On Hofmann's bilinear estimate

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ABSTRACT. Using the framework of a previous article joint with Axelsson and McIntosh, we extend to systems two results of S. Hofmann for real symmetric equations and their perturbations going back to a work of B. Dahlberg for Laplace's equation on Lipschitz domains, The first one is a certain bilinear estimate for a class of weak solutions and the second is a criterion which allows to identify the domain of the generator of the semi-group yielding such solutions.

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1. INTRODUCTION

S. Hofmann proved in [11] that weak solutions of

$$(1) \qquad \mathrm{div}_{t,x} A(x) \nabla_{t,x} U(t,x) = \sum_{i,j=0}^{n} \partial_i A_{i,j}(x) \partial_j U(t,x) = 0$$

on the upper half space $\mathbf{R}_+^{1+n} := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n \; ; \; t > 0\}$, $n \geq 1$, where the matrix $A = (A_{i,j}(x))_{i,j=0}^n \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{1+n}))$ is assumed to be t-independent and within some small L_∞ neighborhood of a real symmetric strictly elliptic t-independent matrix, obey the following bilinear estimate

$$\left| \iint_{\mathbf{R}_+^{1+n}} \nabla_{t,x} U \cdot \overline{\mathbf{v}} \, dt dx \right| \leq C \|U_0\|_2 (\|\|t \nabla \mathbf{v}\|\| + \|N_* \mathbf{v}\|_2)$$

for all $\mathbf{C}^{1+n}$-valued field $\mathbf{v}$ such that the right-hand side is finite. See below for the definition of the square-function $\|\| \; \|\|$ and the non-tangential maximal operator N_*. The trace of U at $t = 0$ is assumed to be in the sense of non-tangential convergence a.e. and in $L_2(\mathbf{R}^n)$.

In addition, he proves that the solution operator $U_0 \to U(t, \cdot)$ defines a bounded C_0 semi-group on $L_2(\mathbf{R}^n)$ whose infinitesimal generator $\mathcal{A}$ has domain $W^{1,2}(\mathbf{R}^n)$ with $\|\mathcal{A}f\|_2 \sim \|\nabla f\|_2$.

Such results were first proved by B. Dahlberg [8] for harmonic functions on a Lipschitz domain. A version of the bilinear estimate for Clifford-valued monogenic functions was proved by Li-McIntosh-Semmes [16]. A short proof of Dahlberg's estimate for harmonic functions and some applications appear in Mitrea's work [17]. L^p versions are recently discussed by Varopoulos [20].

Hofmann's arguments for variable coefficients rely on the deep results of [1], and in particular Theorem 1.11 there where the boundedness and invertibility of the layer potentials are obtained from a $T(b)$ theorem, Rellich estimates in the case of real symmetric matrices and perturbation. This also generalizes somehow the case where $A_{0,i} = A_{i,0} = 0$ for $i = 1, \ldots, n$ corresponding to the Kato square root problem.

The recent works [3, 4], pursuing ideas in [2], allow us to extend this further to systems, making clear in particular that specificities of real symmetric coefficients and their perturbations and of equations - in particular the De Giorgi-Nash-Moser estimates - are not needed: it only depends on whether the Dirichlet problem is solvable. We use the solution operator constructed in [3] and the proof using $P_t - Q_t$ techniques of Coifman-Meyer from [7] makes transparent the para-product like character of this bilinear estimate. We also establish a necessary and sufficient condition telling when the domain of the infinitesimal generator $\mathcal{A}$ of the Dirichlet semi-group is $W^{1,2}$.

We apologize to the reader for the necessary conciseness of this note and suggests he (or she) has (at least) the references [2, 3, 4] handy. In Section 2, we try to extract from them the relevant information. The proof or the bilinear estimate for variable coefficients systems is in Section 3. Section 4 contains the discussion on the domain of the Dirichlet semi-group.

2. Setting

We begin by giving a precise definition of well-posedness of the Dirichlet problem for systems. Throughout this note, we use the notation $X \approx Y$ and $X \lesssim Y$ for estimates to mean that there exists a constant $C > 0$, independent of the variables in the estimate, such that $X/C \leq Y \leq CX$ and $X \leq CY$, respectively.

We write (t, x) for the standard coordinates for $\mathbf{R}^{1+n} = \mathbf{R} \times \mathbf{R}^n$, t standing for the vertical or normal coordinate. For vectors $\mathbf{v} = (\mathbf{v}_i^\alpha)_{0 \leq i \leq n}^{1 \leq \alpha \leq m} \in \mathbf{C}^{(1+n)m}$, we write $\mathbf{v}_0 \in \mathbf{C}^m$ and $\mathbf{v}_\parallel \in \mathbf{C}^{nm}$ for the normal and tangential parts of $\mathbf{v}$, i.e. $\mathbf{v}_0 = (\mathbf{v}_0^\alpha)^{1 \leq \alpha \leq m}$ whereas $\mathbf{v}_\parallel = (\mathbf{v}_i^\alpha)_{1 \leq i \leq n}^{1 \leq \alpha \leq m}$.

For systems, gradient and divergence act as $(\nabla_{t,x} U)_i^\alpha = \partial_i U^\alpha$ and $(\mathrm{div}_{t,x} \mathbf{F})^\alpha = \sum_{i=0}^n \partial_i \mathbf{F}_i^\alpha$, with correponding tangential versions $\nabla_x U = (\nabla_{t,x} U)_\parallel$ and $(\mathrm{div}_x \mathbf{F})^\alpha = \sum_{i=1}^n \partial_i \mathbf{F}_i^\alpha$. With $\mathrm{curl}_x \mathbf{F}_\parallel = 0$, we understand $\partial_j \mathbf{F}_i^\alpha = \partial_i \mathbf{F}_j^\alpha$, for all $i, j = 1, \ldots, n, \alpha = 1, \ldots, m$.

We consider divergence form second order elliptic systems

$$(2) \qquad \sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i A_{i,j}^{\alpha,\beta}(x) \partial_j U^\beta(t,x) = 0, \qquad \alpha = 1, \ldots, m,$$

on the half space $\mathbf{R}_+^{1+n} := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n \; ; \; t > 0\}$, $n \geq 1$, where the matrix $A = (A_{ij}^{\alpha,\beta}(x))_{i,j=0,\ldots,n}^{\alpha,\beta=1,\ldots,m} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ is assumed to be t-independent with complex coefficients and *strictly accretive* on $\mathsf{N}(\mathrm{curl}_\parallel)$, in the sense that there exists $\kappa > 0$ such that

$$(3) \qquad \sum_{i,j=0}^n \sum_{\alpha,\beta=1}^m \int_{\mathbf{R}^n} \mathrm{Re}(A_{i,j}^{\alpha,\beta}(x) \mathbf{f}_j^\beta(x) \overline{\mathbf{f}_i^\alpha(x)}) dx \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbf{R}^n} |\mathbf{f}_i^\alpha(x)|^2 dx,$$

for all $\mathbf{f} \in \mathsf{N}(\mathrm{curl}_\parallel) := \{\mathbf{g} \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) \; ; \; \mathrm{curl}_x(\mathbf{g}_\parallel) = 0\}$. This is nothing but ellipticity in the sense of Gårding. See the discussion in [3]. By changing m to $2m$ we could assume that the coefficients are real-valued. But this does not simplify matters and we need the complex hermitean structure of our L_2 space anyway.

Definition 2.1. The Dirichlet problem (Dir-A) is said to be *well-posed* if for each $u \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a unique function

$$U_t(x) = U(t,x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$$

such that $\nabla_x U \in C^0(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{nm}))$, where U satisfies (2) for $t > 0$, $\lim_{t \to 0} U_t = u$, $\lim_{t \to \infty} U_t = 0$, $\lim_{t \to \infty} \nabla_{t,x} U_t = 0$ in L_2 norm, and $\int_{t_0}^{t_1} \nabla_x U_s \, ds$ converges in L_2 when $t_0 \to 0$ and $t_1 \to \infty$. More precisely, by U satisfying (2), we mean that $\int_t^\infty ((A\nabla_{s,x} U_s)_\parallel, \nabla_x v) ds = -((A\nabla_{t,x} U_t)_0, v)$ for all $v \in C_0^\infty(\mathbf{R}^n; \mathbf{C}^m)$.

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [11] . However, a sufficient condition is provided in [3] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem (see Section 4).

Remark 2.2. In the case of block matrices, ie $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \le i \le n, 1 \le \alpha, \beta \le m$, the second order system (2) can be solved using semi-group theory: $V(t, \cdot) = e^{-tL^{1/2}} u_0$ for $L = -A_{00}^{-1} \mathrm{div}_x A_{\parallel\parallel} \nabla_x$ acting as an unbounded operator on $L_2(\mathbf{R}^n, C^{nm})$ (See below for the notation). This solution satisfies $V_t = V(t, \cdot) \in C^2(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m)) \cap C^1(\mathbf{R}_+, \mathsf{D}(L^{1/2}))$, $\lim_{t \to 0} V_t = u_0$, $\lim_{t \to \infty} V_t = 0$ in L_2 norm, and (2) holds in the strong sense in $\mathbf{R}^n$ for all $t > 0$ (and in the sense of distributions in $\mathbf{R}_+^{1+n}$). Hence, the two notions of solvability are not *a priori* equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for L: $\mathsf{D}(L^{1/2}) = W^{1,2}(\mathbf{R}^n, C^{nm})$ with $\|L^{1/2} f\|_2 \sim \|\nabla_x f\|_2$. See [6] where this is explicitly proved when $A_{00} \ne I$.

The following result is Corollary 3.4 of [3] (which, as we recall, furnishes a different proof of results obtained by combining [12] and [9] in the case of real symmetric matrices equations ($m = 1$)).

Theorem 2.3. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$ and assume that (Dir-A) is well-posed. Then any function $U_t(x) = U(t, x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$ solving (2), with properties as in Definition 2.1, has estimates*

$$\int_{\mathbf{R}^n} |u|^2 dx \approx \sup_{t>0} \int_{\mathbf{R}^n} |U_t|^2 dx \approx \int_{\mathbf{R}^n} |\widetilde{N}_*(U)|^2 dx \approx \||t\nabla_{t,x} U\||^2,$$

where $u = U|_{\mathbf{R}^n}$. If furthermore A is real (not necessarily symmetric) and $m = 1$, then Moser's local boundedness estimate [18] gives the pointwise estimate $\widetilde{N}_*(U)(x) \approx N_*(U)(x)$, where the standard non-tangential maximal function is $N_*(U)(x) := \sup_{|y-x|<ct} |U(t, y)|$, for fixed $0 < c < \infty$.

We use the square-function norm

$$\||F_t\||^2 := \int_0^\infty \|F_t\|_2^2 \frac{dt}{t} = \iint_{\mathbf{R}_+^{1+n}} |F(t, x)|^2 \frac{dt dx}{t}$$

and the following version $\widetilde{N}_*(F)$ of the modified *non-tangential maximal function* introduced in [13]

$$\widetilde{N}_*(F)(x) := \sup_{t>0} t^{-(1+n)/2} \|F\|_{L_2(Q(t,x))},$$

where $Q(t, x) := [(1 - c_0)t, (1 + c_0)t] \times B(x; c_1 t)$, for some fixed constants $c_0 \in (0, 1)$, $c_1 > 0$.

Next is Theorem 3.2 of [3], specialized to the Dirichlet problem.

Theorem 2.4. *The set of matrices A for which (Dir-A) is well-posed is an open subset of $L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$. Furthermore, it contains*

(i) *all Hermitean matrices $A(x) = A(x)^*$ (and in particular all real symmetric matrices),*
(ii) *all block matrices where $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \le i \le n, 1 \le \alpha, \beta \le m$, and*
(iii) *all constant matrices $A(x) = A$.*

More importantly is the solution algorithm using an "infinitesimal generator" T_A. Write $\mathbf{v} \in \mathbf{C}^{(1+n)m}$ as $\mathbf{v} = [\mathbf{v}_0, \mathbf{v}_\|]^t$, where $\mathbf{v}_0 \in \mathbf{C}^m$ and $\mathbf{v}_\| \in \mathbf{C}^{nm}$, and introduce the auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{00} & A_{0\|} \\ 0 & I \end{bmatrix}, \quad \underline{A} := \begin{bmatrix} 1 & 0 \\ A_{\|0} & A_{\|\|} \end{bmatrix}, \quad \text{if } A = \begin{bmatrix} A_{00} & A_{0\|} \\ A_{\|0} & A_{\|\|} \end{bmatrix}$$

in the normal/tangential splitting of $\mathbf{C}^{(1+n)m}$. The strict accretivity of A on $\mathsf{N}(\mathrm{curl}_\|)$, as in (3), implies the pointwise strict accretivity of the diagonal block A_{00}. Hence A_{00} is invertible, and consequently $\overline{A}$ is invertible [This is not necessarily true for $\underline{A}$.] We define

$$T_A = \overline{A}^{-1} D \underline{A}$$

as an unbounded operator on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ with D the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

Proposition 2.5. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\|)$.*

(1) *The operator T_A has quadratic estimates and a bounded holomorphic functional calculus on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$. In particular, for any holomorphic function ψ on the left and right open half planes, with $z\psi(z)$ and $z^{-1}\psi(z)$ qualitatively bounded, one has*

$$\|\psi(tT_A)\mathbf{f}\| \lesssim \|\mathbf{f}\|_2.$$

(2) *The Dirichlet problem (Dir-A) is well-posed if and only if the operator*

$$\mathcal{S} : \overline{R(\chi_+(T_A))} \to L_2(\mathbf{R}^n, \mathbf{C}^m), \mathbf{f} \mapsto \mathbf{f}_0$$

is invertible. Here, $\chi_+ = 1$ on the right open half plane and 0 on the left open half plane.

Item (1) is [3, Corollary 3.6] (and see [4] for an explicit direct proof) and item (2) can be found in [3, Section 4, proof of Theorem 2.2].

Lemma 2.6. *Assume that (Dir-A) is well-posed. Let $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$. Then the solution U of (Dir-A) in the sense of Definition 2.1 is given by*

$$U(t, \cdot) = (e^{-tT_A}\mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1}u_0 \in \overline{R(\chi_+(T_A))}$$

and furthermore

$$\nabla_{t,x}U(t, \cdot) = \partial_t e^{-tT_A}\mathbf{f}.$$

Proof. [3, Lemma 4.2] (See also [2, Lemma 2.55] with a slightly different formulation of the Dirichlet problem). □

3. THE BILINEAR ESTIMATE

We are now in position to state and prove the generalisation of Hofmann's result.

Theorem 3.1. *Assume that (Dir-A) is well-posed. Let $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$ and U be the solution to (Dir-A) in the sense of Definition 2.1. Then for all $\mathbf{v} \colon \mathbf{R}_+^{1+n} \to \mathbf{C}^{(1+n)m}$ such that the right-hand side is finite,*

$$\left| \iint_{\mathbf{R}_+^{1+n}} \nabla_{t,x} U \cdot \overline{\mathbf{v}} \, dt dx \right| \leq C \|u_0\|_2 (\|\|t\nabla_{t,x}\mathbf{v}\|\| + \|N_*\mathbf{v}\|_2).$$

The pointwise values of $\mathbf{v}(t,x)$ in the non-tangential control $N_*\mathbf{v}$ can be slightly improved to L^1 averages on balls having radii $\sim t$ for each fixed t. See the end of proof.

Proof. It follows from the previous result that there exists $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$ such that $U(t, \cdot) = (e^{-tT_A}\mathbf{f})_0$ and

$$\nabla_{t,x} U(t, \cdot) = \partial_t \mathbf{F} = -T_A e^{-tT_A}\mathbf{f}, \quad \mathbf{F} = e^{-tT_A}\mathbf{f}.$$

Integrating by parts with respect to t, we find

$$\iint_{\mathbf{R}_+^{1+n}} \nabla U \cdot \overline{\mathbf{v}} \, dt dx = -\iint_{\mathbf{R}_+^{1+n}} t\partial_t \mathbf{F} \cdot \overline{\partial_t \mathbf{v}} \, dt dx - \iint_{\mathbf{R}_+^{1+n}} t\partial_t^2 \mathbf{F} \cdot \overline{\mathbf{v}} \, dt dx.$$

The boundary term vanishes because $t\partial_t \mathbf{F}$ goes to 0 in L_2 when $t \to 0, \infty$ (this uses $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$) and $\sup_{t>0} \|\mathbf{v}(t, \cdot)\|_2 < \infty$ from $\|N_*\mathbf{v}\|_2 < \infty$.

For the first term, we use Cauchy-Schwarz inequality and that $\|\|t\partial_t \mathbf{F}\|\| \lesssim \|u_0\|_2$ from Theorem 2.3.

For the second term, we use the following identity: $T_A = \overline{A}^{-1} DB\overline{A}$ with $B = A\overline{A}^{-1}$ which, by [3, Proposition 3.2], is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$, and observe that

$$\begin{aligned} t^2 \partial_t^2 \mathbf{F} &= \overline{A}^{-1}(tDB)^2 e^{-tDB}(\overline{A}\mathbf{f}) \\ &= \overline{A}^{-1}(tDB)(I + (tDB)^2)^{-1}\psi(tDB)(\overline{A}\mathbf{f}) \\ &= \overline{A}^{-1}(tDB)(I + (tDB)^2)^{-1}\overline{A}\psi(tT_A)(\mathbf{f}) \end{aligned}$$

with

$$\psi(z) = z(1 + z^2)e^{-(\mathrm{sgn}\,\mathrm{Re}\,z)z}.$$

Thus,

$$\iint_{\mathbf{R}_+^{1+n}} t\partial_t^2 \mathbf{F} \cdot \overline{\mathbf{v}} \, dt dx = \iint_{\mathbf{R}_+^{1+n}} \overline{A}\psi(tT_A)(\mathbf{f}) \cdot \overline{Q_t \mathbf{v}_t} \, \frac{dt dx}{t}$$

with $Q_t = \Theta_t \overline{A}^{-1*}$ and $\Theta_t = (tB^*D)(I + (tB^*D)^2)^{-1}$ acting on $\mathbf{v}_t \equiv \mathbf{v}(\mathbf{t}, \cdot)$ for each fixed t [The notation $\overline{A}$ has nothing to do with complex conjugate and we apologize for any conflict this may cause.] It follows from the quadratic estimates of Proposition 2.5 that

$$\|\|\psi(tT_A)(\mathbf{f})\|\| \lesssim \|\mathbf{f}\|_2.$$

It remains to estimate $\|\|Q_t \mathbf{v}_t\|\|$. To do that we follow the principal part approximation of [4] - which is an elaboration of the so-called Coifman-Meyer trick [7] - applied

to Q_t instead of Θ_t there. That is, we write

$$(4) \qquad Q_t\mathbf{v}_t = Q_t\left(\frac{I - P_t}{t(-\Delta)^{1/2}}\right)t(-\Delta)^{-1/2}\mathbf{v}_t + (Q_tP_t - \gamma_tS_tP_t)\mathbf{v}_t + \gamma_tS_tP_t\mathbf{v}_t$$

where Δ is the Laplacian on $\mathbf{R}^n$, P_t is a nice scalar approximation to the identity acting componentwise on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ and γ_t is the element of $L_{\text{loc}}^2(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ given by

$$\gamma_t(x)\mathbf{w} := (Q_t\mathbf{w})(x)$$

for every $\mathbf{w} \in \mathbf{C}^{(1+n)m}$. We view $\mathbf{w}$ on the right-hand side of the above equation as the constant function valued in $\mathbf{C}^{(1+n)m}$ defined on $\mathbf{R}^n$ by $\mathbf{w}(x) := \mathbf{w}$. We identify $\gamma_t(x)$ with the (possibly unbounded) multiplication operator $\gamma_t : f(x) \mapsto \gamma_t(x)f(x)$. Finally, the *dyadic averaging operator* $S_t : L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m}) \to L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ is given by

$$S_t\mathbf{u}(x) := \frac{1}{|Q|}\int_Q \mathbf{u}(y)\,dy$$

for every $x \in \mathbf{R}^n$ and $t > 0$, where Q is the unique dyadic cube in $\mathbf{R}^n$ that contains x and has side length ℓ with $\ell/2 < t \le \ell$.

With this in hand, we apply the triple bar norm to (4).

Using the uniform L_2 boundedness of Q_t and that of $\frac{1-P_t}{t(-\Delta)^{1/2}}$, the first term in the RHS is bounded by $\||t(-\Delta)^{1/2}\mathbf{v}_t\|| \le \||t\nabla_x\mathbf{v}_t\||$.

Following exactly the computation of Lemma 3.6 in [4], the second term in the RHS is bounded by $C\||t\nabla_xP_t\mathbf{v}_t\|| \le C\||t\nabla_x\mathbf{v}_t\||$ using the uniform L_2 boundedness of P_t. This computation makes use of the off-diagonal estimates of Θ_t, hence of Q_t, proved in [4, Proposition 3.11].

For the third term in the RHS, we observe that $\gamma_t(x)\mathbf{w} = \Theta_t(\overline{A}^{-1*}\mathbf{w})(x)$. Hence, the square-function estimate on Θ_t proved in [4, Theorem 1.1], the off-diagonal estimates of Θ_t and the fact that $\overline{A}^{-1}$ is bounded imply that $|\gamma_t(x)|^2\frac{dtdx}{t}$ is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes $\|N_*(S_tP_t\mathbf{v})\|_2$, which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening. $\qquad\square$

4. The domain of the Dirichlet semi-group

Assume (Dir-A) in the sense of Definition 2.1 is well-posed. If we set

$$\mathcal{P}_tu_0 = (e^{-tT_A}\mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1}u_0 \in \overline{\mathsf{R}(\chi_+(T_A))}$$

for all $t > 0$, then Lemma 2.6 implies that $(\mathcal{P}_t)_{t>0}$ is a bounded C_0-semigroup on $L_2(\mathbf{R}^n, \mathbf{C}^m)$ [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].

Furthermore, with our definition of well-posedness of the Dirichlet problem, the domain of the infinitesimal generator $\mathcal{A}$ of this semi-group is contained in the Sobolev space $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ and $\|\nabla_xu_0\|_2 \lesssim \|\mathcal{A}u_0\|_2$. Indeed, from Lemma 2.6 we have for all $t > 0$, $\partial_te^{-tT_A}\mathbf{f} = \nabla_{t,x}U(t,\cdot)$. Also $\partial_te^{-tT_A}\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$ and the invertibility of $\mathcal{S}$ tells that $\nabla_{t,x}U(t,\cdot) = \mathcal{S}^{-1}(\partial_tU(t,\cdot))$. Therefore

$$\|\nabla_xU(t,\cdot))\|_2 \lesssim \|\partial_tU(t,\cdot)\|_2.$$

By definition of $\mathcal{A}$, $\partial_tU(t,\cdot) = \mathcal{A}U(t,\cdot)$, thus we have for all $t > 0$

$$\|\nabla_xU(t,\cdot))\|_2 \lesssim \|\mathcal{A}U(t,\cdot)\|_2.$$

The conclusion for the domain follows easily.

The question of whether this domain coincides with $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ is answered by the following theorem

Theorem 4.1. *Assume that (Dir-A) and (Dir-A^*) are well-posed. Then the domain of the infinitesimal generator $\mathcal{A}$ of $(\mathcal{P}_t)_{t>0}$ coincides with the Sobolev space $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ and $\|\nabla_x u_0\|_2 \sim \|\mathcal{A} u_0\|_2$.*

This theorem applies to the three situations listed in Theorem 2.4.

Proof. Combining [4, Lemma 4.2] (which says that (Dir-A^*) is equivalent to an auxiliary Neumann problem for A^*), [2, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for A: this is non trivial) with the proof ot Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for A), we have that (Dir-A^*) is well-posed if and only if

$$\mathcal{R} : \overline{\mathsf{R}(\chi_+(T_A))} \to L_2(\mathbf{R}^n, \mathbf{C}^{nm}), \mathbf{f} \mapsto \mathbf{f}_\parallel$$

is invertible. This implies that for $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$, we have that

$$\|\mathbf{f}\|_2 \sim \|\mathbf{f}_\parallel\|_2.$$

Therefore, the conjunction of well-posedness for (Dir-A) and (Dir-A^*) gives

$$\|\mathbf{f}_0\|_2 \sim \|\mathbf{f}_\parallel\|_2, \quad \mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}.$$

From this, it is easy to identify the domain of $\mathcal{A}$ by an argument as before. $\square$

We have seen that invertibility of $\mathcal{S}$ reduces to that of $\mathcal{R}$ (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that $\|\mathbf{f}_\parallel\|_2 \sim \|(A\mathbf{f})_0\|_2$ for all $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$. This method was first used in the context of Laplace equation on Lipschitz domains by Verchota [21]. This depends strongly of A. Various relations between Dirichlet, regularity and Neumann problems for L^p data in the sense of non tangential approach for second order real symmetric equations are studied in [13, 14] and more recently in [15, 19].

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