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# On the Infrared Problem for the Dressed Non-Relativistic Electron in a Magnetic Field

Laurent Amour, Jérémy Faupin, Benoît Grébert, and Jean-Claude Guillot

**ABSTRACT.** We consider a non-relativistic electron interacting with a classical magnetic field pointing along the  $x_3$ -axis and with a quantized electromagnetic field. The system is translation invariant in the  $x_3$ -direction and we consider the reduced Hamiltonian  $H(P_3)$  associated with the total momentum  $P_3$  along the  $x_3$ -axis. For a fixed momentum  $P_3$  sufficiently small, we prove that  $H(P_3)$  has a ground state in the Fock representation if and only if  $E'(P_3) = 0$ , where  $P_3 \mapsto E'(P_3)$  is the derivative of the map  $P_3 \mapsto E(P_3) = \inf \sigma(H(P_3))$ . If  $E'(P_3) \neq 0$ , we obtain the existence of a ground state in a non-Fock representation. This result holds for sufficiently small values of the coupling constant.  
**MSC:** 81V10; 81Q10; 81Q15

## 1. Introduction

In this paper we pursue the analysis of a model considered in [AGG1], describing a non-relativistic particle (an electron) interacting both with the quantized electromagnetic field and a classical magnetic field pointing along the  $x_3$ -axis. An ultraviolet cutoff is imposed in order to suppress the interaction between the electron and the photons of energies bigger than a fixed, arbitrary large parameter  $\Lambda$ . The total system being invariant by translations in the  $x_3$ -direction, it can be seen (see [AGG1]) that the corresponding Hamiltonian admits a decomposition of the form  $H \simeq \int_{\mathbb{R}}^{\oplus} H(P_3) dP_3$  with respect to the spectrum of the total momentum along the  $x_3$ -axis that we denote by  $P_3$ . For any given  $P_3$  sufficiently close to 0, the existence of a ground state for  $H(P_3)$  is proven in [AGG1] provided an infrared regularization is introduced (besides a smallness assumption on the coupling parameter). Our aim is to address the question of the existence of a ground state without requiring any infrared regularization.

The model considered here is closely related to similar non-relativistic QED models of freely moving electrons, atoms or ions, that have been studied recently (see [BCFS, FGS1, Hi, CF, Ch, HH, CFP, FP] for the case of one single electron, and [AGG2, LMS, FGS2, HH, LMS2] for atoms or ions). In each of these papers, the physical systems are translation invariant, in the sense that the associated Hamiltonian  $H$  commutes with the operator of total momentum  $P$ . As a consequence,  $H \simeq \int_{\mathbb{R}^3} H(P) dP$ , and one is led to study the spectrum of the fiber Hamiltonian  $H(P)$  for fixed  $P$ 's.

For the one-electron case, an aspect of the so-called *infrared catastrophe* lies in the fact that, for  $P \neq 0$ ,  $H(P)$  does not have a ground state in the Fock space

(see [CF, Ch, HH, CFP]). More precisely, if an infrared cutoff of parameter  $\sigma$  is introduced in the model in order to remove the interaction between the electron and the photons of energies less than  $\sigma$ , the associated Hamiltonian  $H_\sigma(P)$  does have a ground state  $\Phi_\sigma(P)$  in the Fock space. Nevertheless as  $\sigma \rightarrow 0$ , it is shown that  $\Phi_\sigma(P)$  “leaves” the Fock space. Physically this can be interpreted by saying that a free moving electron in its ground state is surrounded by a cloud of infinitely many “soft” photons.

For negative ions, the absence of a ground state for  $H(P)$  is established in [HH] under the assumption  $\nabla E(P) \neq 0$ , where  $E(P) = \inf \sigma(H(P))$ .

In [CF], with the help of operator-algebra methods, a representation of a *dressed 1-electron state* non-unitarily equivalent to the usual Fock representation of the canonical commutation relations is given. We shall obtain in this paper a related result, following a different approach, under the further assumption that the electron interact with a classical magnetic field and an electrostatic potential.

We shall first provide a necessary and sufficient condition for the existence of a ground state for  $H(P_3)$ . Namely we shall prove that the bottom of the spectrum,  $E(P_3) = \inf \sigma(H(P_3))$ , is an eigenvalue of  $H(P_3)$  if and only if  $E'(P_3) = 0$  where  $E'(P_3)$  denotes the derivative of the map  $P_3 \mapsto E(P_3)$ . In the case  $E'(P_3) \neq 0$ , thanks to a (non-unitary) Bogoliubov transformation, in the same way as in [Ar, DG2], we shall define a “renormalized” Hamiltonian  $H^{\text{ren}}(P_3)$  which can be seen as an expression of the physical Hamiltonian in a non-Fock representation. Then we shall prove that  $H^{\text{ren}}(P_3)$  has a ground state. These results have been announced in [AFGG].

The regularity of the map  $P_3 \mapsto E(P_3)$  plays a crucial role in our proof. Adapting [Pi, CFP] we shall see that  $P_3 \mapsto E(P_3)$  is of class  $C^{1+\gamma}$  for some strictly positive  $\gamma$ . Let us also mention that our method can be adapted to the case of free moving hydrogenoid ions without spin, the condition  $E'(P_3) = 0$  being replaced by  $\nabla E(P) = 0$  (see Subsection 1.2 for a further discussion on this point).

The remainder of the introduction is organized as follows: In Subsection 1.1, a precise definition of the model considered in this paper is given, next, in Subsection 1.2, we state our results and compare them to the literature.

**1.1. The model.** We consider a non-relativistic electron of charge  $e$  and mass  $m$  interacting with a classical magnetic field pointing along the  $x_3$ -axis, an electrostatic potential, and the quantized electromagnetic field in the Coulomb gauge. The Hilbert space for the electron and the photon field is written as

$$(1.1) \quad \mathcal{H} = \mathcal{H}_{\text{el}} \otimes \mathcal{H}_{\text{ph}},$$

where  $\mathcal{H}_{\text{el}} = L^2(\mathbb{R}^3; \mathbb{C}^2)$  is the Hilbert space for the electron, and  $\mathcal{H}_{\text{ph}}$  is the symmetric Fock space over  $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$  for the photons,

$$(1.2) \quad \mathcal{H}_{\text{ph}} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S_n \left[ L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n} \right].$$

Here  $S_n$  denotes the orthogonal projection onto the subspace of symmetric functions in  $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n}$  in accordance with Bose-Einstein statistics. We shall use the notation  $\mathbf{k} = (k, \lambda)$  for any  $(k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$ , and

$$(1.3) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk.$$

Likewise, the scalar product in  $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$  is defined by

$$(1.4) \quad (h_1, h_2) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \bar{h}_1(\mathbf{k}) h_2(\mathbf{k}) d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{h}_1(k, \lambda) h_2(k, \lambda) dk.$$

The position and the momentum of the electron are denoted respectively by  $x = (x_1, x_2, x_3)$  and  $p = (p_1, p_2, p_3) = -i\nabla_x$ . The classical magnetic field is of the form  $(0, 0, b(x'))$ , where  $x' = (x_1, x_2)$  and  $b(x') = (\partial a_2 / \partial x_1)(x') - (\partial a_1 / \partial x_2)(x')$ . Here  $a_j(x')$ ,  $j = 1, 2$ , are real functions in  $C^1(\mathbb{R}^2)$ . The electrostatic potential is denoted by  $V(x')$ . The quantized electromagnetic field in the Coulomb gauge is defined by

$$(1.5) \quad \begin{aligned} A(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho^\Lambda(k) \left[ e^{-ik \cdot x} a^*(\mathbf{k}) + e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k}, \\ B(x) &= -\frac{i}{\sqrt{2\pi}} \int |k|^{1/2} \left( \frac{k}{|k|} \wedge \epsilon^\lambda(k) \right) \rho^\Lambda(k) \left[ e^{-ik \cdot x} a^*(\mathbf{k}) - e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k}, \end{aligned}$$

where  $\rho^\Lambda(k)$  denotes the characteristic function  $\rho^\Lambda(k) = \mathbf{1}_{|k| \leq \Lambda}(k)$  and  $\Lambda$  is an arbitrary large positive real number. Note that this explicit choice of the ultraviolet cutoff function  $\rho^\Lambda$  is made mostly for convenience. Our results would hold without change for any  $\rho^\Lambda$  satisfying  $\int_{|k| \leq 1} |k|^{-2} |\rho^\Lambda(k)|^2 d^3k + \int_{|k| \geq 1} |k| |\rho^\Lambda(k)|^2 d^3k < \infty$ . The vectors  $\epsilon^1(k)$  and  $\epsilon^2(k)$  in (1.5) are real polarization vectors orthogonal to each other and to  $k$ . Besides  $a^*(\mathbf{k})$  and  $a(\mathbf{k})$  are the usual creation and annihilation operators obeying the canonical commutation relations

$$(1.6) \quad [a^\#(\mathbf{k}), a^\#(\mathbf{k}')] = 0 \quad , \quad [a(\mathbf{k}), a^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}') = \delta_{\lambda\lambda'} \delta(k - k').$$

The Pauli Hamiltonian  $H_g$  associated with the system we consider is formally given by

$$(1.7) \quad \begin{aligned} H_g &= \frac{1}{2m} \left( p - ea(x') - gA(x) \right)^2 - \frac{e}{2m} \sigma_3 b(x') \\ &\quad - \frac{g}{2m} \sigma \cdot B(x) + V(x') + H_{\text{ph}}, \end{aligned}$$

where the charge of the electron is replaced by a coupling parameter  $g$  in the terms containing the quantized electromagnetic field. The Hamiltonian for the photons in the Coulomb gauge is given by

$$(1.8) \quad H_{\text{ph}} = d\Gamma(|k|) = \int |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Finally  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  is the 3-component vector of the Pauli matrices.

Noting that  $H_g$  formally commutes with the operator of total momentum in the direction  $x_3$ ,  $P_3 = p_3 + d\Gamma(k_3)$ , one can consider the reduced Hamiltonian associated with  $P_3 \in \mathbb{R}$  that we denote by  $H_g(P_3)$ . For any fixed  $P_3$ ,  $H_g(P_3)$  acts on  $L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$  and is formally given by

$$(1.9) \quad \begin{aligned} H_g(P_3) &= \frac{1}{2m} \sum_{j=1,2} \left( p_j - ea_j(x') - gA_j(x', 0) \right)^2 - \frac{e}{2m} \sigma_3 b(x') + V(x') \\ &\quad + \frac{1}{2m} \left( P_3 - d\Gamma(k_3) - gA_3(x', 0) \right)^2 - \frac{g}{2m} \sigma \cdot B(x', 0) + H_{\text{ph}}. \end{aligned}$$

We define the infrared cutoff Hamiltonian  $H_g^\sigma(P_3)$  by replacing  $A(x)$  in (1.5) with

$$(1.10) \quad A_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) \left[ e^{-ik \cdot x} a^*(\mathbf{k}) + e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k},$$

where  $\rho_\sigma^\Lambda = \mathbf{1}_{\sigma \leq |k| \leq \Lambda}$ , and similarly for  $B_\sigma(x)$ . We set  $E_g(P_3) = \inf \sigma(H_g(P_3))$  and  $E_{g\sigma}(P_3) = \inf \sigma(H_g^\sigma(P_3))$ .

The electronic Hamiltonian  $h(b, V)$  on  $L^2(\mathbb{R}^2; \mathbb{C}^2)$  is defined by

$$(1.11) \quad h(b, V) = \sum_{j=1,2} \frac{1}{2m} (p_j - e a_j(x'))^2 - \frac{e}{2m} \sigma_3 b(x') + V(x').$$

Let  $e_0 = \inf \sigma(h(b, V))$ . We make the following hypothesis:

**(H<sub>0</sub>)**  $h(b, V)$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$  and  $e_0$  is an isolated eigenvalue of multiplicity 1.

We refer to **[AHS, So, IT, Ra]** for possible choices of  $b, V$  satisfying Hypothesis **(H<sub>0</sub>)**. The following proposition is established in **[AGG1, Theorem 2.3]**:

**PROPOSITION 1.1.** *Suppose Hypothesis **(H<sub>0</sub>)**. For sufficiently small values of  $|g|$ ,  $H_g$  is self-adjoint with domain  $D(H_g) = D(H_0)$ , and for any  $\sigma \geq 0$  and  $P_3 \in \mathbb{R}$ ,  $H_g^\sigma(P_3)$  identifies with a self-adjoint operator with domain  $D(H_g^\sigma(P_3)) = D(H_0(P_3))$ . Moreover  $H_g$  admits the decomposition*

$$(1.12) \quad H_g = \int_{\mathbb{R}}^{\oplus} H_g(P_3) dP_3.$$

**1.2. Results and comments.** The key ingredient that we shall need in order to prove our main theorem (see Theorem 1.3 below) lies in the regularity of the map  $P_3 \mapsto E'_{g\sigma}(P_3)$  uniformly in  $\sigma \geq 0$ .

**THEOREM 1.2.** *Assume that **(H<sub>0</sub>)** holds. There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that for all  $|g| \leq g_0$ , for all  $0 \leq \sigma \leq \sigma_0$ , for all  $P_3, k_3$  such that  $|P_3| \leq P_0$ ,  $|P_3 + k_3| \leq P_0$ , for all  $\delta > 0$ ,*

$$(1.13) \quad |E'_{g\sigma}(P_3 + k_3) - E'_{g\sigma}(P_3)| \leq C_\delta |k_3|^{\frac{1}{4} - \delta},$$

where  $C_\delta$  is a positive constant depending only on  $\delta$ .

Similar results for a free electron (that is for  $b = V = 0$ ) interacting with the quantized electromagnetic field have been obtained recently (see **[Ch, CFP, FP]**). The model studied in the latter papers is technically simpler than the one considered here in that the fiber Hamiltonian  $H(P)$  associated with a free electron does not contain the electronic part  $h(b, V)$  and its (minimal) coupling to the quantized electromagnetic field. In particular the operator  $H(P)$  in **[Ch, CFP, FP]** acts only on the Fock space, whereas in our case  $H_{g\sigma}(P_3)$  still contains interactions between the electromagnetic field and the electronic degrees of freedom. We shall use the exponential decay of the ground states  $\Phi_g^\sigma(P_3)$  in  $x'$  in order to overcome this difficulty.

It is proved in **[Ch]** (for a free electron) that  $P \mapsto E(P) = \inf \sigma(H(P))$  is of class  $C^2$  in a neighborhood of 0 thanks to a renormalization group analysis (see also **[BCFS]**). The author also shows that, still in a neighborhood of  $P = 0$ , the derivative  $\nabla E(P)$  vanishes only at  $P = 0$ . In **[CFP]**, with the help of what the authors call “iterative analytic perturbation theory”, following a multiscale

analysis developed in **[Pi]**, it is proved, among other results, that  $P \mapsto E(P)$  is of class  $C^{5/4-\delta}$  for arbitrary small  $\delta > 0$ . The method has later been improved in **[FP]** leading to the  $C^2$  property of  $P \mapsto E(P)$ .

In order to establish our main theorem, Theorem 1.3, the “degree of regularity” we need is reached as soon as  $P_3 \mapsto E_{g\sigma}(P_3)$  is at least of order  $C^{1+\gamma}$ , uniformly in  $\sigma$ , for some  $\gamma > 0$ . Therefore, although one can conjecture that  $P_3 \mapsto E_{g\sigma}(P_3)$  is of class  $C^2$  uniformly in  $\sigma$ , Theorem 1.2 is sufficient for our purpose. In order to prove it we shall adapt **[Pi, CFP]**: First, we shall give a short proof of the existence of a spectral gap for  $H_g^\sigma(P_3)$  (restricted to the space of photons of energies bigger than  $\sigma$ ) above the non-degenerate eigenvalue  $E_{g\sigma}(P_3)$ . Next we shall apply “iterative analytic perturbation theory”.

We postpone the proof of Theorem 1.2 to the appendix. Since several parts are taken from **[Pi, CFP]**, we shall not give all the details, rather we shall emphasize the differences with **[Pi, CFP]**.

For  $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ , let us define the field operator  $\Phi(h)$  by

$$(1.14) \quad \Phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h)),$$

where the creation operator  $a^*(h)$  and the annihilation operator  $a(h)$  are defined respectively by

$$(1.15) \quad a^*(h) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} h(\mathbf{k}) a^*(\mathbf{k}) d\mathbf{k}, \quad a(h) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \bar{h}(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Hence, letting  $h_{j,\sigma}(x')$  and  $\tilde{h}_{j,\sigma}(x')$  for  $j = 1, 2, 3$  be defined respectively by

$$(1.16) \quad \begin{aligned} h_{j,\sigma}(x', \mathbf{k}) &= \pi^{-1/2} \frac{\epsilon_j^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) e^{ik' \cdot x'}, \\ \tilde{h}_{j,\sigma}(x', \mathbf{k}) &= -i\pi^{-1/2} |k|^{1/2} \left( \frac{k}{|k|} \wedge \epsilon^\lambda(k) \right)_j \rho_\sigma^\Lambda(k) e^{ik' \cdot x'}, \end{aligned}$$

where  $k' = (k_1, k_2)$ , we have  $A_{j,\sigma}(x', 0) = \Phi(h_{j,\sigma}(x'))$  and  $B_{j,\sigma}(x', 0) = \Phi(\tilde{h}_{j,\sigma}(x'))$ . The Weyl operator associated with  $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$  is denoted by  $W(h) = e^{i\Phi(h)}$ . Let  $f_\sigma : \mathbb{R}^3 \times \mathbb{Z}_2 \rightarrow \mathbb{C}$  be defined by

$$(1.17) \quad f_\sigma(\mathbf{k}) = -\frac{g}{\sqrt{4\pi}} \frac{\rho_\sigma^\Lambda(k) \epsilon_\lambda^3(k)}{k_3 |k|^{1/2}} \frac{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3)}{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) + |k|}.$$

If  $\sigma = 0$  we remove the subindex  $\sigma$  in the preceding notations. We recall from **[AGG1, Lemma 4.3]** that for  $g, \sigma, P_3$  and  $|k|$  sufficiently small,

$$(1.18) \quad E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) \geq -\frac{3}{4}|k|.$$

Hence in particular for  $\sigma > 0$ , we have  $f_\sigma \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ , whereas if  $\sigma = 0$  and  $P_3 \mapsto E_g(P_3)$  is of class  $C^{1+\gamma}$  with  $\gamma > 0$ , then

$$(1.19) \quad f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \iff E'_g(P_3) = 0.$$

Similarly as in **[Ar]** (see also **[DG2, Pa]**), we define the “renormalized” (Bogoliubov transformed) Hamiltonian  $H_{g\sigma}^{\text{ren}}(P_3)$  by the expression

$$(1.20) \quad H_{g\sigma}^{\text{ren}}(P_3) = W(if_\sigma) H_g^\sigma(P_3) W(if_\sigma)^*.$$

Notice that the identity (1.20) might only be formal for  $\sigma = 0$  since in this case, by (1.19),  $f$  might not be in  $L^2$ . Nevertheless using usual commutation relations (see for instance [DG1]), we define for any  $\sigma \geq 0$ :

$$\begin{aligned} H_{g\sigma}^{\text{ren}}(P_3) &= \frac{1}{2m} \sum_{j=1,2} \left( p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), f_\sigma) \right)^2 \\ &+ \frac{1}{2m} \left( P_3 - d\Gamma(k_3) - \Phi(k_3 f_\sigma) - \frac{1}{2}(k_3 f_\sigma, f_\sigma) - gA_{3,\sigma}(x', 0) + g\text{Re}(h_{3,\sigma}(x'), f_\sigma) \right)^2 \\ &- \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left( B_\sigma(x', 0) - \text{Re}(\tilde{h}_\sigma(x'), f_\sigma) \right) + V(x') \\ &+ H_f + \Phi(|k|f_\sigma) + \frac{1}{2}(|k|f_\sigma, f_\sigma). \end{aligned}$$

In the same way as for  $H_g^\sigma(P_3)$  (see Proposition 1.1), one can verify that  $H_{g\sigma}^{\text{ren}}(P_3)$  is self-adjoint with domain  $D(H_{g\sigma}^{\text{ren}}(P_3)) = D(H_0(P_3))$  for any  $\sigma \geq 0$ . Besides for  $\sigma > 0$ , we have that  $H_{g\sigma}^{\text{ren}}(P_3)$  is unitarily equivalent to  $H_g^\sigma(P_3)$ , whereas for  $\sigma = 0$ , one can verify that  $H_g^{\text{ren}}(P_3)$  is unitarily equivalent to  $H_g(P_3)$  if and only if  $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ . Our main result is:

**THEOREM 1.3.** *Suppose Hypothesis  $(\mathbf{H}_0)$ . There exist  $g_0 > 0$  and  $P_0 > 0$  such that for all  $0 \leq |g| \leq g_0$  and  $0 \leq |P_3| \leq P_0$ ,*

- (i)  $H_g(P_3)$  has a ground state if and only if  $E'_g(P_3) = 0$ ,
- (ii)  $H_g^{\text{ren}}(P_3)$  has a ground state.

The proof of Theorem 1.3 can be adapted to the case of free moving hydrogenoid ions without spins<sup>1</sup>, the condition  $E'_g(P_3) = 0$  being replaced by  $\nabla E_g(P) = 0$ , where  $E_g(P)$  denotes the bottom of the spectrum of the fiber Hamiltonian  $H_g(P)$ . The existence of ground states for atoms has been obtained in [AGG2] thanks to a Power-Zienau-Wooley transformation and the crucial property  $Q = 0$  (here  $Q$  denotes the total charge of the atomic system). Indeed, in [HH], it is proved that for negative ions ( $Q < 0$ )  $H_g(P)$  does not have a ground state if  $\nabla E_g(P) \neq 0$ . Let us also mention [LMS] where the existence of ground states for atoms is proven for any value of the coupling constant  $g$ , by adapting [GLL], under the further assumption  $E_g(P) \geq E_g(0)$  which has not been proven yet. Thus in addition to these results, our method provides the existence of ground states for spinless hydrogenoid ions, both for  $H_g(P)$  in the case  $\nabla E_g(P) = 0$  and for  $H_g^{\text{ren}}(P)$ .

The two statements “ $H_g(P_3)$  has a ground state if  $E'_g(P_3) = 0$ ” and “ $H_g^{\text{ren}}(P_3)$  has a ground state” shall be established following the same standard procedure: An infrared cutoff  $\sigma$  is introduced into the model so that the Hamiltonian  $H_g^\sigma(P_3)$  (respectively  $H_{g\sigma}^{\text{ren}}(P_3)$ ) has a ground state  $\Phi_g^\sigma(P_3)$  (respectively  $\Phi_{g\sigma}^{\text{ren}}(P_3)$ ). We then need to prove that  $\Phi_g^\sigma(P_3)$  and  $\Phi_{g\sigma}^{\text{ren}}(P_3)$  converge strongly as  $\sigma \rightarrow 0$ . To this end we control the number of photons in the states  $\Phi_g^\sigma(P_3)$  and  $\Phi_{g\sigma}^{\text{ren}}(P_3)$  thanks to a pull-through formula and (1.13).

We emphasize that, in the case  $E'_g(P_3) \neq 0$ ,  $H_g^{\text{ren}}(P_3)$  can be seen as an expression of the physical Hamiltonian in a representation of the canonical commutation relations non-unitarily equivalent to the Fock representation. Besides, regarding [Ch] for the case of a single freely moving electron, one can conjecture that for sufficiently small values of  $|P_3|$ ,  $E'_g(P_3) = 0$  if and only if  $P_3 = 0$ .

<sup>1</sup>The hypothesis of simplicity for the electronic ground state  $(\mathbf{H}_0)$  imposes this restriction to hydrogenoid atoms or ions.

Our proof of the absence of a ground state for  $H_g(P_3)$  in the case  $E'_g(P_3) \neq 0$  is based on a contradiction argument and [DG2, Lemma 2.6] (see also Lemma 2.2). Again the result is achieved by deriving a suitable expression of  $a(\mathbf{k})\Phi_g(P_3)$  thanks to a pull-through formula (assuming here that  $H_g(P_3)$  has a ground state  $\Phi_g(P_3)$ ). Note that the regularity property (1.13) appears again as a key property (although here only (1.13) for  $\sigma = 0$  is required).

The paper is organized as follows: In Section 2, we prove Theorem 1.3. Next in the appendix we prove Theorem 1.2.

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## 2. Proof of Theorem 1.3

The following proposition is proven in Subsection A.1 of the appendix.

**PROPOSITION 2.1.** *Assume that  $(\mathbf{H}_0)$  holds. There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that for all  $|g| \leq g_0$ , for all  $0 < \sigma \leq \sigma_0$ , for all  $|P_3| \leq P_0$ ,  $H_{g\sigma}(P_3)$  has a unique normalized ground state  $\Phi_g^\sigma(P_3)$ , i.e.*

$$(2.1) \quad H_g^\sigma(P_3)\Phi_g^\sigma(P_3) = E_{g\sigma}(P_3)\Phi_g^\sigma(P_3), \quad \|\Phi_g^\sigma(P_3)\| = 1.$$

Notice that Proposition 2.1 is also established in [AGG1] under the weaker assumption that  $e_0$  is an isolated eigenvalue of  $h(b, V)$  of finite multiplicity. Let us recall a lemma, due to [DG2], on which is based our proof of the absence of a ground state for  $H_g(P_3)$  in the case  $E'_g(P_3) \neq 0$ .

**LEMMA 2.2.** *Let  $\Psi \in L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$ . Assume that*

$$(2.2) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|(a(\mathbf{k}) - h(\mathbf{k}))\Psi\|^2 d\mathbf{k} < \infty,$$

where  $h$  is a measurable function from  $\mathbb{R}^3 \times \mathbb{Z}_2$  to  $\mathbb{C}$  such that  $h \notin L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ . Then  $\Psi = 0$ .

**PROOF.** See [DG2, Lemma 2.6]. □

Theorem 1.3 shall follow from a suitable decomposition of  $a(\mathbf{k})\Phi_g^\sigma(P_3)$  based on a pull-through formula. The latter is the purpose of the following lemma, where the equalities should be understood as identities between measurable functions from  $\mathbb{R}^3 \times \mathbb{Z}_2$  to  $L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$ . For a rigorous justification of the commutations used in the next proof, we refer for instance to [Ge, HH].

In order to shorten the notations, we shall write

$$(2.3) \quad \begin{aligned} H &= H_g^\sigma(P_3), & E &= E_{g\sigma}(P_3), & \Phi &= \Phi_g^\sigma(P_3), \\ \tilde{H} &= H_g^\sigma(P_3 - k_3), & \tilde{E} &= E_{g\sigma}(P_3 - k_3). \end{aligned}$$

**LEMMA 2.3.** *Let  $\sigma \geq 0$  and let  $\Phi = \Phi_g^\sigma(P_3)$  be a normalized ground state of  $H = H_g^\sigma(P_3)$  (assuming it exists for  $\sigma = 0$ ). We have:*

$$(2.4) \quad a(\mathbf{k})\Phi = L_\sigma(\mathbf{k})\Phi + R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}}f_\sigma(\mathbf{k})\Phi,$$

where  $L_\sigma$  and  $R_\sigma$  are operator-valued functions such that,

$$(2.5) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|L_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq Cg^2,$$



and

$$(2.6) \quad R_\sigma(\mathbf{k}) = -\frac{g}{2\sqrt{2\pi}} \frac{\rho_\sigma^\Lambda(k) \epsilon_3^\lambda(k) |k|^{1/2}}{k_3(\tilde{E} - E + |k|)} \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|}.$$

PROOF. It follows from the canonical commutation relations (1.6) that

$$(2.7) \quad \begin{aligned} a(\mathbf{k})H &= (\tilde{H} + |k|)a(\mathbf{k}) \\ &- \frac{g}{2^{\frac{3}{2}}m} \sum_{j=1,2} \left( h_{j,\sigma}(x', \mathbf{k})(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)) + \sigma_j \tilde{h}_{j,\sigma}(x', \mathbf{k}) \right) \\ &- \frac{g}{2^{\frac{3}{2}}m} \left( h_{3,\sigma}(x', \mathbf{k})(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) + \sigma_3 \tilde{h}_{3,\sigma}(x', \mathbf{k}) \right). \end{aligned}$$

In order to control the term containing  $(p_j - ea_j(x') - gA_{j,\sigma}(x', 0))$  in the right-hand-side of the previous equality, we use that (formally)

$$(2.8) \quad \frac{1}{2m}(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)) = i[H, x'_j],$$

for  $j = 1, 2$ . Notice that an alternative would be to consider the Hamiltonian obtained through a unitary Power-Zienau-Wooley transformation (see for instance [GLL]). For a rigorous justification of (2.8), we refer to [BFP, Theorem II.10] which can easily be adapted to our case. In particular it follows that  $x'_j \Phi \in D(H)$ . Applying (2.7) to  $\Phi$  then yields

$$(2.9) \quad \begin{aligned} a(\mathbf{k})\Phi &= \frac{ig}{2^{\frac{3}{2}}} \sum_{j=1,2} h_{j,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1}(H - E)x'_j \Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} [\tilde{H} - E + |k|]^{-1} \sigma \cdot \tilde{h}_\sigma(x', \mathbf{k}) \Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) \Phi. \end{aligned}$$

Note that the expressions of  $H$  and  $\tilde{H}$  imply

$$(2.10) \quad \tilde{H} - H = -\frac{k_3}{m} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) + \frac{k_3^2}{2m}.$$

From (1.18), we get

$$(2.11) \quad \|[\tilde{H} - E + |k|]^{-1}\| \leq C|k|^{-1}.$$

Moreover it is not difficult to show that

$$(2.12) \quad \left\| (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0))[\tilde{H} - E + |k|]^{-1} \right\| \leq C|k|^{-1},$$

and consequently, by (2.10),

$$(2.13) \quad \left\| (H - E)[\tilde{H} - E + |k|]^{-1} \right\| \leq C.$$

Introducing (2.11)–(2.13) into (2.9) and recalling the definitions (1.16) of  $h_j$  and  $\tilde{h}_j$ , we thus obtain

$$(2.14) \quad \begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(0, \mathbf{k})[\tilde{H} - E + |k|]^{-1} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) \Phi, \end{aligned}$$

where

$$(2.15) \quad \|L_1(\mathbf{k})\Phi\| \leq C|g||k|^{-1/2} (\|\Phi\| + \|x'_1 \Phi\| + \|x'_2 \Phi\|).$$

In passing from (2.9) to (2.14) we used that

$$(2.16) \quad |h_{3,\sigma}(x', \mathbf{k}) - h_{3,\sigma}(0, \mathbf{k})| \leq C|k||x'|.$$

Let us now note the following obvious identity:

$$(2.17) \quad \frac{\tilde{H} - E}{\tilde{H} - E + |k|} = \frac{\tilde{E} - E}{\tilde{E} - E + |k|} + \frac{|k|}{\tilde{E} - E + |k|} \left( \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|} \right).$$

Hence, introducing (2.10) and (2.17) into (2.14) leads to

$$(2.18) \quad \begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi - \frac{gk_3}{2^{\frac{3}{2}}}h_{3,\sigma}(0, \mathbf{k})[\tilde{H} - E + |k|]^{-1}\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3} \frac{\tilde{E} - E}{\tilde{E} - E + |k|} h_{3,\sigma}(0, \mathbf{k})\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3} \frac{|k|}{\tilde{E} - E + |k|} h_{3,\sigma}(0, \mathbf{k}) \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|} \Phi. \end{aligned}$$

We conclude the proof using again that  $\|x'_j\Phi\| < \infty$ .  $\square$

The following lemma shows in particular that if the map  $P_3 \mapsto E(P_3)$  is sufficiently regular, then  $\mathbf{k} \mapsto \|R(\mathbf{k})\Phi\|$  is in  $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ , where  $R(\mathbf{k})$  denotes the operator defined in (2.6) for  $\sigma = 0$ .

LEMMA 2.4. *Let the parameters  $g, \sigma, P_3$  be fixed. Assume that there exist  $\gamma > 0$ ,  $P_0 > 0$  and a positive constant  $C$  independent of  $\sigma \geq 0$  such that for all  $|k_3| \leq P_0$ ,*

$$(2.19) \quad |E'_{g\sigma}(P_3 + k_3) - E'_{g\sigma}(P_3)| \leq C|k_3|^\gamma.$$

*Then there exists a positive constant  $C'$ , independent of  $\sigma$ , such that*

$$(2.20) \quad \left\| (H_g^\sigma(P_3 - k_3) - E_{g\sigma}(P_3 - k_3))^{1/2}\Phi \right\| \leq C'|k_3|^{\frac{1+\gamma}{2}}.$$

PROOF. We use again the notations (2.3) and let in addition  $E' = E'_{g\sigma}(P_3)$ . By (2.10), we have

$$(2.21) \quad \tilde{E} - E \leq (\Phi, (\tilde{H} - H)\Phi) = -\frac{k_3}{m}(\Phi, (P_3 - d\Gamma(k_3) - A_{3,\sigma}(x', 0))\Phi) + \frac{k_3^2}{2m}.$$

Dividing by  $-k_3$  and letting  $k_3 \rightarrow 0$  (distinguishing the cases  $k_3 > 0$  and  $k_3 < 0$ ), we obtain the Feynman-Hellman formula:

$$(2.22) \quad E' = \frac{1}{m}(\Phi, (P_3 - d\Gamma(k_3) - A_{3,\sigma}(x', 0))\Phi).$$

Hence, by (2.10),

$$(2.23) \quad \begin{aligned} \left| (\Phi, (\tilde{H} - \tilde{E})\Phi) \right| &= \left| (\Phi, (\tilde{H} - H) - (\tilde{E} - E)\Phi) \right| \\ &\leq \left| -k_3 E' - (\tilde{E} - E) \right| + \frac{k_3^2}{2m}. \end{aligned}$$

The lemma then follows from (2.19) and the mean value theorem.  $\square$

We are now ready to prove Theorem 1.3:

PROOF OF THEOREM 1.3. Let us begin with estimating the term  $\|R_\sigma(\mathbf{k})\Phi_g^\sigma(P_3)\|$  appearing in Lemma 2.3. Recalling the notations (2.3), we write

$$(2.24) \quad \|R_\sigma(\mathbf{k})\Phi\| \leq \frac{C|g|}{|k_3||k|^{\frac{1}{2}}} \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k) \left\| \frac{(\tilde{H} - \tilde{E})^{1/2}}{\tilde{H} - E + |k|} \right\| \|(\tilde{H} - \tilde{E})^{1/2}\Phi\|.$$

It follows from the Spectral Theorem and (1.18) that

$$(2.25) \quad \left\| \frac{(\tilde{H} - \tilde{E})^{1/2}}{\tilde{H} - E + |k|} \right\| = \sup_{r \geq 0} \left| \frac{r^{\frac{1}{2}}}{r + \tilde{E} - E + |k|} \right| \leq \sup_{r \geq 0} \left| \frac{r^{\frac{1}{2}}}{r + |k|/4} \right| \leq \frac{C}{|k|^{\frac{1}{2}}}.$$

Thus, Theorem 1.2 together with Lemma 2.4 yield

$$(2.26) \quad \|R_\sigma(\mathbf{k})\Phi\| \leq \frac{C|g|}{|k_3|^{\frac{1}{2}-\frac{\gamma}{2}}|k|^{\frac{1}{2}}} \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k),$$

where  $\gamma = 1/4 - \delta$ , and where  $\delta$  in Theorem 1.2 is chosen such that  $0 < \delta < 1/4$ . Hence

$$(2.27) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|R_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq Cg^2.$$

Let us now prove (i). First assume that  $E'_g(P_3) = 0$ . In order to get the existence of a ground state for  $H_g(P_3)$  our aim is to prove that  $\Phi_g^\sigma(P_3)$  converges strongly as  $\sigma \rightarrow 0$ . Using Lemma A.7 (see also Remark A.8), we obtain from (1.17) that

$$(2.28) \quad |f_\sigma(\mathbf{k})| \leq C \left( \frac{g^2\sigma}{|k_3||k|^{\frac{3}{2}}} + \frac{|g|(E_g(P_3 - k_3) - E_g(P_3))}{|k_3||k|^{\frac{3}{2}}} \right) \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k).$$

Hence, since  $E'_g(P_3) = 0$  by assumption, (1.13) implies

$$(2.29) \quad |f_\sigma(\mathbf{k})| \leq C \left( \frac{g^2\sigma}{|k_3||k|^{3/2}} + \frac{|g|k_3^{\frac{1}{4}-\delta}}{|k|^{\frac{3}{2}}} \right) \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k).$$

Therefore

$$(2.30) \quad \|f_\sigma\|_{L^2(\mathbb{R}^3 \times \mathbb{Z}_2)} \leq C|g|.$$

Combining Lemma 2.3 with (2.30) and (2.27), we obtain

$$(2.31) \quad (\Phi_g^\sigma(P_3), \mathcal{N}\Phi_g^\sigma(P_3)) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|a(\mathbf{k})\Phi_g^\sigma(P_3)\|^2 d\mathbf{k} \leq Cg^2,$$

where  $\mathcal{N} = d\Gamma(I)$  denotes the number operator. For a sufficiently small fixed  $|g|$ , the strong convergence of  $\Phi_g^\sigma(P_3)$  as  $\sigma \rightarrow 0$  is then obtained by following for instance [BFS], showing that  $|(\Phi_g^\sigma(P_3), \Phi_{\text{el}} \otimes \Omega)| \geq C > 0$  uniformly in  $\sigma \geq 0$ . Here  $\Phi_{\text{el}}$  denotes a normalized ground state of  $h(b, V)$ .

Assume next that  $E'_g(P_3) \neq 0$  and let us prove that  $H_g(P_3)$  does not have a ground state. By Lemmata 2.2, 2.3 and Estimate (2.27), it suffices to prove that  $f \notin L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ . The latter follows from the fact that

$$(2.32) \quad \left| \frac{E_g(P_3 - k_3) - E_g(P_3)}{k_3} \right| \geq C > 0$$

uniformly for small  $k_3$  since  $E'_g(P_3) \neq 0$ . Hence Theorem 1.3(i) is proven.

Let us finally prove (ii). For  $\sigma > 0$ , we set

$$(2.33) \quad \Phi^{\text{ren}} = W(if_\sigma)\Phi_g^\sigma(P_3).$$

Obviously  $\Phi^{\text{ren}}$  is a normalized ground state of  $H_{g\sigma}^{\text{ren}}(P_3)$ . By Lemma 2.3 we have

$$\begin{aligned} a(\mathbf{k})\Phi^{\text{ren}} &= W(if_\sigma)a(\mathbf{k})\Phi + [a(\mathbf{k}), W(if_\sigma)]\Phi \\ &= W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}}f_\sigma(\mathbf{k})\Phi^{\text{ren}} + [a(\mathbf{k}), W(if_\sigma)]\Phi. \end{aligned}$$

One can compute the commutator  $[a(\mathbf{k}), W(if_\sigma)] = -2^{-1/2}f_\sigma(\mathbf{k})$ , so that

$$(2.34) \quad a(\mathbf{k})\Phi^{\text{ren}} = W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi.$$

Therefore, since  $W(if_\sigma)$  is unitary,  $\|a(\mathbf{k})\Phi^{\text{ren}}\|$  can be estimated in the same way as  $\|a(\mathbf{k})\Phi\|$  (in the case  $E'_g(P_3) = 0$ ), using (2.5) and (2.27). This leads to the existence of a ground state for  $H_g^{\text{ren}}(P_3)$  and concludes the proof of Theorem 1.3.  $\square$

### Appendix A. Uniform regularity of the map $P_3 \mapsto E_{g\sigma}(P_3)$

In this appendix we shall prove Theorem 1.2. The structure follows [Pi] and [CFP]: First, we give a simple proof of the existence of a spectral gap for the infrared cutoff Hamiltonian  $H_g^\sigma(P_3)$ , considered as an operator on the space of photons of energies  $\geq \sigma$ . Our proof is based on the min-max principle. Then we establish (1.13) by adapting [Pi, CFP] (see also [BFP]). In comparison to [CFP], the main technical difference comes from the terms in  $H_g(P_3)$  containing the interaction between the electronic variables  $x'_j$  and the quantized electromagnetic field. This shall be handled in Lemma A.11 below thanks to the exponential decay of  $\Phi_g^\sigma(P_3)$  in  $x'_j$ .

In some parts of our presentation, we shall only sketch the proof, emphasizing the differences that we have to include, and referring otherwise to [Pi], [BFP], or [CFP].

Let us begin with some definitions and notations. Henceforth we remove the subindex  $g$  to simplify the notations, and for  $\sigma \geq 0$ , we replace  $H^\sigma(P_3)$  by its Wick-ordered version  $H^\sigma(P_3) - \frac{g^2}{2m}(\Lambda^2 - \sigma^2)$  (which we still denote by  $H^\sigma(P_3)$ ). Note that this shall not affect our discussion below on the regularity of the ground state energy since the two operators only differ by a constant. We decompose

$$(A.1) \quad H^\sigma(P_3) = h_0(P_3) + H_I^\sigma(P_3),$$

where

$$(A.2) \quad h_0(P_3) = h(b, V) \otimes \mathbf{1} + \mathbf{1} \otimes \left[ \frac{1}{2m} \left( P_3 - d\Gamma(k_3) \right)^2 + H_f \right],$$

and

$$\begin{aligned} (A.3) \quad H_I^\sigma(P_3) &= -\frac{g}{m} \sum_{j=1,2} \left( A_{j,\sigma}(x', 0) \left( p_j - ea_j(x') \right) + \frac{g^2}{2m} A_{j,\sigma}(x', 0)^2 \right) \\ &\quad - \frac{g}{2m} A_{3,\sigma}(x', 0) \left( P_3 - d\Gamma(k_3) \right) - \frac{g}{2m} \left( P_3 - d\Gamma(k_3) \right) A_{3,\sigma}(x', 0) \\ &\quad + \frac{g^2}{2m} A_{3,\sigma}(x', 0)^2 - \frac{g}{2m} \sigma \cdot B_\sigma(x', 0) - \frac{g^2}{2m} (\Lambda^2 - \sigma^2). \end{aligned}$$

Let  $\Phi_{\text{el}}$  denote a normalized ground state of  $h(b, V)$ . For any  $|P_3| < m$ , one can easily check that  $\Phi_{\text{el}} \otimes \Omega$  is a ground state of  $h_0(P_3)$ , with ground state energy

$e_0(P_3) = e_0 + P_3^2/2m$ . Note that for  $\tau \leq \sigma$ , we have

$$\begin{aligned}
& H^\tau(P_3) - H^\sigma(P_3) \\
&= -\frac{g}{m} \sum_{j=1,2} A_{j,\tau}^\sigma(x', 0) \left( p_j - e a_j(x') - g A_{j,\sigma}(x', 0) \right) - \frac{g^2}{2m} (\sigma^2 - \tau^2) \\
&+ \frac{g^2}{2m} A_\tau^\sigma(x', 0)^2 - \frac{g}{2m} A_{3,\tau}^\sigma(x', 0) \left( P_3 - d\Gamma(k_3) - g A_{3,\sigma}(x', 0) \right) \\
&- \frac{g}{2m} \left( P_3 - d\Gamma(k_3) - g A_{3,\sigma}(x', 0) \right) A_{3,\tau}^\sigma(x', 0) - \frac{g}{2m} \sigma \cdot B_\tau^\sigma(x', 0),
\end{aligned} \tag{A.4}$$

where

$$A_\tau^\sigma(x', 0) = \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho_\tau^\sigma(k) \left[ e^{-ik' \cdot x'} a_\lambda^*(k) + e^{ik' \cdot x'} a_\lambda(k) \right] d\mathbf{k}, \tag{A.5}$$

and likewise for  $B_\tau^\sigma(x', 0)$ . Let  $\mathcal{H}_\sigma = L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{F}_\sigma$ , where  $\mathcal{F}_\sigma$  denotes the symmetric Fock space over  $L^2(\{\mathbf{k} \in \mathbb{R}^3 \times \mathbb{Z}_2, |k| \geq \sigma\})$ . The restriction of  $H^\sigma(P_3)$  to  $\mathcal{H}_\sigma$  is denoted by  $H_\sigma(P_3)$ :

$$H_\sigma(P_3) = H^\sigma(P_3)|_{\mathcal{H}_\sigma}, \tag{A.6}$$

and, similarly,

$$h_{0,\sigma}(P_3) = h_0(P_3)|_{\mathcal{H}_\sigma} \quad , \quad H_{I,\sigma}(P_3) = H_I^\sigma(P_3)|_{\mathcal{H}_\sigma}. \tag{A.7}$$

Let  $\Omega_\sigma$  be the vacuum in  $\mathcal{F}_\sigma$ . Then for  $|P_3| < m$ ,  $\Phi_{\text{el}} \otimes \Omega_\sigma$  is a ground state of  $h_{0,\sigma}(P_3)$  with ground state energy  $e_0(P_3)$ , and

$$\text{Gap}(h_{0,\sigma}(P_3)) \geq \left(1 - \frac{|P_3|}{m}\right) \sigma, \tag{A.8}$$

where  $\text{Gap}(H) = \inf(\sigma(H) \setminus \{E(H)\}) - \inf(\sigma(H))$  for any self-adjoint and semi-bounded operator  $H$  with ground state energy  $E(H)$ . We also define

$$H_\tau^\sigma(P_3) = (H^\tau(P_3) - H^\sigma(P_3))|_{\mathcal{H}_\tau}. \tag{A.9}$$

The symmetric Fock space over  $L^2(\{\mathbf{k} \in \mathbb{R}^3 \times \mathbb{Z}_2, \tau \leq |k| \leq \sigma\})$  is denoted by  $\mathcal{F}_\tau^\sigma$ . Note that there exists a unitary operator  $\mathcal{V} : \mathcal{H}_\tau \rightarrow \mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$ . We shall identify  $\mathcal{H}_\tau$  and  $\mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$  in the sequel in order to simplify the notations. We let  $\Omega_\tau^\sigma$  be the vacuum in  $\mathcal{F}_\tau^\sigma$ .

### A.1. Existence of a spectral gap.

LEMMA A.1. *There exist  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that the following holds: Let  $|g| \leq g_0$ ,  $0 \leq \sigma \leq \sigma_0$  and  $|P_3| \leq P_0$  be such that  $H_\sigma(P_3)$  has a normalized ground state  $\Phi_\sigma(P_3)$  and  $\text{Gap}(H_\sigma(P_3)) \geq \gamma\sigma$  for some  $\gamma > 0$ . Then for all  $0 \leq \tau \leq \sigma$ ,  $\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma$  is a normalized ground state of  $H^\sigma(P_3)|_{\mathcal{H}_\tau}$ , and*

$$\text{Gap}(H^\sigma(P_3)|_{\mathcal{H}_\tau}) \geq \min(\gamma\sigma, \tau/4). \tag{A.10}$$

PROOF. To simplify the notations, let us remove the dependence on  $P_3$  throughout the proof. First, one can readily check that  $\Phi_\sigma \otimes \Omega_\tau^\sigma$  is an eigenstate of  $H^\sigma|_{\mathcal{H}_\tau}$

associated with the eigenvalue  $E_\sigma$ . For any  $v$  we let  $[v]$  and  $[v]^\perp$  denote respectively the subspace spanned by  $v$  and its orthogonal complement. We write

$$\begin{aligned} & \inf_{\Phi \in [\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \\ & \geq \min \left( \inf_{\Phi \in [\Phi_\sigma]^\perp \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi), \inf_{\Phi \in \mathcal{H}_\sigma \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \right). \end{aligned}$$

The assumption  $\text{Gap}(H_\sigma) \geq \gamma\sigma$  implies

$$\inf_{\Phi \in [\Phi_\sigma]^\perp \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \geq E_\sigma + \gamma\sigma.$$

On the other hand, using that the number operator  $\int_{\tau \leq |k| \leq \sigma} a^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}$  commutes with  $H^\sigma |_{\mathcal{H}_\tau}$ , one can prove as in [P1] that

$$\inf_{\Phi \in \mathcal{H}_\sigma \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \geq \inf_{\tau \leq |k| \leq \sigma} (E_\sigma(P_3 - k_3) - E_\sigma(P_3) + |k|).$$

We conclude the proof thanks to (1.18)  $\square$

**COROLLARY A.2.** *Under the conditions of Lemma A.1, for all  $0 \leq \tau \leq \sigma$ ,*

$$(A.11) \quad E_\tau(P_3) \leq E_\sigma(P_3) \leq e_0(P_3).$$

**PROOF.** It follows from Lemma A.1 that

$$(A.12) \quad \begin{aligned} E_\tau(P_3) & \leq (\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma, H_\tau(P_3) \Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma) \\ & = (\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma, H^\sigma(P_3) |_{\mathcal{H}_\tau} \Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma) = E_\sigma(P_3). \end{aligned}$$

Hence the first inequality in (A.11) is proven. To prove the second one, it suffices to write similarly

$$(A.13) \quad \begin{aligned} E_\sigma(P_3) & \leq (\Phi_{\text{el}} \otimes \Omega_\sigma, H_\sigma(P_3) \Phi_{\text{el}} \otimes \Omega_\sigma) \\ & = (\Phi_{\text{el}} \otimes \Omega_\sigma, h_{0,\sigma}(P_3) \Phi_{\text{el}} \otimes \Omega_\sigma) = e_0(P_3). \end{aligned}$$

$\square$

We shall establish the existence of a spectral gap of order  $O(\sigma)$  above the bottom of the spectrum of  $H_\sigma(P_3)$  by induction. More precisely, let  $\mathbf{Gap}(\sigma)$  denote the assertion

$$\mathbf{Gap}(\sigma) \left\{ \begin{array}{l} \text{(i)} \quad E_\sigma(P_3) \text{ is a simple eigenvalue of } H_\sigma(P_3), \\ \text{(ii)} \quad \text{Gap}(H_\sigma(P_3)) \geq \sigma/8. \end{array} \right.$$

We shall prove

**PROPOSITION A.3.** *There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that, for all  $|g| \leq g_0$ ,  $0 < \sigma \leq \sigma_0$  and  $|P_3| \leq P_0$ , the assertion  $\mathbf{Gap}(\sigma)$  above holds.*

Let us begin with two preliminary useful estimates:

**LEMMA A.4.** *Fix the parameters  $g$ ,  $\sigma$  and  $P_3$  such that  $0 \leq |g| \leq g_0$ ,  $0 \leq \sigma \leq \sigma_0$  and  $0 \leq |P_3| \leq P_0$ , for some sufficiently small  $g_0$ ,  $\sigma_0$  and  $P_0$ . For any  $0 < \rho < 1$ ,*

$$(A.14) \quad \begin{aligned} & \left\| [h_{0,\sigma}(P_3) - e_0(P_3) + \rho]^{-1/2} H_{I,\sigma}(P_3) [h_{0,\sigma}(P_3) - e_0(P_3) + \rho]^{-1/2} \right\| \\ & \leq C|g|\rho^{-1/2}, \end{aligned}$$

where  $C$  is a positive constant (depending only on  $\Lambda$ ). Likewise,

$$(A.15) \quad \left\| [H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho]^{-1/2} H_\tau^\sigma(P_3) [H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho]^{-1/2} \right\| \leq C|g|\sigma^{1/2}\rho^{-1/2}.$$

PROOF. Let us prove (A.15), Estimate (A.14) would follow similarly. We introduce the expression of  $H_\tau^\sigma(P_3)$  given by (A.4) and (A.9) and estimate each term separately. Consider for instance

$$(A.16) \quad |g| \left\| \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \right. \\ \left. \left( P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x', 0) \right) \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \right\|.$$

Using that

$$(A.17) \quad \left\| \left( P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x', 0) \right) \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \right\| \leq C\rho^{-1/2},$$

we get

$$(A.16) \leq C|g|\rho^{-1/2} \left\| \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \right\|.$$

Moreover, for any  $\Phi \in D(H^\sigma(P_3)|_{\mathcal{H}_\tau})$ ,

$$\left\| \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Phi \right\|^2 \\ \leq \int_{\tau \leq |k|, |\tilde{k}| \leq \sigma} \frac{C}{|k|^{1/2} |\tilde{k}|^{1/2}} \left| \left( \Phi, a(\mathbf{k}) \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1} a^*(\tilde{\mathbf{k}}) \Phi \right) \right| d\mathbf{k} d\tilde{\mathbf{k}}.$$

Now, for any  $\mathbf{k}$  such that  $\tau \leq |k| \leq \sigma$ , we have the pull-through formula

$$(A.18) \quad a(\mathbf{k}) H^\sigma(P_3)|_{\mathcal{H}_\tau} = \left[ H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} + |k| \right] a(\mathbf{k}),$$

since  $a(\mathbf{k})$  commutes with  $A_\sigma(x', 0)$ . Hence

$$\left( \Phi, a(\mathbf{k}) \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1} a^*(\tilde{\mathbf{k}}) \Phi \right) \\ = \delta(\mathbf{k} - \tilde{\mathbf{k}}) \left( \Phi, \left[ H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho \right]^{-1} \Phi \right) \\ + \left( a(\tilde{\mathbf{k}}) \Phi, \left[ H^\sigma(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho \right]^{-1} a(\mathbf{k}) \Phi \right).$$

Using that  $H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| \geq |k|/4$  for any  $k$  sufficiently small (see (1.18)), we get

$$\left\| \left[ H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho \right]^{-1} \right\| \leq \frac{C}{|k|}.$$

Let  $H_{f,\tau}^\sigma = \int_{\tau \leq |k| \leq \sigma} |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}$ . As in [Pi, Lemma 1.1], it follows from the proof of Lemma A.1 that  $H_{f,\tau}^\sigma \leq C(H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3))$  for any  $P_3$  sufficiently

small. This yields

$$\left\| \left[ H_{f,\tau}^\sigma + |k| + |\tilde{k}| \left[ H^\sigma(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho \right]^{-1} \right\| \leq C.$$

Thus, combining the previous estimates we obtain

$$\begin{aligned} & \left\| \left[ H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Phi \right\|^2 \\ & \leq C \int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^2} + C \left[ \int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^{1/2}} \left\| [H_{f,\tau}^\sigma + |k|]^{-1/2} a(\mathbf{k}) \Phi \right\| \right]^2 \leq C\sigma. \end{aligned}$$

Since  $D(H^\sigma(P_3)|_{\mathcal{H}_\tau})$  is dense in  $\mathcal{H}_\tau$ , the result is proven as for the term we have chosen to consider, that is (A.16)  $\leq C|g|\sigma^{1/2}\rho^{-1/2}$ . Since the other terms in the expression of  $H_\tau^\sigma$  given by (A.4) can be treated in the same way, the lemma is established.  $\square$

The next lemma corresponds to the root in the induction procedure leading to the proof of Proposition A.3.

LEMMA A.5. *There exist  $g_0 > 0$ ,  $\sigma_0 > 0$ ,  $P_0 > 0$  and a positive constant  $C_0$  such that for all  $|g| \leq g_0$  and  $|P_3| \leq P_0$ , for all  $\sigma$  such that  $C_0 g^2 \leq \sigma \leq \sigma_0$ , the assertion **Gap**( $\sigma$ ) holds.*

PROOF. To simplify the notations, we write  $H_\sigma$  for  $H_\sigma(P_3)$ ,  $E_\sigma$  for  $E_\sigma(P_3)$ , and similarly for other quantities depending on  $P_3$ . Let  $\mu_\sigma$  denote the first point above  $E_\sigma$  in the spectrum of  $H_\sigma$ . By the min-max principle,

$$(A.19) \quad \mu_\sigma \geq \inf_{\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp, \|\psi\|=1} (\psi, H_\sigma \psi),$$

where  $[v]^\perp$  denotes the orthogonal complement of the vector space spanned by  $v$ . It follows from (A.14) that for any  $\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp$ ,  $\|\psi\| = 1$ , and any  $\rho > 0$ ,

$$(A.20) \quad \begin{aligned} (\psi, H_\sigma \psi) & \geq (\psi, H_{0,\sigma} \psi) - C|g|\rho^{-1/2} (\psi, [h_{0,\sigma} - e_0(P_3) + \rho] \psi) \\ & \geq \left(1 - C|g|\rho^{-1/2}\right) (\psi, H_{0,\sigma} \psi) + C|g|\rho^{-1/2} e_0(P_3) - C|g|\rho^{1/2}. \end{aligned}$$

By (A.8), for any  $\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp$ ,  $(\psi, h_{0,\sigma} \psi) \geq e_0(P_3) + (1 - |P_3|/m)\sigma$  provided that  $\sigma_0$  is chosen sufficiently small. Hence for any  $\rho$  such that  $\rho^{1/2} > C|g|$ ,

$$(A.21) \quad (\psi, H_\sigma \psi) \geq e_0(P_3) + \left(1 - C|g|\rho^{-1/2}\right) \left(1 - \frac{|P_3|}{m}\right) \sigma - C|g|\rho^{1/2}.$$

Choosing  $\rho^{1/2} = 4C|g|$  and  $P_0$  sufficiently small, by Corollary A.2, we obtain

$$(A.22) \quad \begin{aligned} (\psi, H_\sigma \psi) & \geq E_\sigma + \frac{3}{4} \left(1 - \frac{|P_3|}{m}\right) \sigma - 4C^2 g^2 \\ & \geq E_\sigma + \frac{1}{2} \sigma - 4C^2 g^2. \end{aligned}$$

Together with (A.19), this leads to the statement of the lemma provided that the constant  $C_0$  is chosen such that  $C_0 > 32C^2/3$ .  $\square$

The following lemma corresponds to the induction step of the induction process in the proof of Proposition A.3.



LEMMA A.6. *There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that for all  $|g| \leq g_0$  and  $|P_3| \leq P_0$ , for all  $\sigma$  such that  $0 < \sigma \leq \sigma_0$ ,*

$$\mathbf{Gap}(\sigma) \Rightarrow \mathbf{Gap}(\sigma/2).$$

PROOF. Again, throughout the proof, we drop the dependence on  $P_3$  in all the considered quantities. Let  $\mathbf{Gap}(\sigma)$  be satisfied for some  $0 < \sigma$ , let  $\Phi_\sigma$  be a ground state of  $H_\sigma$ , and let  $\tau = \sigma/2$ . As in the proof of Lemma A.5, let  $\mu_\tau$  denote the first point above  $E_\tau$  in the spectrum of  $H_\tau$ . By the min-max principle,

$$(A.23) \quad \mu_\tau \geq \inf_{\psi \in [\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp, \|\psi\|=1} (\psi, H_\tau \psi),$$

where  $\Omega_\tau^\sigma$  is the vacuum in  $\mathcal{F}_\tau^\sigma$  and where  $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$  denotes the orthogonal complement of the vector space spanned by  $\Phi_\sigma \otimes \Omega_\tau^\sigma$  in  $\mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$ . It follows from (A.15) that for any  $\rho > 0$ ,

$$\begin{aligned} (\psi, H_\tau \psi) &\geq (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) + (\psi, H_\tau^\sigma \psi) \\ &\geq \left[1 - C|g|\sigma^{1/2}\rho^{-1/2}\right] (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) + C|g|\sigma^{1/2}\rho^{-1/2}E_\sigma - C|g|\sigma^{1/2}\rho^{1/2}. \end{aligned}$$

Next, from  $\mathbf{Gap}(\sigma)$  and Property (A.10), since  $\tau = \sigma/2$ , we obtain that for any  $\psi$  in  $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$ ,  $\|\psi\| = 1$ ,

$$(A.24) \quad (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) \geq E_\sigma + \min\left(\frac{\sigma}{8}, \frac{\tau}{4}\right) \geq E_\sigma + \sigma/8,$$

provided that  $|g|$  is sufficiently small. Hence for any  $\rho > 0$  such that  $\rho^{1/2} > C|g|\sigma^{1/2}$ ,

$$(A.25) \quad (\psi, H_\tau \psi) \geq E_\sigma + \left[1 - C|g|\sigma^{1/2}\rho^{-1/2}\right] \frac{\sigma}{8} - C|g|\sigma^{1/2}\rho^{1/2}.$$

Choosing  $\rho^{1/2} = 4C|g|\sigma^{1/2}$ , by Corollary A.2, we get

$$(A.26) \quad (\psi, H_\tau \psi) \geq E_\sigma + \frac{3}{32}\sigma - 4C^2g^2\sigma \geq E_\tau + \frac{3}{16}\tau - 8C^2g^2\tau.$$

Hence  $\mu_\tau \geq E_\tau + \tau/8$  provided that  $|g| \leq (8C)^{-1}$ , which proves the lemma.  $\square$

PROOF OF PROPOSITION A.3 As mentioned above, Proposition A.3 easily follows from Lemmata A.5 and A.6, and an induction argument.  $\square$

Let us conclude this Subsection with a bound on the difference  $|E_\tau - E_\sigma|$ .

LEMMA A.7. *Under the conditions of Proposition A.3, there exists a positive constant  $C$  such that for all  $0 \leq \tau \leq \sigma \leq \sigma_0$ ,*

$$(A.27) \quad |E_\tau(P_3) - E_\sigma(P_3)| \leq C|g|\sigma.$$

PROOF. By Corollary A.2, we already have  $E_\tau(P_3) \leq E_\sigma(P_3)$ . The inequality  $E_\sigma(P_3) \leq E_\tau(P_3) + C|g|\sigma$  follows similarly, using (A.15) and a variational argument.  $\square$

REMARK A.8. Lemma A.7 remains true if the operators under consideration are not Wick-ordered. More precisely in this case we have

$$(A.28) \quad E_\tau(P_3) \leq E_\sigma(P_3) + Cg^2\sigma \leq E_\tau(P_3) + C|g|\sigma.$$

**A.2. Proof of Theorem 1.2.** The key property used in the proof of Theorem 1.2 lies in the estimate of  $|E'_\tau(P_3) - E'_\sigma(P_3)|$  for  $\tau \leq \sigma$ .

PROPOSITION A.9. *There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that for all  $0 < |g| \leq g_0$  and  $|P_3| \leq P_0$ , for all  $\sigma, \tau > 0$  such that  $\tau \leq \sigma \leq \sigma_0$ , for all  $\delta > 0$ ,*

$$|E'_\tau(P_3) - E'_\sigma(P_3)| \leq C_\delta \sigma^{1/2-\delta},$$

where  $C_\delta$  is a positive constant depending only on  $\delta$ .

We shall divide the main part of the proof of Proposition A.9 into two lemmata. Let us begin with some definitions and notations. For  $\sigma > 0$  and  $\rho \geq 0$ , we define the function  $g_{\sigma,\rho} \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$  by

$$g_{\sigma,\rho}(\mathbf{k}) = g \mathbf{1}_{\sigma \leq |\mathbf{k}| \leq \Lambda}(k) \frac{\epsilon_\lambda^3(k)}{\sqrt{2\pi} |k|^{1/2}} \frac{\rho}{|k| - k_3 \rho}.$$

Depending on the context, the Weyl operator  $W(ig_{\sigma,\rho})$  will represent an operator on  $\mathcal{H}_\sigma$ ,  $\mathcal{H}_\tau$  (for  $\tau \leq \sigma$ ), or  $\mathcal{H}$ .

From now on, to simplify the notations, we drop the dependence on  $P_3$  everywhere unless a confusion may arise. For  $g$ ,  $\sigma$  and  $P_3$  as in Proposition A.3, let  $\Phi_\sigma$  denote a normalized ground state of  $H_\sigma$ . Define

$$H_{\sigma,\rho}^{\text{ren}} = W(ig_{\sigma,\rho}) H_\sigma W(ig_{\sigma,\rho})^*, \quad \Phi_{\sigma,\rho}^{\text{ren}} = W(ig_{\sigma,\rho}) \Phi_\sigma,$$

and let  $P_{\sigma,\rho}^{\text{ren}}$  be the orthogonal projection onto the vector space spanned by  $\Phi_{\sigma,\rho}^{\text{ren}}$ . Note that  $\Phi_{\sigma,\rho}^{\text{ren}}$  is a normalized, non-degenerate ground state of  $H_{\sigma,\rho}^{\text{ren}}$ , associated with the ground state energy  $E_\sigma$ . Recall that, by Lemma A.1,  $[\Phi_\sigma \otimes \Omega_\tau^\sigma]$  is a ground state of  $H^\sigma|_{\mathcal{H}_\tau}$ . We set

$$H_{\sigma,\rho,\tau}^{\text{ren}} = W(ig_{\sigma,\rho}) H^\sigma|_{\mathcal{H}_\tau} W(ig_{\sigma,\rho})^*, \quad \Phi_{\sigma,\rho,\tau}^{\text{ren}} = W(ig_{\sigma,\rho}) [\Phi_\sigma \otimes \Omega_\tau^\sigma],$$

and the projection onto the vector space spanned by  $\Phi_{\sigma,\rho,\tau}^{\text{ren}}$  is denoted by  $P_{\sigma,\rho,\tau}^{\text{ren}}$ . Since  $W(ig_{\sigma,\rho}) = e^{i\Phi(ig_{\sigma,\rho}) \otimes \mathbf{1}}$ , it can be seen that  $\Phi_{\sigma,\rho,\tau}^{\text{ren}} = [W(ig_{\sigma,\rho}) \Phi_\sigma] \otimes \Omega_\tau^\sigma = \Phi_{\sigma,\rho}^{\text{ren}} \otimes \Omega_\tau^\sigma$ .

LEMMA A.10. *There exists  $g_0 > 0$ ,  $\sigma_0 > 0$  and  $P_0 > 0$  such that for all  $0 < |g| \leq g_0$  and  $|P_3| \leq P_0$ , for all  $\sigma, \tau > 0$  such that  $\tau \leq \sigma \leq \sigma_0$ ,*

$$(A.29) \quad |E'_\sigma - E'_\tau| \leq C \left[ \left\| P_{\sigma,E'_\sigma,\tau}^{\text{ren}} - P_{\tau,E'_\sigma}^{\text{ren}} \right\| + g^2 \sigma \right],$$

where  $C$  is a positive constant.

PROOF. By the Feynman-Hellman formula (see (2.22)),

$$(A.30) \quad E'_\sigma = \frac{1}{m} (\Phi_\sigma, [P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)] \Phi_\sigma)_{\mathcal{H}_\sigma}.$$

It follows from (A.30) and commutation relations with  $W(ig_{\sigma,E'_\sigma})$  that

$$(A.31) \quad E'_\sigma = \frac{1}{m} \left( \Phi_{\sigma,E'_\sigma}^{\text{ren}}, \left[ P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma,E'_\sigma}) - \frac{1}{2} (k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3,\sigma}(x', 0) + g\text{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma}) \right] \Phi_{\sigma,E'_\sigma}^{\text{ren}} \right)_{\mathcal{H}_\sigma},$$

Consequently, for  $\tau \leq \sigma$ , we can write

$$(A.32) \quad E'_\sigma = \frac{1}{m} \left( \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}, \left[ P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + g\text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma}) \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right)_{\mathcal{H}_\tau},$$

whereas

$$(A.33) \quad E'_\tau = \frac{1}{m} \left( \Phi_{\tau, E'_\sigma}^{\text{ren}}, \left[ P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + g\text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}) \right] \Phi_{\tau, E'_\sigma}^{\text{ren}} \right)_{\mathcal{H}_\tau}.$$

The expression into brackets being uniformly bounded with respect to  $H_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ , one can prove that

$$(A.34) \quad \left\| \left[ P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + \text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma}) \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right\| \leq C,$$

and likewise with  $\Phi_{\tau, E'_\sigma}^{\text{ren}}$  replacing  $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ . In addition, we have

$$(A.35) \quad |(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) - (k_3 g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma})| \leq Cg^2\sigma,$$

and, similarly,

$$(A.36) \quad \left\| [\text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}) - \text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma})] \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\| \leq C|g|\sigma.$$

Estimating the difference of (A.32) and (A.33) then leads to

$$(A.37) \quad |E'_\sigma - E'_\tau| \leq C \left[ \left\| \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} - \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\|_{\mathcal{H}_\tau} + g^2\sigma \right]$$

The statement of the lemma now follows by choosing the non-degenerate ground states  $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$  and  $\Phi_{\tau, E'_\sigma}^{\text{ren}}$  in such a way that

$$(A.38) \quad \left\| \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} - \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\|_{\mathcal{H}_\tau} \leq C \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\|.$$

Note that this choice is indeed possible due to the non-degeneracy of the ground states  $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$  and  $\Phi_{\tau, E'_\sigma}^{\text{ren}}$ .  $\square$

For  $g, P_3, \sigma, \rho$  as above, let us define the operator  $\nabla H_{\tau, \rho}^{\text{ren}}$  by

$$\nabla H_{\sigma, \rho}^{\text{ren}} = \frac{1}{m} W(ig_{\sigma, \rho}) [P_3 - d\Gamma(k_3) - gA_{3, \sigma}(x', 0)] W(ig_{\sigma, \rho})^* \\ = \frac{1}{m} \left[ P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma, \rho}) - \frac{1}{2}(k_3 g_{\sigma, \rho}, g_{\sigma, \rho}) \right. \\ \left. - gA_{3, \sigma}(x', 0) + g\text{Re}(h_{3, \sigma}(x'), g_{\sigma, \rho}) \right].$$

LEMMA A.11. *Let  $\Gamma_{\sigma,\mu}$  be the curve  $\Gamma_{\sigma,\mu} = \{\mu\sigma e^{i\nu}, \nu \in [0, 2\pi[ \}$ . There exist  $g_0 > 0$ ,  $\sigma_0 > 0$ ,  $\mu > 0$  and  $P_0 > 0$ , such that for all  $0 < |g| \leq g_0$ ,  $|P_3| \leq P_0$ , for all  $\sigma > 0$  and  $\tau > 0$  such that  $\sigma/2 \leq \tau \leq \sigma \leq \sigma_0$ ,*

$$(A.39) \quad \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| \leq C|g|^{1/2} \sigma^{1/2} \sup_{z \in \Gamma_{\sigma,\mu}} \left[ 1 + \left| \left( \nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}}, \right. \right. \\ \left. \left. \left[ H_{\sigma, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1} \left( \nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}} \right|^{1/2} \right],$$

where  $C$  is a positive constant.

PROOF. By [BFP, Lemma II.11],

$$(A.40) \quad \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| = \left| \left( \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} [P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}}] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right) \right|^{1/2}.$$

It follows from Lemma A.1 and Proposition A.3 that  $\text{Gap}(H_{\sigma, E'_\sigma, \tau}^{\text{ren}}) \geq \sigma/8$  and  $\text{Gap}(H_{\tau, E'_\sigma}^{\text{ren}}) \geq \tau/8 \geq \sigma/16$ . Therefore, since  $|E_\sigma - E_\tau| \leq C|g|\sigma$  by Lemma A.7, we can write

$$P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} = \frac{i}{2\pi} \oint_{\Gamma_{\sigma,\mu}} \left( \left[ H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} - \left[ H_{\tau, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1} \right) dz,$$

provided  $\mu < 1/16$  and  $|g|$  is sufficiently small. Expanding  $\left[ H_{\tau, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1}$  into a (convergent) Neumann series yields

$$P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} = \frac{i}{2\pi} \sum_{n \geq 1} \oint_{\Gamma_{\sigma,\mu}} (-1)^n \left[ H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \\ \left( \left[ H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right] \left[ H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \right)^n dz.$$

Let us compute the difference  $H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}}$  explicitly. We have:

$$H_{\sigma, E'_\sigma, \tau}^{\text{ren}} = \frac{1}{2m} \sum_{j=1,2} \left( p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), g_{\sigma, E'_\sigma}) \right)^2 \\ + \frac{m}{2} (\nabla H_{\sigma, E'_\sigma}^{\text{ren}})^2 - \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left( B_\sigma(x', 0) - \text{Re}(\tilde{h}_\sigma(x'), g_{\sigma, E'_\sigma}) \right) \\ + V(x') + H_f + \Phi(|k|g_{\sigma, E'_\sigma}) + \frac{1}{2} (|k|g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) - \frac{g^2}{2m} (\Lambda^2 - \sigma^2),$$

and

$$H_{\tau, E'_\sigma}^{\text{ren}} = \frac{1}{2m} \sum_{j=1,2} \left( p_j - ea_j(x') - gA_{j,\tau}(x', 0) + g\text{Re}(h_{j,\tau}(x'), g_{\tau, E'_\sigma}) \right)^2 \\ + \frac{m}{2} (\nabla H_{\tau, E'_\sigma}^{\text{ren}})^2 - \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left( B_\sigma(x', 0) - \text{Re}(\tilde{h}_\tau(x'), g_{\tau, E'_\sigma}) \right) \\ + V(x') + H_f + \Phi(|k|g_{\tau, E'_\sigma}) + \frac{1}{2} (|k|g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma}) - \frac{g^2}{2m} (\Lambda^2 - \tau^2).$$

Let us decompose:

$$(A.41) \quad H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}} = [a] + [b] + [c] + [d] + [e],$$

with

$$\begin{aligned}
[a] &= \frac{1}{m} \sum_{j=1,2} \left( -gA_{j,\tau}^\sigma(0,0) + g\operatorname{Re}(h_{j,\tau}(0), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left( p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\operatorname{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right), \\
[b] &= \frac{1}{2m} \sum_{j=1,2} \left( -gA_{j,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{j,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right)^2 - \frac{g^2}{2m} (\sigma^2 - \tau^2) \\
&\quad + \frac{1}{2m} \left( -\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right)^2, \\
&\quad + \frac{g}{2m} \sigma \cdot \operatorname{Re} \left( \tilde{h}_\tau(x') - \tilde{h}_\sigma(x'), g_{\tau,E'_\sigma} \right) \\
[c] &= \frac{1}{m} \sum_{j=1,2} \left( -g(A_{j,\tau}^\sigma(x',0) - A_{j,\tau}^\sigma(0)) + g\operatorname{Re}(h_{j,\tau}(x') - h_{j,\tau}(0), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left( p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\operatorname{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right) \\
&\quad - gE'_\sigma [A_{3,\tau}^\sigma(x',0) - A_{3,\tau}^\sigma(0,0)] + gE'_\sigma \operatorname{Re}(h_{3,\tau}(x') - h_{3,\tau}(0), g_{\tau,E'_\sigma}^\sigma), \\
[d] &= gE'_\sigma (h_{3,\tau}(0), g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2} E'_\sigma (k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma), \\
[e] &= \frac{1}{2} \left( -\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left( \nabla H_{\sigma,E'_\sigma}^{\operatorname{ren}} - E'_\sigma \right) + \frac{1}{2} \left( \nabla H_{\sigma,E'_\sigma}^{\operatorname{ren}} - E'_\sigma \right) \\
&\quad \times \left( -\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right).
\end{aligned}$$

Note that we have added and subtracted  $E'_\sigma$ , using the identity  $(E'_\sigma k_3 - |k|)g_{\sigma,E'_\sigma} = -gE'_\sigma h_{3,\sigma}(0)$  and likewise with  $g_{\tau,E'_\sigma}$  replacing  $g_{\sigma,E'_\sigma}$ . Let us now consider, for some  $n \geq 1$ ,

$$\begin{aligned}
(A.42) \quad & \oint_{\Gamma_{\sigma,\mu}} \left( \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}}, [H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z]^{-1} \right. \\
& \left. \left( [H_{\tau,E'_\sigma}^{\operatorname{ren}} - H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}}] [H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z]^{-1} \right)^n \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \right).
\end{aligned}$$

We insert (A.41) into the right-hand side of (A.42), thus obtaining a sum of terms that we estimate separately. We claim that all the terms where at least one of the operators  $[a]$ ,  $[b]$ , or  $[c]$  appear, are bounded by  $C\sigma(C'|g|)^n$  where  $C, C'$  are two positive constants. The latter can be proven by means of rather standard estimates involving pull-through formulas (see for instance [**BFS**, **Pi**, **BFP**, **CFP**]), so we shall not give all the details. Let us still emphasize that in order to deal with  $[a]$  or  $[c]$  we need to use the exponential decay of  $\Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}}$  in  $x'$  (proven in [**AGG2**, Appendix A]). This is the main difficulty we encounter compared to the proof of [**CFP**]. In order to overcome it, we adapt a method due to [**Si**] (see also [**AFFS**, Section 5]). Let us give an example: Consider

$$(A.43) \quad \left( \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}}, [e] [H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z]^{-1} [a] [H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z]^{-1} [e] \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \right).$$

We shall take advantage of the identity

$$(A.44) \quad \left( p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right) = 2i \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j \right]$$

which holds in the sense of quadratic forms on  $D(H_{\sigma,E'_\sigma,\tau}^{\text{ren}}) \cap D(x'_j)$ . The field operator  $A_{j,\sigma}^\tau(0,0) = \Phi(h_{j,\sigma}^\tau)$  in  $[a]$  decompose into a sum of a creation operator and an annihilation operator that are estimated separately. Take for instance the creation operator. Using a pull-through formula, we have to bound:

$$(A.45) \quad g \int h_{j,\tau}^\sigma(\mathbf{k}) \left( \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}[e] a^*(\mathbf{k}) \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - z \right]^{-1} \right. \\ \left. \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j \right] \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} [e] \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) d\mathbf{k}.$$

Let  $\gamma > 0$  be such that  $\|e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}\| < \infty$ . Undoing the commutator  $[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j]$  gives two terms. We write the first one under the form

$$g \int h_{j,\tau}^\sigma(\mathbf{k}) \left( \left( H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma \right) \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - \bar{z} \right]^{-1} a(\mathbf{k}) [e]^* \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}, \right. \\ \left. x'_j e^{-\gamma\langle x' \rangle} e^{\gamma\langle x' \rangle} \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} e^{-\gamma\langle x' \rangle} [e] e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) d\mathbf{k}.$$

Now we have the following estimates:

$$(A.46) \quad \left\| e^{\gamma\langle x' \rangle} \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} e^{-\gamma\langle x' \rangle} [e] e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right\| \leq C|g|,$$

$$(A.47) \quad \left\| x'_j e^{-\gamma\langle x' \rangle} \right\| \leq C,$$

$$(A.48) \quad \left\| \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - z \right]^{-1} \left( H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma \right) \right\| \leq C,$$

$$(A.49) \quad \left\| a(\mathbf{k}) [e]^* \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right\| \leq C|g| |k|^{-1/2}.$$

Note that in (A.48) and (A.49), we used that  $\tau \leq |k| \leq \sigma$ , and thus in particular that  $a(\mathbf{k}) \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} = 0$ . Since the other term coming from the commutator  $[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j]$  can be estimated in the same way, this yields

$$(A.50) \quad |(A.45)| \leq C|g|^3 \int |h_{j,\tau}^\sigma(\mathbf{k})| |k|^{-1/2} d\mathbf{k} \leq C|g|^3 \sigma^2.$$

Taking into account the factor  $\sigma$  coming from the integration in (A.42) would finally lead to our claim in the case of the example (A.43). The same holds for the terms containing  $[c]$  at least once (except that the use of (A.44) is then not required). Besides, since  $[d]$  is constant,

$$\oint_{\Gamma_{\sigma,\mu}} \left( \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \left( [d] \left[ H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \right)^n \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) = 0.$$

Therefore it remains to consider the terms containing only  $[d]$  or  $[e]$ , with  $[e]$  appearing at least in one factor. One can prove that this leads to

$$\begin{aligned} \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| &\leq C |g|^{1/2} \sigma^{1/2} \sup_{z \in \Gamma_{\sigma, \mu}} \left[ 1 + \sigma^{-1} \left\| H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right\|^{-1/2} \right. \\ &\quad \left. \left( -\Phi(k_3 g_{\tau, E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau, E'_\sigma}^\sigma, g_{\tau, E'_\sigma}^\sigma) - g A_{3, \tau}^\sigma(x', 0) + g \text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}^\sigma) \right) \right. \\ &\quad \left. \left[ \nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right\| \Bigg|. \end{aligned}$$

Using again the exponential decay of  $\Phi_{\sigma, E'_\sigma}^{\text{ren}}$  in  $x'$ , we may replace  $\text{Re}(f_{3, \tau}(x'), g_{\tau, E'_\sigma}^\sigma)$  with  $\text{Re}(f_{3, \tau}(0), g_{\tau, E'_\sigma}^\sigma)$  in the previous expression. Proceeding then as in [CFP, Lemma A.3], since both

$$(\Phi(k_3 g_{\tau, E'_\sigma}^\sigma) + g A_{3, \tau}^\sigma(x', 0))(\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \quad \text{and} \quad (\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$$

are orthogonal to  $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ , we obtain Inequality (A.39) (notice in particular that  $\sigma_0$  and  $\mu$  must be fixed sufficiently small to pass from the last estimate to (A.39)).  $\square$

PROOF OF PROPOSITION A.9 To conclude the proof of Proposition A.9, in view of Lemmata A.10 and A.11, it suffices to show that

$$\left| \left( (\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma}^{\text{ren}}, [H_{\sigma, E'_\sigma}^{\text{ren}} - E_\sigma - z]^{-1} (\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma}^{\text{ren}} \right) \right| \leq \frac{C_\delta}{|g| \sigma^{2\delta}},$$

for any  $z \in \Gamma_{\sigma, \mu}$  and any  $\delta > 0$ . This corresponds to the bound (IV.68) in [CFP] and can be proven in the same way as in [CFP, Subsection IV.5, step (4)], using an induction procedure. We therefore refer the reader to [CFP] for a proof.  $\square$

PROOF OF THEOREM 1.2 Fix  $P_3$  and  $k_3$  such that  $|P_3| \leq P_0$ ,  $|P_3 + k_3| \leq P_0$ . One can see that there exist positive constants  $C_0$  and  $C$  such that, for any  $0 < \beta < 1$  and  $\sigma \geq C_0 |k_3|^\beta$ ,

$$(A.51) \quad |E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \leq C |k_3|^{\frac{1}{2}(1-\beta)}.$$

This can be proven by estimating  $|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)|$  in terms of  $\|\Phi_\sigma(P_3 + k_3) - \Phi_\sigma(P_3)\|$ , then using the second resolvent equation to estimate  $\|[H_\sigma(P_3 + k_3) - z]^{-1} - [H_\sigma(P_3) - z]^{-1}\|$ . Now, for  $\sigma \leq C_0 |k_3|^\beta$ , we use Proposition A.9, which yields

$$\begin{aligned} &|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \\ &\leq \left| E'_\sigma(P_3 + k_3) - E'_{C_0 |k_3|^\beta}(P_3 + k_3) \right| + \left| E'_{C_0 |k_3|^\beta}(P_3 + k_3) - E'_{C_0 |k_3|^\beta}(P_3) \right| \\ &\quad + \left| E'_\sigma(P_3) - E'_{C_0 |k_3|^\beta}(P_3) \right| \\ &\leq C_\delta \left[ |k_3|^{\frac{1}{2}(1-\beta)} + |k_3|^{\frac{1}{2}\beta(1-\delta)} \right]. \end{aligned}$$

The theorem follows by choosing  $\beta = [2 - \delta]^{-1}$ .  $\square$

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