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# ON HOFMANN'S BILINEAR ESTIMATE

PASCAL AUSCHER

ABSTRACT. Using the framework of a previous article joint with Axelsson and McIntosh, we extend to systems two results of S. Hofmann for real symmetric equations and their perturbations going back to a work of B. Dahlberg for Laplace's equation on Lipschitz domains, The first one is a certain bilinear estimate for a class of weak solutions and the second is a criterion which allows to identify the domain of the generator of the semi-group yielding such solutions.

MSC classes: 35J25, 35J55, 47N20, 42B25

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## 1. INTRODUCTION

S. Hofmann proved in [11] that weak solutions of

$$(1) \quad \operatorname{div}_{t,x} A(x) \nabla_{t,x} U(t, x) = \sum_{i,j=0}^n \partial_i A_{i,j}(x) \partial_j U(t, x) = 0$$

on the upper half space  $\mathbf{R}_+^{1+n} := \{(t, x) \in \mathbf{R} \times \mathbf{R}^n ; t > 0\}$ ,  $n \geq 1$ , where the matrix  $A = (A_{i,j}(x))_{i,j=0}^n \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{1+n}))$  is assumed to be  $t$ -independent and within some small  $L_\infty$  neighborhood of a real symmetric strictly elliptic  $t$ -independent matrix, obey the following bilinear estimate

$$\left| \iint_{\mathbf{R}_+^{1+n}} \nabla_{t,x} U \cdot \bar{\nabla} t dtdx \right| \leq C \|U_0\|_2 (\|t \nabla \mathbf{v}\| + \|N_* \mathbf{v}\|_2)$$

for all  $\mathbf{C}^{1+n}$ -valued field  $\mathbf{v}$  such that the right-hand side is finite. See below for the definition of the square-function  $\| \cdot \|$  and the non-tangential maximal operator  $N_*$ . The trace of  $U$  at  $t = 0$  is assumed to be in the sense of non-tangential convergence a.e. and in  $L_2(\mathbf{R}^n)$ .

In addition, he proves that the solution operator  $U_0 \rightarrow U(t, \cdot)$  defines a bounded  $C_0$  semi-group on  $L_2(\mathbf{R}^n)$  whose infinitesimal generator  $\mathcal{A}$  has domain  $W^{1,2}(\mathbf{R}^n)$  with  $\|\mathcal{A}f\|_2 \sim \|\nabla f\|_2$ .

Such results were first proved by B. Dahlberg [8] for harmonic functions on a Lipschitz domain. A version of the bilinear estimate for Clifford-valued monogenic functions was proved by Li-McIntosh-Semmes [16]. A short proof of Dahlberg's estimate for harmonic functions and some applications appear in Mitrea's work [17].  $L^p$  versions are recently discussed by Varopoulos [20].

Hofmann's arguments for variable coefficients rely on the deep results of [1], and in particular Theorem 1.11 there where the boundedness and invertibility of the layer potentials are obtained from a  $T(b)$  theorem, Rellich estimates in the case of real symmetric matrices and perturbation. This also generalizes somehow the case where  $A_{0,i} = A_{i,0} = 0$  for  $i = 1, \dots, n$  corresponding to the Kato square root problem.

The recent works [3, 4], pursuing ideas in [2], allow us to extend this further to systems, making clear in particular that specificities of real symmetric coefficients and their perturbations and of equations - in particular the De Giorgi-Nash-Moser estimates - are not needed: it only depends on whether the Dirichlet problem is solvable. We use the solution operator constructed in [3] and the proof using  $P_t - Q_t$  techniques of Coifman-Meyer from [7] makes transparent the para-product like character of this bilinear estimate. We also establish a necessary and sufficient condition telling when the domain of the infinitesimal generator  $\mathcal{A}$  of the Dirichlet semi-group is  $W^{1,2}$ .

We apologize to the reader for the necessary conciseness of this note and suggests he (or she) has (at least) the references [2, 3, 4] handy. In Section 2, we try to extract from them the relevant information. The proof of the bilinear estimate for variable coefficients systems is in Section 3. Section 4 contains the discussion on the domain of the Dirichlet semi-group.

## 2. SETTING

We begin by giving a precise definition of well-posedness of the Dirichlet problem for systems. Throughout this note, we use the notation  $X \approx Y$  and  $X \lesssim Y$  for estimates to mean that there exists a constant  $C > 0$ , independent of the variables in the estimate, such that  $X/C \leq Y \leq CX$  and  $X \leq CY$ , respectively.

We write  $(t, x)$  for the standard coordinates for  $\mathbf{R}^{1+n} = \mathbf{R} \times \mathbf{R}^n$ ,  $t$  standing for the vertical or normal coordinate. For vectors  $\mathbf{v} = (\mathbf{v}_i^\alpha)_{0 \leq i \leq n}^{1 \leq \alpha \leq m} \in \mathbf{C}^{(1+n)m}$ , we write  $\mathbf{v}_0 \in \mathbf{C}^m$  and  $\mathbf{v}_\parallel \in \mathbf{C}^{nm}$  for the normal and tangential parts of  $\mathbf{v}$ , i.e.  $\mathbf{v}_0 = (\mathbf{v}_0^\alpha)_{1 \leq \alpha \leq m}$  whereas  $\mathbf{v}_\parallel = (\mathbf{v}_i^\alpha)_{1 \leq i \leq n}^{1 \leq \alpha \leq m}$ .

For systems, gradient and divergence act as  $(\nabla_{t,x} U)_i^\alpha = \partial_i U^\alpha$  and  $(\operatorname{div}_{t,x} \mathbf{F})^\alpha = \sum_{i=0}^n \partial_i \mathbf{F}_i^\alpha$ , with corresponding tangential versions  $\nabla_x U = (\nabla_{t,x} U)_\parallel$  and  $(\operatorname{div}_x \mathbf{F})^\alpha = \sum_{i=1}^n \partial_i \mathbf{F}_i^\alpha$ . With  $\operatorname{curl}_x \mathbf{F}_\parallel = 0$ , we understand  $\partial_j \mathbf{F}_i^\alpha = \partial_i \mathbf{F}_j^\alpha$ , for all  $i, j = 1, \dots, n, \alpha = 1, \dots, m$ .

We consider divergence form second order elliptic systems

$$(2) \quad \sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i A_{i,j}^{\alpha,\beta}(x) \partial_j U^\beta(t, x) = 0, \quad \alpha = 1, \dots, m,$$

on the half space  $\mathbf{R}_+^{1+n} := \{(t, x) \in \mathbf{R} \times \mathbf{R}^n ; t > 0\}$ ,  $n \geq 1$ , where the matrix  $A = (A_{i,j}^{\alpha,\beta}(x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$  is assumed to be  $t$ -independent with complex coefficients and *strictly accretive* on  $\mathbf{N}(\operatorname{curl}_\parallel)$ , in the sense that there exists  $\kappa > 0$  such that

$$(3) \quad \sum_{i,j=0}^n \sum_{\alpha,\beta=1}^m \int_{\mathbf{R}^n} \operatorname{Re}(A_{i,j}^{\alpha,\beta}(x) \mathbf{f}_j^\beta(x) \overline{\mathbf{f}_i^\alpha(x)}) dx \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbf{R}^n} |\mathbf{f}_i^\alpha(x)|^2 dx,$$

for all  $\mathbf{f} \in \mathbf{N}(\operatorname{curl}_\parallel) := \{\mathbf{g} \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) ; \operatorname{curl}_x(\mathbf{g}_\parallel) = 0\}$ . This is nothing but ellipticity in the sense of Gårding. See the discussion in [3]. By changing  $m$  to  $2m$  we could assume that the coefficients are real-valued. But this does not simplify matters and we need the complex hermitean structure of our  $L_2$  space anyway.

**Definition 2.1.** The Dirichlet problem (Dir- $A$ ) is said to be *well-posed* if for each  $u \in L_2(\mathbf{R}^n; \mathbf{C}^m)$ , there is a unique function

$$U_t(x) = U(t, x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$$

such that  $\nabla_x U \in C^0(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{nm}))$ , where  $U$  satisfies (2) for  $t > 0$ ,  $\lim_{t \rightarrow 0} U_t = u$ ,  $\lim_{t \rightarrow \infty} U_t = 0$ ,  $\lim_{t \rightarrow \infty} \nabla_{t,x} U_t = 0$  in  $L_2$  norm, and  $\int_{t_0}^{t_1} \nabla_x U_s ds$  converges in  $L_2$  when  $t_0 \rightarrow 0$  and  $t_1 \rightarrow \infty$ . More precisely, by  $U$  satisfying (2), we mean that  $\int_t^\infty ((A \nabla_{s,x} U_s)_\parallel, \nabla_x v) ds = -((A \nabla_{t,x} U_t)_0, v)$  for all  $v \in C_0^\infty(\mathbf{R}^n; \mathbf{C}^m)$ .

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [11]. However, a sufficient condition is provided in [3] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem (see Section 4).

**Remark 2.2.** In the case of block matrices, ie  $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$ ,  $1 \leq i \leq n$ ,  $1 \leq \alpha, \beta \leq m$ , the second order system (2) can be solved using semi-group theory:  $V(t, \cdot) = e^{-tL^{1/2}} u_0$  for  $L = -A_{00}^{-1} \operatorname{div}_x A_{\parallel\parallel} \nabla_x$  acting as an unbounded operator on  $L_2(\mathbf{R}^n, \mathbf{C}^{nm})$  (See below for the notation). This solution satisfies  $V_t = V(t, \cdot) \in C^2(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m)) \cap C^1(\mathbf{R}_+, D(L^{1/2}))$ ,  $\lim_{t \rightarrow 0} V_t = u_0$ ,  $\lim_{t \rightarrow \infty} V_t = 0$  in  $L_2$  norm, and (2) holds in the strong sense in  $\mathbf{R}^n$  for all  $t > 0$  (and in the sense of distributions in  $\mathbf{R}_+^{1+n}$ ). Hence, the two notions of solvability are not *a priori* equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for  $L$ :  $D(L^{1/2}) = W^{1,2}(\mathbf{R}^n, \mathbf{C}^{nm})$  with  $\|L^{1/2} f\|_2 \sim \|\nabla_x f\|_2$ . See [6] where this is explicitly proved when  $A_{00} \neq I$ .

The following result is Corollary 3.4 of [3] (which, as we recall, furnishes a different proof of results obtained by combining [12] and [9] in the case of real symmetric matrices equations ( $m = 1$ )).

**Theorem 2.3.** *Let  $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$  be a  $t$ -independent, complex matrix function which is strictly accretive on  $N(\operatorname{curl}_\parallel)$  and assume that  $(\operatorname{Dir}-A)$  is well-posed. Then any function  $U_t(x) = U(t, x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$  solving (2), with properties as in Definition 2.1, has estimates*

$$\int_{\mathbf{R}^n} |u|^2 dx \approx \sup_{t>0} \int_{\mathbf{R}^n} |U_t|^2 dx \approx \int_{\mathbf{R}^n} |\tilde{N}_*(U)|^2 dx \approx \|t \nabla_{t,x} U\|^2,$$

where  $u = U|_{\mathbf{R}^n}$ . If furthermore  $A$  is real (not necessarily symmetric) and  $m = 1$ , then Moser's local boundedness estimate [18] gives the pointwise estimate  $\tilde{N}_*(U)(x) \approx N_*(U)(x)$ , where the standard non-tangential maximal function is  $N_*(U)(x) := \sup_{|y-x|<ct} |U(t, y)|$ , for fixed  $0 < c < \infty$ .

We use the square-function norm

$$\|F_t\|^2 := \int_0^\infty \|F_t\|_2^2 \frac{dt}{t} = \iint_{\mathbf{R}_+^{1+n}} |F(t, x)|^2 \frac{dt dx}{t}$$

and the following version  $\tilde{N}_*(F)$  of the modified *non-tangential maximal function* introduced in [13]

$$\tilde{N}_*(F)(x) := \sup_{t>0} t^{-(1+n)/2} \|F\|_{L_2(Q(t,x))},$$

where  $Q(t, x) := [(1 - c_0)t, (1 + c_0)t] \times B(x; c_1 t)$ , for some fixed constants  $c_0 \in (0, 1)$ ,  $c_1 > 0$ .

Next is Theorem 3.2 of [3], specialized to the Dirichlet problem.

**Theorem 2.4.** *The set of matrices  $A$  for which  $(\text{Dir-}A)$  is well-posed is an open subset of  $L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ . Furthermore, it contains*

- (i) *all Hermitean matrices  $A(x) = A(x)^*$  (and in particular all real symmetric matrices),*
- (ii) *all block matrices where  $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$ ,  $1 \leq i \leq n, 1 \leq \alpha, \beta \leq m$ , and*
- (iii) *all constant matrices  $A(x) = A$ .*

More importantly is the solution algorithm using an “infinitesimal generator”  $T_A$ . Write  $\mathbf{v} \in \mathbf{C}^{(1+n)m}$  as  $\mathbf{v} = [\mathbf{v}_0, \mathbf{v}_\parallel]^t$ , where  $\mathbf{v}_0 \in \mathbf{C}^m$  and  $\mathbf{v}_\parallel \in \mathbf{C}^{nm}$ , and introduce the auxiliary matrices

$$\bar{A} := \begin{bmatrix} A_{00} & A_{0\parallel} \\ 0 & I \end{bmatrix}, \quad \underline{A} := \begin{bmatrix} 1 & 0 \\ A_{\parallel 0} & A_{\parallel\parallel} \end{bmatrix}, \quad \text{if } A = \begin{bmatrix} A_{00} & A_{0\parallel} \\ A_{\parallel 0} & A_{\parallel\parallel} \end{bmatrix}$$

in the normal/tangential splitting of  $\mathbf{C}^{(1+n)m}$ . The strict accretivity of  $A$  on  $\mathbf{N}(\text{curl}_\parallel)$ , as in (3), implies the pointwise strict accretivity of the diagonal block  $A_{00}$ . Hence  $A_{00}$  is invertible, and consequently  $\bar{A}$  is invertible [This is not necessarily true for  $\underline{A}$ .] We define

$$T_A = \bar{A}^{-1} D \underline{A}$$

as an unbounded operator on  $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$  with  $D$  the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{bmatrix} 0 & \text{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

**Proposition 2.5.** *Let  $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$  be a  $t$ -independent, complex matrix function which is strictly accretive on  $\mathbf{N}(\text{curl}_\parallel)$ .*

- (1) *The operator  $T_A$  has quadratic estimates and a bounded holomorphic functional calculus on  $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ . In particular, for any holomorphic function  $\psi$  on the left and right open half planes, with  $z\psi(z)$  and  $z^{-1}\psi(z)$  qualitatively bounded, one has*

$$\|\psi(tT_A)\mathbf{f}\| \lesssim \|\mathbf{f}\|_2.$$

- (2) *The Dirichlet problem  $(\text{Dir-}A)$  is well-posed if and only if the operator*

$$\mathcal{S} : \overline{R(\chi_+(T_A))} \rightarrow L_2(\mathbf{R}^n, \mathbf{C}^m), \mathbf{f} \mapsto \mathbf{f}_0$$

*is invertible. Here,  $\chi_+ = 1$  on the right open half plane and 0 on the left open half plane.*

Item (1) is [3, Corollary 3.6] (and see [4] for an explicit direct proof) and item (2) can be found in [3, Section 4, proof of Theorem 2.2].

**Lemma 2.6.** *Assume that  $(\text{Dir-}A)$  is well-posed. Let  $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$ . Then the solution  $U$  of  $(\text{Dir-}A)$  in the sense of Definition 2.1 is given by*

$$U(t, \cdot) = (e^{-tT_A}\mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1}u_0 \in \overline{R(\chi_+(T_A))}$$

*and furthermore*

$$\nabla_{t,x} U(t, \cdot) = \partial_t e^{-tT_A} \mathbf{f}.$$

*Proof.* [3, Lemma 4.2] (See also [2, Lemma 2.55] with a slightly different formulation of the Dirichlet problem).  $\square$

## 3. THE BILINEAR ESTIMATE

We are now in position to state and prove the generalisation of Hofmann's result.

**Theorem 3.1.** *Assume that (Dir-A) is well-posed. Let  $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$  and  $U$  be the solution to (Dir-A) in the sense of Definition 2.1. Then for all  $\mathbf{v}: \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^{(1+n)m}$  such that the right-hand side is finite,*

$$\left| \iint_{\mathbf{R}_+^{1+n}} \nabla_{t,x} U \cdot \bar{\mathbf{v}} \, dt dx \right| \leq C \|u_0\|_2 (\|t \nabla_{t,x} \mathbf{v}\| + \|N_* \mathbf{v}\|_2).$$

The pointwise values of  $\mathbf{v}(t, x)$  in the non-tangential control  $N_* \mathbf{v}$  can be slightly improved to  $L^1$  averages on balls having radii  $\sim t$  for each fixed  $t$ . See the end of proof.

*Proof.* It follows from the previous result that there exists  $\mathbf{f} \in \overline{\mathbf{R}(\chi_+(T_A))}$  such that  $U(t, \cdot) = (e^{-tT_A} \mathbf{f})_0$  and

$$\nabla_{t,x} U(t, \cdot) = \partial_t \mathbf{F} = -T_A e^{-tT_A} \mathbf{f}, \quad \mathbf{F} = e^{-tT_A} \mathbf{f}.$$

Integrating by parts with respect to  $t$ , we find

$$\iint_{\mathbf{R}_+^{1+n}} \nabla U \cdot \bar{\mathbf{v}} \, dt dx = - \iint_{\mathbf{R}_+^{1+n}} t \partial_t \mathbf{F} \cdot \bar{\partial_t \mathbf{v}} \, dt dx - \iint_{\mathbf{R}_+^{1+n}} t \partial_t^2 \mathbf{F} \cdot \bar{\mathbf{v}} \, dt dx.$$

The boundary term vanishes because  $t \partial_t \mathbf{F}$  goes to 0 in  $L_2$  when  $t \rightarrow 0, \infty$  (this uses  $\mathbf{f} \in \overline{\mathbf{R}(\chi_+(T_A))}$  and  $\sup_{t>0} \|\mathbf{v}(t, \cdot)\|_2 < \infty$  from  $\|N_* \mathbf{v}\|_2 < \infty$ ).

For the first term, we use Cauchy-Schwarz inequality and that  $\|t \partial_t \mathbf{F}\| \lesssim \|u_0\|_2$  from Theorem 2.3.

For the second term, we use the following identity:  $T_A = \bar{A}^{-1} D B \bar{A}$  with  $B = \underline{A} \bar{A}^{-1}$  which, by [3, Proposition 3.2], is strictly accretive on  $\mathbf{N}(\text{curl}_\parallel)$ , and observe that

$$\begin{aligned} t^2 \partial_t^2 \mathbf{F} &= \bar{A}^{-1} (t D B)^2 e^{-t D B} (\bar{A} \mathbf{f}) \\ &= \bar{A}^{-1} (t D B) (I + (t D B)^2)^{-1} \psi(t D B) (\bar{A} \mathbf{f}) \\ &= \bar{A}^{-1} (t D B) (I + (t D B)^2)^{-1} \bar{A} \psi(t T_A) (\mathbf{f}) \end{aligned}$$

with

$$\psi(z) = z(1 + z^2)e^{-(\text{sgn Re } z)z}.$$

Thus,

$$\iint_{\mathbf{R}_+^{1+n}} t \partial_t^2 \mathbf{F} \cdot \bar{\mathbf{v}} \, dt dx = \iint_{\mathbf{R}_+^{1+n}} \bar{A} \psi(t T_A) (\mathbf{f}) \cdot \overline{Q_t \mathbf{v}_t} \frac{dt dx}{t}$$

with  $Q_t = \Theta_t \bar{A}^{-1*}$  and  $\Theta_t = (t B^* D) (I + (t B^* D)^2)^{-1}$  acting on  $\mathbf{v}_t \equiv \mathbf{v}(\mathbf{t}, \cdot)$  for each fixed  $t$  [The notation  $\bar{A}$  has nothing to do with complex conjugate and we apologize for any conflict this may cause.] It follows from the quadratic estimates of Proposition 2.5 that

$$\|\psi(t T_A) (\mathbf{f})\| \lesssim \|\mathbf{f}\|_2.$$

It remains to estimate  $\|Q_t \mathbf{v}_t\|$ . To do that we follow the principal part approximation of [4] - which is an elaboration of the so-called Coifman-Meyer trick [7] - applied

to  $Q_t$  instead of  $\Theta_t$  there. That is, we write

$$(4) \quad Q_t \mathbf{v}_t = Q_t \left( \frac{I - P_t}{t(-\Delta)^{1/2}} \right) t(-\Delta)^{-1/2} \mathbf{v}_t + (Q_t P_t - \gamma_t S_t P_t) \mathbf{v}_t + \gamma_t S_t P_t \mathbf{v}_t$$

where  $\Delta$  is the Laplacian on  $\mathbf{R}^n$ ,  $P_t$  is a nice scalar approximation to the identity acting componentwise on  $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$  and  $\gamma_t$  is the element of  $L_{\text{loc}}^2(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$  given by

$$\gamma_t(x) \mathbf{w} := (Q_t \mathbf{w})(x)$$

for every  $\mathbf{w} \in \mathbf{C}^{(1+n)m}$ . We view  $\mathbf{w}$  on the right-hand side of the above equation as the constant function valued in  $\mathbf{C}^{(1+n)m}$  defined on  $\mathbf{R}^n$  by  $\mathbf{w}(x) := \mathbf{w}$ . We identify  $\gamma_t(x)$  with the (possibly unbounded) multiplication operator  $\gamma_t : f(x) \mapsto \gamma_t(x)f(x)$ . Finally, the *dyadic averaging operator*  $S_t : L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m}) \rightarrow L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$  is given by

$$S_t \mathbf{u}(x) := \frac{1}{|Q|} \int_Q \mathbf{u}(y) dy$$

for every  $x \in \mathbf{R}^n$  and  $t > 0$ , where  $Q$  is the unique dyadic cube in  $\mathbf{R}^n$  that contains  $x$  and has side length  $\ell$  with  $\ell/2 < t \leq \ell$ .

With this in hand, we apply the triple bar norm to (4).

Using the uniform  $L_2$  boundedness of  $Q_t$  and that of  $\frac{1-P_t}{t(-\Delta)^{1/2}}$ , the first term in the RHS is bounded by  $\| \|t(-\Delta)^{1/2} \mathbf{v}_t\| \| \leq \| \|t \nabla_x \mathbf{v}_t\| \|$ .

Following exactly the computation of Lemma 3.6 in [4], the second term in the RHS is bounded by  $C \| \|t \nabla_x P_t \mathbf{v}_t\| \| \leq C \| \|t \nabla_x \mathbf{v}_t\| \|$  using the uniform  $L_2$  boundedness of  $P_t$ . This computation makes use of the off-diagonal estimates of  $\Theta_t$ , hence of  $Q_t$ , proved in [4, Proposition 3.11].

For the third term in the RHS, we observe that  $\gamma_t(x) \mathbf{w} = \Theta_t(\bar{A}^{-1*} \mathbf{w})(x)$ . Hence, the square-function estimate on  $\Theta_t$  proved in [4, Theorem 1.1], the off-diagonal estimates of  $\Theta_t$  and the fact that  $\bar{A}^{-1}$  is bounded imply that  $|\gamma_t(x)|^2 \frac{dtdx}{t}$  is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes  $\|N_*(S_t P_t \mathbf{v})\|_2$ , which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening.  $\square$

#### 4. THE DOMAIN OF THE DIRICHLET SEMI-GROUP

Assume (Dir-A) in the sense of Definition 2.1 is well-posed. If we set

$$\mathcal{P}_t u_0 = (e^{-tT_A} \mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1} u_0 \in \overline{\mathbf{R}(\chi_+(T_A))}$$

for all  $t > 0$ , then Lemma 2.6 implies that  $(\mathcal{P}_t)_{t>0}$  is a bounded  $C_0$ -semigroup on  $L_2(\mathbf{R}^n, \mathbf{C}^m)$  [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].

Furthermore, with our definition of well-posedness of the Dirichlet problem, the domain of the infinitesimal generator  $\mathcal{A}$  of this semi-group is contained in the Sobolev space  $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$  and  $\|\nabla_x u_0\|_2 \lesssim \|\mathcal{A} u_0\|_2$ . Indeed, from Lemma 2.6 we have for all  $t > 0$ ,  $\partial_t e^{-tT_A} \mathbf{f} = \nabla_{t,x} U(t, \cdot)$ . Also  $\partial_t e^{-tT_A} \mathbf{f} \in \overline{\mathbf{R}(\chi_+(T_A))}$  and the invertibility of  $\mathcal{S}$  tells that  $\nabla_{t,x} U(t, \cdot) = \mathcal{S}^{-1}(\partial_t U(t, \cdot))$ . Therefore

$$\|\nabla_x U(t, \cdot)\|_2 \lesssim \|\partial_t U(t, \cdot)\|_2.$$

By definition of  $\mathcal{A}$ ,  $\partial_t U(t, \cdot) = \mathcal{A} U(t, \cdot)$ , thus we have for all  $t > 0$

$$\|\nabla_x U(t, \cdot)\|_2 \lesssim \|\mathcal{A} U(t, \cdot)\|_2.$$

The conclusion for the domain follows easily.

The question of whether this domain coincides with  $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$  is answered by the following theorem

**Theorem 4.1.** *Assume that  $(\text{Dir-}A)$  and  $(\text{Dir-}A^*)$  are well-posed. Then the domain of the infinitesimal generator  $\mathcal{A}$  of  $(\mathcal{P}_t)_{t>0}$  coincides with the Sobolev space  $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$  and  $\|\nabla_x u_0\|_2 \sim \|\mathcal{A}u_0\|_2$ .*

This theorem applies to the three situations listed in Theorem 2.4.

*Proof.* Combining [4, Lemma 4.2] (which says that  $(\text{Dir-}A^*)$  is equivalent to an auxiliary Neumann problem for  $A^*$ ), [2, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for  $A$ : this is non trivial) with the proof of Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for  $A$ ), we have that  $(\text{Dir-}A^*)$  is well-posed if and only if

$$\mathcal{R} : \overline{\mathcal{R}(\chi_+(T_A))} \rightarrow L_2(\mathbf{R}^n, \mathbf{C}^{nm}), \mathbf{f} \mapsto \mathbf{f}_\parallel$$

is invertible. This implies that for  $\mathbf{f} \in \overline{\mathcal{R}(\chi_+(T_A))}$ , we have that

$$\|\mathbf{f}\|_2 \sim \|\mathbf{f}_\parallel\|_2.$$

Therefore, the conjunction of well-posedness for  $(\text{Dir-}A)$  and  $(\text{Dir-}A^*)$  gives

$$\|\mathbf{f}_0\|_2 \sim \|\mathbf{f}_\parallel\|_2, \quad \mathbf{f} \in \overline{\mathcal{R}(\chi_+(T_A))}.$$

From this, it is easy to identify the domain of  $\mathcal{A}$  by an argument as before. □

We have seen that invertibility of  $\mathcal{S}$  reduces to that of  $\mathcal{R}$  (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that  $\|\mathbf{f}_\parallel\|_2 \sim \|(A\mathbf{f})_0\|_2$  for all  $\mathbf{f} \in \overline{\mathcal{R}(\chi_+(T_A))}$ . This method was first used in the context of Laplace equation on Lipschitz domains by Verchota [21]. This depends strongly of  $A$ . Various relations between Dirichlet, regularity and Neumann problems for  $L^p$  data in the sense of non tangential approach for second order real symmetric equations are studied in [13, 14] and more recently in [15, 19].

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