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# A POOR MAN'S SQUARE FUNCTION ESTIMATE ON DOMAINS

O.IVANOVICI AND F.PLANCHON

ABSTRACT. The purpose of this note is to provide an elementary proof of a weak version of the square function estimate on  $L^p(\Omega)$ , which is often used in the context of wave or Schrödinger equations to reduce estimates for data in Sobolev spaces to spectrally localized data. In the process we obtain useful  $L^p(\Omega)$  bounds for the heat kernel and its derivatives, through a simple argument which holds irrespective of the domain.

## 1. INTRODUCTION

Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with smooth boundary  $\partial\Omega$ . Let  $\Delta_D$  denote the Laplace operator on  $\Omega$  with Dirichlet boundary conditions, acting on  $L^2(\Omega)$ , with domain  $H^2(\Omega) \cap H_0^1(\Omega)$ .

Our main result reads as follows

**Theorem 1.1.** *Let  $f \in C^\infty(\Omega)$  and  $\Psi \in C_0^\infty(\mathbb{R}^*)$  such that*

$$(1.1) \quad \sum_{j \in \mathbb{Z}} \Psi(2^{-2j} \lambda) = 1, \quad \lambda \in \mathbb{R}.$$

*Then for all  $p \in [2, \infty)$  we have*

$$(1.2) \quad \|f\|_{L^p(\Omega)} \leq C_p \left( \sum_{j \in \mathbb{Z}} \|\Psi(-2^{-2j} \Delta_D) f\|_{L^p(\Omega)}^2 \right)^{1/2},$$

*where the operator  $\Psi(-2^{-2j} \Delta_D)$  is defined by (2.1) below.*

Readers who are familiar with functional spaces theory will have recognized the embedding  $\dot{B}_p^{0,2}(\Omega) \hookrightarrow L^p(\Omega)$ , where the Besov space is defined using the right hand-side of (1.2) as a norm. One expects, by analogy with the  $\mathbb{R}^n$  case, the much stronger equivalence

$$\|f\|_{L^p(\Omega)} \approx \left\| \left( \sum_j |\Psi(-2^{-2j} \Delta_D) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

to hold for all  $1 < p < +\infty$ ; in other words,  $L^p(\Omega)$  and the Triebel-Lizorkin space  $\dot{F}_p^{0,2}(\Omega)$  coincide. Such an equivalence (and much more !) is proven in [20, 21, 22], though one has to reconstruct it from several different sections (functional spaces are defined differently, only the inhomogeneous ones are treated, among other things). As such, the casual user with mostly a PDE background might find it difficult to reconstruct the argument for his own sake without digesting the whole theory. In contrast, the present note aims at giving a self-contained proof, with “acceptable” black boxes, namely complex interpolation and spectral calculus. In fact, if one

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accepts to replace the spectral localization by the heat flow, the proof can be made entirely self-contained, relying only on integration by parts. Our strategy to prove Theorem 1.1 is indeed to reduce matters to an estimate involving the heat flow, by proving almost orthogonality between spectral projectors and heat flow localization; this only requires basic parabolic estimates in  $L^p(\Omega)$ , together with a little help from spectral calculus.

*Remark 1.2.* For compact manifolds without boundaries, one may find a direct proof of (1.2) (with  $\Delta_D$  replaced by the Laplace-Beltrami Laplacian) in [5], which proceeds by reduction to the  $\mathbb{R}^n$  case using standard pseudo-differential calculus. Our elementary approach provides an alternative direct proof.

*Remark 1.3.* One can also adapt the proof to the case of Neumann boundary conditions, provided special care is taken of the zero frequency (note that on an exterior domain, a decay condition at infinity solves the issue).

*Remark 1.4.* Theorem 1.1 is useful, among other things, when dealing with  $L^p$  estimates for wave or dispersive evolution equations. For such equations, one naturally considers initial data in Sobolev spaces, and spectral localization conveniently reduces matters to data in  $L^2$ , and helps with finite speed of propagation arguments. One however wants to sum eventually over all frequencies, if possible without loss. Recent examples may be found in [13] or [17], as well as in [14].

Even in the  $\mathbb{R}^n$  case, proving the strong version of the square function estimate requires a lot more technology (more or less some Hilbert-valued version of Calderon-Zygmund theory), and as such, in the case of domains, one may easily see the benefit of adopting the functional spaces point of view and follow Triebel's path. Alternatively, one can derive everything from adapting to the domain case the theory which ultimately led to the proof of the Kato conjecture ([4, 3]). Such a possible development is pointed out by P. Auscher in [2] (chap. 7, p. 66) and was originally our starting point; eventually we were led to the elementary approach we present here, but we provide a sketch of an alternate proof (see Remark 3.2), which was kindly outlined to us by Pascal Auscher.

## 2. FUNCTIONAL CALCULUS

We start by recalling the Dynkin-Helffer-Sjöstrand formula ([11, 12]) and refer to the appendix of [15] for a nice presentation of the use of almost-analytic extensions in the context of functional calculus. In what follows we will also rely on Davies' presentation ([9]) from which we will use a couple of useful lemma.

**Definition 2.1.** (see [15, Lemma A.1]) Let  $\Psi \in C_0^\infty(\mathbb{R})$ , possibly complex valued. We assume that there exists  $\tilde{\Psi} \in C_0^\infty(\mathbb{C})$  such that  $|\bar{\partial}\tilde{\Psi}(z)| \leq C|\text{Im}z|$  and  $\tilde{\Psi}|_{\mathbb{R}} = \Psi$ . Then we have (as a bounded operator in  $L^2(\Omega)$ )

$$(2.1) \quad \Psi(-h^2\Delta_D) = \frac{i}{2\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{\Psi}(z)(z + h^2\Delta_D)^{-1} d\bar{z} \wedge dz.$$

The next result ensures the existence of  $\tilde{\Psi}$  in the previous definition ( see [15, Lemma A.2] and [19], where it is linked with Hadamard's problem of finding a smooth function with prescribed derivatives at a given point):

**Lemma 2.2.** *If  $\Psi$  belongs to  $C_0^\infty(\mathbb{R})$  there exists  $\tilde{\Psi} \in C_0^\infty(\mathbb{C})$  such that  $\tilde{\Psi}|_{\mathbb{R}} = \Psi$  and*

$$(2.2) \quad |\bar{\partial}\tilde{\Psi}(z)| \leq C_{N,\Psi} |\operatorname{Im}z|^N, \quad \forall z \in \mathbb{C}, \quad \forall N \in \mathbb{N}.$$

Moreover, if  $\Psi$  belongs to a bounded subset of  $C_0^\infty(\mathbb{R})$  (elements of  $\mathcal{B}$  are supported in a given compact subset of  $\mathbb{R}$  with uniform bounds), then the mapping  $\mathcal{B} \ni \Psi \rightarrow \tilde{\Psi} \in C_0^\infty(\mathbb{C})$  is continuous and  $C_{N,\Psi}$  can be chosen uniformly w.r.t  $\Psi \in \mathcal{B}$ .

*Remark 2.3.* Estimate (2.2) simply means that  $\bar{\partial}\tilde{\Psi}(z)$  vanishes at any order on the real axis. Precisely, if  $z = x + iy$

$$\partial_y^N \tilde{\Psi}|_{\mathbb{R}} = (i\partial_x)^N \tilde{\Psi}|_{\mathbb{R}} = (i\partial_x)^N \Psi|_{\mathbb{R}}.$$

In particular if  $\langle x \rangle = (1 + x^2)^{1/2}$  then for any given  $N \geq 0$ , a useful example of an almost analytic extension of  $\Psi \in C_0^\infty(\mathbb{R})$  is given by

$$\tilde{\Psi}(x + iy) = \left( \sum_{m=0}^N \partial^m \Psi(x) (iy)^m / m! \right) \tau\left(\frac{y}{\langle x \rangle}\right),$$

where  $\tau$  is a non-negative  $C^\infty$  function such that  $\tau(s) = 1$  if  $|s| \leq 1$  and  $\tau(s) = 0$  if  $|s| \geq 2$ . For later purposes, we also set

$$\|\Psi\|_N \stackrel{\text{def}}{=} \sum_{m=0}^N \int_{\mathbb{R}} |\partial^m \Psi(x)| \langle x \rangle^{m-1} dx.$$

Our next lemma lets us deal with Lebesgue spaces.

**Lemma 2.4.** *Let  $z \notin \mathbb{R}$  and  $|\operatorname{Im}z| \lesssim |\operatorname{Re}z|$ , then  $\Delta_D$  satisfies*

$$(2.3) \quad \|(z - \Delta_D)^{-1}\|_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq \frac{c}{|\operatorname{Im}z|} \left( \frac{|z|}{|\operatorname{Im}z|} \right)^\alpha, \quad \forall z \notin \mathbb{R}$$

for  $1 \leq p \leq +\infty$ , with a constant  $c = c(p) > 0$  and  $\alpha = \alpha(n, p) > n|\frac{1}{2} - \frac{1}{p}|$ .

*Remark that, for all  $h \in (0, 1]$ , the operator  $h^2 \Delta_D$  satisfies (2.3) with the same constants  $c$  and  $\alpha$  (this is nothing but scale invariance).*

For  $p = 2$  the proof of Lemma 2.4 is trivial by multiplying the resolvent equation  $-\Delta_D u + zu = f$  by  $\bar{u}$  and we get  $\alpha = 0$ ; however for  $p \neq 2$  it requires a non trivial argument which we postpone to Appendix 2.

**Corollary 2.5.** *For  $N \geq \alpha + 1$  the integral (2.1) is norm convergent and  $\forall h \in (0, 1]$*

$$(2.4) \quad \|\Psi(-h^2 \Delta_D)\|_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq c \|\Psi\|_{N+1},$$

for some constant  $c$  independent of  $h$ .

*Proof:* By scale invariance it is enough to prove (2.4) for  $h = 1$ . The integrand in (2.1) is norm continuous for  $z \notin \mathbb{R}$ . If we set

$$U \stackrel{\text{def}}{=} \{z = x + iy | \langle x \rangle < |y| < 2\langle x \rangle\}, \quad V \stackrel{\text{def}}{=} \{z = x + iy | 0 < |y| < 2\langle x \rangle\},$$

then the norm of the integrand is dominated by

$$\begin{aligned} & c \sum_{m=0}^N |\partial^m \Psi(x)| \frac{2^m}{m!} \langle x \rangle^{m-2} \|\partial\tau\|_{L^\infty([1,2])} 1_U(x + iy) + \\ & + c |\partial^{N+1} \Psi(x)| \frac{2^N}{N!} |y|^N \left( \frac{\langle x \rangle}{|y|} \right)^\alpha \|\tau\|_{L^\infty([0,2])} 1_V(x + iy). \end{aligned}$$

Integrating with respect to  $y$  for  $N \geq \alpha + 1$  yields the bound

$$\begin{aligned} \|\Psi(-\Delta_D)\|_{L^p(\Omega) \rightarrow L^p(\Omega)} &\lesssim \int_{\mathbb{R}} \left( \sum_{m=0}^N |\partial^m \Psi(x)| \langle x \rangle^{m-1} + \right. \\ &\quad \left. + |\partial^{N+1} \Psi(x)| \langle x \rangle^N \right) dx = \|\Psi\|_{N+1}. \end{aligned}$$

One may then prove that the operator  $\Psi(-\Delta_D)$ , acting on  $L^p(\Omega)$ , is independent of  $N \geq 1 + n/2$  and of the cut-off function  $\tau$  in the definition of  $\tilde{\Psi}$ , see [9].

We now recall two lemma which will be useful when composing operators.

**Lemma 2.6** (Lemma 2.2.5,[9]). *If  $\Psi \in C_0^\infty(\mathbb{R})$  has support disjoint from the spectrum of  $-h^2\Delta_D$  then  $\Psi(-h^2\Delta_D) = 0$ .*

**Lemma 2.7** (Lemma 2.2.6, [9]). *If  $\Psi_1, \Psi_2 \in C_0^\infty(\mathbb{R})$ , then  $(\Psi_1\Psi_2)(-h^2\Delta_D) = \Psi_1(-h^2\Delta_D)\Psi_2(-h^2\Delta_D)$ .*

### 3. HEAT EQUATION

We consider the linear heat equation on  $\Omega$  with Dirichlet boundary conditions and initial data  $f$

$$(3.1) \quad \partial_t u - \Delta_D u = 0, \text{ on } \Omega \times \mathbb{R}_+; \quad u|_{t=0} = f \in C^\infty(\Omega); \quad u|_{\partial\Omega} = 0.$$

We denote the solution  $u(t, x) = S(t)f(x)$ , where we set  $S(t) = e^{t\Delta_D}$ . For the sake of simplicity  $\Delta_D$  has constant coefficients, but the same method applies in the case when the coefficients belong to a bounded set of  $C^\infty$  and the principal part is uniformly elliptic (one may lower the regularity requirements on both the coefficients and the boundary, and a nice feature of the proof which follows is that counting derivatives is relatively straightforward).

**Proposition 3.1.** Let  $2 \leq p < +\infty$ , then we have

$$(3.2) \quad \|f\|_{L^p(\Omega)} \leq c_p \left( \int_0^\infty |\nabla S(t)f|^2 dt \right)^{1/2} \|_{L^p(\Omega)} \leq C_p \left( \int_0^\infty \|\nabla S(t)f\|_{L^p(\Omega)}^2 dt \right)^{1/2}.$$

*Remark 3.2.* The main drawback from (3.2) is the presence of  $\nabla S(t)$  on the right hand-side: one is leaving the functional calculus of  $\Delta_D$ , and in fact for domains with Lipschitz boundaries the operator  $\nabla S(t)$  may not even be bounded. As such, a suitable alternative is to replace  $\nabla S(t)$  by  $\sqrt{\partial_t} S(t)$ . Then the corresponding estimate may be obtained following [2] as follows:

- prove that the associated square function in time is bounded by the  $L^p$  norm, for all  $1 < p \leq 2$ , essentially following step 3 in chapter 6, page 55 in [2]. This requires very little on the semi-group, and Gaussian bounds on  $S(t)$  and  $\partial_t S(t)$  ([8]) are more than enough to apply the weak (1,1) criterion from [2] (Theorem 1.1, chapter 1). Moreover, the argument can be extended to domains with Lipschitz boundaries, assuming the Laplacian is defined through the associated Dirichlet form;
- by duality, we get the square function bound for  $p > 2$  (step 5, page 56 in [2]);
- from now on one proceeds as in the remaining part of our paper to obtain the bound with spectral localization, and almost orthogonality (3.8) is even easier because we stay in the functional calculus. One has, however, to be

careful if one is willing to extend this last step to Lipschitz boundaries, as this would most likely require additional estimates on the resolvent to deal with the  $\Delta_j$ .

*Remark 3.3.* We will later obtain (3.2) with  $\sqrt{t}\partial_t S(t)$  replacing  $\nabla S(t)$ , see Remark 4.3, but the outline from the previous remark would work for this slightly different square function as well.

In order to prove Proposition 3.1 we need the following lemma.

**Lemma 3.4.** *For all  $1 \leq p \leq +\infty$ , we have*

$$(3.3) \quad \|S(t)f\|_{L^p(\Omega)} \rightarrow_{t \rightarrow \infty} 0,$$

$$(3.4) \quad \sup_{t \geq 0} \|S(t)f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.$$

Moreover,

$$(3.5) \quad \left\| \sup_{t \geq 0} |S(t)f| \right\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.$$

*Proof:* The estimate (3.4) clearly follows from (3.5), which in turn is a direct consequence of the Gaussian nature of the Dirichlet heat kernel, see [7]. The same Gaussian estimate implies (3.3). However we do not need such a strong fact to prove (3.4), which will follow from the next computation as well (see (3.6)). Estimate (3.3) can also be obtained through elementary arguments. We defer such a proof to Appendix 1.

*Proof of Proposition 3.1*

If  $p = 2$  the proof is nothing more than the energy inequality, combined with (3.3). In fact, for  $p = 2$ , we have equality in (3.2) with  $C_2 = 2$ . We now take  $p = 2m$  where  $m \geq 2$ . Multiplying equation (3.1) by  $\bar{u}|u|^{p-1}$  and taking the integral over  $\Omega$  and  $[0, T]$ ,  $T > 0$  yields, taking advantage of the Dirichlet boundary condition,

$$(3.6) \quad \frac{2}{p} \int_0^T \partial_t \|u\|_{L^p(\Omega)}^p dt + 2 \int_0^T \int_{\Omega} |\nabla u|^2 |u|^{p-2} dx dt \\ + \frac{(p-2)}{2} \int_0^T \int_{\Omega} (\nabla(|u|^2))^2 |u|^2 |u|^{p-4} dx dt = 0,$$

from which we can estimate either  $\|u\|_{L^p(\Omega)}^p(T) \leq \|f\|_{L^p(\Omega)}^p$  (which is (3.4)) or

$$\|f\|_{L^p(\Omega)}^p \leq \|u\|_{L^p(\Omega)}^p(T) + p(p-1) \int_0^T \int_{\Omega} |\nabla u|^2 |u|^{p-2} dx dt.$$

Letting  $T$  go to infinity and using (3.3) from Lemma 3.4 and Hölder inequality we find

$$\|f\|_{L^p(\Omega)}^p \leq p(p-1) \left( \int_{\Omega} \left( \int_0^{\infty} |\nabla u|^2 dt \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \left( \int_{\Omega} \left( \sup_t |u|^{p-2} \right)^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}}.$$

The proof follows using again Lemma 3.4, as

$$\|f\|_{L^p(\Omega)}^p \leq C_p \left\| \left( \int_0^{\infty} |\nabla u|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \left( \left\| \sup_{t \geq 0} |u| \right\|_{L^p(\Omega)} \right)^{p-2}.$$

Note that we may prove the weaker part of (3.2), without assuming the maximal in time bound, by reversing the order of integration in our argument. This would keep

the argument essentially self-contained, without any need for Gaussian bounds on the heat kernel.

*Remark 3.5.* We do not claim novelty here: our argument follows closely (a dual version of) the proof of a classical square function bound for the Poisson kernel in the whole space, see [18].

At this point, we have essentially proved Theorem 1.1, but with the  $\Psi$  operator replaced by the gradient heat kernel and the discrete parameter  $2^{-2j}$  by the continuous parameter  $t$ . The rest of this note is devoted to proving the equivalence between the Besov norms which are defined by the heat kernel or the spectral localization.

**Lemma 3.6.** *Let  $1 \leq p \leq +\infty$  and  $Q_t \stackrel{\text{def}}{=} t^{1/2} \nabla S(t)$ . We have the following equivalence between dyadic and continuous versions of the Besov norm:*

$$\frac{3}{4} \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|_{L^p(\Omega)}^2 \leq \int_0^\infty \|Q_t f\|_{L^p(\Omega)}^2 \frac{dt}{t} \leq 3 \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|_{L^p(\Omega)}^2.$$

This follows at once from factoring the semi-group: for  $2^{-2j} \leq t \leq 2^{-2(j-1)}$ , write  $S(t) = S(t - 2^{-2j})S(2^{-2j})$  and use (3.4).

**Lemma 3.7.** *Let  $1 < p < +\infty$ . The operators  $Q_t$ ,  $\mathbf{Q}_t \stackrel{\text{def}}{=} t \Delta_D S(t)$  are bounded on  $L^p(\Omega)$ , uniformly in  $t \geq 0$ .*

We postpone the proof of the lemma to Appendix 1, Section 4 and turn to the proof of Theorem 1.1. Let  $\Psi \in C_0^\infty(\mathbb{R}^*)$  satisfying (1.1) and denote  $\Delta_j f \stackrel{\text{def}}{=} \Psi(2^{-2j} \Delta_D) f$ , where  $\Psi(2^{-2j} \Delta_D) f$  is given by the Dynkin-Helffer-Sjöstrand formula (2.1). From Proposition 3.1 and Lemma 3.6 we have

$$(3.7) \quad \|f\|_{L^p(\Omega)} \leq 3C_p \left( \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|_{L^p(\Omega)}^2 \right)^{1/2}$$

and we will show that (3.7) implies (1.2): it suffices to prove the following almost orthogonality property between localization operators  $\Delta_j$  and  $Q_{2^{-2k}}$ :

$$(3.8) \quad \forall k, j \in \mathbb{Z}, \quad \|Q_{2^{-2k}} \Delta_j f\|_{L^p(\Omega)} \lesssim 2^{-|j-k|} \|\Delta_j f\|_{L^p(\Omega)}.$$

Then, from  $(2^{-|j-k|})_k \in l^1$  and  $(\|\Delta_j f\|_{L^p(\Omega)})_j \in l^2$  we estimate

$$(3.9) \quad \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|_{L^p(\Omega)}^2 = \sum_{k \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{Z}} Q_{2^{-2k}} \Delta_j f \right\|_{L^p(\Omega)}^2$$

as an  $l^1 * l^2$  convolution and conclude using Lemma 3.7. It remains to show (3.8):

- for  $k < j$  we write

$$Q_{2^{-2k}} \Delta_j f = 2^{3/2} 2^{-2(j-k)} \left( 2^{-(2k+1)/2} \nabla S(2^{-(2k+1)}) \right) \left( 2^{-(2k+1)} \Delta_D S(2^{-(2k+1)}) \right) \check{\Psi}(-2^{-2j} \Delta_D) \Psi(-2^{-2j} \Delta_D) f,$$

where we set  $\check{\Psi}(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda} \tilde{\Psi}(\lambda)$ , and  $\tilde{\Psi} \in C_0^\infty$ ,  $\tilde{\Psi} = 1$  on  $\text{supp} \Psi$ . By Lemma 3.7, the operators  $Q_{2^{-(2k+1)}} = 2^{-(2k+1)/2} \nabla S(2^{-(2k+1)})$  and  $\mathbf{Q}_{2^{-(2k+1)}} = 2^{-(2k+1)} \Delta_D S(2^{-(2k+1)})$  are bounded on  $L^p(\Omega)$  and we obtain (3.8) using Corollary 2.5 for  $\check{\Psi}$ .

- for  $k \geq j$  we set  $\Psi_1(\xi) = \tilde{\Psi}(\xi) \exp(\xi)$ ,  $\Psi_2(\xi) = \Psi(\xi)$ , and we use again Lemma 2.7 to write (slightly abusing the notation as  $2^{-2k} - 2^{-2j} < 0$ )

$$(3.10) \quad S(2^{-2k} - 2^{-2j})\Delta_j f = S(2^{-2k})\Psi_1(-2^{-2j}\Delta_D)\Psi_2(-2^{-2j}\Delta_D)f.$$

Then

$$Q_{2^{-2k}}\Delta_j f = 2^{-(k-j)}\left(2^{-j}\nabla S(2^{-2j})\right)\left(S(2^{-2k} - 2^{-2j})\Delta_j f\right),$$

and using again Lemma 3.7 we see that the operator  $2^{-j}\nabla S(2^{-2j})$  is bounded while the remaining operator (3.10) is bounded by Corollary 2.5. This ends the proof.

*Remark 3.8.* One may prove a similar bound with  $Q_{2^{-2k}}$  and  $\Delta_j$  reversed, either directly or by duality. Hence Besov norms based on  $\Delta_j$  or  $Q_{2^{-2k}}$  are equivalent.

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#### 4. APPENDIX 1

We now return to the proof of Lemma 3.7. For  $\mathbf{Q}_t$ , boundedness on all  $L^p$  spaces, including  $p = 1, +\infty$ , follows once again from a Gaussian upper bound on  $\partial_t S(t)$  (see [8] or [10]). However the subsequent Gaussian bound on the gradient  $\nabla_x S(t)$  in [8] is a direct consequence of the Li-Yau inequality, which holds only inside convex domains. We were unable to find a reference which would provide the desired bound for  $Q_t$  in the context of the exterior domain. Therefore we provide an elementary detailed proof for  $Q_t$ . Furthermore, we only deal with  $1 < p < 2$  or powers of two,  $p = 2^m$ ,  $m \in \mathbb{N}^*$ : complex interpolation takes care of remaining values of  $p$ , though one could adapt the following argument to generic values  $p > 2$ , at the expense of lengthier computations.

Set  $v(x, t) = Q_t f = t^{1/2}\nabla u(x, t)$  and assume without loss of generality that  $v$  is real: we multiply the equation satisfied by  $v$  by  $v|v|^{p-2}$  and integrate over  $\Omega$ ,

$$(4.1) \quad \partial_t \left( \frac{1}{p} \|v\|_{L^p(\Omega)}^p \right) - \int_{\partial\Omega} ((\vec{n} \cdot \nabla)v) \cdot v |v|^{p-2} d\sigma \\ + (p-1) \int_{\Omega} |\nabla v|^2 |v|^{p-2} dx = \frac{1}{2t} \|v\|_{L^p(\Omega)}^p,$$

where  $\vec{n}$  is the outgoing unit normal vector to  $\partial\Omega$  and  $d\sigma$  is the surface measure on  $\partial\Omega$ . We claim that the second term in the left hand side vanishes: in fact we write

$$\int_{\partial\Omega} (\vec{n} \cdot \nabla v) \cdot v |v|^{p-2} d\sigma = \frac{t^{p/2}}{2} \int_{\partial\Omega} \partial_n (|\partial_n u|^2 + |\nabla_{\text{tang}} u|^2) (|\partial_n u|^2 + |\nabla_{\text{tang}} u|^2)^{(p-2)/2} d\sigma,$$

and from  $u|_{\partial\Omega} = 0$  the time and tangential derivative  $(\partial_t, \nabla_{\text{tang}})u|_{\partial\Omega}$  vanishes; furthermore, using the equation,  $\partial_n^2 u = 0$  on  $\partial\Omega$ .



*Remark 4.1.* Notice that this term vanishes as well with Neumann boundary conditions.

Now, if  $1 < p < 2$ , multiply by  $\|v\|_{L^p(\Omega)}^{2-p}$  and integrate over  $[0, T]$ ,

$$\|v\|_{L^p(\Omega)}^2(T) \lesssim \int_0^T \|Q_t f\|_{L^p(\Omega)}^2 \frac{dt}{t} \lesssim \|f\|_p^2,$$

where the last inequality is the dual of (3.2). Hence we are done with  $1 < p < 2$ .

*Remark 4.2.* We ignored the issue of  $v$  vanishing in the third term in (4.1). This is easily fixed by replacing  $|v|^{p-2}$  by  $(\sqrt{\varepsilon + |v|^2})^{p-2}$  and proceeding with the exact same computation. Then let  $\varepsilon$  go to 0 after dropping the positive term on the left handside of (4.1).

Now let  $p = 2^m$  with  $m \geq 1$ : we proceed directly by integrating (4.1) over  $[0, T]$ , to get

$$(4.2) \quad \frac{1}{p} \|v\|_{L^p(\Omega)}^p(T) + (p-1) \int_0^T \int_{\Omega} |\nabla v|^2 |v|^{p-2} dx dt = \int_0^T \frac{1}{2t} \|v\|_{L^p(\Omega)}^p dt,$$

On the other hand (recall (3.6)),

$$(4.3) \quad \frac{1}{p} \|u\|_{L^p(\Omega)}^p(T) + (p-1) \int_0^T \int_{\Omega} |\nabla u|^2 |u|^{p-2} dx dt = \frac{1}{p} \|f\|_{L^p(\Omega)}^p.$$

If  $p = 2$  the estimates are trivial since from (4.2), (4.3),

$$\frac{1}{2} \|v\|_{L^2(\Omega)}^2(T) \leq \int_0^T \frac{1}{2t} \|v\|_{L^2(\Omega)}^2 dt = \frac{1}{2} \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 dt \leq \frac{1}{4} \|f\|_{L^2(\Omega)}^2.$$

Now, let  $p \geq 4$ ; for convenience, denote by  $J$  the time integral on the left hand-side of (4.2),

$$J = \int_0^{+\infty} \int_{\Omega} |\nabla v|^2 |v|^{p-2} dx dt \approx \int_0^{+\infty} |\nabla^2 u|^2 |\nabla u|^{p-2} t^{\frac{p}{2}} dx dt,$$

and set

$$I_k = \int_0^T \int_{\Omega} |\nabla u|^{2k} |u|^{p-2k} t^{k-1} dx dt \text{ where } 2 \leq 2k \leq p.$$

For our purposes, it suffices to estimate the right hand-side of (4.2), which rewrites

$$(4.4) \quad \frac{1}{2} \int_0^{+\infty} t^{\frac{p}{2}-1} \|\nabla_x u\|_{L^p(\Omega)}^p dt = \frac{1}{2} I_{\frac{p}{2}}.$$

Integrate by parts the inner (space) integral in  $I_k$ , the boundary term vanishes and

$$(4.5) \quad I_x = (p-2k+1) \int_{\Omega} \nabla u \nabla u |\nabla u|^{2(k-1)} |u|^{p-2k} dx \\ \leq (2k-1) \int_{\Omega} |\nabla^2 u| |\nabla u|^{2k-2} |u|^{p-2k+1} dx.$$

By Cauchy-Schwarz,

$$I_x \lesssim \left( \int_{\Omega} |\nabla^2 u|^2 |\nabla u|^{p-2} dx \right)^{1/2} \left( \int_{\Omega} |\nabla u|^{4k-4-(p-2)} |u|^{2p+2-4k} dx \right)^{1/2},$$

therefore

$$I_k \lesssim J^{\frac{1}{2}} I_{2k-\frac{p}{2}-1}^{\frac{1}{2}}.$$

We aim at controlling  $I_m$  by  $J^{1-\eta}I_1^\eta$ , for some  $\eta > 0$  which depends on  $m$  (notice that when  $p = 4$ , which is  $m = 2$ , we are already done, using  $k = 2$ !). Set  $k = \frac{p}{2} - (2^j - 1)$  with  $j \leq m - 2$ ,

$$I_{2^{m-1}-(2^j-1)} \lesssim J^{\frac{1}{2}} I_{2^{m-1}-(2^{j+1}-1)}^{\frac{1}{2}},$$

and iterating  $m - 2$  times, we finally control  $I_{\frac{p}{2}}$  by  $J^{1-\eta}I_1^\eta$ , which proves that  $Q_t$  is bounded on  $L^p(\Omega)$ .

We now proceed to obtain boundedness of  $\mathbf{Q}_t$  on  $L^p(\Omega)$  from the  $Q_t$  bound; this is worse than using the Gaussian properties of its kernel, as the constants blow up when  $p \rightarrow 1, +\infty$ . It is, however, quite simple. By duality  $Q_t^*$  is bounded on  $L^p(\Omega)$ , and

$$\mathbf{Q}_t = t\partial_t S(t) = tS\left(\frac{t}{2}\right)\Delta S\left(\frac{t}{2}\right) = 2\sqrt{\frac{t}{2}}S\left(\frac{t}{2}\right)\nabla \cdot \sqrt{\frac{t}{2}}\nabla S\left(\frac{t}{2}\right) = 2Q_{\frac{t}{2}}^* Q_{\frac{t}{2}},$$

and we are done with Lemma 3.7.

*Remark 4.3.* From the previous decomposition, we also obtain

$$\|\mathbf{Q}_t f\|_{L^p(\Omega)} \lesssim \|Q_t f\|_{L^p(\Omega)},$$

which implies that any Besov norm defined with  $\mathbf{Q}_t$  is bounded by the corresponding norm for  $Q_t$ . The reverse bound is true as well, though slightly more involved. We provide the proof for completeness. Consider  $f, h \in C_0^\infty(\Omega)$  and  $\langle f, g \rangle = \int_\Omega fg$ . Then

$$\begin{aligned} \langle f, g \rangle &= - \int_0^{+\infty} \langle \partial_t S(t)f, h \rangle dt = -2 \int_0^{+\infty} \langle \partial_t S(t)f, S(t)h \rangle dt \\ &= 2 \int_{t < s} \langle \partial_t S(t)f, \partial_s S(s)h \rangle dt ds = 4 \int_0^{+\infty} \langle \nabla S(s)\partial_t S(t)f, \nabla S(s)h \rangle dt ds \\ &\lesssim \int_s \left\| \int_0^s \nabla S(t)\partial_s S(s)f dt \right\|_p \|\nabla S(s)h\|_{p'} ds \lesssim \int_s \sqrt{s} \|\partial_s S(s)f\|_p \|\nabla S(s)h\|_{p'} ds \end{aligned}$$

where we used our bound on  $\sqrt{t}\nabla S(t)$  at fixed  $t$ . Then

$$\langle f, h \rangle \lesssim \int_s \|\mathbf{Q}_s f\|_p \|Q_s h\|_{p'} \frac{ds}{s}$$

from which we are done by Hölder.

We now return to Lemma 3.4 and provide an elementary proof of (3.3): while we only write  $p = 2$ , there is nothing specific to the  $L^2$  case in what follows. Let  $\chi$  be a smooth cut-off near the boundary  $\partial\Omega$ . Then  $v = (1 - \chi)u$  solves the heat equation in the whole space, with source term  $[\chi, \Delta]u$ :

$$(1 - \chi)u = S_0(t)(1 - \chi)u_0 + \int_0^t S_0(t - s)[\chi, \Delta]u(s) ds,$$

where  $S_0$  is the free heat semi-group. We have, taking advantage of the localization near the boundary,

$$\|[\chi, \Delta]u\|_{L_t^2(L^{\frac{2n}{n+2}})} \lesssim C(\chi, \chi') \|\nabla u\|_{L_t^2(L^2)} < +\infty,$$

by the energy inequality (3.6). The integral equation on  $(1 - \chi)u$  features  $S_0$  for which we have trivial Gaussian estimates, and both the homogeneous and inhomogeneous terms are  $C_t(L^2)$  and go to zero as time goes to  $+\infty$ . On the other hand, by Poincaré inequality (or Sobolev),

$$\int_0^t \|\chi u\|_2^2 ds \lesssim \int_0^t \|\nabla u\|_2^2 ds,$$

which ensures that  $\|\chi u\|_2$  goes to zero as well at  $t = +\infty$ .

## 5. APPENDIX 2

Let us prove the resolvent estimate (2.3) from Lemma 2.4. If  $\operatorname{Re} z > 0$ , this is nothing but a standard elliptic estimate. The trouble comes with  $\operatorname{Re} z < 0$  and getting close to the spectrum. In  $\mathbb{R}^n$ , one may evaluate directly the convolution operator by proving its kernel to be in  $L^1$ : this follows from

$$|z + |\xi|^2|^2 = \sin^2 \frac{(\pi - \theta)}{2} (|z| + |\xi|^2)^2 + \cos^2 \frac{(\pi - \theta)}{2} (|\xi|^2 - |z|)^2, \quad \text{with } z = |z|e^{i\theta},$$

and a direct computation of  $L^2$  norms of  $\partial^\alpha (z + |\xi|^2)^{-1}$ . By reflection, one then extends this estimate to the half-space case, with both Dirichlet and Neumann boundary conditions. By localizing  $L^p$  estimates close to the boundary and flattening, one may then obtain the desired estimate (2.3); such an approach is carried out in [1] in a greater generality (systems of Laplace equations, mixed boundary conditions), at the expense of fixing the angle  $\theta$  and not tracking explicit dependances on  $|z|$  and  $\theta$ . While (relatively) elementary, such a proof is, out of necessity, filled with lengthy calculations and most certainly does not provide the sharpest constant. It is worth noting, however, that it relies on standard elliptic techniques.

To keep in line with the parabolic approach, we present a short proof, relying on the holomorphic nature of  $S(w)$  in the half-plane  $\operatorname{Re} w > 0$ . Remark that by our  $L^p$  bound on  $S(t)$ ,  $t \in \mathbb{R}_+$ , the trivial  $L^2$  bound on  $S(w)$ ,  $\operatorname{Re} w \geq 0$ , and Stein's parameter version of complex interpolation, one may easily derive that  $S(w)$  is holomorphic in a sector around the positive real axis; but its angle will narrow with large or small  $p$ . However the argument may be refined and  $S(w)$  was proved to be holomorphic in the whole right half-plane in [16], using in a crucial way the Gaussian nature of the heat kernel on domains ([7]). This was extended to more general settings in [6], where an explicit bound is stated:

$$(5.1) \quad \|S(w)\|_{L^p \rightarrow L^p} \leq C_\varepsilon \left( \frac{|w|}{|\operatorname{Re} w|} \right)^{n|\frac{1}{2} - \frac{1}{p}| + \varepsilon}.$$

Then (2.3) follows easily: recall the following formula, which is simply a Laplace transform,

$$(5.2) \quad (z - \Delta_D)^{-1} = \int_L e^{w\Delta_D - wz} dw,$$

where  $L$  can be chosen to be a half ray from the origin. Set  $z = re^{i\theta}$ ,  $w = \rho e^{i\phi}$ , then

$$(z - \Delta_D)^{-1} = \int_0^{+\infty} e^{\rho \exp(i\phi)\Delta_D - r\rho \exp i(\theta+\phi)} d\rho.$$

Now, if  $\operatorname{Re} z > 0$ , we may take  $\phi = 0$  and use estimates for the semi-group  $S(\rho)$ . We would like to extend the range to the  $\operatorname{Re} z < 0$  region, up to a thin sector around the

negative real axis ( $|\pi - \theta| < \epsilon$ ); getting close to the spectrum is required if we want to define  $\Psi(-\Delta_D)$  with  $\Psi \in C_0^\infty([0, +\infty[)$ . One picks  $\phi$  such that  $2|\theta + \phi| < \pi$ , which ensures a decaying exponential in 5.2, provided we bound  $S(w)$  in  $L^p$ . But the condition on  $\phi$  yields  $|\phi| < \pi/2$ , and the bound amounts to the holomorphy of  $S(w)$ . The constant in (5.1) translates into a  $(|z|/|\operatorname{Im}z|)^\alpha$  factor, while integration over  $\rho$  provides the remaining  $1/|\operatorname{Im}z|$  in (2.3). This concludes the proof.

*Remark 5.1.* This seems the only place where the Gaussian bound on the heat kernel is required. Since we emphasize the elementary nature of our proofs, it should be noted that such a Gaussian bound is only a few steps away from the parabolic estimates we obtained in Appendix 1, Section 4. In fact, the usual hypercontractivity bound is obtained from  $L^2$  regularity estimates and Sobolev embedding; then off-diagonal kernel bounds follow from a clever use of weights, see [7].

## REFERENCES

1. T. Akiyama, H. Kasai, Y. Shibata, and M. Tsutsumi, *On a resolvent estimate of a system of Laplace operators with perfect wall condition*, Funkcial. Ekvac. **47** (2004), no. 3, 361–394. MR MR2126323 (2005m:35076)
2. Pascal Auscher, *On necessary and sufficient conditions for  $L^p$ -estimates of Riesz transforms associated to elliptic operators on  $\mathbb{R}^n$  and related estimates*, Mem. Amer. Math. Soc. **186** (2007), no. 871, xviii+75.
3. Pascal Auscher, Alan McIntosh, and Philippe Tchamitchian, *Heat kernels of second order complex elliptic operators and applications*, J. Funct. Anal. **152** (1998), no. 1, 22–73.
4. Pascal Auscher and Philippe Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque (1998), no. 249, viii+172.
5. N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605.
6. Gilles Carron, Thierry Coulhon, and El-Maati Ouhabaz, *Gaussian estimates and  $L^p$ -boundedness of Riesz means*, J. Evol. Equ. **2** (2002), no. 3, 299–317. MR MR1930609 (2003i:35116)
7. E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.
8. ———, *Pointwise bounds on the space and time derivatives of heat kernels*, J. Operator Theory **21** (1989), no. 2, 367–378. MR MR1023321 (90k:58214)
9. ———, *The functional calculus*, J. London Math. Soc. (2) **52** (1995), no. 1, 166–176.
10. ———, *Non-Gaussian aspects of heat kernel behaviour*, J. London Math. Soc. (2) **55** (1997), no. 1, 105–125. MR MR1423289 (97i:58169)
11. E. M. Dyn'kin, *An operator calculus based on the Cauchy-Green formula*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **30** (1972), 33–39, Investigations on linear operators and the theory of functions, III. MR MR0328640 (48 #6982)
12. B. Helffer and J. Sjöstrand, *Équation de Schrödinger avec champ magnétique et équation de Harper*, Schrödinger operators (Sønderborg, 1988), Lecture Notes in Phys., vol. 345, Springer, Berlin, 1989, pp. 118–197. MR MR1037319 (91g:35078)
13. Oana Ivanovici, *On Schrödinger equation outside strictly convex obstacles*, 2008, arXiv:math.AP/0809.1060.
14. Gilles Lebeau, *Estimation de dispersion pour les ondes dans un convexe*, Journées “Équations aux Dérivées Partielles” (Evian, 2006), 2006.
15. Francis Nier, *A variational formulation of Schrödinger-Poisson systems in dimension  $d \leq 3$* , Comm. Partial Differential Equations **18** (1993), no. 7-8, 1125–1147. MR MR1233187 (94i:35057)
16. El-Maati Ouhabaz, *Gaussian estimates and holomorphy of semigroups*, Proc. Amer. Math. Soc. **123** (1995), no. 5, 1465–1474. MR MR1232142 (95f:47068)
17. Fabrice Planchon and Luis Vega, *Bilinear identities and applications*, 2007, to appear in Ann. Sci. E.N.S., arXiv:math.AP/0712.4076.
18. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

19. François Trèves, *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*, Plenum Press, New York, 1980, Pseudodifferential operators, The University Series in Mathematics. MR MR597144 (82i:35173)
20. H. Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
21. Hans Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
22. ———, *Theory of function spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.

*E-mail address:* `oana.ivanovici@math.u-psud.fr`

UNIVERSITÉ PARIS SUD, MATHÉMATIQUES, BÂT 430, 91405 ORSAY CEDEX

*E-mail address:* `fab@math.univ-paris13.fr`

UNIVERSITÉ PARIS 13, L.A.G.A., UMR 7539, INSTITUT GALILÉE, 99 AVENUE J.B. CLÉMENT,  
F-93430 VILLETANEUSE