H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C}) \simeq \ker \mathcal{G}_\nu(f)/B_\nu$$

where B_ν is the set of 1-forms in $\ker \mathcal{G}_\nu(f)$ that are exact. Basically, the elements in B_ν are of the form $d\left(\frac{P}{f^s}\right)$ for some $P \in \mathbb{K}[X,Y]$ and $s \in \mathbb{N}$. However, we claim that the following equality holds

$$B_\nu = \left\{ w = \frac{1}{f}(G\mathbf{d}X + H\mathbf{d}Y), (G,H) \in \mathbb{K}[X,Y]_{\leq \nu-1} \times \mathbb{K}[X,Y]_{\leq \nu-1} \right. \\ \left. \text{such that } \exists P \in K[X,Y]_{\leq \nu} \text{ with } \mathbf{d}\left(\frac{P}{f}\right) = w \right\} \tag{1.1}$$

It is a consequence of the following technical results.

Lemma 1. *Let p, q be polynomials in $\mathbb{K}[X,Y]$ such that p divides $q\mathbf{d}p$. Then each irreducible factor of p divides q.*

Proof. Let $p_1, \ldots, p_r$ be distinct irreducible factors of p such that $p = \prod_{i=1}^r p_i^{e_i}$. Then the equality

$$\frac{\mathbf{d}p}{p} = \sum_{i=1}^r e_i \frac{\mathbf{d}p_i}{p_i}$$

together with our hypothesis implies that $p_i^{e_i}$ divides $q\sum_{j=1}^r e_j \frac{p}{p_j}\mathbf{d}p_j$. We deduce that $p_i^{e_i}$ must divide $q\frac{p}{p_i}\mathbf{d}p_i$ and therefore that p_i divides q. □

Lemma 2. *Let $f \in \mathbb{K}[X,Y]$ of degree d and $G, H \in \mathbb{K}[X,Y]$ of degree $\leq \nu - 1$ with $\nu \geq d$. If $P \in \mathbb{K}[X,Y]$ and $s \in \mathbb{N}$ are such that*

$$\mathbf{d}\left(\frac{P}{f^s}\right) = \frac{1}{f}(G\mathbf{d}X + H\mathbf{d}Y)$$

and f does not divide P if $s \geq 1$, then either $s = 1$ and $\deg(P) \leq \nu$ or either $s = 0$ and $\deg(P) \leq \nu - d$.

Proof. This proof is inspired by [Sch07, Lemma 8.10]. Since

$$\mathbf{d}\left(\frac{P}{f^s}\right) = \frac{f\mathbf{d}P - sP\mathbf{d}f}{f^{s+1}}$$

we have

$$f\mathbf{d}P - sP\mathbf{d}f = f^s(G\mathbf{d}X + H\mathbf{d}Y) \tag{1.2}$$

Assume that $s \geq 2$ and denote by $f = \prod_{i=1}^r f_i^{e_i}$ an irreducible factorization of f. Equation (1.2) implies that f divides $P\mathbf{d}f$ and therefore, by Lemma 1, that f_i divides P for all $i = 1, \ldots, r$. Furthermore, since

$$\frac{\mathbf{d}f}{f} = \sum_{i=1}^r e_i \frac{\mathbf{d}f_i}{f_i}$$

we get

$$f\mathbf{d}P - sP\mathbf{d}f = f\mathbf{d}P - sPf\sum_{i=1}^r e_i\frac{\mathbf{d}f_i}{f_i} = f(\mathbf{d}P - s\sum_{i=1}^r e_i\frac{P}{f_i}\mathbf{d}f_i)$$

But f^s divides $f\mathbf{d}P - sP\mathbf{d}f$ by (1.2), so we deduce that

$$f^{s-1} \,|\, \mathbf{d}P - s\sum_{i=1}^r e_i\frac{P}{f_i}\mathbf{d}f_i$$

Define $Q := \gcd(f, P) = \prod_{i=1}^r f_i^{\mu_i}$ with $1 \leq \mu_i \leq e_i$ for all $i = 1, \ldots, r$ and set $R := P/Q$. We obtain that f^{s-1} divides

$$Q\mathbf{d}R + R\mathbf{d}Q - s\sum_{i=1}^r e_i\frac{P}{f_i}\mathbf{d}f_i = Q\mathbf{d}R + \sum_{i=1}^r(\mu_i - se_i)R\frac{Q}{f_i}\mathbf{d}f_i$$

since

$$\frac{\mathbf{d}Q}{Q} = \sum_{i=1}^r \mu_i\frac{\mathbf{d}f_i}{f_i}$$

As $s \geq 2$, $\mu_i - se_i < 0$ for all i and hence $f_i^{\mu_i}$ divides $R\frac{Q}{f_i}\mathbf{d}f_i$. It follows that f_i divides $R\mathbf{d}f_i$ and therefore that f_i divides R by Lemma 1. But then $f_i^{\mu_i+1}$ divides P which implies that $\mu_i = e_i$ for all i. Therefore, we conclude that if $s \geq 2$ then necessarily f divides P: a contradiction with our hypotheses. So we must have $0 \leq s \leq 1$.

Suppose that $s = 0$. Then

$$G\mathbf{d}X + H\mathbf{d}Y = f\mathbf{d}P = f\partial_X P\mathbf{d}X + f\partial_Y P\mathbf{d}Y$$

and hence $\deg(P) \leq \nu - d$.

Now, assume that $s = 1$. We have

$$fG\mathbf{d}X + fH\mathbf{d}Y = f\mathbf{d}P - P\mathbf{d}f = (f\partial_X P - P\partial_X f)\mathbf{d}X + (f\partial_Y P - P\partial_Y f)\mathbf{d}Y$$

Denote by δ the degree of P and by P_δ, resp. f_d, the homogeneous part of highest degree of P, resp. f. If $f_d\partial_X P_\delta - P_\delta\partial_X f_d \neq 0$ or $f_d\partial_Y P_\delta - P_\delta\partial_Y f_d \neq 0$ then necessarily $\delta \leq \nu$ since $\deg(fG) \leq \nu + d - 1$ and $\deg(fH) \leq \nu + d - 1$. Otherwise, we obtain that

$$\mathbf{d}\left(\frac{P_\delta}{f_d}\right) = \frac{f_d\mathbf{d}P_\delta - P_\delta\mathbf{d}f_d}{(f_d)^2} = 0$$

and hence that $\delta = d \leq \nu$. □

We are now ready to compute the dimension of the kernel of the $\mathbb{K}$-linear map $\mathcal{G}_\nu(f)$ for all $\nu \geq d$.

Proposition 3. *Let $f(X,Y) \in \mathbb{K}[X,Y]$ of degree d such that $f = f_1^{e_1} \cdots f_r^{e_r}$ is a factorization of f where each polynomial f_i is irreducible of degree d_i. Then, for all $\nu \geq d$ we have*

$$\dim_{\mathbb{K}} \ker \mathcal{G}_\nu(f) = r - 1 + \binom{2 + \nu - d + \sum_{i=1}^r d_i(e_i - 1)}{2}$$

Proof. From the above discussion on the interpretation of $\ker \mathcal{G}_\nu(f)$ in terms of 1-differential forms, we know that

$$\dim_{\mathbb{K}} \ker \mathcal{G}_\nu(f) = \dim_{\mathbb{K}} H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C}) + \dim_{\mathbb{K}} B_\nu$$

where B_ν is defined by (1.1). Since we also know that $\dim_{\mathbb{K}} H^1(\mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{C}) = r$, it remains to compute the dimension of B_ν. For that purpose, observe that the condition $\mathbf{d}\left(\frac{P}{f}\right) = w$ in the definition of B_ν is equivalent to the system of equations

$$\begin{cases} f\partial_X P - P\partial_X f - Gf = 0 \\ f\partial_Y P - P\partial_Y f - Hf = 0 \end{cases}$$

with the constraints $\deg(G) \leq \nu - 1$, $\deg(H) \leq \nu - 1$ and $\deg(P) \leq \nu$.

Denote by L_ν the vector space of those triples (G,H,P) solution of this system. The canonical projection $(G,H,P) \mapsto (G,H)$ sends L_ν to B_ν. Moreover, the kernel of this projection are the triples $(0,0,P)$ satisfying the condition $\mathbf{d}\left(\frac{P}{f}\right) = 0$ which implies that P is equal to f up to multiplication by an element in $\mathbb{K}$. Therefore, $\dim_{\mathbb{K}} B_\nu = \dim_{\mathbb{K}} L_\nu - 1$ and we are left with the computation of the dimension of L_ν.

The first equation defining L_ν, that can be rewritten as $f(\partial_X P - G) = P\partial_X f$, implies that P must be of the form

$$P = Q_1 \frac{f}{\gcd(f, \partial_X f)}$$

where Q_1 is a polynomial of degree less or equal to $\nu - d + \deg \gcd(f, \partial_X f)$. Moreover, any such polynomial P provides a couple (P,G) that is solution of the above equation – once P is fixed then so does for G. A similar reasoning with the second defining equation of L_ν shows that its solutions are in correspondence with the polynomials P of the form $Q_2 f / \gcd(f, \partial_Y f)$ where Q_2 is any polynomial of degree less or equal to $\nu - d + \deg \gcd(f, \partial_Y f)$.

Now, to obtain the common solutions of the two defining equations of L_ν we have to solve the equation

$$Q_2 \gcd(f, \partial_X f) = Q_1 \gcd(f, \partial_Y f)$$

But again, with similar arguments and using the fact that

$$\gcd\left(\gcd(f, \partial_X f), \gcd(f, \partial_Y f)\right) = \gcd(f, \partial_X f, \partial_Y f)$$

we get that

$$Q_1 = Q \frac{\gcd(f, \partial_X f)}{\gcd(f, \partial_X f, \partial_Y f)}, \quad Q_2 = Q \frac{\gcd(f, \partial_Y f)}{\gcd(f, \partial_X f, \partial_Y f)}$$

where Q is any polynomial in $\mathbb{K}[X,Y]$ of degree less or equal to

$$\nu - d + \deg \gcd(f, \partial_X f, \partial_Y f) = \nu - d + \sum_{i=1}^{r} d_i(e_i - 1) \tag{1.3}$$

Therefore, we deduce that the dimension of L_ν is equal to the dimension of the $\mathbb{K}$-vector space of polynomials in $\mathbb{K}[X,Y]$ of degree less or equal to the quantity (1.3), that is to say

$$\binom{2 + \nu - d + \sum_{i=1}^{r} d_i(e_i - 1)}{2}$$

and the claimed formula is proved. □

Following Ruppert's approach in [Rup86], we introduce a new $\mathbb{K}$-linear map which is similar to $\mathcal{G}_\nu(f)$ but with a source of smaller dimension. This property will be very important in the next section. To be more precise, for all positive integer ν consider the $\mathbb{K}$-vector space

$$E_\nu = \{(G,H) \in \mathbb{K}[X,Y]_{\leq \nu-1} \times \mathbb{K}[X,Y]_{\leq \nu-1} \text{ such that } \deg(XG + YH) \leq \nu - 1\}$$

It is of dimension $\nu^2 - 1$ and has the following property.

Lemma 4. *Let $f \in \mathbb{K}[X,Y]$ of degree d. For all positive integer ν and all couple $(G,H) \in E_\nu$, the polynomial*

$$f^2 \left(\partial_Y \left(\frac{G}{f} \right) - \partial_X \left(\frac{H}{f} \right) \right)$$

has degree at most $\nu + d - 3$

Proof. Denote by $G_{\nu-1}$, resp. $H_{\nu-1}$, f_d, the homogeneous component of G, resp. H, f of degree $\nu - 1$, resp. $\nu - 1$, d. We have

$$\begin{aligned}
& f_d^2\, \mathbf{d}\left(\frac{XG_{\nu-1} + YH_{\nu-1}}{f_d} \right) \\
&= f_d \mathbf{d}(XG_{\nu-1} + YH_{\nu-1}) - (XG_{\nu-1} + YH_{\nu-1})\mathbf{d}f_d \\
&= f_d(G_{\nu-1} + X\partial_X G_{\nu-1} + Y\partial_X H_{\nu-1})\mathbf{d}X \\
&\quad + f_d(H_{\nu-1} + X\partial_Y G_{\nu-1} + Y\partial_Y H_{\nu-1})\mathbf{d}Y \\
&\quad - (XG_{\nu-1}\partial_X f_d + YH_{\nu-1}\partial_X f_d)\mathbf{d}X - (YH_{\nu-1}\partial_Y f_d + XG_{\nu-1}\partial_Y f_d)\mathbf{d}Y
\end{aligned}$$

So, using Euler's relation the coefficient of $\mathbf{d}X$ is

$$f_d(\nu G_{\nu-1} - Y\partial_Y G_{\nu-1} + Y\partial_X H_{\nu-1}) - G_{\nu-1}(df_d - Y\partial_Y f_d) - YH_{\nu-1}\partial_X f_d$$

that is to say

$$(\nu - d) f_d G_{\nu-1} - Y f_d^2 \left(\partial_Y \left(\frac{G_{\nu-1}}{f_d} \right) - \partial_X \left(\frac{H_{\nu-1}}{f_d} \right) \right) \tag{1.4}$$

Similarly, the coefficient of $\mathbf{d}Y$ is

$$(\nu - d) f_d H_{\nu-1} - X f_d^2 \left(\partial_Y \left(\frac{G_{\nu-1}}{f_d} \right) - \partial_X \left(\frac{H_{\nu-1}}{f_d} \right) \right) \tag{1.5}$$

Now, since $(G,H) \in E_\nu$ we have $XG_{\nu-1}+YH_{\nu-1}=0$. Therefore the quantities (1.4) and (1.5) are both equal to zero. It follows that

$$0 = X \times (1.4) + Y \times (1.5) =$$
$$(\nu-d)f_d(XG_{\nu-1}+YH_{\nu-1}) - XYf_d^2\left(\partial_Y\left(\frac{G_{\nu-1}}{f_d}\right) - \partial_X\left(\frac{H_{\nu-1}}{f_d}\right)\right) =$$
$$- XYf_d^2\left(\partial_Y\left(\frac{G_{\nu-1}}{f_d}\right) - \partial_X\left(\frac{H_{\nu-1}}{f_d}\right)\right)$$

and the lemma is proved. □

Let $f(X,Y) \in \mathbb{K}[X,Y]$ of degree d. For all integer $\nu \geq d$ we define the $\mathbb{K}$-linear map

$$\mathcal{R}_\nu(f) : E \longrightarrow \mathbb{K}[X,Y]_{\leq \nu+d-3} : (G,H) \mapsto f^2\left(\partial_Y\left(\frac{G}{f}\right) - \partial_X\left(\frac{H}{f}\right)\right)$$

Point out that the operator $\mathcal{R}_\nu(-)$ is $\mathbb{K}$-linear, that is to say that for all couples $(f,g) \in \mathbb{K}[X,Y]_{\leq \nu}$ and all couple $(u,v) \in \mathbb{K}^2$, we have

$$\mathcal{R}_\nu(uf+vg) = u\mathcal{R}_\nu(f) + v\mathcal{R}_\nu(g)$$

Of course, a similar property holds for the operator $\mathcal{G}_\nu(f)$.

Proposition 5. *Let $f(X,Y) \in \mathbb{K}[X,Y]$ of degree d. Then*

$$\dim_{\mathbb{K}} \ker \mathcal{R}_d(f) = \dim_{\mathbb{K}} \ker \mathcal{G}_d(f) - 1$$

and for all $\nu > d$

$$\dim_{\mathbb{K}} \ker \mathcal{R}_\nu(f) = \dim_{\mathbb{K}} \ker \mathcal{G}_{\nu-1}(f)$$

Proof. Denote by $G_{\nu-1}$, resp. $H_{\nu-1}$, f_d, the homogeneous component of G, resp. H, f of degree $\nu-1$, resp. $\nu-1$, d.

First, notice that for all integer $\nu \geq d$ and all couple $(G,H) \in \ker \mathcal{G}_\nu(f)$ we have

$$\mathbf{d}\left(\frac{XG_{\nu-1}+YH_{\nu-1}}{f_d}\right) = (\nu-d)\frac{G_{\nu-1}\mathbf{d}X + H_{\nu-1}\mathbf{d}Y}{f_d} \tag{1.6}$$

Indeed, this follows from the computation we did in the proof of Lemma 4, more precisely the coefficients (1.4) and (1.5).

Now, let $f = f_1^{e_1}\cdots f_r^{e_r}$ be a factorization of f where each polynomial f_i is irreducible of degree d_i. By definition of both maps $\mathcal{G}_d(f)$ and $\mathcal{R}_d(f)$, it is obvious to notice that any element in the kernel of $\mathcal{R}_d(f)$ is also in the kernel of $\mathcal{G}_d(f)$. Moreover, it is easy to check that

$$\left(\frac{f}{f_1}\partial_X f_1, \frac{f}{f_1}\partial_Y f_1\right) \in \ker \mathcal{G}_d(f)$$

but does not belong to the kernel of $\mathcal{R}_d(f)$ because

$$X\frac{f}{f_1}\partial_X f_1 + Y\frac{f}{f_1}\partial_Y f_1 = \frac{f}{f_1}(X\partial_X f_1 + Y\partial_Y f_1) = d_1 f + \frac{f}{f_1}\tilde{f}_1$$

where $\deg(\tilde{f}_1) < d_1$ (by Euler's relation). Nevertheless, for all couple $(G,H) \in \ker \mathcal{G}_d(f)$, Equation (1.6) shows that there exists $\alpha \in \mathbb{K}$ such that

$$XG_{d-1} + YH_{d-1} = \alpha f_d$$

It follows that

$$(G,H)+\frac{\alpha}{d_1}\left(\frac{f}{f_1}\partial_X f_1,\frac{f}{f_1}\partial_Y f_1\right)\in\ker\mathcal{R}_d(f)$$

and therefore

$$\dim_{\mathbb{K}}\ker\mathcal{R}_d(f)=\dim_{\mathbb{K}}\ker\mathcal{G}_d(f)-1$$

To finish the proof, fix an integer $\nu>d$. It is clear from the definitions that

$$\ker\mathcal{G}_{\nu-1}(f)\subseteq\ker\mathcal{R}_\nu(f)\subseteq\ker\mathcal{G}_{\nu+1}(f)$$

Pick a couple $(G,H)\in\ker\mathcal{R}_\nu(f)$. It satisfies $XG_{\nu-1}+YH_{\nu-1}=0$. Therefore, using (1.6) we deduce that

$$\frac{G_{\nu-1}\mathbf{d}X+H_{\nu-1}\mathbf{d}Y}{f_d}=0$$

that is to say that $G_{\nu-1}=H_{\nu-1}=0$. It follows that $(G,H)\in\ker\mathcal{G}_{\nu-1}(f)$. □

Corollary 6. *Let $f(X,Y)\in\mathbb{K}[X,Y]$ of degree d such that $f=f_1^{e_1}\cdots f_r^{e_r}$ is a factorization of f where each polynomial f_i is irreducible of degree d_i. Then*

$$\dim_{\mathbb{K}}\ker\mathcal{R}_d(f)=r-2+\binom{2+\sum_{i=1}^r d_i(e_i-1)}{2}$$

In particular, $f(X,Y)$ is irreducible if and only if $\dim_{\mathbb{K}}\ker\mathcal{R}_d(f)=0$.

Remark 7. As already mentioned, a basis of the kernel of $\mathcal{G}_d(f)$ is known in the case where f is a square-free polynomial. Under the same hypothesis, it is not hard to check that the set

$$\left\{\left(-d_i\frac{f}{f_1}\partial_X f_1+d_1\frac{f}{f_i}\partial_X f_i,-d_i\frac{f}{f_1}\partial_Y f_1+d_1\frac{f}{f_i}\partial_Y f_i\right),\ i=2,\dots,r\right\}$$

form a basis of the kernel of $\mathcal{R}_d(f)$.

Since we will often deal with homogeneous polynomials in the rest of this paper, we need to extend Corollary 6 to the case of a homogeneous polynomial. To proceed, it is first necessary to define Ruppert's matrix in this setting. If $f(X,Y,Z)\in\mathbb{K}[X,Y,Z]$ is a homogenous polynomial of degree d, we define

$$\mathcal{R}(f):E\longrightarrow\mathbb{K}[X,Y,Z]_{2d-3}:(G,H)\mapsto\frac{1}{Z}f^2\left(\partial_Y\left(\frac{G}{f}\right)-\partial_X\left(\frac{H}{f}\right)\right)$$

where

$$E=\{(G,H)\in\mathbb{K}[X,Y,Z]_{d-1}\times\mathbb{K}[X,Y,Z]_{d-1}\text{ such that }Z|XG+YH\}$$

Observe that the division by Z in this definition is justified by Lemma 4. Here is the main result of this section.

Theorem 8. *Let $f(X,Y,Z)\in\mathbb{K}[X,Y,Z]$ homogeneous of degree d and suppose that $f=f_1^{e_1}\cdots f_r^{e_r}$ where each polynomial $f_i(X,Y,Z)$ is irreducible and homogeneous of degree d_i. Then*

$$\dim_{\mathbb{K}}\ker\mathcal{R}(f)=r-2+\binom{2+\sum_{i=1}^r d_i(e_i-1)}{2}$$

In particular, $f(X,Y,Z)$ is irreducible if and only if $\dim_{\mathbb{K}}\ker\mathcal{R}(f)=0$.

Proof. Denote $\tilde{f}(X,Y) = f(X,Y,1) \in \mathbb{K}[X,Y]$ and consider the map $\mathcal{R}_d(\tilde{f})$. We claim that the kernels of $\mathcal{R}(f)$ and $\mathcal{R}_d(\tilde{f})$ are isomorphic $\mathbb{K}$-vector spaces.

Indeed, let $(\tilde{G}(X,Y), \tilde{H}(X,Y)) \in \ker \mathcal{R}_d(\tilde{f})$ and set

$$G(X,Y,Z) = Z^{d-1}\tilde{G}\left(\frac{X}{Z},\frac{Y}{Z}\right), \quad H(X,Y,Z) = Z^{d-1}\tilde{H}\left(\frac{X}{Z},\frac{Y}{Z}\right)$$

Multiplying by Z^{2d-2} the equality

$$\begin{aligned}\tilde{f}\left(\frac{X}{Z},\frac{Y}{Z}\right)\partial_Y\tilde{G}\left(\frac{X}{Z},\frac{Y}{Z}\right) - \tilde{G}\left(\frac{X}{Z},\frac{Y}{Z}\right)\partial_Y\tilde{f}\left(\frac{X}{Z},\frac{Y}{Z}\right)\\ - \tilde{f}\left(\frac{X}{Z},\frac{Y}{Z}\right)\partial_X\tilde{H}\left(\frac{X}{Z},\frac{Y}{Z}\right) + \tilde{H}\left(\frac{X}{Z},\frac{Y}{Z}\right)\partial_X\tilde{f}\left(\frac{X}{Z},\frac{Y}{Z}\right) = 0\end{aligned}$$

we get

$$f\partial_Y G - G\partial_Y f - f\partial_X H + H\partial_X f = \mathcal{R}(f)(G,H) = 0$$

Moreover, since $\deg(X\tilde{G} + Y\tilde{H}) \leq d-1$ we deduce that Z divides $XG + YH$ and conclude that (G,H) belongs to the kernel of $\mathcal{R}(f)$. Similarly, if $(G,H) \in \ker \mathcal{R}(f)$ then $(\tilde{G},\tilde{H}) = (G(X,Y,1), H(X,Y,1)) \in \ker \mathcal{R}_d(\tilde{f})$. Therefore, we have proved that

$$\dim_{\mathbb{K}} \ker \mathcal{R}(f) = \dim_{\mathbb{K}} \ker \mathcal{R}_d(\tilde{f})$$

From here, the claimed equality follows from Corollary 6 if $\deg(\tilde{f}) = d$. Otherwise, if $\deg(\tilde{f}) < d$, it is a consequence of Proposition 3 and Proposition 5. □

2. An upper bound for the total order of reducibility

In this section, given a non-composite rational function $r = f/g \in \mathbb{K}(X,Y)$ we establish an upper bound for its total order of reducibility counting multiplicities $m(f,g)$ (recall that if r is composite then $\sigma(f,g)$ is not a finite set). Surprisingly, this upper bound is the same as the known upper bound for the usual total order of reducibility $\rho(f,g)$ [Lor93, Vis93]. Notice that we will actually prove a stronger result by considering a quantity which is bigger than $m(f,g)$. To proceed, we first need some notations.

Throughout this section, we will assume that the algebraically closed field $\mathbb{K}$ has characteristic $p = 0$.

Given a non-composite rational function $r = f/g \in \mathbb{K}(X,Y)$ of degree d, define the two homogeneous polynomials of degree d in $\mathbb{K}[X,Y,Z]$

$$f^\sharp(X,Y,Z) = Z^d f\left(\frac{X}{Z},\frac{Y}{Z}\right), \quad g^\sharp(X,Y,Z) = Z^d g\left(\frac{X}{Z},\frac{Y}{Z}\right)$$

If $(\mu:\lambda) \in \sigma(f,g)$ and

$$\mu f^\sharp(X,Y,Z) + \lambda g^\sharp(X,Y,Z) = \prod_{i=1}^{n(\mu:\lambda)} P_{(\mu:\lambda),i}^{e_{(\mu:\lambda),i}} \tag{2.1}$$

where each polynomial $P_{(\mu:\lambda),i}$ is irreducible and homogeneous in $\mathbb{K}[X,Y,Z]$, then

$$\rho(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \big(n(\mu:\lambda) - 1\big)$$

and

$$m(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} (m(\mu:\lambda)-1) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \left(\left(\sum_{i=1}^{n(\mu:\lambda)} e_{(\mu:\lambda),i} \right) - 1 \right)$$

The number of *multiple factors* of $\mu f^\sharp(X,Y,Z) + \lambda g^\sharp(X,Y,Z)$, counted with multiplicity, is

$$\sum_{i=1}^{n(\mu:\lambda)} \left(e_{(\mu:\lambda),i} - 1\right)$$

In the sequel we will actually balance each multiplicity in this sum with the degree of its corresponding factor, that is to say, we will rather consider the number

$$\omega(\mu:\lambda) = \sum_{i=1}^{n(\mu:\lambda)} \deg(P_{(\mu:\lambda),i}) \left(e_{(\mu:\lambda),i} - 1\right) \geq \sum_{i=1}^{n(\mu:\lambda)} \left(e_{(\mu:\lambda),i} - 1\right)$$

Consequently, we define

$$\omega(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \omega(\mu:\lambda)$$

Before going further in the notation, let us make a digression on the interesting quantity $\omega(f,g)$ that first appears in the works of Darboux [Dar78] and Poincaré [Poi91] on the qualitative study of first order differential equations. In particular, they knew the following result:

Lemma 9. *Let $r = f/g \in \mathbb{K}(X,Y)$ a non-composite reduced rational function of degree d. Then,*

$$\omega(f,g) \leq 2d-2$$

Proof. See [Jou79, Chapitre 2, Corollaire 3.5.6] for a detailed proof of this result valid with an arbitrary number of variables. □

It is also interesting to emphasize how Lemma 9 implies that the cardinal of the set

$$\gamma(f,g) := \Big\{ (\mu:\lambda) \in \mathbb{P}^1_{\mathbb{K}} \text{ such that } \mu f^\sharp + \lambda g^\sharp = P_{(\mu:\lambda)}^{e_{(\mu:\lambda)}} \\ \text{with } e_{(\mu:\lambda)} \geq 2 \text{ and } P_{(\mu:\lambda)} \in \mathbb{K}[X,Y,Z] \text{ irreducible } \Big\} \subset \mathbb{P}^1_{\mathbb{K}}$$

that is to say of the set of geometrically irreducible but reduced fibers[2], is less or equal to 3. Indeed, Lemma 9 yields

$$\sum_{(\mu:\lambda)\in\gamma(f,g)} \deg(P_{(\mu:\lambda)})(e_{(\mu:\lambda)} - 1) \leq 2d-2$$

But obviously, $\deg(P_{(\mu:\lambda)}) \leq \frac{d}{2}$ for all $(\mu:\lambda) \in \gamma(f,g)$ and, denoting by $|\gamma(f,g)|$ the cardinal of $\gamma(f,g)$, it follows that

$$d\,|\gamma(f,g)| = \sum_{(\mu:\lambda)\in\gamma(f,g)} e_{(\mu:\lambda)} \deg(P_{(\mu:\lambda)}) \\ \leq 2d-2 + \sum_{(\mu:\lambda)\in\gamma(f,g)} \deg(P_{(\mu:\lambda)}) \leq 2d-2+\frac{d}{2}\,|\gamma(f,g)|$$

[2]Notice that these fibers appear in the work of Poincaré [Poi91] as the *critical remarkable values of fifth types*.

Therefore, since d is a positive integer we have $|\gamma(f,g)| \leq 3$.

Mention that one can also be interested in fibers that are non reduced and geometrically irreducible on the affine space $\mathbb{A}^2_{\mathbb{K}}$, say with variables X, Y, that is to say fibers of the pencil of curves $\mu f^\sharp + \lambda g^\sharp$ of the form $Z^{e_\infty} P^e$ where P is an irreducible and homogeneous polynomial and $e\deg(P) + e_\infty = d$. Since there is at most one point $(\mu : \lambda) \in \mathbb{P}^1_{\mathbb{K}}$ such that Z divides $\mu f^\sharp + \lambda g^\sharp$, we deduce from the inequality $|\gamma(f,g)| \leq 3$ that the number of such fibers is at most 4. This property actually appears in [AHS03, General Mixed Primset Theorem, p 74].

Closing this parenthesis on the quantity $\omega(f,g)$, we finish with the notation by defining from (2.1) the quantity

$$\theta(\mu,\lambda) = \binom{\omega(\mu:\lambda)+1}{2} - \sum_{i=1}^{n(\mu:\lambda)} (e_{(\mu:\lambda),i} - 1)$$

which is positive since

$$\theta(\mu,\lambda) \geq \binom{\omega(\mu:\lambda)+1}{2} - \omega(\mu:\lambda) = \binom{\omega(\mu:\lambda)-1}{2}$$

Finally, we set

$$\theta(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}^1_{\mathbb{K}}} \theta(\mu:\lambda)$$

It is important to notice that we defined $\theta(\mu,\lambda)$ in order to have the equality

$$m(\mu:\lambda) - 1 + \omega(\mu:\lambda) + \theta(\mu:\lambda) = \dim\ker \mathcal{R}(\mu f^\sharp + \lambda g^\sharp) \tag{2.2}$$

according to Theorem 8.

Theorem 10. *Let $r = f/g \in \mathbb{K}(X,Y)$ a non-composite reduced rational function and set $d = \deg(r) = \max(\deg(f),\deg(g))$. We have*

$$0 \leq \rho(f,g) \leq m(f,g) + \omega(f,g) + \theta(f,g) \leq d^2 - 1$$

Proof. For all $(\mu:\lambda) \in \mathbb{P}^1_{\mathbb{K}}$, consider the linear map

$$\mathcal{R}(\mu f^\sharp + v g^\sharp) = \mu\mathcal{R}(f^\sharp) + v\mathcal{R}(g^\sharp)$$

and, choosing bases for the $\mathbb{K}$-vector spaces E and $\mathbb{K}[X,Y,Z]_{2d-3}$, the corresponding matrix

$$\mathrm{M}(\mu f^\sharp + v g^\sharp) = \mu\mathrm{M}(f^\sharp) + v\mathrm{M}(g^\sharp)$$

They form a pencil of matrices that has $d^2 - 1$ columns and more rows. We define the polynomial $\mathrm{Spect}(U,V) \in \mathbb{K}[U,V]$ as the greatest common divisor of all the (d^2-1)-minors of the matrix

$$U\mathrm{M}(f^\sharp) + V\mathrm{M}(g^\sharp) \tag{2.3}$$

It is a homogeneous polynomial of degree $\leq d^2 - 1$, since each entry of (2.3) is a linear form in $\mathbb{K}[U,V]$.

First, notice that $\mathrm{Spect}(U,V)$ is nonzero. Indeed, since $r = f/g$ is reduced and non-composite, the spectrum $\sigma(f,g)$ is finite and hence there exists $(\mu:\lambda) \notin \sigma(f,g)$. By Theorem 8, it follows that $\ker \mathrm{M}(\mu f^\sharp + \lambda g^\sharp) = \{0\}$ and therefore that at least one of the (d^2-1)-minors of (2.3) is nonzero since it has to be nonzero after the specializations of U to μ and V to λ.

Now, let $(\mu:\lambda) \in \sigma(f,g)$. By Theorem 8

$$\dim\ker \mathrm{M}(\mu f^\sharp + \lambda g^\sharp) = m(\mu:\lambda) - 1 + \omega(\mu:\lambda) + \theta(\mu:\lambda) > 0 \tag{2.4}$$

Therefore, $(\mu : \lambda)$ is a root of $\mathrm{Spect}(U, V)$. Moreover, by a well-known property of characteristic polynomials, $(\mu : \lambda)$ is a root of $\mathrm{Spect}(U, V)$ of multiplicity at least (2.4). Summing all these multiplicities over all the elements in the spectrum $\sigma(f, g)$, we obtain the quantity $m(f, g) + \omega(f, g) + \theta(f, g)$. It is bounded above by $d^2 - 1$ because $\mathrm{Spect}(U, V)$ is a polynomial of degree less or equal to $d^2 - 1$. □

It is remarkable that the term $m(f, g)+\omega(f, g)+\theta(f, g)$ depends quadratically on the degrees and on the multiplicities of the irreducible components of the reducible curves in the pencil $\mu f^\sharp + \lambda g^\sharp$. This has to be compared with the bound $d^2 - 1$ which depends quadratically on the total degree d of the pencil.

As mentioned earlier, the inequality $\rho(f, g) \leq d^2 - 1$ has been proved in [Lor93, Vis93]. This bound is known to be reached only for $d = 1, 2, 3$ and several authors raised the question of the optimality of this bound for an arbitrary degree d (see for instance [AHS03, Question 1, p. 79] or [Vis93, top of p. 254]). Coming back to the total order of reducibility counting multiplicities, we do not know whether the bound $d^2 - 1$ given in Theorem 10 is optimal. Of course, it is optimal for $d = 1, 2, 3$ since this is the case for the bound $\rho(f, g) \leq d^2 - 1$. Nevertheless, as a consequence of Theorem 10 we obtain the

Corollary 11. *Let $r = f/g \in \mathbb{K}(X, Y)$ a non-composite reduced rational function of degree d. If $\rho(f, g) = d^2 - 1$ then $\omega(f, g) = 0$.*

In other words, if there exists a pencil of curves with total order of reducibility equal to $d^2 - 1$ then it must have all its reducible members scheme-theoretically reduced.

In the same spirit, given a polynomial $f \in \mathbb{K}[X, Y]$ of degree d, one may ask if there exists a sharper bound for the spectrum $m(f) := m(f, 1)$ than $d^2 - 1$. Indeed, as a consequence of a result of Stein [Ste89] (see also [Lor93] and [AHS03]), such a phenomenon appears when multiplicities of the irreducible factors are not considered; one has $\rho(f) \leq d - 1$ (and this bound is reached). The technique we used for proving Theorem 10 allows to show that

$$m(f) + \omega(f) + \theta(f) \leq d(d - 1)/2 \tag{2.5}$$

providing f is a non-composite polynomial. Indeed, (2.5) follows from the fact that $\mathcal{R}_d(1)$ has rank $d(d-1)/2$. This rank can be easily computed since this linear map sends a couple (G, H) to the difference $\partial_Y G - \partial_X H$. However, one can expect that a bound linear in the degree d holds for $m(f)$.

Although beyond the scope of this paper, we would like to mention that our approach can be directly applied for a collection of polynomials $(f_1, \ldots, f_r)$ rather than a couple of polynomials (f, g). The problem is then to investigate the variety $\mathcal{S}$ of points $(\lambda_1, \ldots, \lambda_r)$ such that the polynomial $\lambda_1 f_1^\sharp + \cdots + \lambda_r f_r^\sharp$ is reducible, assuming that this latter is generically irreducible. As an immediate consequence of our approach, the degree of $\mathcal{S}$ is less or equal to $d^2 - 1$. Notice that the study of $\mathcal{S}$ has already been considered in [BDN08] in arbitrary characteristic.

Finally, before closing this section we establish a result similar to Theorem 10 in the multivariate case. This kind of result is based on a classical use of Bertini's Theorem under the following form.

Lemma 12. *Let*

$$f = \sum_{|\underline{e}|\leq d} c_{e_1,\ldots,e_n} X_1^{e_1} \ldots X_n^{e_n} \in \mathbb{K}[X_1,\ldots,X_n]$$

set $|\underline{e}| = e_1 + \cdots + e_n$ *and*

$$\mathbb{L} := \mathbb{K}(U_1,\ldots,U_n,V_1,\ldots,V_n,W_1,\ldots,W_n)$$

where $U_1,\ldots,U_n,V_1,\ldots,V_n,W_1,\ldots,W_n$ *are algebraically independent indeterminates.*

Then, the bivariate polynomial

$$\tilde{f}(X,Y) = f(U_1X + V_1Y + W_1,\ldots,U_nX + V_nY + W_n) \in \mathbb{L}[X,Y]$$

is irreducible in $\overline{\mathbb{L}}[X,Y]$ *if and only if* f *is irreducible in* $\mathbb{K}[X_1,\ldots,X_n]$.

Proof. See [Kal95, lemma 7]. See also [Jou83] for a complete treatment of Bertini's Theorem. □

In the following theorem, the quantities $m(f,g)$, $\omega(f,g)$ and $\theta(f,g)$ that we have defined for a rational function $r = f/g$ in two variables are straightforwardly extended to a rational function in several variables, denoting by X_0 the homogenizing variable.

Theorem 13. *Let* $r = f/g \in \mathbb{K}(X_1,\ldots,X_n)$ *a non-composite reduced rational function of degree* d. *We have*

$$m(f,g) + \omega(f,g) + \theta(f,g) \leq d^2 - 1$$

Proof. Given $(\mu:\lambda) \in \mathbb{P}^1_{\mathbb{K}}$, Lemma 12 implies that

$$\mu f^\sharp + \lambda g^\sharp = \prod_{i=1}^{n(\mu:\lambda)} P_{(\mu:\lambda),i}^{e_{(\mu:\lambda),i}}$$

with $P_{(\mu:\lambda),i}$ homogeneous and irreducible in $\mathbb{K}[X_0,X_1,\ldots,X_n]$, if and only if

$$\mu \tilde{f}^\sharp + \lambda \tilde{g}^\sharp = \prod_{i=1}^{n(\mu:\lambda)} \tilde{P}_{(\mu:\lambda),i}^{e_{(\mu:\lambda),i}}$$

with $\tilde{P}_{(\mu:\lambda),i}$ homogeneous irreducible in $\overline{\mathbb{L}}[X,Y,Z]$. Therefore, $m(f,g) = m(\tilde{f},\tilde{g})$, $\omega(f,g) = \omega(\tilde{f},\tilde{g})$ and $\theta(f,g) = \theta(\tilde{f},\tilde{g})$. The claimed result then follows from Theorem 10 applied to the rational function $r = \tilde{f}/\tilde{g} \in \mathbb{K}(X,Y)$. □

3. Exploiting Newton's polygon

In the previous section we considered rational functions f/g with a certain fixed degree. In this section, we will refine this characterization by considering the Newton's polygons of f and g. In this way, we will give an upper bound for the total order of reducibility counting multiplicities $m(f,g)$ that improves the one of Theorem 10 in many cases. In particular, an example for which this bound is almost reached for an arbitrary degree is presented.

To obtain this upper bound, we will follow a more basic approach than in Section 2. Indeed, instead of using Theorem 8 we will exhibit explicit elements in the kernel of a suitable Ruppert's linear map and show that they are linearly independent. This has the advantage to allow us working in non-zero characteristic, but has the

disadvantage to provide a bound for the quantity $m(f,g)$ and not $m(f,g)+\omega(f,g)+\theta(f,g)$ as in Theorem 8.

Before going further into details, mention that a bound for the total order of reducibility $\rho(f,g)$ related to the Newton's polygons of f and g is contained in the result of Vistoli [Vis93, Theorem 2.2] since this amounts to homogenize the corresponding pencil of curves over a certain toric variety which is built from the Newton's polygons of f and g. The bound provided in [Vis93] is then expressed in terms of invariants of this variety and of the pencil of curves that are very hard to make explicit.

Hereafter, p stands for the characteristic of the algebraically closed field $\mathbb{K}$. We begin with some notation and preliminary materials.

Given a polynomial $f(X,Y) \in \mathbb{K}[X,Y]$, its support is the set S_f of integer points (i,j) such that the monomial X^iY^j appears in f with a non zero coefficient. The convex hull, in the real space $\mathbb{R}^2$, of S_f is denoted $N(f)$ and called the Newton's polygon of f. It is contained in the first quadrant of the plane $\mathbb{R}^2$.

The superior envelop of $N(f)$, that we will denote $N^+(f)$, is the smallest convex set that contains $N(f)$ and the origin, and that is bordered by edges having non-positive slopes (horizontal and vertical edges are hence allowed). It is also contained in the first quadrant of the plane $\mathbb{R}^2$. Moreover, it will be useful in the sequel to notice that if $f,g \in \mathbb{K}[X,Y]$ are such that g divides f then clearly $N(f/g) \subset N^+(f)$.

Recall that the Minkowski sum $A+B$ of two sets A and $B \in \mathbb{R}^2$ is the set of all elements $a+b$ with $a \in A$ and $b \in B$. We have the following classical result due to Ostrowski: let $f, f_1, \ldots, f_r$ be polynomials in $\mathbb{K}[X,Y]$ such that $f = f_1 \ldots f_r$, then

$$N(f) = N(f_1) + \cdots + N(f_r) \tag{3.1}$$

The notion of total degree of a polynomial $f \in \mathbb{K}[X,Y]$ can be refined in many ways in the sparse context. For instance, if $f(X,Y) = \sum_{i,j} f_{i,j}X^iY^j$ in $\mathbb{K}[X,Y]$, given a couple $(a,b) \in \mathbb{Z}^2$ the (a,b)-weighted degree, or simply weighted degree, of f is defined by

$$d_{a,b}(f) = \max_{(i,j)\in\mathbb{N}^2}\{ai+bj \mid f_{i,j} \neq 0\}$$

Thus, the total degree of a f is nothing but $d_{1,1}(f)$ and the degree of f with respect to the variable X, resp. Y, corresponds to $\deg_{1,0}(f)$, resp. $\deg_{0,1}(f)$.

If $\mathcal{E}$ is an edge of a given convex set $\mathcal{N}$, denote by $a_\mathcal{E}X + b_\mathcal{E}Y = c_\mathcal{E}$ one of its integer equation. Then, it is clear that $d_{a_\mathcal{E},b_\mathcal{E}}(m) = d_{a_\mathcal{E},b_\mathcal{E}}(n)$ if $m,n \in \mathcal{E}$, and that $d_{a_\mathcal{E},b_\mathcal{E}}(m) \neq d_{a_\mathcal{E},b_\mathcal{E}}(n)$ if $m \notin \mathcal{E}$, $n \in \mathcal{E}$. In what follows we will use this remark for particular edges that we will call *good edges*.

Definition 14. Suppose given a convex set $\mathcal{N}$ in the first quadrant of the plane. An edge $\mathcal{E}$ of $\mathcal{N}$ is called a good edge if the two following conditions hold:

- there exists $(a_\mathcal{E}, b_\mathcal{E}) \in \mathbb{N}^2 \setminus (0,0)$ and $c_\mathcal{E} \in \mathbb{N}$ such that $a_\mathcal{E}X + b_\mathcal{E}Y = c_\mathcal{E}$ is an equation of $\mathcal{E}$,
- if $n \in \mathcal{N}$, $n \notin \mathcal{E}$ and $m \in \mathcal{E}$ then $0 \leq d_{a_\mathcal{E},b_\mathcal{E}}(n) < d_{a_\mathcal{E},b_\mathcal{E}}(m)$.

Notice that a good edge does not always exist, as the reader can convince itself very easily.

We are now ready to state the main result of this section.

Theorem 15. *Let $\mathcal{N}$ be a convex set in $\mathbb{R}^2$. Denote by N its number of interior integral points and by N_X, resp. N_Y, the number of points in $\mathcal{N}$ lying on the X-axis, resp. Y-axis. If $\mathcal{N}$ possesses a good edge $\mathcal{E}$, then $N_{\mathcal{E}}$ stands for the number of integral points in $\mathcal{N}$ lying on $\mathcal{E}$; otherwise set $N_{\mathcal{E}} = 0$.*

Suppose given a non-composite reduced rational function $r = f/g \in \mathbb{K}(X,Y)$ of degree d, assume that $\mathcal{N} \subseteq N\left((1+X+Y)^d\right)$ and that $p = 0$ or $p > d(d-1)$.

- *If $N(f) \subset \mathcal{N}$ and $N(g) \subset \mathcal{N}$ then*

$$\rho(f,g) \leq 2N - N_X - N_Y - N_{\mathcal{E}} + \kappa \tag{3.2}$$

- *If $N^+(f) \subset \mathcal{N}$ and $N^+(g) \subset \mathcal{N}$ then*

$$m(f,g) \leq 2N - N_X - N_Y - N_{\mathcal{E}} + \kappa \tag{3.3}$$

- *If $N(f) \subset \mathcal{N}$, $N(g) \subset \mathcal{N}$ and $(-g(0,0) : f(0,0)) \notin \sigma(f,g)$ then*

$$m(f,g) \leq 2N - N_X - N_Y - N_{\mathcal{E}} + \kappa \tag{3.4}$$

where $\kappa = \max(e_\infty - 1, 0)$ with e_∞ the multiplicity (possibly 0) of the line at infinity $\{Z = 0\}$ in the pencil of curves $\mu f^\sharp + \lambda g^\sharp$.

Before proceeding with the proof of this theorem, we comment it and illustrate it through three examples. First, consider the dense case which corresponds to the situation studied in Section 2. Here, we have

$$\mathcal{N} = N(f) = N^+(f) = N(g) = N^+(g) = N((1+X+Y)^d)$$

and therefore $N = (d+2)(d+1)/2$, $N_X = N_Y = N_{\mathcal{E}} = d+1$, the good edge $\mathcal{E}$ being the diagonal joining the vertices $(0,d)$ and $(d,0)$. Moreover, since a linear change of coordinates leaves invariant $\rho(f,g)$, $m(f,g)$ and the Newton's polygons $N(f)$ and $N(g)$, we can assume that $\kappa = 0$, that is to say that the line at infinity does not belong to the pencil of curves $\mu f^\sharp + \lambda g^\sharp$. It follows that we obtain the expected bounds $\rho(f,g) \leq d^2 - 1$ and $m(f,g) \leq d^2 - 1$.

Our next example, taken from [Lor93, Remark 5], is to show that the bound (3.3) is almost reached for an arbitrary degree d. Indeed, set

$$f(X,Y) = X(X+1)\cdots(X+d-2)Y + X, \quad g(X,Y) = 1$$

It is not hard to check that $r = f/g$ is non-composite (see [Lor93, Remark 5]) and that $m(f,g) \geq 2d-2$. Now, defining $\mathcal{N} = N^+(f+g)$ wich is a rectangle with vertices $(0,0)$, $(d-1,0)$, $(d-1,1)$ and $(0,1)$ we have $N = 2d$, $N_X = N_Y = d$. Futhermore, we choose the horizontal good edge corresponding to $a_\varepsilon = 0$, $b_\varepsilon = 1$, $c_\varepsilon = 1$ and obtain $N_{\mathcal{E}} = d$. Since $\kappa = d-1$, the bound given in (3.3) is equal to $2d-1$ and we obtain

$$2d - 2 \leq m(f,g) \leq 2d - 1$$

Finally, our last example is to justify why we chose to state (3.4) despite the technical hypothesis requiring that the projective point $(-g(0,0) : f(0,0))$ does not belong to the spectrum of f/g. Consider the example

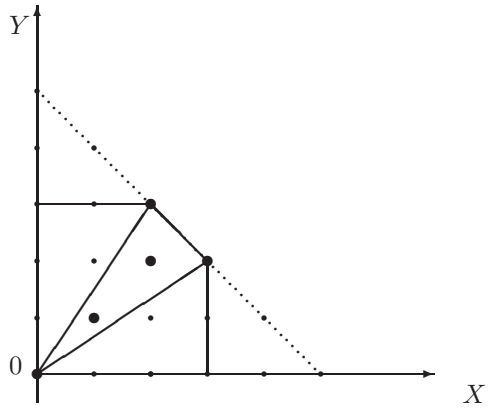
$$\begin{aligned} f(X,Y) &= a_0 + a_1XY + a_2X^2Y^2 + a_3X^3Y^2 + a_4X^2Y^3 \\ g(X,Y) &= b_0 + b_1XY + b_2X^2Y^2 + b_3X^3Y^2 + b_4X^2Y^3 \end{aligned}$$

where the coefficients a_i's and b_j's are all assumed to be nonzero and such that the above mentioned hypothesis is satisfied. We have $N(g) = N(f)$ and it is clear that $N(f) \subsetneq N^+(f)$. Taking $\mathcal{N} = N^+(f)$ and defining the good edge $\mathcal{E}$ as, for instance,

the top horizontal edge of $N^+(f)$, we get $N = 15$, $N_x = 4$, $N_y = 4$, $N_{\mathcal{E}} = 3$, $d = 5$. Therefore, (3.3) yields

$$m(f,g) \leq 2N - N_x - N_y - N_{\mathcal{E}} = 19 < d^2 - 1 = 24 \tag{3.5}$$

Now, choosing $\mathcal{N} = N(f)$ there is only one choice for the good edge $\mathcal{E}$ and we obtain $N = 5$, $N_x = 1$, $N_y = 1$, $N_{\mathcal{E}} = 2$, $d = 5$. Consequently, (3.4) gives $m(f,g) \leq 10$, to be compared with (3.5). The following picture shows the different polytopes involved in this example.



We now turn to the proof of Theorem 15. We begin with the following preliminary definition and result.

Definition 16. Let $f(X,Y) \in \mathbb{K}[X,Y]$, let $(a_{\mathcal{E}}, b_{\mathcal{E}}) \in \mathbb{Z}^2$ and denote $f = f_1^{e_1} \cdots f_r^{e_r}$ a factorization of f where each polynomial f_i is irreducible. For all $i = 2, \ldots, r$, we set

$$\begin{aligned} \mathcal{G}_i^{(1)} &= -d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f_i)\frac{f}{f_1}\partial_X f_1 + d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f_1)\frac{f}{f_i}\partial_X f_i \\ \mathcal{H}_i^{(1)} &= -d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f_i)\frac{f}{f_1}\partial_Y f_1 + d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f_1)\frac{f}{f_i}\partial_Y f_i \end{aligned}$$

and for all $i = 1, \ldots, r$ and $k = 2, \ldots, e_i$ we set

$$\mathcal{G}_i^{(k)} = \frac{f}{f_i^k}\partial_X f_i, \quad \mathcal{H}_i^{(k)} = \frac{f}{f_i^k}\partial_Y f_i.$$

Proposition 17. *Let $f(X,Y) \in \mathbb{K}[X,Y]$ of degree d, denote $f = f_1^{e_1} \cdots f_r^{e_r}$ a factorization of f where each polynomial f_i is irreducible and assume that $p = 0$ or $p > d$.*

(i) *For all $i = 2, \ldots, r$ and all $(a_{\mathcal{E}}, b_{\mathcal{E}}) \in \mathbb{Z}^2$,*

$$N\left(X\mathcal{G}_i^{(1)}\right) \subset N(f) \text{ and } N\left(Y\mathcal{H}_i^{(1)}\right) \subset N(f)$$

(ii) *For all* $i = 1, \dots r$ *and all* $k = 2, \dots, e_i$,

$$N\Big(X\mathcal{G}_i^{(k)}\Big) \subset N^+(f) \text{ and } N\Big(Y\mathcal{H}_i^{(k)}\Big) \subset N^+(f)$$

Furthermore, if $f(0,0) \neq 0$ *then for all* $i = 1, \dots r$ *and all* $k = 2, \dots, e_i$,

$$N\Big(X\mathcal{G}_i^{(k)}\Big) \subset N(f) \text{ and } N\Big(Y\mathcal{H}_i^{(k)}\Big) \subset N(f)$$

(iii) *If* $\mathcal{E}$ *is a good edge of* $N(f)$ *with equation* $a_\mathcal{E}X + b_\mathcal{E}Y = c_\mathcal{E}$, *then for all* $i = 1, \dots r$ *and all* $k = 1, \dots, e_i$,

$$d_{a_\mathcal{E},b_\mathcal{E}}\Big(a_\mathcal{E}X\mathcal{G}_i^{(k)} + b_\mathcal{E}Y\mathcal{H}_i^{(k)}\Big) \leq d_{a_\mathcal{E},b_\mathcal{E}}(f) - 1$$

(iv) *If* $p = 0$ *or* $p > d$ *then the* $(\sum_{i=1}^r e_i) - 1$ *elements*

$$\Big(\mathcal{G}_i^{(k)}, \mathcal{G}_i^{(k)}\Big), \ i = 1, \dots, r, \ k = 1, \dots, e_i, \ (i,k) \neq (1,1)$$

are $\mathbb{K}$*-linearly independent.*

Proof. We begin with the proof of (i) and (ii). For all $i = 1, \dots, r$ and $k = 1, \dots, e_i$, set

$$g_i^{(k)} = \frac{f}{f_i^k}\partial_X f_i, \quad h_i^{(k)} = \frac{f}{f_i^k}\partial_Y f_i \tag{3.6}$$

By (3.1), $N(Xg_i^{(k)}) = N\Big(\frac{f}{f_i^k}\Big) + N(X\partial_X f_i)$, and since $N(X\partial_X f_i) \subset N(f_i)$ we get

$$N\Big(X\frac{f}{f_i^k}\partial_X f_i\Big) \subset N\Big(\frac{f}{f_i^k}\Big) + N(f_i) = N\Big(\frac{f}{f_i^{k-1}}\Big) \subset N^+(f) \tag{3.7}$$

If $k = 1$, Equation (3.7) shows that $N(Xg_i^{(1)}) \subset N(f)$ for all $i = 1, \dots, r$ and hence that $N(X\mathcal{G}_i^{(1)}) \subset N(f)$ for all $i = 2, \dots, r$.

If $k > 1$ then, by (3.1) we have

$$N(f) = N\Big(\frac{f}{f_i^{k-1}}\Big) + N(f_i^{k-1})$$

So, if $f(0,0) \neq 0$ then $f_i^{k-1}(0,0) \neq 0$ and hence $(0,0) \in N(f_i^{k-1})$. It follows that

$$N\Big(\frac{f}{f_i^{k-1}}\Big) \subset N(f)$$

proving that $N\Big(X\mathcal{G}_i^{(k)}\Big) \subset N(f)$ for all $i = 1, \dots r$ and all $k = 2, \dots, e_i$.

We can proceed similarly with the polynomials $h_i^{(k)}$ and conclude this way the proof of (i) and (ii).

We turn to the proof of (iii). If $k > 1$, then by Definition 14 we clearly have

$$d_{a_\mathcal{E},b_\mathcal{E}}(a_\mathcal{E}X\mathcal{G}_i^{(k)} + b_\mathcal{E}Y\mathcal{H}_i^{(k)}) \leq d_{a_\mathcal{E},b_\mathcal{E}}(f) - 1$$

If $k = 1$, denote by f^{top} the homogeneous part of f with maximum weighted degree $d_{a_\mathcal{E},b_\mathcal{E}}(f)$. Then, Euler's relation

$$a_\mathcal{E}X\partial_X f^{\text{top}} + b_\mathcal{E}Y\partial_Y f^{\text{top}} = d_{a_\mathcal{E},b_\mathcal{E}}(f)f^{\text{top}}$$

allows to conclude.

It remains to prove (iv). For this purpose, we will prove that the $\sum_{i=1}^{r} e_i$ elements $(g_i^{(k)}, h_i^{(k)})$ defined by (3.6) are $\mathbb{K}$-linearly independent. So suppose that there exists a collection of $\lambda_{i,k} \in \mathbb{K}$ such that

$$\sum_{i=1}^{r}\sum_{k=1}^{e_i} \lambda_{i,k} g_i^{(k)} = 0, \quad \sum_{i=1}^{r}\sum_{k=1}^{e_i} \lambda_{i,k} h_i^{(k)} = 0$$

and choose an integer $j \in \{1, \dots, r\}$. We have

$$\sum_{i=1}^{r}\sum_{k=1}^{e_i} \lambda_{i,k} g_i^{(k)} g_j^{(e_j)} = 0$$

and since f divides $g_i^{(k)} g_j^{(e_j)}$ for all $i \neq j$, we deduce that

$$\sum_{k=1}^{e_j} \lambda_{j,k} g_j^{(k)} g_j^{(e_j)} = 0 \mod f$$

Equivalently, there exists a polynomial $T \in \mathbb{K}[X, Y]$ such that

$$fT = \lambda_{j,1} \frac{f}{f_j} \frac{f}{f_j^{e_j}} (\partial_X f_j)^2 + \cdots + \lambda_{j,e_j} \frac{f}{f_j^{e_j}} \frac{f}{f_j^{e_j}} (\partial_X f_j)^2$$

that is to say, such that

$$f_j^{e_j} T = (\partial_X f_j)^2 \frac{f}{f_j^{e_j}} (\lambda_{j,1} f_j^{e_j - 1} + \cdots + \lambda_{j,e_j}).$$

It follows that either f_j divides $\partial_X f_j$ or either $f_j^{e_j}$ divides $(\lambda_{j,1} f_j^{e_j-1} + \cdots + \lambda_{j,e_j})$.

A similar reasoning by replacing $g_i^{(k)}$ with $h_i^{(k)}$ shows that either f_j divides $\partial_Y f_j$ or either $f_j^{e_j}$ divides $(\lambda_{j,1} f_j^{e_j-1} + \cdots + \lambda_{j,e_j})$. Now, since $\deg f_j \leq d < p$ and f is not a constant polynomial, $(\partial_X f_j, \partial_Y f_j) \neq (0, 0)$ and hence f_j cannot divide one of its partial derivative. It follows that necessarily $f_j^{e_j}$ divides $\lambda_{j,1} f_j^{e_j-1} + \cdots + \lambda_{j,e_j}$. But since $\deg f_j^{e_j} > \deg \lambda_{j,1} f_j^{e_j-1} + \cdots + \lambda_{j,e_j}$ we must have $\lambda_{j,1} f_j^{e_j-1} + \cdots + \lambda_{j,e_j} = 0$. Furthermore $\deg f_j^{e_j} > \deg f_j^{e_j-1} > \cdots > \deg f_j$, so that $\lambda_{j,k} = 0$ for all j, k. This proves that the $(g_i^{(k)}, h_i^{(k)})$, hence the $(G_i^{(k)}, H_i^{(k)})$, are linearly independent over $\mathbb{K}$. □

Observe that the technical hypothesis $f(0, 0) \neq 0$ in (ii) is necessary. Indeed, if $f = X^3(Y^2 + X + 1)$, $f_1 = X$ and $e_1 = 3$ then $X\mathcal{G}_1^{e_1} = X(Y^2 + X + 1)$ and its Newton's polygon is not included in the Newton's polygon of f.

Proof of Theorem 15. We will proceed similarly to what we did to prove Theorem 10. Given a polynomial $h(X, Y) \in \mathbb{K}[X, Y]$, we define the $\mathbb{K}$-linear map

$$\begin{array}{rcl} \mathcal{SR}(h) : E_{\mathcal{N}} & \longrightarrow & \mathbb{K}[X, Y] \\ (G, H) & \longmapsto & h^2 \left(\partial_Y \left(\dfrac{G}{h} \right) - \partial_X \left(\dfrac{H}{h} \right) \right) \end{array}$$

where

$$E_{\mathcal{N}} = \{(G, H) \in \mathbb{K}[X, Y] \times \mathbb{K}[X, Y] \text{ such that } N(XG) \subset \mathcal{N}, N(YH) \subset \mathcal{N}, \\ d_{a_\mathcal{E}, b_\mathcal{E}}(a_\mathcal{E} XG + b_\mathcal{E} YH) \leq d_{a_\mathcal{E}, b_\mathcal{E}}(h) - 1\}$$

Notice that the last condition in the above definition has to be forgotten if there does not exist a good edge $\mathcal{E}$. Also, observe that all the elements introduced in Definition 16 belong to the kernel of $\mathcal{SR}(f)$, keeping the notation of loc. cit.

Since we assumed that $\mathcal{N} \subseteq N\left((1+X+Y)^d\right)$, $E_{\mathcal{N}}$ is a subvector space of the $\mathbb{K}$-vector space $\mathbb{K}[X,Y]_{\leq d-1} \times \mathbb{K}[X,Y]_{\leq d-1}$, so that $\mathcal{SR}(h)$ is a restriction of the $\mathbb{K}$-linear map $\mathcal{G}_d(h)$ introduced in Section 1, to $E_{\mathcal{N}}$. Let us compute the dimension of this latter vector space. Pick $(G,H) \in E_{\mathcal{N}}$. We have $N(XG) \subset \mathcal{N}$, hence XG has at most $N - N_x$ nonzero coefficients and so does G, because XG and G have the same number of nonzero coefficients. Similarly, we get that H has $N - N_y$ nonzero coefficients. The condition

$$d_{a_{\mathcal{E}},b_{\mathcal{E}}}(a_{\mathcal{E}}XG + b_{\mathcal{E}}YH) \leq d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f) - 1$$

means that the weighted homogeneous part of highest degree of G and H are related. That is to say, we can write the homogeneous part of weighted degree $d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f)-1$ of H in terms of the homogeneous part of weighted degree $d_{a_{\mathcal{E}},b_{\mathcal{E}}}(f) - 1$ of G. Consequently, we obtain

$$\dim_{\mathbb{K}} E_{\mathcal{N}} = 2N - N_x - N_y - N_{\mathcal{E}} \tag{3.8}$$

Now, for all $(\mu : \lambda) \in \mathbb{P}^1_{\mathbb{K}}$, consider the linear map

$$\mathcal{SR}(\mu f + \lambda g) = \mu\mathcal{SR}(f) + \lambda\mathcal{SR}(g)$$

and, choosing bases for $E_{\mathcal{N}}$ and $\mathbb{K}[X,Y]_{\leq 2d-2}$, the corresponding matrix

$$\mathrm{M}(\mu f + \lambda g) = \mu\mathrm{M}(f) + \lambda\mathrm{M}(g)$$

They form a pencil of matrices that has (3.8) columns and more rows. Then, define the polynomial $\mathrm{Spect}(U,V) \in \mathbb{K}[U,V]$ as the greatest common divisor of all the minors of size (3.8) of the matrix

$$U\mathrm{M}(f) + V\mathrm{M}(g)$$

It is a homogeneous polynomial of degree lower or equal to (3.8).

The polynomial $\mathrm{Spect}(U,V)$ is nonzero for the same reason that the one given in Theorem 10, since the linear maps $\mathcal{SR}(-)$ are restrictions of the linear maps $\mathcal{G}_d(-)$. The fact that this property remains valid if $p > d(d-1)$ is a consequence of [Gao03, Lemma 2.4] where Gao studied the property of the linear map $\mathcal{G}_d(-)$ for square-free polynomials in positive characteristic.

Now, let $(\mu : \lambda) \in \sigma(f,g)$. Then $\dim\ker \mathrm{M}(\mu f + \lambda g) > 0$ and $(\mu : \lambda)$ is root of $\mathrm{Spect}(U,V)$ of multiplicity, say η. Since $\eta \geq \dim\ker \mathrm{M}(\mu f + \lambda g)$, Proposition 17 gives some lower bounds for η that allow to conclude the proof of this theorem as follows.

- If $N(f), N(g) \subset \mathcal{N}$, then Proposition 17,(i),(iii) and (iv) implies that $\eta \geq n(\mu:\lambda)-1$ if $\deg(\mu f+\lambda g) = d$ or $\eta+\kappa \geq n(\mu:\lambda)-1$ otherwise. Summing over all the elements in $\sigma(f,g)$ we deduce the bound (3.2).
- If $N(f)^+, N(g)^+ \subset \mathcal{N}$, then Proposition 17,(i), (ii) - first part, (iii) and (iv) implies that $\eta \geq m(\mu:\lambda) - 1$ if $\deg(\mu f + \lambda g) = d$ or $\eta + \kappa \geq m(\mu:\lambda) - 1$ otherwise. Summing over all the elements in $\sigma(f,g)$ we deduce the bound (3.3).

- If $N(f), N(g) \subset \mathcal{N}$ and $(-g(0,0) : f(0,0)) \notin \sigma(f,g)$, then Proposition 17,(i), (ii) - second part, (iii) and (iv) implies that $\eta \geq m(\mu : \lambda) - 1$ if $\deg(\mu f + \lambda g) = d$ or $\eta + \kappa \geq m(\mu : \lambda) - 1$ otherwise. Summing over all the elements in $\sigma(f,g)$ we deduce the bound (3.3).

Notice that we used the fact that the polynomial $\mathrm{Spect}(U,V)$ is of degree lower or equal to (3.8). □

To finish, point out that we can not state a result similar to Theorem 15 in terms of the Newton's polygon of f and $g \in \mathbb{K}[X_1, \ldots, X_n]$ following the above strategy because we are not able to preserve the sparsity of the polynomials through Bertini's Theorem.

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