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Universal convex coverings

Roland Bacher

December 18, 2008

Abstract

In every dimension $d \geq 1$, we establish the existence of a positive finite constant v_d and of a subset \mathcal{U}_d of \mathbb{R}^d such that the following holds: $\mathcal{C} + \mathcal{U}_d = \mathbb{R}^d$ for every convex set $\mathcal{C} \subset \mathbb{R}^d$ of volume at least v_d and \mathcal{U}_d contains at most $\log(r)^{d-1}r^d$ points at distance at most r from the origin, for every large r .¹

1 Introduction

Fix a dimension $d \geq 1$ and consider the volume associated to the standard Lebesgue measure on \mathbb{R}^d . For any positive v , a subset \mathcal{U} of \mathbb{R}^d is a *universal convex covering of \mathbb{R}^d for the volume v* if, for every convex subset \mathcal{C} of \mathbb{R}^d of volume strictly greater than v , $\mathcal{C} + \mathcal{U} = \mathbb{R}^d$, where $\mathcal{C} + \mathcal{U}$ denotes the set of points $P + Q$ with P in \mathcal{C} and Q in \mathcal{U} .

For every positive t , a subset \mathcal{U} of \mathbb{R}^d is a universal convex covering for the volume v if and only if $t\mathcal{U}$ is a universal convex covering for the volume $t^d v$. The value of v is thus irrelevant for the properties we are interested in. We call \mathcal{U} a *universal convex covering of \mathbb{R}^d* if there exists a positive v such that \mathcal{U} is a universal convex covering of \mathbb{R}^d for the volume v .

Our main result is the following.

Theorem 1.1. *Let $d \geq 1$. Any universal convex covering of \mathbb{R}^d has at least a multiple of $\ell_d(r) = r^d$ points at distance at most r from the origin. There exists a universal convex covering \mathcal{U}_d of \mathbb{R}^d with at most a multiple of $u_d(r) = \log(r)^{d-1}r^d$ points at distance at most r from the origin, for every large r .*

The first part of theorem 1.1 is obvious since, in any dimension d , one can consider the unit cube as the convex set \mathcal{C} . In dimension $d = 1$, the second part is obvious as well since $\mathcal{I} + \mathbb{Z} = \mathbb{R}$ for every interval \mathcal{I} of length strictly

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greater than 1, hence one can choose $\mathcal{U}_1 = \mathbb{Z}$. But in dimension $d \geq 2$, there is a gap of \log^{d-1} between the easy lower bound ℓ_d and the upper bound u_d .

It is surely possible to lower this gap somewhat but I would be very surprised by the existence of universal coverings in dimension d achieving the lower bound ℓ_d for $d \geq 2$.

I am not aware of any results in the literature closely related to theorem 1.1. Universal convex coverings have of course some common points with sphere coverings, see [1] for an overview. There is also some literature on possible densities and multiplicities for a convex covering of \mathbb{R}^d with a given fixed convex shape, see for example [3] and [2].

This paper is organized as follows. In section 2 we collect some preliminary facts. In section 3 we construct recursively a sequence of sets $(\mathcal{U}_d)_{d \geq 1}$ such that $\mathcal{U}_1 = \mathbb{Z}$ and $\mathcal{U}_d \subset \mathbb{R}^d$ for every $d \geq 1$, and we show by induction on $d \geq 1$ that \mathcal{U}_d is a universal convex covering of \mathbb{R}^d . In section 4, we define growth classes of functions and we introduce a natural equivalence relation on them, which is compatible with the natural partial order on increasing positive functions. Finally, we show in section 5 that the growth class of the universal convex covering \mathcal{U}_d of section 3 is represented by u_d , a fact which implies theorem 1.1 by a suitable scaling of \mathcal{U}_d .

2 Preliminaries

For any subset \mathcal{S} of \mathbb{R}^d , let $-\mathcal{S}$ denote the opposite set of \mathcal{S} , that is,

$$-\mathcal{S} = \{-Q \in \mathbb{R}^d \mid Q \in \mathcal{S}\}.$$

Proposition 2.1. *Choose a positive v and a subset \mathcal{U} of \mathbb{R}^d . Then, \mathcal{U} is a universal convex covering for the volume v if and only if every convex subset of \mathbb{R}^d with volume at least v intersects \mathcal{U} non-trivially.*

Proof. Let \mathcal{C} denote a convex set with volume at least v . Then $-\mathcal{C}$ is a convex set of the same volume. For any point Q in \mathbb{R}^d , Q belongs to $\mathcal{C} + \mathcal{U}$ if and only if $-\mathcal{C} + Q$ intersects \mathcal{U} . \square

The growth function $f_{\mathcal{S}}$ of subset $\mathcal{S} \subset \mathbb{R}^d$ without accumulation points is defined as follows: for r an arbitrary positive real number, $f_{\mathcal{S}}(r)$ denotes the number of points of \mathcal{S} at distance at most r from the origin.

Universal convex coverings are stable under affine bijections. Universal convex coverings for a given volume are stable under affine volume-preserving bijections. Thus we consider the growth class with respect to the equivalence relation \sim defined as follows: for any increasing nonnegative functions

f and g , $f \sim g$ if there exists a real number $t \geq 1$ such that, for every $r \geq t$, $f(r) \leq g(tr) \leq f(t^2r)$.

Growth functions of sets without accumulation points related by affine bijections are equivalent under this equivalence relation.

For any nonzero integer n , $v_2(n)$ denotes the 2-valuation of n : this is the unique nonzero integer k such that n is 2^k times an odd integer. Write any point x of \mathbb{R}^d as $x = (x_i)_{1 \leq i \leq d}$, use the coordinate functions π_i defined as $\pi_i(x) = x_i$ and let $\rho^{(i)}(x)$ denote the point of \mathbb{R}^{d-1} such that $\rho^{(i)}(x) = (x_j)_{j \neq i}$.

3 From dimension d to dimension $d + 1$

Let \mathcal{U} denote a subset of \mathbb{R}^d . For every $1 \leq i \leq d+1$, let $\varphi_i^d(\mathcal{U})$ denote the set of points $x = (x_j)_{1 \leq j \leq d+1}$ in \mathbb{R}^{d+1} such that $x_i \in \mathbb{Z} \setminus \{0\}$ and $2^{v_2(x_i)/d} \rho^{(i)}(x)$ belongs to \mathcal{U} . Finally, let

$$\varphi_d(\mathcal{U}) = \bigcup_{i=1}^{d+1} \varphi_i^d(\mathcal{U}).$$

For example, $\varphi_1(\mathbb{Z}) \subset \mathbb{R}^2$ is the set of all points $(x, y) \in (\mathbb{Z}[\frac{1}{2}])^2$ such that $xy \in \mathbb{Z} \setminus \{0\}$ or $xy = 0$ and $x+y \in \mathbb{Z} \setminus \{0\}$. Otherwise stated, a point (x, y) of $\varphi_1(\mathbb{Z})$ is either a non-zero element of \mathbb{Z}^2 or it has two non-zero coordinates and belongs to the set $\bigcup_{n=0}^{\infty} (2^n \mathbb{Z}) \times (\frac{1}{2^n} \mathbb{Z}) \cup (\frac{1}{2^n} \mathbb{Z}) \times (2^n \mathbb{Z})$.

Proposition 3.1. *Let \mathcal{U} denote any universal convex covering of \mathbb{R}^d for the volume v . Then $\varphi_d(\mathcal{U})$ is a universal covering of \mathbb{R}^{d+1} for the volume*

$$v' = 4^{d+1} \max(1, 4v).$$

The value of v' in proposition 3.1 is not optimal and can easily be improved.

Let $(\mathcal{U}_d)_{d \geq 1}$ denote the sequence of sets defined recursively by $\mathcal{U}_1 = \mathbb{Z}$ and, for every $d \geq 1$,

$$\mathcal{U}_{d+1} = \varphi_d(\mathcal{U}_d).$$

Proposition 3.1 implies the following result.

Corollary 3.2. *For every $d \geq 1$, \mathcal{U}_d is a universal convex covering of \mathbb{R}^d .*

Proof of proposition 3.1. By proposition 2.1, it is enough to show that the volume of any convex set \mathcal{C} which does not intersect $\varphi_d(\mathcal{U})$ is bounded by v' . Without loss of generality, we assume that \mathcal{C} is open. Let L denote the diameter of \mathcal{C} with respect to the L^∞ norm of \mathbb{R}^{d+1} defined by $\|x\|_\infty = \max_{1 \leq i \leq d+1} |x_i|$. Two cases arise.

First, if $L \leq 4$, the volume of \mathcal{C} is $|\mathcal{C}| \leq 4^{d+1} \leq v'$.

The remaining case is when $L > 4$. Hence we assume that $L > 4$ and we must show that the volume $|\mathcal{C}|$ of \mathcal{C} is at most $4^{d+2}v$.

Note that there exists an index $1 \leq i \leq d+1$ such that $\pi_i(\mathcal{C}) =]a, b[\subset \mathbb{R}$ is an interval of length L . Thus one can pick two real numbers α and β such that

$$a < a + \frac{L}{4} \leq \alpha < \alpha + \frac{L}{4} \leq \beta < \beta + \frac{L}{4} \leq b$$

and $\alpha\beta \geq 0$ (or, equivalently, α and β are of the same sign).

Then the interval $] \alpha, \beta [$ contains an integer k such that $|k| = 2^m \geq L/8$. This implies that $\mathcal{C}' = \pi_i^{-1}(\{k\})$ is a convex set of \mathbb{R}^d which does not intersect $2^{-m/d}\mathcal{U}$. By proposition 2.1, the volume $|\mathcal{C}'|$ of \mathcal{C}' is at most $v/2^m \leq 8v/L$.

Let \mathcal{C}_- denote the set of points x in \mathcal{C} such that $x_i \leq k$, and let \mathcal{C}_+ denote the set of points x in \mathcal{C} such that $x_i \geq k$. Then,

$$|\mathcal{C}_-| \leq (k-a)|\mathcal{C}'| \left(\frac{k-a}{b-k} \right)^d < L \frac{8v}{L} \left(\frac{L}{L/4} \right)^d \leq 2 \cdot 4^{d+1}v.$$

The same bound holds for $|\mathcal{C}_+|$. Since $|\mathcal{C}| = |\mathcal{C}_+| + |\mathcal{C}_-|$, this implies that $|\mathcal{C}| \leq 4^{d+2}v \leq v'$, which concludes the proof. \square

4 Growth classes

Let \mathcal{G}_0 denote the set of positive and increasing functions f defined on an interval $[M(f), +\infty[$, where $M(f)$ is a finite real number which may depend on f . Then \mathcal{G}_0 is equipped with a preorder relation \preceq defined by $f \preceq g$ if there exists $t \geq 1$ such that $f(x) \leq g(tx)$ for every $x \geq t$.

The set \mathcal{G} of (*affine*) *growth classes* is the quotient set of \mathcal{G}_0 by the equivalence relation \sim defined by $f \sim g$ if there exists $t \geq 1$ such that, for every $x \geq t$,

$$g(x) \leq f(tx) \leq g(t^2x).$$

The preorder relation \preceq on \mathcal{G}_0 induces a partial order on \mathcal{G} .

Recall that for every $a > 0$ and $x > 0$, $\ell_a(x) = x^a$, hence $\ell_a \in \mathcal{G}_0$. A function $f \in \mathcal{G}$ is *polynomially bounded* if there exists $a > 0$ such that $f \preceq \ell_a$. If f is polynomially bounded, f has *critical exponent* $a > 0$ if $\ell_b \preceq f \preceq \ell_c$ for every positive b and c such that $b < a < c$. Additionally, f has *critical exponent* 0 if $f \preceq \ell_b$ for every $b > 0$.

Polynomially bounded functions can have no critical exponent. This happens when $\sup\{a \mid \ell_a \preceq f\} < \inf\{a \mid f \preceq \ell_a\}$. Any function f with critical exponent a can be written as $f = \ell_a h$, where the (not necessarily ultimately

increasing) function h is such that, for every $b > 0$, there exists a finite x_b such that $x^{-b} \leq h(x) \leq x^b$ for every $x \geq x_b$.

The notions of polynomial boundedness and critical exponent of functions in \mathcal{G}_0 are well behaved with respect to the preorder relation \preceq on \mathcal{G}_0 , hence these can also be defined on suitable growth classes in \mathcal{G} .

Given $\mathcal{S} \subset \mathbb{R}^d$, the choice of a norm on \mathbb{R}^d yields an increasing non-negative function $f_{\mathcal{S}}$ such that $f_{\mathcal{S}}(r)$ is the number of elements of \mathcal{S} whose norm is at most r . The growth class of $f_{\mathcal{S}}$ is independent of the norm, hence one can call it the growth class of \mathcal{S} . Two subsets of \mathbb{R}^d related by a translation belong to the same growth class. Growth classes are invariant under the action of the group of affine bijections of \mathbb{R}^d .

A set $S \subset \mathbb{R}^d$ is *sparse* if its growth class is strictly smaller than ℓ_d . We say that $S \subset \mathbb{R}^d$ is *nearly uniform* if it has a polynomially bounded growth class of critical exponent d . The growth class of a nearly uniform set can be represented by a function $h\ell_d$, where h encodes the ‘‘asymptotic density’’ of S up to affine bijections.

For example, \mathbb{Z}^d and \mathbb{N}^d are both nearly uniform sets and in the same growth class ℓ_d .

A more concise and less precise reformulation of theorem 1.1 is as follows.

Theorem 4.1. *In every dimension, there exists nearly uniform universal convex coverings.*

We conclude this section with a remark.

One can also define growth classes for measurable subsets $\mathcal{A} \subset \mathbb{R}^d$ and for any given measure μ and norm on \mathbb{R}^d , by replacing $f_{\mathcal{A}}$ by the function $f_{\mathcal{A}}^{\mu}$ such that $f_{\mathcal{A}}^{\mu}(r)$ denotes the μ measure of the intersection of \mathcal{A} with the ball of radius r centered at the origin.

5 Growth class of \mathcal{U}_d

Recall that $u_d(r) = \log(r)^{d-1}r^d$ for every $r \geq 1$.

Proposition 5.1. *The universal convex covering \mathcal{U}_d defined in corollary 3.2 belongs to the growth class of u_d .*

Proof of theorem 1.1. By proposition 5.1, there exists a constant c_d such that the set \mathcal{U}_d constructed in corollary 3.2 has at most $c_d u_d(r)$ elements at distance at most r from the origin. Considering the rescaled set $t\mathcal{U}_d$ for $t > c_d^{1/d}$ ends the proof. \square

Proof of proposition 5.1. We proceed by induction on the dimension d . For $d = 1$, $u_1(r) = r$ hence $\mathcal{U}_1 = \mathbb{Z}$ belongs to the growth class of u_1 .

We assume now that \mathcal{U}_d is of growth class u_d for some $d \geq 1$. Then, the growth class of \mathcal{U}_{d+1} is described by the set $\mathcal{B} \subset \mathbb{N} \times \mathbb{R}^d$ defined as

$$\mathcal{B} = \bigcup_{m \geq 1} (m, 2^{-v_2(m)/d} \mathcal{U}_d) = \bigcup_{n \geq 0} \bigcup_{k \geq 0} (2^n(2k+1), 2^{-n/d} \mathcal{U}_d).$$

Neglecting an asymptotic factor and using the induction hypothesis, the growth class of \mathcal{B} is thus represented by

$$\sum_{n \geq 0} \sum_{k \geq 0} u_d \left(2^{n/d} \sqrt{r^2 - (2^n k)^2} \right),$$

where the sums above run over the integers (n, k) such that $r^2 \geq (2^n k)^2$. Introducing the set

$$\mathcal{A}(r) = \{(n, k) \in \mathbb{N}^2 \mid r^2 \geq (2^n k)^2\} \subset \mathbb{N}^2,$$

we have

$$\begin{aligned} & \sum_{(n,k) \in \mathcal{A}(r)} u_d \left(2^{n/d} \sqrt{r^2 - (2^n k)^2} \right) \\ & \leq \left(1 + \frac{1}{d} \right)^{d-1} \log(r)^{d-1} \sum_{(n,k) \in \mathcal{A}(r)} 2^{n(d+1)} \left((r/2^n)^2 - k^2 \right)^{d/2} \end{aligned}$$

and

$$\sum_{(n,k) \in \mathcal{A}(r)} 2^{n(d+1)} \left((r/2^n)^2 - k^2 \right)^{d/2}$$

is asymptotically equal to $\frac{1}{2} \log_2(r) r^{d+1}$ times the volume $\pi^{(d+1)/2} / ((d+1)/2)!$ of a $(d+1)$ -dimensional unit ball in a $(d+1)$ -dimensional Euclidean space.

Hence the growth class of \mathcal{B} and thus of \mathcal{U}_{d+1} are bounded above by the function $u_{d+1}(r) = \log(r)^d r^{d+1}$. A slightly more careful argument shows that they are exactly represented (up to equivalence) by u_{d+1} . This concludes the proof. \square

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