On Hofmann's bilinear estimate

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Abstract. We generalize a result of S. Hofmann to systems, using the framework of a previous paper with Axelsson and McIntosh.

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1. Introduction

S. Hofmann proved in [10] that weak solutions of

$$\operatorname{div}_{t,x} A(x)\nabla_{t,x} U(t,x) = \sum_{i,j=0}^{n} \partial_i A_{i,j}(x)\partial_j U(t,x) = 0 \tag{1}$$

on the upper half space $\mathbb{R}^{1+n}_+ := \{(t,x)\in \mathbb{R}\times\mathbb{R}^n; t>0\}$, $n\geq 1$, where the matrix $A = (A_{i,j}(x))_{i,j=0}^n \in L_1(\mathbb{R}^n; L(\mathbb{C}^{1+n}))$ is assumed to be t-independent and within some small L_1 neighborhood of a real symmetric strictly elliptic t-independent matrix, obey the following bilinear estimate

$$\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} U\, \overline{v}|\, dtdx \leq C\|U_0\|_2 (\|\nabla v\| + \|N v\|_2).$$

The trace of U at $t=0$ is assumed to be in the sense of non-tangential convergence a.e. and in $L_2(\mathbb{R}^n)$. See below for the definition of the square function $\|\|$ and the non-tangential maximal operator N. In addition, he proves that the solution operator $U_0 \to U(t,\cdot)$ defines a bounded C_0 semi-group on $L_2(\mathbb{R}^n)$ whose infinitesimal generator A has domain $W^{1,2}(\mathbb{R}^n)$ with $\|Af\|_2 \approx \|\nabla f\|_2$. This generalizes results of B. Dahlberg [7] corresponding to the case where $\operatorname{div} A\nabla$ is the pullback of the Laplace operator on a special Lipschitz domain. Hofmann's results utilize the deep results of [1] and in particular Theorem 1.11 where the boundedness and invertibility of the layer potentials are obtained from a T(b) theorem, Rellich estimates in the case of real symmetric matrices and perturbation. This also generalizes somehow the case where $A_{0,i} = A_{i,0} = 0$ for $i = 1,\dots,n$ corresponding to the Kato square root problem.

The recent papers [3, 4] allows us to extend this further to systems, making clear in particular that specificities of real symmetric coefficients and of equations are not needed: it only depends on whether the Dirichlet problem is solvable. We use the solution operator constructed in [3] and this also makes transparent the para-product like character of this bilinear estimate. We also state a necessary and sufficient condition telling when the domain of the infinitesimal generator A is $W^{1,2}$.

2. Setting

We begin by giving the precise definition of well posedness of the BVPs for systems. Throughout this note, we use the notation $X \lesssim Y$ and $X \simeq Y$ for estimates to mean that there exists a constant $C > 0$, independent of the variables in the estimate, such that $X \le C Y$ and $X \ge C Y$, respectively.

We write (t,x) for the standard coordinates for $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$, t standing for the vertical or normal coordinate. For vectors $v = (v_i)_{0 \le i \le n}^{1 \le \alpha \le m} \in \mathbb{C}^{(1+n)m}$, we write $v_0 \in \mathbb{C}^m$ and $v_k \in \mathbb{C}^{nm}$ for the normal and tangential parts of v, i.e. $v_0 = (v_0^\alpha)_{1 \le \alpha \le m}$ whereas $v_k = (v_i^\alpha)_{1 \le i \le n}^{1 \le \alpha \le m}$.

For systems, gradient and divergence act as $(\nabla_{t,x} U)_i = \partial_i U$ and $(\mathrm{div}_{t,x} F) = \sum_{i=0}^n \partial_i F_i$, with corresponding tangential versions $\nabla_x U = (\nabla_{t,x} U)_k$ and $(\mathrm{div}_x F) = \sum_{i=1}^n \partial_i F_i$. With $\mathrm{curl}_x F_k = 0$, we understand $\partial_j F_i = \partial_i F_j$, for all $i,j = 1,\dots,n$; $\alpha = 1,\dots,m$.

We consider divergence form second order elliptic systems

$$\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i A_{i,j}^{\alpha,\beta}(x) \partial_j U^\beta(t,x) = 0, \qquad \alpha = 1,\dots,m, \tag{2}$$

on the half space $\mathbb{R}^{1+n}_+ := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n ; t > 0\}$, $n \ge 1$, where the matrix $A = (A_{ij}^{\alpha,\beta}(x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ is assumed to be t-independent with complex coefficients and strictly accretive on $N(\mathrm{curl}_x)$, in the sense that there exists $\kappa > 0$ such that

$$\sum_{i,j=0}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathbb{R}^n} \mathrm{Re}(A_{i,j}^{\alpha,\beta}(x) f_j^\beta(x) \overline{f_i^\alpha(x)}) dx \ge \kappa \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 dx, \tag{3}$$

for all $f \in N(\mathrm{curl}_x) := \{g \in L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m}) ; \mathrm{curl}_x(g_k) = 0\}$. By changing m to $2m$ we could assume that the coefficients are real-valued. But this does not simplify matters and we need the complex hermitean structure of our L_2 space anyway.

Definition 2.1. The Dirichlet problem (Dir-A) is said to be well posed if for each $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$, there is a unique function

$$U_t(x) = U(t,x) \in C^1(\mathbb{R}_+; L_2(\mathbb{R}^n; \mathbb{C}^m))$$

such that $\nabla_x U \in C^0(\mathbb{R}_+; L_2(\mathbb{R}^n; \mathbb{C}^{nm}))$, where U satisfies (2) for $t > 0$, $\lim_{t \to 0} U_t = u$, $\lim_{t \to \infty} U_t = 0$, $\lim_{t \to \infty} \nabla_{t,x} U_t = 0$ in L_2 norm, and $\int_{t_0}^{t_1} \nabla_x U_s ds$ converges in L_2 when $t_0 \to 0$ and $t_1 \to \infty$. More precisely, by U satisfying (2), we mean that $\int_t^\infty ((A \nabla_{s,x} U_s)_k ; \nabla_x v) ds = ((A \nabla_{t,x} U_t)_0 ; v)$ for all $v \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^m)$.

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [10]. However, a sufficient condition is provided in [3] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem (see Section 4).

Remark 2.2. In the case of block matrices, ie $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \le i \le n$; $1 \le \alpha, \beta \le m$, the second order system (2) can be solved using semi-group theory: $V(t,\cdot) = e^{-tL^{1/2}} u_0$ for $L = -A_{00}^{-1} \mathrm{div}_x A_{kk} \nabla_x$ acting as an unbounded operator on $L_2(\mathbb{R}^n; \mathbb{C}^{nm})$ (See below for the notation). This solution satisfies $V_t = V(t,\cdot) \in C^2(\mathbb{R}_+; L_2(\mathbb{R}^n; \mathbb{C}^m)) \cap C^1(\mathbb{R}_+; D(L^{1/2}))$; $\lim_{t \to 0} V_t = u_0$, $\lim_{t \to \infty} V_t = 0$ in L_2 norm,

and (2) holds in the strong sense in $\mathbb{R}^n$ for all $t > 0$ (and in the sense of distributions in $\mathbb{R}^{1+n}_+$). Hence, the two notions of solvability are not a priori equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for $L : D(L^{1/2}) = W^{1,2}(\mathbb{R}^n; \mathbb{C}^{nm})$ with $\|L^{1/2} f\|_2 \approx \|\nabla_x f\|_2$. See [6] where this is explicitly proved when $A_{00} \equiv I$.

The following result is Corollary 3.4 of [3] (which, as we recall, furnishes a different proof of results obtained by combining [11] and [8] in the case of real symmetric matrices equations ($m = 1$)).

Theorem 2.3. Let $A \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathcal{N}(\mathrm{curl}_x)$ and assume that (Dir-A) is well-posed. Then any function $U_t(x) = U(t,x) \in C^1(\mathbb{R}_+; L_2(\mathbb{R}^n; \mathbb{C}^m))$ solving (2), with properties as in Definition 2.1, has estimates

$$\int_{\mathbb{R}^n} |u|^2 dx \approx \sup_{t>0} \int_{\mathbb{R}^n} |U_t|^2 dx \approx \int_{\mathbb{R}^n} |\mathcal{N}(U)|^2 dx \approx \|\nabla_{t,x} U\|^2,$$

where $u = U|_{\mathbb{R}^n}$. If furthermore A is real (not necessarily symmetric) and $m = 1$, then Moser's local boundedness estimate [14] gives the pointwise estimate $\mathcal{N}(U)(x) \lesssim N(U)(x)$, where the standard non-tangential maximal function is $N(U)(x) := \sup_{|y-x|<ct} |U(t,y)|$, for fixed $0 < c < 1$.

We use the square function norm

$$\|F\|^2 := \int_0^\infty \|F_t\|_2^2 \frac{dt}{t} = \int\!\!\int_{\mathbb{R}^{1+n}_+} |F(t,x)|^2 \frac{dtdx}{t}$$

and the norm $\|\mathcal{N}(F)\|_2$, using the modified non-tangential maximal function

$$\mathcal{N}(F)(x) := \sup_{t>0} t^{-(1+n)/2} \|F\|_{L_2(Q(t,x))},$$

where $Q(t,x) := [(1-c_0)t, (1+c_0)t] \times B(x, c_1 t)$, for some fixed constants $c_0 \in (0,1)$, $c_1 > 0$.

Next is Theorem 3.2 of [3], specialized to the Dirichlet problem.

Theorem 2.4. The set of matrices A for which (Dir-A) is well-posed is an open subset of $L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$. Furthermore, it contains

(i) all Hermitean matrices $A(x) = A(x)^*$ (and in particular all real symmetric matrices),

(ii) all block matrices where $A_{0,i}(x) = 0 = A_{i,0}(x)$, $1 \le i \le n$; $1 \le \ell \le m$, and

(iii) all constant matrices $A(x) = A$.

More importantly is the solution algorithm using an "infinitesimal generator" T_A. Write $v \in \mathbb{C}^{(1+n)m}$ as $v = [v_0, v_k]^t$, where $v_0 \in \mathbb{C}^m$ and $v_k \in \mathbb{C}^{nm}$, and introduce the auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{00} & A_{0k} \\ 0 & I \end{bmatrix}; \quad \underline{A} := \begin{bmatrix} 1 & 0 \\ A_{k0} & A_{kk} \end{bmatrix}; \quad \text{if } A = \begin{bmatrix} A_{00} & A_{0k} \\ A_{k0} & A_{kk} \end{bmatrix}$$

in the normal/tangential splitting of $\mathbb{C}^{(1+n)m}$. The strict accretivity of A on $\mathcal{N}(\mathrm{curl}_x)$, as in (3), implies the pointwise strict accretivity of the diagonal block A_{00}. Hence

A_{00} is invertible, and consequently $\overline{A}$ is invertible [This is not necessarily true for $\underline{A}$.] We define

$$T_A = \overline{A}^{-1} D \underline{A}$$

as an unbounded operator on $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ with D the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{pmatrix} 0 & \operatorname{div}_x \\ \nabla_x & 0 \end{pmatrix}.$$

Proposition 2.5. Let $A \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $N(\operatorname{curl}_x)$.

(1) The operator T_A has quadratic estimates and a bounded holomorphic functional calculus on $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$. In particular, for any holomorphic function ψ on the left and right open half planes, with $z\psi(z)$ and $z^{-1}\psi(z)$ qualitatively bounded, one has

$$\|\psi(tT_A) f\|_2 \lesssim \|f\|_2.$$

(2) The Dirichlet problem (Dir-A) is well-posed if and only if the operator

$$S : \overline{R(\Psi_+(T_A))} \to L_2(\mathbb{R}^n; \mathbb{C}^m); f \mapsto f_0$$

is invertible. Here, $\Psi_+ = 1$ on the right open half plane and 0 on the left open half plane.

Item (1) is [3, Corollary 3.6] (and see [4] for an explicit direct proof) and item (2) can be found in [3, Section 4, proof of Theorem 2.2].

Lemma 2.6. Assume that (Dir-A) is well-posed. Let $u_0 \in L_2(\mathbb{R}^n; \mathbb{C}^m)$. Then the solution U of (Dir-A) in the sense of definition 2.1 is given by

$$U(t, \cdot) = (e^{tT_A} f)_0, \quad f = S^{-1} u_0 \in \overline{R(\Psi_+(T_A))}$$

and furthermore

$$\nabla_{t,x} U(t, \cdot) = (e^{tT_A} f)_1.$$

Proof. [3, Lemma 4.2] (See also [2, Lemma 2.55] with a slightly different formulation of the Dirichlet problem).

3. The bilinear estimate

We are now in position to state and prove the generalisation of Hofmann's result.

Theorem 3.1. Assume that (Dir-A) is well-posed. Let $u_0 \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ and U be the solution to (Dir-A) in the sense of definition 2.1. Then for all $v : \mathbb{R}^{1+n}_+ \to \mathbb{C}^{(1+n)m}$ such that the right hand side is finite,

$$\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} U \cdot \overline{v}| \, dt dx \lesssim \|u_0\|_2 (\|\nabla_{t,x} v\|_+ + \|N v\|_2).$$

The pointwise values of $v(t,x)$ in the non-tangential control Nv can be slightly improved to L^1 averages on balls having radii t for each fixed t. See the end of proof.

Proof. It follows from the previous result that there exists $f \in \overline{R(\Pi_+(T_A))}$ such that $U(t,\cdot) = (e^{tT_A}f)_0$ and

$$\nabla_{t,x} U(t,\cdot) = \mathcal{E}F = \Pi_A e^{tT_A} f, \quad F = e^{tT_A} f.$$

Integrating by parts with respect to t, we find

$$\iint_{\mathbb{R}^{1+n}_+} \nabla U \cdot \overline{v}\, dtdx = \iint_{\mathbb{R}^{1+n}_+} t\partial_t F \cdot \overline{\mathcal{E}v}\, dtdx - \iint_{\mathbb{R}^{1+n}_+} t\partial_t^2 F \cdot \overline{v}\, dtdx.$$

The boundary term vanishes because $t\partial_t F$ goes to 0 in L_2 when $t \to 0,\infty$ (this uses $f \in \overline{R(\Pi_+(T_A))}$) and $\sup_{t>0} \|v(t,\cdot)\|_2 < \infty$ from $\|N_+ v\|_2 < \infty$.

For the first term, we use Cauchy-Schwarz inequality and that $\|t\partial_t F\|_2 \lesssim \|u_0\|_2$ from Theorem 2.3.

For the second term, we use the following identity: $T_A = \overline{A}^{-1} D_B \overline{A}$ with $B = \underline{A}\overline{A}^{-1}$, which by [3, Proposition 3.2] is strictly accretive on $N(\mathrm{curl}_x)$, and

$$\begin{aligned} t^2\partial_t^2 F &= \overline{A}^{-1}(tD_B)^2 e^{-tD_B}(\overline{A}f) \\ &= \overline{A}^{-1}(tD_B)(I+(tD_B)^2)^{-1}(tD_B)(\overline{A}f) \\ &= \overline{A}^{-1}(tD_B)(I+(tD_B)^2)^{-1}\overline{A}\,(t\Pi_A)(f) \end{aligned}$$

with

$$\psi(z) = z(1+z^2)e^{-(\mathrm{sgn}\,\mathrm{Re}\, z)z}.$$

Thus,

$$\iint_{\mathbb{R}^{1+n}_+} t\partial_t^2 F \cdot \overline{v}\, dtdx = \iint_{\mathbb{R}^{1+n}_+} \overline{A}\,(t\Pi_A)(f) \cdot \overline{Q_t^* v_t}\, \frac{dtdx}{t}$$

with $Q_t = \psi_t \overline{A}^{-1}$ and $\Gamma_t = (tB_{D^*})(I+(tB_{D^*})^2)^{-1}$ acting on $v_t := v(t,\cdot)$ for each fixed t. [The notation $\overline{A}$ has nothing to do with complex conjugate and we apologize for any confusion this may cause.] It follows from the quadratic estimates proved in Proposition 2.5 that

$$\|(t\Pi_A)(f)\|_2 \lesssim \|f\|_2.$$

It remains to estimate $\|Q_t v_t\|_2$. To do that we follow the principal part approximation of [4] applied to Q_t instead of Γ_t there. That is, we write

$$Q_t v_t = Q_t \frac{I - P_t}{t(-\Delta)^{1/2}}\, t(-\Delta)^{1/2} v_t + (Q_t P_t - \Gamma_t S_t P_t) v_t + \Gamma_t S_t P_t v_t \tag{4}$$

where Δ is the Laplacian on $\mathbb{R}^n$, P_t is a nice scalar approximation to the identity acting componentwise on $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ and Γ_t is the element of $L^2_{\mathrm{loc}}(\mathbb{R}^n; L(\mathbb{C}^{(1+n)m}))$ given by

$$\Gamma_t(x) w := (Q_t w)(x)$$

for every $w \in \mathbb{C}^{(1+n)m}$. We view w on the right-hand side of the above equation as the constant function valued in $\mathbb{C}^{(1+n)m}$ defined on $\mathbb{R}^n$ by $w(x) := w$. We identify $\Gamma_t(x)$ with the (possibly unbounded) multiplication operator $\Gamma_t : f(x) \mapsto \Gamma_t(x) f(x)$. Finally, the dyadic averaging operator $S_t : L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m}) \to L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ is given by

$$S_t u(x) := \frac{1}{|Q|} \int_Q u(y)\, dy$$

for every $x \in \mathbb{R}^n$ and $t > 0$, where Q is the unique dyadic cube in $\mathbb{R}^n$ that contains x and has side length ℓ with $\ell/2 < t \le \ell$.

With this in hand, we apply the triple bar norm to (4).

Using the uniform L_2 boundedness of Q_t and that of $\frac{1}{t(\quad)^{1/2}}$, the first term in the RHS is bounded by $k|t(\quad)^{1/2}v_t|\quad k|tr_x v_t|k|$.

Following exactly the computation of Lemma 3.6 in [4], the second term in the RHS is bounded by $Ck|tr_x P_t v_t|k|\quad Ck|tr_x v_t|k|$ using the uniform L_2 boundedness of P_t. This computation makes use of the off-diagonal estimates of $_t$, hence of Q_t, proved in [4, Proposition 3.11].

For the third term in the RHS, we observe that $_t(x)w = \ _t(\overline{A}^{\ 1}w)(x)$. Hence, the square function estimate on $_t$ proved in [4, Theorem 1.1], the off-diagonal estimates of $_t$ and the fact that $\overline{A}^{\ 1}$ is bounded imply that $|_t(x)|^2\frac{dtdx}{t}$ is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes to $kN\ (S_tP_tv)k_2$, which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening.

4. The domain of the Dirichlet semigroup

Assume (Dir-A) in the sense of definition 2.1 is well-posed. If we set

$$P_tu_0 = (e^{\ tT_A}f)_0; \quad f = S^{\ 1}u_0 \ 2\ \overline{R(\ _+(T_A))}$$

for all $t > 0$, then Lemma 2.6 implies that $(P_t)_{t>0}$ is a bounded C_0-semigroup on $L_2(R^n;C^m)$ [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].

Furthermore, with our definition of the Dirichlet problem, the domain of the infinitesimal generator A of this semi-group is contained the Sobolev space $W^{1,2}(R^n;C^m)$ and $kr_x u_0k_2 \ . \ kA\ u_0k_2$. Indeed, from Lemma 2.6 we have for all $t > 0$, $@_te^{\ tT_A}f = r_{t,x}U(t;\ ):$ Also $@_te^{\ tT_A}f\ 2\ \overline{R(\ _+(T_A))}$ and the invertibility of S tells that $r_{t,x}U(t;\ ) = S^{\ 1}(@_tU(t;\ ))$. Therefore

$$kr_xU(t;\ ))k_2 \ . \ k@_tU(t;\ )k_2:$$

By definition of A, $@_tU(t;\ ) = AU(t;\ )$, thus we have for all $t > 0$

$$kr_xU(t;\ ))k_2 \ . \ kAU(t;\ )k_2$$

The conclusion for the domain follows easily.

The question of whether this domain coincides with $W^{1,2}(R^n;C^m)$ is answered by the following theorem

Theorem 4.1. *Assume that (Dir-A) and (Dir-A) are well-posed. Then the domain of the infinitesimal generator A of $(P_t)_{t>0}$ coincides with the Sobolev space $W^{1,2}(R^n;C^m)$ and $kr_xu_0k_2 \quad kA\ u_0k_2$.*

This theorem applies to the three situations listed in Theorem 2.4.

Proof. Combining [4, Lemma 4.2] (which says that (Dir-A) is equivalent to an auxiliary Neumann problem for A), [2, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for A : this is non trivial) with the proof ot Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for A), we have that (Dir-A) is well-posed if and only if

$$R\ :\overline{R(\ _+(T_A))}\ !\ \ L_2(R^n;C^{nm});f\ 7!\ f_x$$

is invertible. This implies that for $f \in \overline{R(\ _+(T_A))}$, we have that

$$\|f\|_2 \quad \|f_\|_2 :$$

Therefore, the conjunction of well-posedness for (Dir-A) and (Dir-A) gives

$$\|f_0\|_2 \quad \|f_\|_2; \quad f \in \overline{R(\ _+(T_A))}:$$

From this, it is easy to conclude for the domain of A by an argument as before.

We have seen that invertibility of S reduces to that of R (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that $\|f_\|_2 \quad \|(A f)_0\|_2$ for all $f \in \overline{R(\ _+(T_A))}$: This method was first used in the context of Laplace equation on Lipschitz domains by Verchota [15]. This depends strongly of A. Various relations between Dirichlet, regularity and Neumann problems in the sense of non tangential approach for second order real symmetric equations are studied in [12, 13].

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