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A simple model for Carnot heat engines

Jacques ARNAUD ^{*}, Laurent CHUSSEAU [†], Fabrice PHILIPPE [‡]

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Abstract

Contrary to what some texts are saying, the theory of heat engines and heat pumps presented by Carnot around 1824 is entirely accurate. We offer in this paper a simple mechanical model consisting of reservoirs at altitude ϵ containing N ball locations and n weight-1 balls. The reservoir energy is $Q = n\epsilon$. We consider particularly two such reservoirs, with the label l referring to the lower reservoir and the label h to the higher reservoir. A cycle consists of exchanging balls between the reservoirs. It is straightforward to show that the efficiency, defined as the ratio of the work produced to the energy lost by the higher reservoir is $\eta = 1 - \epsilon_l/\epsilon_h$. To relate this energy-generating device to a *heat* engine, we introduce the entropy, defined as the logarithm of the number of ball configurations in a reservoir, $S(n) = \ln[N!/n!(N-n)!]$. The absolute temperature is then defined as $T = [(Q(n+1) - Q(n))/(S(n+1) - S(n))]$, and the large n limit is assumed. It follows that when $n_l \approx n_h$, the system efficiency $\eta = 1 - \epsilon_l/\epsilon_h = 1 - T_l/T_h$ is the Carnot efficiency. Because the concept of time is not involved, a treatment such as the one presented here may be given early in Physics courses.

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1 Introduction

The purpose of this paper is to give newcomers to the field of Thermodynamics a feel for the concepts involved in the simplest possible manner, the concept of time being not involved. Our opinion is that these Thermodynamical concepts are so simple that they could be presented early in Physics courses. The first section of the paper is historical. It is often said in the literature that Carnot discovered the second law of thermodynamics while ignoring the first one, and furthermore, that he confused entropy and heat. As a matter of fact, the Carnot theory is entirely accurate¹. An empirical result employed in the Carnot theory is that reversibility is broken when two bodies at different temperatures are contacted. The present treatment follows from purely logical reasoning. As such, it is valid for quantum as well as classical working substances, and could even be realized at the macroscopic level, using lottery-type containers. In a subtle way, concepts that were introduced only at the end of the 19th century by Boltzmann are involved. The paper is directly applicable to the so-called Otto cycles with electrons in a magnetic field as a working substance. A more general discussion is in [1]. Quan [2, 3] shows that the maximum efficiency of every quantum heat engine is the Carnot efficiency. A good list of references is also given in his papers.

Some historical observations are made in Section 2. We consider in Section 3 reservoirs at two different altitudes in the earth gravitational field, containing weight-1 balls. Work is produced when balls at one location are being transferred from one reservoir to the other. The relation of such a device to heat engines is explained in Sections 4 and 5. Fluctuations are considered in the appendix.

2 History

Carnot [4, 5, 6] suggested around 1824 that heat engines are in some sense analogous to water mills, water-reservoir altitudes being analogous to heat-reservoir absolute temperatures. The efficiency of water mills is the ratio of the work actually performed and the work that would be performed if the consumed higher-reservoir fluid had dropped all the way down to some lower reference level. This efficiency is less than unity, even under ideal

¹One reason for the misunderstanding is that part of the Carnot contribution appeared in print only decades after his early death. A second one is that his work was popularized by Clapeyron in a partly erroneous manner. A third one is the unfortunate use by Carnot of the word “calorique” to designate what Clausius later on called “entropy”. The word “calorique” had been formerly employed to designate some hypothetical heat substance.

conditions. Heat engine efficiency, on the other hand, is defined as the ratio of the work performed and the higher temperature reservoir heat consumption, both being expressed in the same energy units, e.g., in joules. In order to evaluate the work produced and the efficiency, the first and second laws of thermodynamics are needed. Both were established by Carnot. The first law asserts that heat is a form of energy and that isolated systems energies do not vary in the course of time. Let us cite Carnot [4] : “Heat is nothing but motive power, or rather another form of motion. Wherever motive power is destroyed, heat is generated in precise proportion to the quantity of motive power destroyed; conversely, wherever heat is destroyed, motive power is generated”. Carnot calculated that 1 calorie of heat is equivalent to 3.27 J, instead of the modern value: 4.18 J. He proved that engine efficiencies reach their maximum value when they are reversible, from the consideration that energy cannot be obtained for free. He therefore looked for processes that could work in a reversed manner, ending up with the celebrated “Carnot cycle”.

On the other hand, it is a well-known empirical fact that heat may flow spontaneously (i.e., with no work involved) from a body to another having a lower temperature, but that the converse never occurs. Eventually, the temperatures of the two bodies equalize. Carnot established that the maximum efficiency of heat engines operating between baths at absolute temperatures T_l and T_h , respectively, is $\eta = 1 - T_l/T_h$, and that this efficiency may be reached under some idealized conditions (Carnot cycle). He also established that the work produced is $W = (T_h - T_l)S$, where S denote the entropy transferred from the hot bath to the cold bath. Because there is some confusion in the literature concerning the significance of the Carnot contribution on that respect, let us quote Zemansky and Dittman [7] : “Carnot used *chaleur* when referring to heat in general, but when referring to the motive power of fire that is brought about when heat enters an engine at high temperature and leaves at low temperature, he uses the expression *chute de calorique*, never *chute de chaleur*[...]. Carnot had in the back of his mind the concept of entropy, for which he reserved the term *calorique*”.

3 Exchange of balls between two reservoirs

Consider two reservoirs with N possible ball locations, at altitude ϵ_l and $\epsilon_h > \epsilon_l$, respectively, as shown in Fig. 1. For each reservoir every ball configuration is equally likely to occur in view of the fact that the energies are the same. A “cycle” consists of carrying the left ball (if there is one) from one reservoir to the other. It is straightforward to show that, among

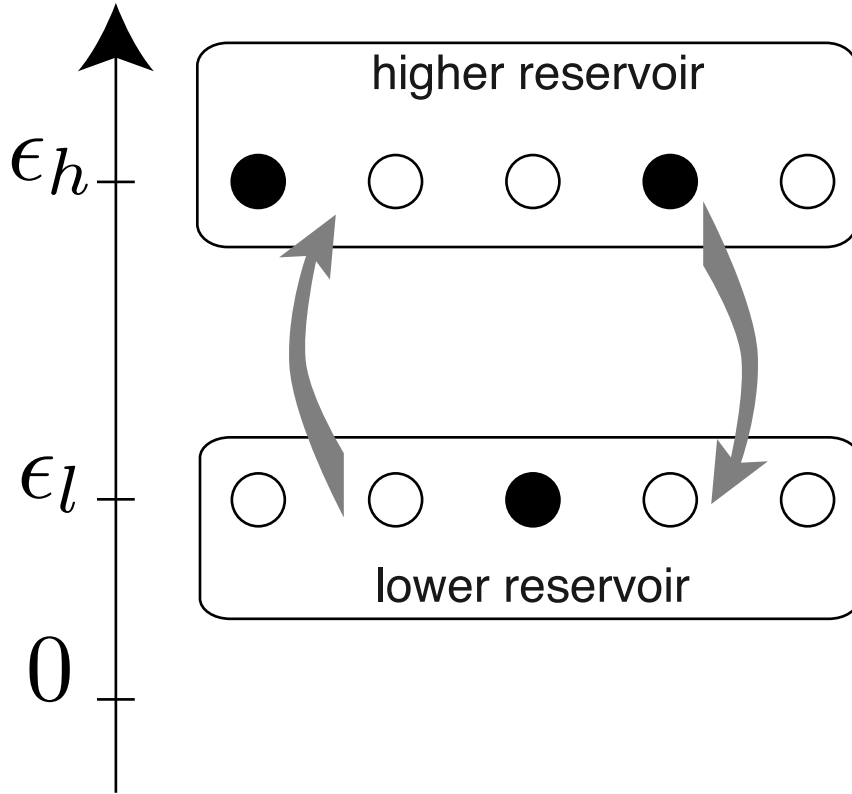


Figure 1: Schematic representation of an engine that converts potential energy into mechanical work. The figure represents two reservoirs located at low and high altitudes, ϵ_l , ϵ_h respectively, with N possible ball locations ($N = 5$ in the figure). The number of weight-1 balls (black circles) is n_l in the lower reservoir, and n_h in the higher reservoir ($n_l = 1$, $n_h = 2$ in the figure). For each reservoir, every ball configuration is equally likely considering that the energies are the same. A cycle consists of simultaneously carrying the left ball from one reservoir (if there is one) to the other reservoir. In the case of the figure, this would amount to increment the number of balls in the lower reservoir from 1 to 2, and the number of balls in the higher reservoir from 2 to 1. Averages are understood to be taken over all ball configurations. This is equivalent to picking up a ball at random with probability n/N , $n = n_l, n_h$.

the $N!/n!(N-n)!$ ball configurations, the fraction of these that have a ball at a specific location is n/N , a most intuitive result. It is then easy to show that the efficiency, defined as the ratio of the average work produced to the average energy lost by the higher reservoir, is $\eta = 1 - \epsilon_l/\epsilon_h$ (see below).

A reservoir energy is obviously $Q = n\epsilon$ (kinetic energy being not considered, the total energy coincides with the potential energy). The letter Q is employed anticipating a correspondence with heat. When a weight-1 ball is added to a reservoir at altitude ϵ the reservoir energy is incremented by ϵ . On the other hand, if a ball is randomly picked up from a reservoir containing n weight-1 balls and subsequently carried to a reservoir at altitude ϵ , the latter reservoir average energy is incremented by $\Delta Q = \epsilon n/N$. From now on the word "average" is omitted for brevity in the main text.

Consider now two such reservoirs. One at altitude ϵ_l (lower reservoir) and containing n_l weight-1 balls. The other at altitude ϵ_h (higher reservoir) containing n_h weight-1 balls. A cycle consists of exchanging balls picked up at random between the two reservoirs. From what has just been said and if we set $l \equiv n_l/N$, $h \equiv n_h/N$, the energies added to the lower and higher reservoirs read respectively

$$\begin{aligned}\Delta Q_l &= \epsilon_l(h-l), \\ \Delta Q_h &= -\epsilon_h(h-l).\end{aligned}\tag{1}$$

The work performed follows from the law of conservation of energy

$$W = -\Delta Q_l - \Delta Q_h = (\epsilon_h - \epsilon_l)(h-l).\tag{2}$$

The engine efficiency, defined as the ratio of the work performed W and the energy $-\Delta Q_h$ lost by the higher reservoir, is therefore

$$\eta \equiv \frac{W}{-\Delta Q_h} = 1 - \frac{\epsilon_l}{\epsilon_h}.\tag{3}$$

The purpose of the next section is to relate the engine described above to heat engines. We will show that when $h \approx l$ the efficiency given in (3) coincides with the Carnot efficiency and the work given in (2) coincides with the expression given by Carnot.

4 Heat engines

We consider again reservoirs containing N locations and n weight-1 balls. As said above, a cycle consists of exchanging randomly-picked balls. To relate this device to *heat* engines, let us first recall that the number of ball

configurations is $N!/n!(N-n)!$. For example, if $N = 3$ and $n = 1$, there are $3!/1!2! = 3$ configurations, namely: 100, 010 and 001. Next, we define the entropy as the logarithm of the number of configurations, the Boltzmann constant being set equal to unity, that is

$$S(n) = \ln \left(\frac{N!}{n!(N-n)!} \right). \quad (4)$$

Note that

$$\begin{aligned} S(n+1) - S(n) &= \ln \left(\frac{N!}{(n+1)!(N-n-1)!} \right) - \ln \left(\frac{N!}{n!(N-n)!} \right) \\ &= \ln \left(\frac{N-n}{n+1} \right) \approx \ln \left(\frac{N}{n} - 1 \right), \end{aligned} \quad (5)$$

$$\begin{aligned} S(n-1) - S(n) &= \ln \left(\frac{N!}{(n-1)!(N-n+1)!} \right) - \ln \left(\frac{N!}{n!(N-n)!} \right) \\ &= \ln(nN - n + 1) \approx -\ln \left(\frac{N}{n} - 1 \right). \end{aligned} \quad (6)$$

for large n .

The absolute temperature of a reservoir is then defined as

$$T(n) = \frac{Q(n+1) - Q(n)}{S(n+1) - S(n)} \approx \frac{\epsilon}{\ln(\frac{N}{n} - 1)}. \quad (7)$$

Temperature is an *intensive* quantity. For example, the temperature of two identical bodies at temperature T , considered together, is again T . Because heat has the nature of an energy and is an extensive quantity, it is required that S be also an extensive quantity. Since the number of configurations in two separate bodies is the *product* of the configurations (for each configuration of one body one must consider all the configurations of the other body) and the logarithmic function has the property that $\ln(ab) = \ln(a) + \ln(b)$, the above definitions do ensure that T be an intensive quantity.

The cycle efficiency given in (3) may now be written in terms of temperatures as

$$\eta = 1 - \frac{\epsilon_l}{\epsilon_h} = 1 - \frac{T_l \ln(\frac{1}{l} - 1)}{T_h \ln(\frac{1}{h} - 1)}. \quad (8)$$

Thus, when $l \approx h$, the last fraction in the above equation drops out and the Carnot efficiency is indeed obtained. In the limit $l \approx h$ the work W

produced per cycle is very small. However, one may always add up the work contributions of any number of similar devices having the same reservoir temperatures (but possibly different values of ϵ , n), and achieve any specified work at the Carnot efficiency.

The ball exchange discussed above may increment the reservoir entropies. The number of balls in a reservoir may indeed be incremented by one, remain the same, or be decremented by one. From what was said before, the probability that a weight-1 ball be transferred from the high reservoir to the lower one is $h \equiv n_h/N$, and the probability that a weight-1 ball be transferred from the low reservoir to the higher one is $l \equiv n_l/N$. Since these events are independent, the lower reservoir entropy increment reads

$$\Delta S_l = h(1-l)[S(n_l+1) - S(n_l)] + l(1-h)[S(n_l-1) - S(n_l)]. \quad (9)$$

Using (6) we obtain

$$\Delta S_l = (h-l) \ln \left(\frac{1}{l} - 1 \right). \quad (10)$$

The increment of the higher reservoir entropy is obtained by exchanging the h and l labels in the above expression, that is

$$\Delta S_h = -(h-l) \ln \left(\frac{1}{h} - 1 \right). \quad (11)$$

We thus find that in the limit $n_l \approx n_h$ (or $l \approx h$), $\Delta S_l \approx -\Delta S_h \equiv S$ so that there is no net entropy produced. Entropy is just carried from the higher reservoir to the lower one. The Carnot expression for the work recalled in Section 2 may thus be written as

$$W = (T_h - T_l)\Delta S_l = -(T_h - T_l)\Delta S_h \approx \left(\frac{\epsilon_h}{\ln \left(\frac{1}{h} - 1 \right)} - \frac{\epsilon_l}{\ln \left(\frac{1}{l} - 1 \right)} \right) \Delta S_l \quad (12)$$

$$\approx (\epsilon_h - \epsilon_l)(h - l), \quad (13)$$

so that the Carnot general formula for W recalled in Section 2 indeed coincides with the expression for the work performed per cycle evaluated for our model from simple reasoning.

5 Relation to practical heat engines

Real heat engines retrieve energy from two arbitrarily large baths, one at temperature T_l and the other at temperature T_h . A working agent such as

a gas-filled cylinder of length L is put in contact with the hot bath. When an equilibrium has been reached, the working agent is carried to the low-temperature bath while the cylinder length is allowed to increase. Once equilibrium with the low-temperature bath has been reached, the working agent is carried back to the hot bath while the cylinder length is reduced to its initial value. It is true (but not obvious) that net work may be obtained in that manner.

The reservoirs considered in this paper model may reach equilibrium with baths located at the same altitude ϵ provided they have the same h or l values. This is achieved by allowing balls to be displaced from the reservoir and the bath. Usually, working agents and baths exchange energy, but not particles. If we insist on the condition that only energy be exchanged, our model needs be slightly modified. Instead of balls one should consider electrons immersed in a magnetic field. An electron has two energy levels depending on its spin state. The lower one may be set at 0 (corresponding to empty locations) and the other at 1 (corresponding to a weight-1 ball). What we previously called the reservoir “altitude” corresponds here to the magnetic field in which the electrons are being immersed.

There is an alternative model close to the one shown in the figure. Instead of weight-1 balls in the earth gravitational field, one may have electrons in a constant electrical field V/d created by two parallel conducting plates spaced a distance d apart, with a potential difference V . The potential between the plates is a constant if they are connected to an arbitrarily large capacitance. As in the figure, there is a collection of n_l electrons at altitude ϵ_l and a collection of n_h electrons at altitude ϵ_h . When an electron drops from altitude ϵ_h to altitude ϵ_l , it delivers an energy $eV(\epsilon_h - \epsilon_l)/d$. Conversely, when at altitude ϵ_l is raised to altitude ϵ_h , it absorbs an energy $eV(\epsilon_h - \epsilon_l)/d$. These energies increment or decrement the large capacitance charge by q , such that qV equals the energy delivered by the electron. In that model, the work W evaluated above appears in the form of an electrical energy corresponding to a current $i = q/\tau$ lasting during the cycle duration τ .

Another model is that of a shallow potential electron trap with a magnetic field. The trap may possess N states having almost the same energy, ϵ_h for one spin state, and $\epsilon_l < \epsilon_h$ for the other spin state. Only $n \leq N$ electrons of a given spin may occupy these states, according to the Pauli principle. If un-polarized electrons are injected in the system, some will end up in the higher energy state and others in the lower energy state, giving rise to a temperature difference. The results presented in that paper should apply approximately to such an electronic system.

6 Conclusion

We have seen that heat engines may be equivalent to mechanical engines of a special kind. Precisely, our model consists of two reservoirs containing N locations and n_l , n_h weight-1 balls, at different altitudes. It involves a concept of probability introduced for the first time in Thermodynamics by Boltzmann at the end of the 19th century. To summarize, the only concepts involved in the present paper are those of potential energy and of probability of picking up specific balls from an ensemble of N identical balls. We have been able to prove that the efficiency and work in our model coincide with the Carnot expressions in the limit where $n_l \approx n_h$. Full Carnot cycles may be generated out of this elementary configuration.

7 Appendix: fluctuations

Recall that in our model a cycle consists of exchanging simultaneously a ball from the higher reservoir (at altitude ϵ_h and containing n_h balls) and a ball from the lower reservoir (at altitude ϵ_l and containing n_l balls). The probability that a ball be picked up from the higher reservoir is $h \equiv n_h/N$ (and therefore the probability that no ball be picked up is $1 - h$), The probability that a ball be picked up from the lower reservoir is $l \equiv n_l/N$ (and therefore the probability that no ball be picked up is $1 - l$). The two events are independent. N is the number of possible ball locations in each reservoir.

Setting $\epsilon \equiv \epsilon_h - \epsilon_l$, we have seen in the main text that the average work produced per cycle is $\langle W \rangle = \epsilon(h-l)$. We now evaluate $\langle W^2 \rangle$. The probability that a ball falls and none is raised is $h(1-l)$. If this event occurs, the work performed squared is equal to ϵ^2 . Conversely, the probability that a ball is raised and none falls is $l(1-h)$. If this event occurs, the work performed squared is again equal to ϵ^2 . It follows that $\langle W^2 \rangle = \epsilon^2[h(1-l) + l(1-h)]$. Therefore, the variance of the work produced reads

$$\text{var}(W) \equiv \langle W^2 \rangle - \langle W \rangle^2 = \epsilon^2[h(1-h) + l(1-l)]. \quad (14)$$

In the limit $h \approx l$ considered in the main text, we have

$$\text{var}(W) \approx 2\epsilon^2l(1-l). \quad (15)$$

Let us now consider the total entropy produced $\Delta S \equiv \Delta S_l + \Delta S_h$. When a ball is being transferred from the high reservoir to the lower one and none from the low reservoir to the higher one, an event that occurs with probability $h(1-l)$, the increment of S_l is, according to (5), $\Delta S(n_l + 1) - \Delta S(n_l) =$

$\ln(\frac{1}{l} - 1)$, and the increment of S_h is $\Delta S(n_h - 1) - \Delta S(n_h) = -\ln(\frac{1}{h} - 1)$. It follows that the increment in total entropy is $\ln(\frac{\frac{1}{l}-1}{\frac{1}{h}-1})$ with probability $h(1-l)$. When a ball is being transferred from the low reservoir to the higher one and none from the high reservoir to the lower one, an event that occurs with probability $l(1-h)$, the increment of S_l is, according to (5), $\Delta S(n_l - 1) - \Delta S(n_l) = -\ln(\frac{1}{l} - 1)$, and the increment of S_h is $\Delta S(n_h + 1) - \Delta S(n_h) = \ln(\frac{1}{h} - 1)$. It follows that the increment in total entropy is $\ln(\frac{\frac{1}{h}-1}{\frac{1}{l}-1})$ with probability $l(1-h)$.

The average increment in total entropy is therefore

$$\begin{aligned}\langle \Delta S \rangle &= h(1-l) \ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) + l(1-h) \ln \left(\frac{\frac{1}{h} - 1}{\frac{1}{l} - 1} \right) \\ &= (h-l) \ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) \geq 0.\end{aligned}\quad (16)$$

As we said in the main text, when $h \approx l$, $\Delta S \approx 0$ and the system tends to be reversible and to achieve the highest efficiency. Note that the entropy increment is non-negative for both a heat engine ($h > l$) and a heat pump ($l > h$). More precisely, noting that to first order in $\delta \equiv h - l$ we have $\ln[(1/l - 1)/(1/h - 1)] \approx \delta/[l(1-l)]$, and thus

$$\langle \Delta S \rangle \approx \frac{\delta^2}{l(1-l)} \quad (17)$$

Finally, we evaluate the variance of the total entropy increment. From the above expressions, it follows that

$$\begin{aligned}\langle (\Delta S)^2 \rangle &= h(1-l) \left[\ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) \right]^2 + l(1-h) \left[\ln \left(\frac{\frac{1}{h} - 1}{\frac{1}{l} - 1} \right) \right]^2 \\ &= (h+l-2lh) \left[\ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) \right]^2,\end{aligned}\quad (18)$$

and the variance reads

$$\begin{aligned}\text{var}(\Delta S) &\equiv \langle (\Delta S)^2 \rangle - \langle \Delta S \rangle^2 \\ &= (h+l-2lh - (h-l)^2) \left[\ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) \right]^2 \\ &= [(h(1-h) + l(1-l))] \left[\ln \left(\frac{\frac{1}{l} - 1}{\frac{1}{h} - 1} \right) \right]^2,\end{aligned}\quad (19)$$

which vanishes, as well as the average entropy produced, when $h \approx l$. To first order in $\delta \equiv h - l$, we have

$$\text{var}(\Delta S) \approx 2 \langle \Delta S \rangle, \quad (20)$$

a remarkably simple result. This result is related to conclusions given in [8]. It has been presented before in [9] just after Eq. (18).

References

- [1] J. Arnaud, L. Chusseau, and F. Philippe, “Mechanical equivalent of quantum heat engines,” *Phys. Rev. E*, vol. 77, p. 061102, 2008.
- [2] H. T. Quan, Y.-X. Liu, C. P. Sun, and F. Nori, “Quantum thermodynamic cycles and quantum heat engines,” *Phys. Rev. E*, vol. 76, p. 031105, 2007.
- [3] H. T. Quan, “Quantum thermodynamic cycles and quantum heat engines (II).” arXiv:0811.2756 [quant-ph], 17 Nov 2008.
- [4] N. S. Carnot, *Réflexions sur la puissance motrice du feu*. Paris: Bachelier, 1824. (facsimile of the original edition by Jacques Gabay ed., Sceaux, 1960; English translation by Dover publ. Inc., New-York, 1960).
- [5] N. S. Carnot, *Biographie et Manuscrit publiés sous le haut patronage de l’Académie des Sciences*. Paris: Gauthier-Villars, 1927. Most historians consider that these manuscripts were written by Carnot around 1824.
- [6] V. M. Brodiansky, *Sadi Carnot*. Perpignan: Presses Universitaires de Perpignan, 2006.
- [7] M. Zemansky and R. Dittman, *Heat and Thermodynamics*. New-York: MacGraw-Hill, 1997.
- [8] C. Jarzynski, “Nonequilibrium equality for free-energy differences,” *Phys. Rev. Lett.*, vol. 78, pp. 2690–2693, 1997. preprint cond-mat/9601209.
- [9] J. Arnaud, L. Chusseau, and F. Philippe, “Carnot cycle for an oscillator,” *Eur. J. Phys.*, vol. 23, pp. 489–500, 2002.