AN EMPIRICAL CENTRAL LIMIT THEOREM IN L^1 FOR STATIONARY SEQUENCES.

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ABSTRACT. In this paper, we derive asymptotic results for the $\mathbf{L}^1$-Wasserstein distance between the distribution function and the corresponding empirical distribution function of a stationary sequence. Next, we give some applications to dynamical systems and causal linear processes. To prove our main result, we give a Central Limit Theorem for ergodic stationary sequences of random variables with values in $\mathbf{L}^1$. The conditions obtained are expressed in terms of projective-type conditions. The main tools are martingale approximations.

1. INTRODUCTION

The Kantorovich or $\mathbf{L}^1$-Wasserstein distance between two probability measures P_1 and P_2 on $\mathbb{R}$ with finite mean, is defined by

$$d_1(P_1, P_2) := \inf \Big\{ \int |x - y| \, d\nu(x, y) : \ \nu \in \mathcal{P}(\mathbb{R}^2) \text{ with marginals } P_1, \, P_2 \Big\},$$

where $\mathcal{P}(\mathbb{R}^2)$ is the space of probability measures on $\mathbb{R}^2$.

Let Λ_1 be the space of 1-Lipschitz functions. It is well known that d_1 can also be written as follows:

$$d_1(P_1, P_2) = \int |F_2(t) - F_1(t)| \, dt = \sup_{f \in \Lambda_1} \Big| \int f dP_1 - \int f dP_2 \Big|,$$

where F_1 (respectively F_2) is the distribution function of P_1 (respectively of P_2).

Let $(X_i)_{i \in \mathbb{Z}}$ be a stationary sequence of real-valued random variables. In this paper, we are concerned with the Central Limit Theorem (CLT) for the $\mathbf{L}^1$-Wasserstein distance, defined by

$$\int_{\mathbb{R}} |F_n(t) - F_X(t)| \, dt, \tag{1.1}$$

where F_X is the common distribution function of the variables X_i, and F_n is the corresponding empirical distribution function (see Section 3).

In the literature, several previous works on the Kantorovich or $\mathbf{L}^1$-Wasserstein distance, have already been done, for a sequence of i.i.d random variables $X =$

$(X_i)_{i \in \mathbb{Z}}$ (see for instance del Barrio, Giné and Matrán [3]). Recall that if X has the distribution function F_X, then the condition

$$\int_{-\infty}^{\infty} \sqrt{F_X(t)(1 - F_X(t))}\, dt < \infty,$$

is equivalent to

$$\Lambda_{2,1}(X) := \int_0^{\infty} \sqrt{\mathbb{P}(|X| > t)}\, dt < \infty.$$

In their Theorem 2.1, del Barrio, Giné and Matrán [3] prove that if $(X_i)_{i \in \mathbb{Z}}$ is i.i.d, then the processes $\sqrt{n}(F_n - F_X)$ converge in law in $\mathbf{L}^1$ to the process $\{B(F(t)), t \in \mathbb{R}\}$, where B is a Brownian bridge, if and only if $\Lambda_{2,1}(X) < \infty$. Our main result extends Theorem 2.1 in del Barrio, Giné and Matràn [3] to the case of stationary sequences, satisfying some appropriate dependence conditions.

Before giving the idea of the proof, let us introduce $\mathbf{L}^1(\mu) = \mathbf{L}^1(\mathbf{T}, \mu)$, where μ is a σ-finite measure, the Banach space of μ-integrable real functions on $\mathbf{T}$, with the norm $\|.\|_{1,\mu}$, defined by $\|x\|_{1,\mu} = \int_{\mathbf{T}} |x(t)|\, \mu(dt)$. Let $\mathbf{L}^{\infty}(\mu)$ be its dual space.

First, we give the Central Limit Theorem (CLT) for ergodic stationary sequences of martingale differences in $\mathbf{L}^1(\mu)$ (see Section 4.1). Then, by martingale approximation (see for instance Volný [19]), we derive a Central Limit Theorem for some ergodic stationary sequences of $\mathbf{L}^1(\mu)$-valued random variables satisfying some projective criteria. This result allows us to get sufficient conditions to derive the asymptotic behavior of (1.1).

The paper is organized as follows. In Section 2, we state our main result. In Section 3, we derive the empirical Central Limit Theorem for statistics of the type (1.1) for a large class of dependent sequences. In particular, the results apply to unbounded functions of expanding maps of the interval, and to causal linear processes.

2. Central Limit Theorem for stationary sequences in $\mathbf{L}^1(\mu)$

From now, we assume that the ergodic stationary sequence $(X_i)_{i \in \mathbb{Z}}$ of centered random variables with values in $\mathbf{L}^1(\mu)$, is given by $X_i = X_0 \circ \mathbb{T}^i$, where $\mathbb{T} : \Omega \longrightarrow \Omega$ is a bijective bimeasurable transformation preserving the probability $\mathbb{P}$ on $(\Omega, \mathcal{A})$. Let $S_n = \sum_{j=1}^n X_j$, be the partial sums. For a subfield $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq \mathbb{T}^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = \mathbb{T}^{-i}(\mathcal{F}_0)$.

Notation 2.1. For any integer $p \geq 1$ and for any real random variable Y, we denote $\|.\|_p$, the $\mathbf{L}^p$-norm defined by $\|Y\|_p = \mathbb{E}(|Y|^p)^{1/p}$, and $\|.\|_{\infty}$ denotes the $\mathbf{L}^{\infty}$-norm, that is the smallest u such that $\mathbb{P}(|Y| > u) = 0$.

Here is our main result:

Theorem 2.2. *Assume that, for any real t, $\mathbb{E}(X_0(t)|\mathcal{F}_{-\infty}) = 0$, $\mathbb{E}(X_0(t)|\mathcal{F}_{\infty}) = X_0(t)$ and*

$$\int_{\mathbf{T}} \|X_0(t)\|_2\, \mu(dt) < \infty. \tag{2.1}$$

Let $P_0(X(t)) = \mathbb{E}(X(t)|\mathcal{F}_0) - \mathbb{E}(X(t)|\mathcal{F}_{-1})$ and assume that

$$\sum_{k \in \mathbb{Z}} \int_{\mathbf{T}} \|P_0(X_k(t))\|_2 \, \mu(dt) < \infty. \tag{2.2}$$

Then

$$n^{-1/2} \sum_{i=1}^{n} X_0 \circ \mathbb{T}^i \underset{n \to \infty}{\longrightarrow} G \text{ in law in } \mathbf{L}^1(\mu), \tag{2.3}$$

where G is a $\mathbf{L}^1(\mu)$-valued centered Gaussian random variable with covariance operator: for any $f \in \mathbf{L}^\infty(\mu)$,

$$\Phi_G(f, f) = \mathbb{E}\Big(\big(f\big(\sum_{k \in \mathbb{Z}} P_0(X_k)\big)\big)^2\Big) = \sum_{k \in \mathbb{Z}} \mathrm{Cov}(f(X_0), f(X_k)). \tag{2.4}$$

As a consequence, we have

Corollary 2.3. *Assume that (2.1) holds. Moreover, suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \|\mathbb{E}(X_n(t) \mid \mathcal{F}_0)\|_2 \, \mu(dt) < \infty, \tag{2.5}$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \|X_{-n} - \mathbb{E}(X_{-n}(t) \mid \mathcal{F}_0)\|_2 \, \mu(dt) < \infty. \tag{2.6}$$

Then, the conclusion of Theorem 2.2 holds.

3. Applications to the empirical distribution function

Let $Y = (Y_i)_{i \in \mathbb{Z}}$ be a sequence of real-valued random variables. We denote their common distribution function by F_Y and by F_n the corresponding empirical distribution function of Y:

$$\forall \, t \in \mathbb{R}, \ F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{Y_i \leq t}.$$

Let λ be the Lebesgue measure on $\mathbb{R}$. If $\mathbb{E}(|Y_1|) < \infty$, the random variable $X_i(.) = \{t \mapsto \mathbf{1}_{Y_i \leq t} - F_Y(t), t \in \mathbb{R}\}$ may be viewed as a centered random variable with values in $\mathbf{L}^1(\lambda)$.

Notation 3.1. Let $F_{Y_k|\mathcal{F}_0}$ be the conditional function of Y_k given $\mathcal{F}_0$, and let $F_{Y_k|\mathcal{F}_{-1}}$ be the conditional function of Y_k given $\mathcal{F}_{-1}$.

With these notations, the following equalities are valid: for every k in $\mathbb{Z}$,

$$\int_{\mathbf{T}} \|P_0(X_k(t))\|_2 \, dt = \int_{\mathbf{T}} \|F_{Y_k|\mathcal{F}_0}(t) - F_{Y_k|\mathcal{F}_{-1}}(t)\|_2 \, dt, \tag{3.1}$$

$$\text{and } \int_{\mathbf{T}} \|\mathbb{E}(X_k(t) \mid \mathcal{F}_0)\|_2 \, dt = \int_{\mathbf{T}} \|F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)\|_2 \, dt. \tag{3.2}$$

3.1. **Dependent sequences.**

As we shall see in this section, applying Corollary 2.3, we can derive sufficient conditions for the convergence in $\mathbf{L}^1(\lambda)$ of the process $\sqrt{n}(F_n - F_Y)$, as soon as the sequence Y satisfies some weak dependence conditions. Set $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$. We first recall the following dependence coefficients as defined in Dedecker and Prieur [9]: for any integer $k \geq 0$,

$$\tilde{\phi}(k) = \sup_{t \in \mathbb{R}} \|\mathbb{P}(Y_k \leq t \mid \mathcal{F}_0) - \mathbb{P}(Y_k \leq t)\|_\infty,$$

and

$$\tilde{\alpha}(k) = \sup_{t \in \mathbb{R}} \|\mathbb{P}(Y_k \leq t \mid \mathcal{F}_0) - \mathbb{P}(Y_k \leq t)\|_1.$$

When the sequence Y is $\tilde{\phi}$-dependent, the following result holds:

Proposition 3.2. *Assume that*

$$\sum_{k \geq 1} \sqrt{\frac{\tilde{\phi}(k)}{k}} < \infty \ and \ \int_0^\infty \sqrt{\mathbb{P}(|Y| > t)} \, dt < \infty, \tag{3.3}$$

then $\{t \mapsto \sqrt{n}(F_n(t) - F_Y(t)), t \in \mathbb{R}\}$ converges in $\mathbf{L}^1(\lambda)$, to a centered Gaussian random variable, with covariance function: for any f, $g \in \mathbf{L}^\infty(\lambda)$,

$$\Phi_\lambda(f, g) = \int_{\mathbb{R}^2} f(s)g(t)C(s,t) \, dt \, ds \tag{3.4}$$

with

$$C(s,t) = F_Y(t \wedge s) - F_Y(t)F_Y(s) + 2\sum_{k \geq 1}(\mathbb{P}(Y_0 \leq t, Y_k \leq s) - F_Y(t)F_Y(s)).$$

Remark 3.3. Proposition 3.2 is also true with the ϕ-mixing coefficient of Ibragimov [12]. Notice that this result contains the i.i.d case, developed in del Barrio, Giné and Matrán [3].

Before giving sufficient conditions when the sequence Y is $\tilde{\alpha}$-dependent, we first recall the following definition:

Definition 3.4. For any nonnegative and integrable random variable Y, define the quantile function Q_Y of $|Y|$, that is the cadlag inverse of the tail function $x \to \mathbb{P}(|Y| > x)$.

Proposition 3.5. *Assume that*

$$\sum_{k \geq 1} \frac{1}{\sqrt{k}} \int_0^{\tilde{\alpha}(k)} \frac{Q_Y(u)}{\sqrt{u}} \, du < \infty, \tag{3.5}$$

then the conclusion of Proposition 3.2 holds.

Remark 3.6. Notice that Proposition 3.5 is also true with strong α-mixing coefficients of Rosenblatt [18]. Notice also that (3.5) is equivalent to

$$\sum_{k \geq 1} \frac{1}{\sqrt{k}} \int_0^\infty \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{\mathbb{P}(|Y| > t)} \, dt < \infty. \tag{3.6}$$

3.1.1. *Application to expanding maps.* Let T be a map from $[0,1]$ to $[0,1]$ preserving a probability μ on $[0,1]$. Recall that the Perron-Frobenius operator K from $\mathbf{L}^1(\mu)$ to $\mathbf{L}^1(\mu)$ is defined via the equality: for any $h \in \mathbf{L}^1(\mu)$ and $f \in \mathbf{L}^\infty(\mu)$,

$$\int_0^1 (Kh)(x)f(x)\mu(dx) = \int_0^1 h(x)(f \circ T)(x)\mu(dx).$$

Here we are interested by giving sufficient conditions for the convergence in $\mathbf{L}^1(\lambda)$ of the empirical distribution function associated to F_Y where the random variables $(Y_i)_{i \in \mathbb{Z}}$ are defined as follows: for a given monotonic function f, let

$$Y_k = f \circ T^k. \tag{3.7}$$

In fact since on the probability $([0,1], \mu)$, the random variable $(T, T^2, ..., T^n)$ is distributed as $(Z_n, Z_{n-1}, ..., Z_1)$, where $(Z_i)_{i \geq 0}$ is a stationary Markov chain with invariant measure μ and transition Kernel K (see Lemma $XI.3$ in Hennion and Hervé [11]), the convergence in $\mathbf{L}^1(\lambda)$ of the empirical distribution function associated to F_Y is reduced to the one of the empirical distribution function associated to $F_{f(Z)}$.

In this section we consider two cases: first the case of a class of BV-contracting maps and secondly the case of a class of intermittent maps.

a) The case of BV-contracting maps. Let BV be the class of bounded variation functions from $[0,1]$ to $\mathbb{R}$. For any $h \in BV$, denote by $\|dh\|$ the variation norm of the measure dh. A Markov kernel K is said to be BV-contracting if there exist $C > 0$ and $\rho \in [0, 1[$ such that

$$\|dK^n(h)\| \leq C\rho^n \|dh\|. \tag{3.8}$$

A map T is then said to be BV-contracting if its Perron-Frobenius operator K is BV-contracting (see for instance Dedecker and Prieur [9], for more details and examples of maps which are BV-contracting).

In this case, the following result holds:

Corollary 3.7. *If T is BV-contracting and f is a monotonic function from $]0, 1[$ to $\mathbb{R}$ satisfying $\int_0^\infty \sqrt{\lambda(|f| > t)}\, dt < \infty$, then the conclusion of Proposition 3.2 holds for the sequence $(Y_k)_{k \in \mathbb{Z}}$ where Y_k is defined by (3.7).*

Remark 3.8. In the particular case when f is positive and non increasing on $]0, 1[$, with $f(x) \leq Dx^{-a}$ for some $a > 0$ and D a constant, we get that

$$\int_0^\infty \sqrt{\lambda(|f| > t)}\, dt \leq C_2 \int_1^\infty \frac{1}{t^{1/(2a)}}\, dt,$$

where C_2 is a constant. Consequently, Corollary 3.7 holds as soon as $a < \frac{1}{2}$ holds.

b) Application to intermittent maps. For γ in $]0, 1[$, we consider the intermittent map T_γ from $[0,1]$ to $[0,1]$, studied for instance by Liverani, Saussol and Vaienti [14], which is a modification of the Pomeau-Manneville map [16]:

$$T_\gamma = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

We denote by ν_γ the unique T_γ-probability measure on $[0, 1]$ and by K_γ the Perron-Frobenius operator of T_γ with respect to ν_γ. For these maps, we obtain the following result:

Corollary 3.9. *For γ in $]0, 1[$, if T_γ is an intermittent map and f is a monotonic function from $]0, 1[$ to $\mathbb{R}$, satisfying*

$$\sum_{k\geq 1} \frac{1}{\sqrt{k}} \int_0^\infty \frac{1}{k^{\frac{1-\gamma}{2\gamma}}} \wedge \sqrt{\nu_\gamma(|f| > t)}\, dt < \infty, \tag{3.9}$$

then the conclusion of Proposition 3.5 holds for the sequence $(Y_k)_{k\in\mathbb{Z}}$ where Y_k is defined by (3.7).

Remark 3.10. In the particular case when f is positive and non increasing on $]0, 1[$, with $f(x) \leq Dx^{-a}$ for some $a > 0$ and g_{ν_γ} the density of ν_γ such that $g_{\nu_\gamma}(x) \leq V_\gamma x^{-\gamma}$ where $V(\gamma)$ is a constant, we can prove that (3.9) holds as soon as $a < \frac{1}{2} - \gamma$ does (see Section 4.8). In his comment after Theorem 3, Gouëzel [10] proved that if $f(x) = x^{-a}$, then $n^{-1/2} \sum_{k=1}^n (f \circ T_\gamma^i - \nu_\gamma(f))$ converges to a normal law if $a < 1/2 - \gamma$, and that there is a convergence to a stable law (with a different normalization) if $a > 1/2 - \gamma$. This example shows that our condition is close to optimality.

3.2. **Causal linear processes.**

We focus here on the stationary sequence

$$Y_k = \sum_{j\geq 0} a_j \varepsilon_{k-j}, \tag{3.10}$$

where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of real-valued i.i.d random variables in $\mathbf{L}^2$ and $\sum_{j\geq 0} |a_j| < +\infty$.

Corollary 3.11. *Assume that, ε_0 has a density bounded by K and that $|a_0| \neq 0$. Moreover, assume that*

$$\sum_{k\geq 0} \int_0^{(a_k)^2} \frac{Q_{|Y_0|}(u)}{\sqrt{u}}\, du < \infty. \tag{3.11}$$

Then the conclusion of Proposition 3.2 holds for the sequence $(Y_k)_{k\in\mathbb{Z}}$ where Y_k is defined by (3.10).

Remark 3.12. Since $\sum_{j\geq 0} |a_j| < \infty$, (3.11) is true provided that

$$\sum_{k\in\mathbb{Z}} \int_0^{|a_k|^2} \frac{Q_{|\varepsilon_0|}(u)}{\sqrt{u}}\, du < \infty. \tag{3.12}$$

As a consequence, we get the following result

Corollary 3.13. *Assume either Item 1 or 2 below:*

   1. *for some $r > 2$, the i.i.d random variables $(\varepsilon_i)_{i\in\mathbb{Z}}$ are in $\mathbf{L}^r$ and ε_0 has a density bounded by K. In addition $|a_0| \neq 0$ and*

$$\sum_{k\geq 0} k^{1/(r-1)} |a_k|^{(r-2)/(r-1)} < \infty. \tag{3.13}$$

2. *for some $r > 2$,*

$$\forall\, x > 0, \;\; \mathbb{P}(|\varepsilon| > x) \leq \left(\frac{c}{x}\right)^r \; \text{where } c \text{ is a positive constant},$$

and ε_0 has a density bounded by K. In addition $|a_0| \neq 0$ and

$$\sum_{k \geq 0} |a_k|^{1-2/r} < \infty. \tag{3.14}$$

Then the conclusion of Proposition 3.2 holds for the sequence $(Y_k)_{k \in \mathbb{Z}}$ where Y_k is defined by (3.10).

4. Proofs

4.1. Central Limit Theorem for $\mathbf{L}^1(\mu)$-valued martingale differences.

We extend to $\mathbf{L}^1(\mu)$-valued martingale differences, a result of Jain [13], which is, for a sequence of i.i.d centered $\mathbf{L}^1(\mu)$-valued random variables $X = (X_i)_{i \in \mathbb{Z}}$, the Central Limit Theorem holds if and only if $\int_{\mathbf{T}} (\mathbb{E}(X_1(t)^2))^{1/2}\, \mu(dt) < \infty$.

Theorem 4.1. *Let $(M_i)_{i \in \mathbb{Z}}$ be a sequence of stationary ergodic martingale differences with values in $\mathbf{L}^1(\mu)$ such that $M_i = M_0 \circ \mathbb{T}^i$. Assume that*

$$\int_{\mathbf{T}} \|M_0(t)\|_2\, \mu(dt) < \infty. \tag{4.1}$$

Then

$$n^{-1/2} \sum_{i=1}^n M_0 \circ \mathbb{T}^i \xrightarrow[n \to \infty]{} G \text{ in law,} \tag{4.2}$$

where G is a $\mathbf{L}^1(\mu)$-valued centered Gaussian random variable with covariance function: for any f in $\mathbf{L}^\infty(\mu)$, $\Phi_G(f, f) = \mathbb{E}(f^2(M_0))$.

Proof of Theorem 4.1. Before giving the proof, we recall some notations and definitions used in the proof.

Notation 4.2. Using the notations of Jain [13], we consider for any real separable Banach space $\mathbb{B}$ and its dual $\mathbb{B}'$,

$$W M_0^2 = \{\nu \text{ probability measures on } \mathbb{B}: \int |f|^2\, d\nu < \infty, \; \int f\, d\nu = 0, \; \forall\, f \in \mathbb{B}'\}.$$

Definition 4.3. If $\nu \in W M_0^2$, its covariance kernel Φ_ν is given by: for any $f, g \in \mathbb{B}'$,

$$\Phi_\nu(f, g) = \int fg\, d\nu.$$

Definition 4.4. $\mu \in W M_0^2$ is pregaussian if there is a Gaussian measure ν, such that $\Phi_\nu = \Phi_\mu$.

By the classical Linderberg's theorem for stationary ergodic martingale differences in Billingsley [2], $n^{-1/2}f(S_n)$ converges in law to the centered gaussian random variable Z in $\mathbb{R}$, with variance $\mathbb{E}(f^2(M_0))$, for each $f \in \mathbf{L}^\infty(\mu)$. Now, we have to prove that the distribution of $n^{-1/2}S_n$ is relatively compact. As $\mathbf{L}^1(\mu)$ is of cotype 2, we use the same approach as in the proof of Theorem 6.4 in de Acosta, Araujo and Giné [1].
By stationarity, it follows that

$$\mathbb{E}\left(\left[f\left(\frac{S_n}{\sqrt{n}}\right)\right]^2\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}([f(M_i)]^2) = \mathbb{E}([f(M_0)]^2).$$

By Theorem 11 in Jain [13], if (4.1) holds, then M_0 is pregaussian, so there exists a $\mathbf{L}^1(\mu)$-valued centered Gaussian random variable G (or a Gaussian measure γ on $\mathbf{L}^1(\mu)$), with covariance operator Φ_G, such that, for any f in $\mathbf{L}^\infty(\mu)$,

$$\mathbb{E}([f(G)]^2) = \Phi_G(f,f) = \mathbb{E}([f(M_0)]^2).$$

By Theorem 5.6 in de Acosta, Araujo and Giné [1], every centered Gaussian measure on $\mathbf{L}^1(\mu)$, is strongly Gaussian which means that there exist a Hilbert space $\mathbb{H}$, a continuous linear map $\mathbf{M} : \mathbb{H} \to \mathbf{L}^1(\mu)$ and a tight centered Gaussian measure ν on $\mathbb{H}$ such that $\gamma = \nu \circ \mathbf{M}^{-1}$. Therefore, we can apply Theorem 6.2 [1], with $\mathfrak{K} = \{\xi$ a probability measure on $\mathbf{L}^1(\mu)$ such that $\Phi_\xi(f,f) \leq \Phi_\gamma(f,f),$ for all $f \in \mathbf{L}^\infty(\mu)\}$, so $\mathfrak{K}$ is relatively compact. We have proved that the distribution of $n^{-1/2}S_n$ is relatively compact. $\square$

4.2. **Proof of Theorem 2.2.**

We construct the martingale

$$M_n = \sum_{i=1}^{n} M_0 \circ \mathbb{T}^i,$$

where $M_0 = \sum_{k\in\mathbb{Z}} P_0(X_k)$. Notice that $(M_0 \circ \mathbb{T}^i)_{i\in\mathbb{Z}}$ is a sequence of a stationary ergodic martingale differences. By triangle inequality,

$$\int_{\mathbf{T}} \|M_0(t)\|_2 \, \mu(dt) = \int_{\mathbf{T}} \Big\| \sum_{k\in\mathbb{Z}} P_0(X_k(t)) \Big\|_2 \mu(dt)$$

$$\leq \sum_{k\in\mathbb{Z}} \int_{\mathbf{T}} \|P_0(X_k(t))\|_2 \, \mu(dt) < \infty.$$

Applying Theorem 4.1, we infer that

$$\frac{1}{\sqrt{n}} M_n \xrightarrow[n\to\infty]{} G \text{ in law in } \mathbf{L}^1(\mu),$$

where G is a $\mathbf{L}^1(\mu)$-valued centered Gaussian random variable such that $\Phi_G(f,f) = \mathbb{E}([f(M_0)]^2)$, for any $f \in \mathbf{L}^\infty(\mu)$.

To conclude the proof, it suffices to prove that

$$\lim_{n\longrightarrow\infty} \int_{\mathbf{T}} \Big\| \frac{S_n(t)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt) = 0. \tag{4.3}$$

The proof is inspired by the proof of Theorem 1 in Dedecker, Merlevède and Volný [8]. By triangle inequality,

$$
\int_{\mathbf{T}} \Big\| \frac{S_n(t)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt)
$$

$$
= \int_{\mathbf{T}} \Big\| \frac{S_n(t)}{\sqrt{n}} - \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n)}{\sqrt{n}} + \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n)}{\sqrt{n}} + \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}}
$$

$$
- \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt)
$$

$$
\leq \int_{\mathbf{T}} \Big\| \frac{S_n(t)}{\sqrt{n}} - \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n)}{\sqrt{n}} \Big\|_2 \mu(dt)
$$

$$
+ \int_{\mathbf{T}} \Big\| \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n)}{\sqrt{n}} - \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt)
$$

$$
+ \int_{\mathbf{T}} \Big\| \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}} \Big\|_2 \mu(dt). \tag{4.4}
$$

It suffices to prove that each term of the right-hand side in Inequality (4.4) tends to 0, as n tends to infinity. Let us first control the second term. Since

$$
\mathbb{E}(S_n(t) \mid \mathcal{F}_n) - \mathbb{E}(S_n(t) \mid \mathcal{F}_0) = \sum_{i=1}^{n} \sum_{k=1}^{n} P_i(X_k(t)),
$$

it follows that, by stationarity and orthogonality,

$$
\int_{\mathbf{T}} \Big\| \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n) - \mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt)
$$

$$
= \int_{\mathbf{T}} \sqrt{ \Big\| \frac{\mathbb{E}(S_n(t) \mid \mathcal{F}_n) - \mathbb{E}(S_n(t) \mid \mathcal{F}_0)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2^2 } \mu(dt)
$$

$$
= \int_{\mathbf{T}} \sqrt{ \frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{k=1}^{n} P_0(X_{k-i}(t)) - M_0(t) \Big\|_2^2 } \mu(dt)
$$

$$
= \int_{\mathbf{T}} \sqrt{ \frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{j=1-i}^{n-i} P_0(X_j(t)) - M_0(t) \Big\|_2^2 } \mu(dt)
$$

$$
\leq \sqrt{2} \Big[\int_{\mathbf{T}} \sqrt{ \frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{j \leq -i} P_0(X_j(t)) \Big\|_2^2 } \mu(dt) + \int_{\mathbf{T}} \sqrt{ \frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2 } \mu(dt) \Big]. \tag{4.5}
$$

Splitting the sum on i of the first term in the right-hand side of Inequality (4.5), we get that

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt)$$

$$= \int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=1}^{N} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2 + \frac{1}{n}\sum_{i=N+1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt)$$

$$\leq \int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=1}^{N} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt) + \int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=N+1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt).$$

Fubini entails that

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=1}^{N} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt) \;\leq\; \frac{1}{\sqrt{n}}\int_{\mathbf{T}} \sum_{i=1}^{N} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2 \, \mu(dt)$$

$$\leq\; \frac{1}{\sqrt{n}}\int_{\mathbf{T}} N \sum_{j\leq -1} \big\| P_0(X_j(t)) \big\|_2 \, \mu(dt)$$

$$\leq\; \frac{1}{\sqrt{n}} N \sum_{j\in\mathbb{Z}} \int_{\mathbf{T}} \| P_0(X_j(t)) \|_2 \, \mu(dt) \xrightarrow[n\to\infty]{} 0.$$

Moreover, since

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=N+1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt) \;\leq\; \int_{\mathbf{T}} \sqrt{\frac{1}{n}(n-N)\Big(\sum_{j\leq -N} \| P_0(X_j(t)) \|_2 \Big)^2}\, \mu(dt)$$

$$\leq\; \int_{\mathbf{T}} \Big(\sum_{j\leq -N} \| P_0(X_j(t)) \|_2 \Big)\, \mu(dt)$$

$$\leq\; \sum_{j\leq -N} \int_{\mathbf{T}} \| P_0(X_j(t)) \|_2 \, \mu(dt),$$

we infer by (2.2) that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=N+1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt) = 0\,.$$

Whence

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n}\sum_{i=1}^{n} \big\| \sum_{j\leq -i} P_0(X_j(t)) \big\|_2^2}\, \mu(dt) \xrightarrow[n\to\infty]{} 0.$$

In the same way, splitting the sum on i of the second term in the right-hand side of Inequality (4.5), we derive that

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt)$$

$$= \int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=1}^{n-N} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2 + \frac{1}{n} \sum_{i=n-N+1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt)$$

$$\leq \int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=1}^{n-N} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt) + \int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=n-N+1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt).$$

Since

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=1}^{n-N} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt) \;\leq\; \int_{\mathbf{T}} \sqrt{\frac{(n-N)}{n}} \Big(\sum_{j \geq N+1} \|P_0(X_j(t))\|_2 \Big) \mu(dt)$$

$$\leq \; \sum_{j \geq N+1} \int_{\mathbf{T}} \|P_0(X_j(t))\|_2 \, \mu(dt),$$

and

$$\int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=n-N+1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt) \;\leq\; \int_{\mathbf{T}} \frac{1}{\sqrt{n}} \sum_{i=n-N+1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2 \, \mu(dt)$$

$$\leq \; \frac{N}{\sqrt{n}} \int_{\mathbf{T}} \sum_{j \in \mathbb{Z}} \|P_0(X_j(t))\|_2 \, \mu(dt),$$

we deduce by (2.2), that

$$\lim_{n \to \infty} \int_{\mathbf{T}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \Big\| \sum_{j \geq n-i+1} P_0(X_j(t)) \Big\|_2^2} \, \mu(dt) = 0.$$

Consequently, we derive that

$$\int_{\mathbf{T}} \Big\| \mathbb{E}\Big(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_n \Big) - \mathbb{E}\Big(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_0 \Big) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_0(t) \circ \mathbb{T}^i \Big\|_2 \mu(dt) \xrightarrow[n \to \infty]{} 0.$$

To prove that the last term of Inequality (4.4) tends to 0 as n tends to infinity, we first write that

$$\int_{\mathbf{T}} \Big\| \mathbb{E}\Big(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_0 \Big) \Big\|_2 \mu(dt) \;\leq\; \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \Big\| \sum_{k=1}^{N} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \Big\|_2 \mu(dt)$$

$$+ \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \Big\| \sum_{k=N+1}^{n} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \Big\|_2 \mu(dt).$$

By orthogonality,

$$\big\| \sum_{k=N+1}^{n} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2^2 \;=\; \sum_{k=N+1}^{n} \sum_{l=N+1}^{n} \mathbb{E}(\mathbb{E}(X_k(t) \mid \mathcal{F}_0)\mathbb{E}(X_l(t) \mid \mathcal{F}_0))$$

$$=\; \sum_{k=N+1}^{n} \sum_{l=N+1}^{n} \mathbb{E}(\sum_{m=0}^{\infty} P_{-m}(X_k(t))P_{-m}(X_l(t))).$$

Using Cauchy-Schwarz inequality and stationarity, it follows that

$$\frac{1}{n}\big\| \sum_{k=N+1}^{n} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2^2 \;\le\; \frac{1}{n}\sum_{m=0}^{\infty} \sum_{k=N+m+1}^{n+m} \sum_{l=N+1+m}^{n+m} \|P_0(X_k(t))\|_2 \|P_0(X_l(t))\|_2$$

$$\le\; \Big(\sum_{k=N+1}^{\infty} \|P_0(X_k(t))\|_2 \Big)^2.$$

Consequently

$$\frac{1}{\sqrt{n}} \int_{\mathbf{T}} \big\| \sum_{k=N+1}^{n} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2 \mu(dt) \le \sum_{k=N+1}^{\infty} \int_{\mathbf{T}} \|P_0(X_k(t))\|_2 \, \mu(dt)\,,$$

and by (2.2), it follows that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \big\| \sum_{k=N+1}^{n} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2 \mu(dt) = 0. \qquad (4.6)$$

On the other hand, by stationarity,

$$\big\| \sum_{k=1}^{N} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2 \;=\; \big\| \sum_{k=1}^{N} \sum_{i\in\mathbb{Z}} \mathbb{E}(P_i(X_k(t)) \mid \mathcal{F}_0) \big\|_2$$

$$\le\; \sum_{k=1}^{N} \big(\sum_{i\in\mathbb{Z}} \|P_i(X_k(t))\|_2 \big)$$

$$\le\; N \sum_{i\in\mathbb{Z}} \|P_0(X_i(t))\|_2\,.$$

Hence by (2.2), we get that

$$\lim_{n\to\infty} \int_{\mathbf{T}} \frac{1}{\sqrt{n}} \big\| \sum_{k=1}^{N} \mathbb{E}(X_k(t) \mid \mathcal{F}_0) \big\|_2 \mu(dt) = 0\,. \qquad (4.7)$$

Therefore, (4.6) and (4.7) imply that

$$\lim_{n\to\infty} \int_{\mathbf{T}} \Big\| \mathbb{E}\Big(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_0 \Big) \Big\|_2 \mu(dt) = 0.$$

To prove that the first term of Inequality (4.4) tends to 0 as n tends to infinity, we write that

$$\int_{\mathbf{T}} \left\| \frac{S_n(t)}{\sqrt{n}} - \mathbb{E}\left(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_n \right) \right\|_2 \mu(dt) \leq \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \left\| \sum_{k=1}^{n-N} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2 \mu(dt)$$

$$+ \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \left\| \sum_{k=n-N+1}^{n} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2 \mu(dt).$$

By orthogonality,

$$\left\| \sum_{k=1}^{n-N} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2^2$$

$$= \sum_{k=1}^{n-N} \sum_{l=1}^{n-N} \mathbb{E}([X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)][X_l(t) - \mathbb{E}(X_l(t) \mid \mathcal{F}_n)])$$

$$= \sum_{k=1}^{n-N} \sum_{l=1}^{n-N} [\mathbb{E}(X_k(t)X_l(t)) - \mathbb{E}(\mathbb{E}(X_k(t) \mid \mathcal{F}_n)\mathbb{E}(X_l(t) \mid \mathcal{F}_n))]$$

$$= \sum_{k=1}^{n-N} \sum_{l=1}^{n-N} \mathbb{E}\left(\sum_{m=n+1}^{\infty} P_m(X_k(t))P_m(X_l(t)) \right).$$

Consequently by using Cauchy-Schwarz inequality and stationarity,

$$\frac{1}{n} \left\| \sum_{k=1}^{n-N} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2^2 \leq \frac{1}{n} \sum_{m=n+1}^{\infty} \sum_{k=1-m}^{n-N-m} \sum_{l=1-m}^{n-N-m} \|P_0(X_k(t))\|_2 \|P_0(X_l(t))\|_2$$

$$\leq \left(\sum_{k=-\infty}^{-(N+1)} \|P_0(X_k(t))\|_2 \right)^2.$$

Hence we get

$$\frac{1}{\sqrt{n}} \int_{\mathbf{T}} \left\| \sum_{k=1}^{n-N} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2 \mu(dt) \leq \sum_{k=-\infty}^{-(N+1)} \int_{\mathbf{T}} \|P_0(X_k(t))\|_2 \mu(dt),$$

and by (2.2), it follows that

$$\lim_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \int_{\mathbf{T}} \left\| \sum_{k=1}^{n-N} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \right\|_2 \mu(dt) = 0. \qquad (4.8)$$

Also by stationarity,

$$\Big\| \sum_{k=n-N+1}^{n} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \Big\|_2 = \Big\| \sum_{k=n-N+1}^{n} \sum_{m=n+1}^{\infty} P_m(X_k(t)) \Big\|_2$$

$$\leq \sum_{k=n-N+1}^{n} \Big(\sum_{m \in \mathbb{Z}} \| P_m(X_k(t)) \|_2 \Big)$$

$$\leq N \sum_{i \in \mathbb{Z}} \| P_0(X_i(t)) \|_2 .$$

Hence

$$\int_{\mathbf{T}} \frac{1}{\sqrt{n}} \Big\| \sum_{k=n-N+1}^{n} [X_k(t) - \mathbb{E}(X_k(t) \mid \mathcal{F}_n)] \Big\|_2 \mu(dt) \leq \frac{N}{\sqrt{n}} \sum_{i \in \mathbb{Z}} \int_{\mathbf{T}} \| P_0(X_i(t)) \|_2 \mu(dt).$$
(4.9)

Therefore, (4.8) and (4.9) imply that

$$\lim_{n \to \infty} \int_{\mathbf{T}} \Big\| \frac{S_n(t)}{\sqrt{n}} - \mathbb{E}\Big(\frac{S_n(t)}{\sqrt{n}} \Big| \mathcal{F}_n \Big) \Big\|_2 \mu(dt) = 0.$$

To end the proof, it remains to prove (2.4). With this aim, we use Corollary 1 in Dedecker, Merlevède and Volný [8]. Hence it suffices to prove that, for any f in $\mathbf{L}^{\infty}(\mu)$,

$$\sum_{k \in \mathbb{Z}} \| P_0(f(X_k)) \|_2 < \infty.$$
(4.10)

As f is a linear form on $\mathbf{L}^1(\mu)$ then f belongs to $\mathbf{L}^{\infty}(\mu)$. It follows that

$$\| P_0(f(X_k)) \|_2 = \| f(P_0(X_k)) \|_2 \leq C(f) \Big\| \int_{\mathbf{T}} | P_0(X_k)(t) | \, \mu(dt) \Big\|_2$$

$$\leq C(f) \int_{\mathbf{T}} \| P_0(X_k)(t) \|_2 \, \mu(dt),$$

where $C(f)$ is a constant depending on f. Consequently, (4.10) holds as soon as (2.2) holds.   □

4.3. **Proof of Corollary 2.3.**

We prove Corollary 2.3 with the same arguments as in the end of the proof of Corollary 2 in Peligrad and Utev [15].

By stationarity and orthogonality, for all k in $\mathbb{Z}$, we have

$$\| \mathbb{E}(X_k \mid \mathcal{F}_0) \|_2^2 = \Big\| \sum_{j=-\infty}^{0} P_j(X_k) \Big\|_2^2 = \sum_{j=k}^{\infty} \| P_{-j}(X_0) \|_2^2$$

and

$$\| X_{-k} - \mathbb{E}(X_{-k} \mid \mathcal{F}_0) \|_2^2 = \Big\| \sum_{j=1}^{\infty} P_j(X_{-k}) \Big\|_2^2 = \sum_{j=1}^{\infty} \| P_j(X_{-k}) \|_2^2 = \sum_{j=k+1}^{\infty} \| P_j(X_0) \|_2^2.$$

Now, applying Lemma $A.2$ in Peligrad and Utev [15], to $a_i := \|P_{-i}(X_0)\|_2$, it follows

$$
\begin{aligned}
\sum_{i=1}^{\infty} \|P_{-i}(X_0)\|_2 &\leq 3\sum_{n=1}^{\infty} n^{-1/2}\Big(\sum_{i=n}^{\infty} \|P_{-i}(X_0)\|_2^2\Big)^{1/2} \\
&\leq 3\sum_{n=1}^{\infty} n^{-1/2}\|\mathbb{E}(X_n \mid \mathcal{F}_0)\|_2,
\end{aligned} \tag{4.11}
$$

and then to $a_i := \|P_i(X_0)\|_2$,

$$
\begin{aligned}
\sum_{i=1}^{\infty} \|P_i(X_0)\|_2 &\leq 3\sum_{n=1}^{\infty} n^{-1/2}\Big(\sum_{i=n}^{\infty} \|P_i(X_0)\|_2^2\Big)^{1/2} \\
&\leq 3\sum_{n=1}^{\infty} n^{-1/2}\|X_{-(n-1)} - \mathbb{E}(X_{-(n-1)}|\mathcal{F}_0)\|_2 \\
&\leq 3\sum_{n=0}^{\infty} (n+1)^{-1/2}\|X_{-n} - \mathbb{E}(X_{-n}|\mathcal{F}_0)\|_2 \\
&\leq 3\|X_0 - \mathbb{E}(X_0|\mathcal{F}_0)\|_2 + 3\sum_{n=1}^{\infty} n^{-1/2}\|X_{-n} - \mathbb{E}(X_{-n}|\mathcal{F}_0)\|_2 \\
&\leq 3\sum_{n=1}^{\infty} n^{-1/2}\|X_{-n} - \mathbb{E}(X_{-n} \mid \mathcal{F}_0)\|_2 + 6\|X_0\|_2.
\end{aligned} \tag{4.12}
$$

Therefore, by (4.11) and (4.12), we deduce that

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_2 &= \sum_{k=1}^{\infty} \|P_{-k}(X_0)\|_2 + \sum_{k=1}^{\infty} \|P_k(X_0)\|_2 + \|P_0(X_0)\|_2 \\
&\leq 3\sum_{n=1}^{\infty} n^{-1/2}\Big(\sum_{i=n}^{\infty} \|P_{-i}(X_0)\|_2^2\Big)^{1/2} \\
&\qquad + 3\sum_{n=1}^{\infty} n^{-1/2}\Big(\sum_{i=n}^{\infty} \|P_i(X_0)\|_2^2\Big)^{1/2} + \|P_0(X_0)\|_2 \\
&\leq 3\Big[\sum_{n=1}^{\infty} n^{-1/2}\|\mathbb{E}(X_n \mid \mathcal{F}_0)\|_2 + \sum_{n=1}^{\infty} n^{-1/2}\|X_{-n} - \mathbb{E}(X_{-n} \mid \mathcal{F}_0)\|_2\Big] + 8\|X_0\|_2.
\end{aligned}
$$

Consequently, (2.5) and (2.6) implies (2.2). $\qquad\square$

4.4. Proof of Proposition 3.2.

We apply Corollary 2.3 to the variables $X_i(.) = \{t \mapsto \mathbf{1}_{Y_i \leq t} - F_Y(t), t \in \mathbb{R}\}$. Let

$$
Z(t) = \frac{F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)}{\|F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)\|_2}.
$$

Obviously,

$$
\forall\, t \in \mathbb{R},\ \|\mathbb{E}(X_k(t) \mid \mathcal{F}_0)\|_2 \leq \|X_k(t)\|_2 \leq \sqrt{F_Y(t)(1 - F_Y(t))}. \tag{4.13}
$$

By Proposition 2.1 in Dedecker [4] and by (4.13), for any k in $\mathbb{Z}$, we derive

$$
\begin{aligned}
\|F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)\|_2 &= \mathbb{E}\Big(\frac{F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)}{\|F_{Y_k|\mathcal{F}_0}(t) - F_Y(t)\|_2}(F_{Y_k|\mathcal{F}_0}(t) - F_Y(t))\Big) \\
&= \operatorname{Cov}(Z(t), X_k(t)) \\
&\leq 2\|Z(t)\|_2 \|X_k(t)\|_2 \sqrt{\tilde{\phi}(k)} \\
&\leq 2\sqrt{F_Y(t)(1 - F_Y(t))}\sqrt{\tilde{\phi}(k)}.
\end{aligned}
$$

Consequently, we deduce that (2.5) holds as soon as (3.3) holds. $\qquad\square$

4.5. **Proof of Proposition 3.5.**

We apply Corollary 2.3 to the random variables $X_i(.) = \{t \mapsto \mathbf{1}_{Y_i \leq t} - F_Y(t), t \in \mathbb{R}\}$.

By Hölder's inequality for any $k \geq 0$, we get

$$
\begin{aligned}
\|\mathbb{E}(X_k \mid \mathcal{F}_0)\|_2 &= \|\mathbb{E}(\mathbf{1}_{Y_k \leq t} \mid \mathcal{F}_0) - \mathbb{P}(Y_k \leq t)\|_2 \\
&\leq \sqrt{\|\mathbb{E}(\mathbf{1}_{Y_k \leq t} \mid \mathcal{F}_0) - \mathbb{P}(Y_k \leq t)\|_1}\sqrt{\|\mathbb{E}(\mathbf{1}_{Y_k \leq t} \mid \mathcal{F}_0) - \mathbb{P}(Y_k \leq t)\|_\infty} \\
&\leq \sqrt{\tilde{\alpha}(k)}. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.14)
\end{aligned}
$$

Using (4.13) and (4.14),

$$
\begin{aligned}
\int_{-\infty}^{\infty} \|\mathbb{E}(X_k(t) \mid \mathcal{F}_0)\|_2\, dt &\leq \int_{\mathbb{R}} \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{F_Y(t)(1 - F_Y(t))}\, dt \\
&\leq \int_0^{\infty} \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{1 - F_Y(t)}\, dt + \int_{-\infty}^0 \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{F_Y(t)}\, dt \\
&\leq \int_0^{\infty} \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{\mathbb{P}(|Y| > t)}\, dt.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\sqrt{\tilde{\alpha}(k)} \wedge \sqrt{\mathbb{P}(|Y| > t)} &= \int_0^1 \mathbf{1}_{u \leq \sqrt{\mathbb{P}(|Y| > t)}}\mathbf{1}_{u \leq \sqrt{\tilde{\alpha}(k)}}\, du \\
&= \int_0^{\sqrt{\tilde{\alpha}(k)}} \mathbf{1}_{u^2 \leq \mathbb{P}(|Y| > t)}\, du \\
&= \int_0^{\sqrt{\tilde{\alpha}(k)}} \mathbf{1}_{Q_Y(u^2) \geq t}\, du.
\end{aligned}
$$

By Fubini and by a change of variable, we derive

$$
\begin{aligned}
\int_0^{+\infty} \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{\mathbb{P}(|Y| > t)}\, dt &= \int_0^{\infty} \Big(\int_0^1 \mathbf{1}_{u \leq \sqrt{\mathbb{P}(|Y| > t)}}\mathbf{1}_{u \leq \sqrt{\tilde{\alpha}(k)}}\, du\Big)\, dt \\
&= \int_0^{\sqrt{\tilde{\alpha}(k)}} \Big(\int_0^{\infty} \mathbf{1}_{Q_Y(u^2) \geq t}\, dt\Big)\, du \\
&= \int_0^{\sqrt{\tilde{\alpha}(k)}} Q_Y(u^2)\, du = \frac{1}{2}\int_0^{\tilde{\alpha}(k)} \frac{Q_Y(u)}{\sqrt{u}}\, du.
\end{aligned}
$$

We deduce that (3.5) implies (2.5). The CLT holds, by applying Corollary 2.3. □

4.6. **Proof of Corollary 3.7.**

Recall that to prove the convergence for the empirical distribution function of Y where $(Y_k)_{k \in \mathbb{Z}}$ is defined by (3.7), it suffices to show the convergence in $\mathbf{L}^1(\lambda)$ of the empirical distribution function of $f(Z)$ where $(Z_i)_{i \in \mathbb{Z}}$ is the stationary Markov chain with transition Kernel K. Hence we shall prove that $f(Z)$ satisfies the conditions of Proposition 3.2.

Using the fact that f is a monotonic function, Dedecker and Prieur [9] proved that

$$\tilde{\phi}(\mathcal{F}_0, f(Z_k)) \leq \tilde{\phi}(\mathcal{F}_0, Z_k).$$

In addition, they proved that if (3.8) holds then

$$\tilde{\phi}(\mathcal{F}_0, Z_k) \leq C_1 \rho^k,$$

with C_1 a positive constant (see [9]). This entails that (3.3) holds. □

4.7. **Proof of Corollary 3.9.**

For the same reasons given in the proof of Corollary 3.7, we shall prove that $f(Z)$ satisfies the conditions of Proposition 3.5. Hence we apply Proposition 3.5 to the variables $X_i(.) = \{t \mapsto \mathbf{1}_{Y_i \leq t} - F_Y(t), t \in \mathbb{R}\}$. Using the fact that f is a monotonic function, Dedecker and Prieur [9] proved that

$$\tilde{\alpha}(\mathcal{F}_0, Y_k) = \tilde{\alpha}(\mathcal{F}_0, f(Z_k)) \leq \tilde{\alpha}(\mathcal{F}_0, Z_k).$$

Recently, Dedecker, Gouëzel and Merlevède [6] proved in Proposition 1.12, that there exists a constant C_γ, such that, for any positive integer k,

$$\tilde{\alpha}(\mathcal{F}_0, Z_k) \leq \frac{C_\gamma}{(n+1)^{\frac{1-\gamma}{\gamma}}}. \tag{4.15}$$

As T_γ is an intermittent map, and f is a monotonic function, it follows by (4.15) that

$$\int_0^\infty \sqrt{\tilde{\alpha}(k)} \wedge \sqrt{\nu_\gamma(|f| > t)}\, dt \leq \int_0^\infty \frac{C_\gamma}{(k+1)^{\frac{1-\gamma}{2\gamma}}} \wedge \sqrt{\nu_\gamma(|f| > t)}\, dt.$$

Consequently, (3.6) holds as soon as (3.9) holds. □

4.8. **Proof of Remark 3.10.**

Since the density g_{ν_γ} of ν_γ is such that $g_{\nu_\gamma}(x) \leq V(\gamma)x^{-\gamma}$, we infer that

$$\nu_\gamma(f > t) \leq \frac{D^{\frac{1-\gamma}{a}}V(\gamma)}{1-\gamma} t^{-\frac{1-\gamma}{a}}, \tag{4.16}$$

where D and $V(\gamma)$ are positive constants. By Fubini and (4.16), we then get that

$$\int_0^{+\infty} k^{-\frac{1-\gamma}{2\gamma}} \wedge \sqrt{\nu_\gamma(|f| > t)}\, dt = \int_0^\infty \left(\int_0^1 \mathbf{1}_{u \leq \sqrt{\nu_\gamma(|f| > t)}} \mathbf{1}_{u \leq k^{-\frac{1-\gamma}{2\gamma}}}\, du \right) dt$$

$$\leq K(\gamma, a) k^{\left(-\frac{(1-\gamma)}{2\gamma}\right)\left(-\frac{2a}{1-\gamma}+1\right)},$$

where $K(\gamma, a)$ is a constant.

Consequently, (3.9) holds as soon as $a < \frac{1}{2} - \gamma$ does. □

4.9. **Proof of Proposition 3.11.**

We apply Theorem 2.2 to the random variables $X_k(t) = \{t \mapsto \mathbf{1}_{Y_k \leq t} - F_Y(t)\}$. Let $\mathcal{M}_i = \sigma(\varepsilon_k, k \leq i)$. By a result in Lemma 6 in Dedecker and Merlevède [7],

$$\|F_{Y_k|\mathcal{M}_0}(t) - F_{Y_k|\mathcal{M}_{-1}}(t)\|_2 \leq K \mid a_0 \mid^{-1} \mid a_k \mid \|\varepsilon_1 - \varepsilon_0\|_2. \qquad (4.17)$$

Moreover, we have

$$\begin{aligned}
\|F_{Y_k|\mathcal{M}_0}(t) - F_{Y_k|\mathcal{M}_{-1}}(t)\|_2 &\leq \|F_{Y_k|\mathcal{M}_0}(t) - F_Y(t)\|_2 + \|F_{Y_k|\mathcal{M}_{-1}}(t) - F_Y(t)\|_2 \\
&\leq 2\sqrt{F_Y(t)(1 - F_Y(t))}.
\end{aligned}$$

We deduce that (2.2) holds as soon as

$$\sum_{k \in \mathbb{Z}} \int_0^\infty (K \mid a_0 \mid^{-1} \mid a_k \mid \|\varepsilon_1 - \varepsilon_0\|_2) \wedge (2\sqrt{\mathbb{P}(|Y_k| > t)})\, dt < \infty,$$

and it may be reduced to

$$\sum_{k \in \mathbb{Z}} \int_0^\infty \mid a_k \mid \wedge \sqrt{\mathbb{P}(|Y_k| > t)}\, dt < \infty \Leftrightarrow \sum_{k \in \mathbb{Z}} \int_0^{|a_k|^2} \frac{Q_{|Y_k|}(u)}{\sqrt{u}}\, du < \infty.$$

Now, from Theorem 2.2, we infer that $\sqrt{n}(F_n - F_Y)$ converges in law to a $\mathbf{L}^1(\lambda)$-valued centered Gaussian random variable G, with covariance operator Φ_μ defined by (3.4). $\qquad\square$

4.10. **Proof of Remark 3.12.**

By using Lemma 2.1 in Rio [17], page 35, we have that

$$\int_0^{|a_k|^2} \frac{Q_{|Y_0|}(u)}{\sqrt{u}}\, du \leq \Big(\sum_{j \geq 0} |a_j|\Big) \int_0^{|a_k|^2} \frac{Q_{|\varepsilon_0|}(u)}{\sqrt{u}}\, du.$$

Consequently since $\sum_{j \geq 0} |a_j| < \infty$, (3.11) is true provided that

$$\sum_{k \in \mathbb{Z}} \int_0^{|a_k|^2} \frac{Q_{|\varepsilon_0|}(u)}{\sqrt{u}}\, du < \infty. \qquad (4.18)$$

$\qquad\square$

4.11. **Proof of Corollary 3.13.**

4.11.1. *Proof of Item 1 of Corollary 3.13.*

To apply Corollary 3.11, it suffices to prove (3.12).

Firstly, recall that, if U is an uniform random variable on $[0,1]$, $Q_{|\varepsilon_0|}^2(U)$ and $|\varepsilon_0|^2$ have the same law.

We proceed as in Rio [17] p 15. By Hölder's inequality on $[0,1]$, it follows that

$$\begin{aligned}
\sum_{k \geq 0} \int_0^{|a_k|^2} \frac{Q_{|\varepsilon_0|}(u)}{\sqrt{u}}\, du &= \int_0^1 Q_{|\varepsilon_0|}(u)\Big(\frac{\sum_{k \geq 0} \mathbf{1}_{\{u \leq |a_k|^2\}}}{\sqrt{u}}\Big) \\
&\leq \Big(\int_0^1 Q_{|\varepsilon_0|}(u)^r\, du\Big)^{1/r} \Big(\int_0^1 \Big(\frac{\sum_{k \geq 0} \mathbf{1}_{\{u \leq |a_k|^2\}}}{\sqrt{u}}\Big)^{r/(r-1)} du\Big)^{(r-1)/r}.
\end{aligned}$$

Using the same notations as in Dedecker and Doukhan [5], let

$$\delta^{-1}(u) = \sum_{k \geq 0} \mathbf{1}_{\{u \leq |a_k|^2\}} \text{ and } f(x) = x^{r/(r-1)}.$$

We infer that

$$
\begin{aligned}
f(\delta^{-1}) &= \sum_{j=0}^{\infty} (f(j+1) - f(j))\mathbf{1}_{\{u \leq |a_j|^2\}} \\
&= \sum_{j=0}^{\infty} ((j+1)^{r/(r-1)} - j^{r/(r-1)})\mathbf{1}_{\{u \leq |a_j|^2\}}.
\end{aligned}
$$

Set $C_r = 1 \vee (\frac{r}{r-1})$. Since $(j+1)^{r/(r-1)} - j^{r/(r-1)} \leq C_r j^{1/(r-1)}$, (3.12) holds as soon as

$$\int_0^1 \sum_{j \geq 0} j^{1/(r-1)} \frac{\mathbf{1}_{\{u \leq |a_j|^2\}}}{u^{\frac{r}{2(r-1)}}} \, du < \infty \, ,$$

which is true provided that

$$\sum_{j \geq 0} j^{1/(r-1)} |a_j|^{\frac{r-2}{r-1}} < \infty.$$

$\square$

4.11.2. *Proof of Item 2 of Corollary 3.13.*
We apply Corollary 3.11, so it suffices to prove (3.12).
Notice that, the quantile function $Q_{|\varepsilon_0|}$, here, is dominated by $cu^{-1/r}$. Thus, we derive

$$
\begin{aligned}
\int_0^{|a_k|^2} \frac{Q_{|\varepsilon_0|}(u)}{\sqrt{u}} \, du &\leq \int_0^{|a_k|^2} \frac{c}{u^{1/2+1/r}} \, du \\
&\leq c \frac{2r}{r-2} |a_k|^{1-2/r}.
\end{aligned}
$$

Consequently, (3.12) holds as soon as (3.14) does. $\square$

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