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Exponential decay for solutions to semilinear damped wave equation

Stéphane Gerbi* and Belkacem Said-Houari†

Abstract

This paper is concerned with decay estimate of solutions to the semilinear wave equation with strong damping in a bounded domain. Introducing an appropriate Lyapunov function, we prove that when the damping is linear, we can find initial data, for which the solution decays exponentially. This result improves an early one in [3].

Keywords: strong damping, stable set, global existence, decay rate source, positive initial energy.

AMS Classification: 35L70, 35B40

1 Introduction

In this paper we are concerned by the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega, t > 0 \\ u(x, t) = 0, & x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & x \in \Omega \end{cases} \quad (1.1)$$

in a bounded regular domain $\Omega \subset \mathbb{R}^n$. Here $p \geq 2$ and ω, μ are positive constants. The present problem has been studied by Gazzola and Squassina [3]. In their work, the authors proved some results on the well-posedness and investigate the asymptotic behavior of solutions of problem (1.1). In particular, they showed the global existence and the polynomial decay property of solutions provided that the initial data is in the potential well, [3, Theorem 3.8]. The proof in [3] is based on a method used in [7] and [6]. In these works, the authors obtain the following differential inequality:

$$\frac{d}{dt} [(1+t)E(t)] \leq E(t) \quad ,$$

where E is the energy of the solution. Unfortunately they obtain a decay rate which is not optimal.

The nonlinear wave equations related to (1.1) has been investigated by many authors [1, 2, 5, 7, 13, 14, 20, 21].

In the absence of the nonlinear source term, it is well known that the presence of one damping term i.e. $\omega > 0$ or $\mu > 0$ ensures global existence and decay of solutions for arbitrary initial data (see [5, 10]). For $\omega = \mu = 0$, the nonlinear term $u|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy (see [1, 9]).

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The interaction between the damping and the source terms was first considered by Levine [11, 12]. He showed that solutions with negative initial energy blows up in finite time. When $\omega = 0$ and the linear term u_t is replaced by $u_t |u_t|^{m-2}$, Georgiev and Todorova [4] extended Levine's result to the case where $m > 2$. In their work, the authors introduced a method different from the one known as the concavity method. They determined suitable relations between m and p , for which there is global existence or alternatively finite time blow-up. Precisely, they showed that the solution continues to exist globally "in time" if $m \geq p$ and blows up in finite time if $p > m$ and the initial energy is sufficiently negative. Vitillaro [19] extended the results in [4] to situations where the damping is nonlinear and the solution has positive initial energy. Similar results have been also established by Todorova [17, 18], for different Cauchy problems.

We recall here that the potential well method introduced by Payne and Sattinger [16] is also useful and widely used in the literature to investigate the local existence, global existence and asymptotic behavior of the solutions to some problems related to problem (1.1) (see [18, 19, 2, 14, 6, 7, 20]). Introducing a strong damping term Δu_t makes the problem different from the one considered in [4]. For this reason less results are, at the present time, known for the wave equation with strong damping and many problems remain unsolved (see [3]).

The purpose of this paper is to obtain a better decay estimate of solutions to the problem (1.1). More precisely we show that we can always find initial data in the stable set for which the solution of problem (1.1) decays exponentially. The key tool in the proof is an idea of Zuazua [21], which is based on the construction of a suitable Lyapunov function.

2 Asymptotic stability

In this section we introduce and prove our main result. For this purpose let us introduce the definition of the solution of problem (1.1) given by Gazzola and Squassina in [3].

Definition 2.1 For $T > 0$, we denote

$$Y_T = \left\{ \begin{array}{l} u \in C^0([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-1}(\Omega)) \\ u_t \in L^2([0, T], L^2(\Omega)) \end{array} \right\}$$

Given $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, a function $u \in Y_T$ is a local solution to (1.1), if $u(0) = u_0$, $u_t(0) = u_1$ and

$$\int_{\Omega} u_{tt} \phi dx + \int_{\Omega} \nabla u \nabla \phi dx + \omega \int_{\Omega} \nabla u_t \nabla \phi dx + \mu \int_{\Omega} u_t \phi dx = \int_{\Omega} |u|^{p-2} u \phi dx,$$

for any function $\phi \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$.

Let us first define the Sobolev critical exponent \bar{p} as:

$$\bar{p} = \begin{cases} \frac{2N}{N-2}, & \text{for } \omega > 0 \text{ and } N \geq 3 \\ \frac{2N-2}{N-2}, & \text{for } \omega = 0 \text{ and } N \geq 3 \end{cases} \quad \text{and } \bar{p} = \infty, \text{ if } N = 1, 2.$$

We first state a local existence theorem whose proof is written by Gazzola and Squassina, [3, Theorem 3.1].

Theorem 2.1 Assume $2 < p \leq \bar{p}$. Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exist $T > 0$ and a unique solution of (1.1) over $[0, T]$ in the sense of definition 2.1.

Definition 2.2 Let $2 \leq p \leq \bar{p}$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We denote u the solution of (1.1). We define:

$$T_{max} = \sup\{T > 0, u = u(t) \text{ exists on } [0, T]\}$$

Since the solution $u \in Y_T$ (the solution is “enough regular”), let us recall that if $T_{max} < \infty$, then

$$\lim_{\substack{t \rightarrow T_{max} \\ t < T_{max}}} \|\nabla u\|_2 + \|u_t\|_2 = +\infty \quad .$$

If $T_{max} < \infty$, we say that the solution of (1.1) blows up and that T_{max} is the blow up time.

If $T_{max} = \infty$, we say that the solution of (1.1) is global.

In order to study the blow up phenomenon or the global existence of the solution of (1.1), we define the following functions:

$$I(t) = I(u(t)) = \|\nabla u\|_2^2 - \|u\|_p^p, \quad (2.1)$$

$$J(t) = J(u(t)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \quad (2.2)$$

and

$$E(u(t)) = E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 \quad (2.3)$$

Let us remark that multiplying (1.1) by u_t , integrating over Ω and using integration by parts we obtain:

$$\frac{dE(t)}{dt} = -\omega\|\nabla u_t\|_2^2 - \mu\|u_t\|_2^2, \quad \forall t \geq 0. \quad (2.4)$$

Thus the function E is decreasing along the trajectories. As in [15], the potential well depth is defined as:

$$d = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{\lambda \geq 0} J(\lambda u). \quad (2.5)$$

We can now define the so called “Nehari manifold” as follows:

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\}; I(t) = 0\}.$$

\mathcal{N} separates the two unbounded sets:

$$\mathcal{N}^+ = \{u \in H_0^1(\Omega); I(t) > 0\} \cup \{0\} \quad \text{and} \quad \mathcal{N}^- = \{u \in H_0^1(\Omega); I(t) < 0\}.$$

The *stable* set \mathcal{W} and *unstable* set \mathcal{U} are defined respectively as:

$$\mathcal{W} = \{u \in H_0^1(\Omega); J(t) \leq d\} \cap \mathcal{N}^+ \quad \text{and} \quad \mathcal{U} = \{u \in H_0^1(\Omega); J(t) \leq d\} \cap \mathcal{N}^-.$$

It is readily seen that the potential depth d is also characterized by

$$d = \min_{u \in \mathcal{N}} J(u).$$

As it was remarked by Gazzola and Squassina in [3], this alternative characterization of d shows that

$$\beta = \text{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|\nabla u\|_2 = \sqrt{\frac{2dp}{p-2}} > 0 \quad . \quad (2.6)$$

In the lemma 2.1, we would like to prove the invariance of the set \mathcal{N}^+ : if the initial data u_0 is in the set \mathcal{N}^+ and if the initial energy $E(0)$ is not large (we will precise exactly how large may be the initial energy), then $u(t)$ stays in \mathcal{N}^+ forever.

For this purpose, as in [3, 19], we denote by C_* the best constant in the Poincaré-Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ defined by:

$$C_*^{-1} = \inf \{ \|\nabla u\|_2 : u \in H_0^1(\Omega), \|u\|_p = 1 \} \quad . \quad (2.7)$$

Let us remark (as in [3, 19]) that if $p < \bar{p}$ the embedding is compact and the infimum in (2.7) (as well as in (2.5)) is attained. In such case (see, e.g. [15, Section 3]), any mountain pass solution of the stationary problem is a minimizer for (2.7) and C_* is related to its energy:

$$d = \frac{p-2}{2p} C_*^{-2p/(p-2)} \quad . \quad (2.8)$$

Remark 2.1 [15, 8] For every solution of (1.1), given by Theorem 2.1, only one of the following assumption holds:

- i) if there exists some $t_0 \geq 0$ such that $u(t_0) \in \mathcal{W}$ and $E(t_0) < d$, then $\forall t \geq t_0$, $u(t) \in \mathcal{W}$ and $E(t) < d$.
- ii) if there exists some $t_0 \geq 0$ such that $u(t_0) \in \mathcal{U}$ and $E(t_0) < d$, then $\forall t \geq t_0$, $u(t) \in \mathcal{U}$ and $E(t) < d$.
- iii) $\forall t \geq 0$, $E(t) \geq d$.

We can now proceed in the global existence result investigation. For this sake, let us state two lemmas.

Lemma 2.1 Assume $2 \leq p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then $u(t, \cdot) \in \mathcal{N}^+$ for each $t \in [0, T)$.

Remark 2.2 Let us remark, that if there exists $t_0 \in [0, T)$ such that

$$E(t_0) < d$$

the same result stays true. It is the reason why we choose $t_0 = 0$.

Moreover, one can easily see that, from (2.8), the condition $E(0) < d$ is equivalent to the inequality:

$$C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1 \quad (2.9)$$

This last inequality will be used in the remaining proofs.

Proof of lemma 2.1: Since $I(u_0) > 0$, then by continuity, there exists $T_* \leq T$ such that $I(u(t, \cdot)) \geq 0$, for all $t \in [0, T_*)$. Since we have the relation:

$$J(t) = \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{1}{p} I(t)$$

we easily obtain :

$$J(t) \geq \frac{p-2}{2p} \|\nabla u\|_2^2, \quad \forall t \in [0, T_*) \quad .$$

Hence we have:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} J(t) \quad .$$

From (2.2) and (2.3), we obviously have $\forall t \in [0, T_*)$, $J(t) \leq E(t)$. Thus we obtain:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} E(t)$$

Since E is a decreasing function of t , we finally have:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} E(0), \forall t \in [0, T_*) . \quad (2.10)$$

By definition of C_* , we have:

$$\|u\|_p^p \leq C_*^p \|\nabla u\|_2^p = C_*^p \|\nabla u\|_2^{p-2} \|\nabla u\|_2^2$$

Using the inequality (2.10), we deduce:

$$\|u\|_p^p \leq C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \|\nabla u\|_2^2 .$$

Now exploiting the inequality on the initial condition (2.9) we obtain:

$$\|u\|_p^p < \|\nabla u\|_2^2 . \quad (2.11)$$

Hence $\|\nabla u\|_2^2 - \|u\|_p^p > 0$, $\forall t \in [0, T_*)$, this shows that $u(t, \cdot) \in \mathcal{N}^+$, $\forall t \in [0, T_*)$. By repeating this procedure, T_* is extended to T . \blacksquare

Lemma 2.2 *Assume $2 \leq p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then the solution of the problem (1.1) is global in time.*

Proof of lemma 2.2: Since the map $t \mapsto E(t)$ is a decreasing function of time t , we have:

$$E(0) \geq E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{(p-2)}{2p} \|\nabla u\|_2^2 + \frac{1}{p} I(t) ,$$

which gives us:

$$E(0) \geq \frac{1}{2} \|u_t\|_2^2 + \frac{(p-2)}{2p} \|\nabla u\|_2^2 .$$

Thus, $\forall t \in [0, T)$, $\|\nabla u\|_2 + \|u_t\|_2$ is uniformly bounded by a constant depending only on $E(0)$ and p . Then by definition 2.2, the solution is global, so $T_{max} = \infty$. \blacksquare

We can now state the asymptotic behavior of the solution of (1.1).

Theorem 2.2 *Assume $2 \leq p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then there exist two positive constants \widehat{C} and ξ independent of t such that:*

$$0 < E(t) \leq \widehat{C} e^{-\xi t}, \forall t \geq 0.$$

Remark 2.3 Let us remark that these inequalities imply that there exist positive constants K and ζ independent of t such that:

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq K e^{-\zeta t}, \forall t \geq 0.$$

Thus, this result improves the decay rate of Gazzola and Squassina [3, Theorem 3.8], in which the authors showed only the polynomial decay. Here we show that we can always find initial data satisfying $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$ which verify the inequality (2.9), such that the solution can decay faster than $1/t$, in fact with an exponential rate. Also, the same situation happens in absence of strong damping ($\omega = 0$).

Proof of theorem 2.2: Since we have proved that $\forall t \geq 0$, $u(t) \in \mathcal{N}^+$, we already have:

$$0 < E(t) \quad \forall t \geq 0.$$

The proof of the other inequality relies on the construction of a Lyapunov functional by performing a suitable modification of the energy. To this end, for $\varepsilon > 0$, to be chosen later, we define

$$L(t) = E(t) + \varepsilon \int_{\Omega} u_t u dx + \frac{\varepsilon \omega}{2} \|\nabla u\|_2^2 \quad . \quad (2.12)$$

It is straightforward to see that $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and $\beta_2 > 0$ depending on ε such that for $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \quad (2.13)$$

By taking the time derivative of the function L defined above in equation (2.12), using problem (1.1), and performing several integration by parts, we get:

$$\begin{aligned} \frac{dL(t)}{dt} &= -\omega \|\nabla u_t\|_2^2 - \mu \|u_t\|_2^2 + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ &\quad + \varepsilon \|u\|_p^p - \varepsilon \mu \int_{\Omega} u_t u dx \quad . \end{aligned} \quad (2.14)$$

Now, we estimate the last term in the right hand side of (2.14) as follows. By using Young's inequality, we obtain, for any $\delta > 0$

$$\int_{\Omega} u_t u dx \leq \frac{1}{4\delta} \|u_t\|_2^2 + \delta \|u\|_2^2 \quad . \quad (2.15)$$

Consequently, inserting (2.15) into (2.14) and using inequality (2.11), we have:

$$\begin{aligned} \frac{dL(t)}{dt} &\leq -\omega \|\nabla u_t\|_2^2 + \left(\varepsilon \left(\frac{1}{4\delta} + 1 \right) - \mu \right) \|u_t\|_2^2 \\ &\quad + \varepsilon \left(\underbrace{C_* \delta + C_*^p \left(\frac{2p}{(p-2)} E(0) \right)^{\frac{p-2}{2}}}_{< 0} - 1 \right) \|\nabla u\|_2^2 \quad . \end{aligned} \quad (2.16)$$

From (2.9), we have

$$C_*^p \left(\frac{2p}{(p-2)} E(0) \right)^{\frac{p-2}{2}} - 1 < 0 \quad .$$

Now, let us choose δ small enough such that:

$$C_* \delta + C_*^p \left(\frac{2p}{(p-2)} E(0) \right)^{\frac{p-2}{2}} - 1 < 0 \quad .$$

From (2.16), we may find $\eta > 0$, which depends only on δ , such that:

$$\frac{dL(t)}{dt} \leq -\omega \|\nabla u_t\|_2^2 + \left(\varepsilon \left(\frac{1}{4\delta} + 1 \right) - \mu \right) \|u_t\|_2^2 - \varepsilon \eta \|\nabla u\|_2^2$$

Consequently, using the definition of the energy (2.3), for any positive constant M , we obtain:

$$\begin{aligned} \frac{dL(t)}{dt} &\leq -M\varepsilon E(t) + \left(\varepsilon \left(\frac{1}{4\delta} + 1 + \frac{M}{2} \right) - \mu \right) \|u_t\|_2^2 - \omega \|\nabla u_t\|_2^2 \\ &\quad + \varepsilon \left(\frac{M}{2} - \eta \right) \|\nabla u\|_2^2 \quad . \end{aligned} \quad (2.17)$$

Now, choosing $M \leq 2\eta$, and ε small enough such that

$$\left(\varepsilon \left(\frac{1}{4\delta} + 1 + \frac{M}{2} \right) - \mu \right) < 0 \quad ,$$

inequality (2.17) becomes:

$$\frac{dL(t)}{dt} \leq -M\varepsilon E(t), \quad \forall t \geq 0.$$

On the other hand, by virtue of (2.13), setting $\xi = -M\varepsilon/\beta_2$, the last inequality becomes:

$$\frac{dL(t)}{dt} \leq -\xi L(t), \quad \forall t \geq 0 \quad . \quad (2.18)$$

Integrating the previous differential inequality (2.18) between 0 and t gives the following estimate for the function L :

$$L(t) \leq Ce^{-\xi t}, \quad \forall t \geq 0 \quad .$$

Consequently, by using (2.13) once again, we conclude

$$E(t) \leq \widehat{C}e^{-\xi t}, \quad \forall t \geq 0 \quad .$$

This completes the proof. ■

Remark 2.4 Note that we can obtain the same results as in Theorem 2.2 in the case $\omega = 0$, by taking the following Lyapunov function

$$L(t) = E(t) + \varepsilon \int_{\Omega} u_t u dx.$$

Remark 2.5 It is clear that the following problem:

$$\begin{cases} u_{tt} - \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega, t > 0 \\ u(x, t) = 0, & x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & x \in \Omega \quad . \end{cases}$$

could be treated with the same method and we obtained also an exponential decay of the solution if the initial condition is in the positive Nehari space and its energy is lower than the potential well depth.

References

- [1] J. Ball. Remarks on blow up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford.*, 28(2):473–486, 1977.
- [2] J. Esquivel-Avila. Qualitative analysis of a nonlinear wave equation. *Discrete. Contin. Dyn. Syst.*, 10:787–804, 2004.
- [3] F. Gazzola and M. Squassina. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. I. H. Poincaré*, 23:185–207, 2006.
- [4] V. Georgiev and G. Todorova. Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Differential Equations*, 109(2):295–308, 1994.

- [5] A. Haraux and E. Zuazua. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Rational Mech. Anal.*, 150:191–206, 1988.
- [6] R. Ikehata. Some remarks on the wave equations with nonlinear damping and source terms. *Nonlinear. Anal.*, 27:1165–1175, 1996.
- [7] R. Ikehata and T. Suzuki. Stable and unstable sets for evolution equations of parabolic and hyperbolic type. *Hiroshima Math. J.*, 26:475–491, 1996.
- [8] Esquivel-Avila J. The dynamics of nonlinear wave equation. *J. Math. Anal. Appl.*, 279:135–150, 2003.
- [9] V. K. Kalantarov and O. A. Ladyzhenskaya. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. *J. Soviet. Math.*, 10:53–70, 1978.
- [10] M. Kopackova. Remarks on bounded solutions of a semilinear dissipative hyperbolic equation. *Comment. Math. Univ. Carolin.*, 30(4):713–719, 1989.
- [11] H. A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, 192:1–21, 1974.
- [12] H. A. Levine. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM J. Math. Anal.*, 5:138–146, 1974.
- [13] S. Messaoudi and B. Said-Houari. Global non-existence of solutions of a class of wave equations with non-linear damping and source terms. *Math. Methods Appl. Sci.*, 27:1687–1696, 2004.
- [14] K. Ono. On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation. *Math. Methods Appl. Sci.*, 20(2):151–177, 1997.
- [15] L. E. Payne and D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel J. Math.*, 22(3-4):273–303, 1975.
- [16] L. E. Payne and D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel. J. Math.*, 22:273–303, 1981.
- [17] G. Todorova. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. *C. R. Acad Sci. Paris Ser.*, 326(1):191–196, 1998.
- [18] G. Todorova. Stable and unstable sets for the cauchy problem for a nonlinear wave with nonlinear damping and source terms. *J. Math. Anal. Appl.*, 239:213–226, 1999.
- [19] E. Vitillaro. Global nonexistence theorems for a class of evolution equations with dissipation. *Arch. Ration. Mech. Anal.*, 149(2):155–182, 1999.
- [20] Z. Yang. Existence and asymptotic behavior of solutions for a class of quasilinear evolution equations with non-linear damping and source terms. *Math. Meth. Appl. Sci.*, 25:795–814, 2002.
- [21] E. Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial. Diff. Eq.*, 15(2):205–235, 1990.