The 3D Finite Larmor Radius Approximation

Daniel Han-Kwan

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Abstract

Following Frénod and Sonnendrücker ([11]), we consider the finite Larmor radius regime for a plasma submitted to a large magnetic field and take into account the quasineutrality of the plasma. We then rigorously establish the asymptotic limit of the rescaled Vlasov-Poisson system in a 3D setting with the help of an averaging lemma.

1 Introduction and main results

1.1 Physical motivation

We are interested in the behaviour of a plasma (*id est* a gaz made of ions with individual charge Ze and mass m_i and electrons with individual charge $-e$ and mass m_e, with $m_i >> m_e$) which is submitted to a large external magnetic field. It is "well-known" that such a field induces fast small oscillations for the particles and consequently introduces a new small time scale which is very restrictive and inconvenient from the numerical point of view. The simulation of such plasmas appears to be primordial since the model can be applied to plasmas in tokamaks from magnetic confinement fusion (like for the ITER project).

1.1.1 Heuristic study

Let us give some heuristic formal arguments to investigate the behaviour of the plasma: if we consider the motion of one particle (of charge $q > 0$, mass m, position x and velocity v) submitted to an external constant field B, the fundamental principle of mechanics gives that:

$$\frac{dx}{dt} = v, \ \frac{dv}{dt} = \frac{q}{m}(v \wedge B) \tag{1.1}$$

Straightforward calculations show first of all that the parallel velocity, denoted by $v_{\|}$ (that is to say the component of the velocity in the direction of the magnetic field) is conserved and so is the norm of the perpendicular velocity $v_{\perp}$ (the component of the velocity in the perpendicular plane). Then, we can see that the particle moves on a helix whose axis is the direction of the magnetic field. The rotation period around the axis is the inverse of the cyclotron frequency Ω:

$$\Omega = \frac{|q||B|}{m} \tag{1.2}$$

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and the radius is the so-called Larmor radius:

$$r_L = \frac{|v_\perp|}{\Omega} \tag{1.3}$$

In the case where the magnetic field is very strong, Ω tends to infinity whereas r_L tends to zero. More precisely, if we take $|B| \sim \frac{1}{\epsilon}$ (with $\epsilon \to 0$) we have:

$$\begin{cases} \Omega \sim \frac{1}{\epsilon} \\ r_L \sim \epsilon \end{cases}$$

The approximation which consists in taking $r_L = 0$ is the classical guiding center approximation ([16]). This means that each particle is assimilated to its "guiding center" (in other words its "instantaneous rotation center"), which is equivalent to neglect the very fast rotation of the particle around the axis.

If one also applies some external constant electric field E, an easy computation shows that there appears:

1. an acceleration $\frac{E.B}{|B|}$ in the direction of B. If we consider $E \sim 1$, then:

$$\frac{E.B}{|B|} \sim 1 \tag{1.4}$$

2. a drift $\frac{E \wedge B}{|B|^2}$ in the orthogonal plane. We have:

$$\frac{E \wedge B}{|B|^2} \sim \epsilon \tag{1.5}$$

This drift, usually called the electric drift is problematic as regards to the problem of plasma confinement. It is negligible compared to the acceleration in the direction of B, but in the time scale for plasma fusion which is expected to be very long, one can not neglect this small drift, since it creates a displacement of order ϵt (t represents the time).

At last, note also that if the fields are not constant, various other drifts may appear, whose order in ϵ is higher than those of the electric drift.

Actually, the fields considered are neither constant, nor external, but self-induced by the plasma itself. The effects we would like to describe are non linear interaction between the particles and the electromagnetic field.

1.1.2 The mathematical model

In all the sequel, we assume that the magnetic field is external and constant and we suppose that the speed of particles in small compared to the speed of light, so that we can use the electrostatic approximation which consists in reducing the Maxwell equations to the Poisson equation. Finally, we decide to choose a kinetic description for the plasma.

The basic model usually considered for the ions is the following Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v.\nabla_x f + (E + v \wedge B).\nabla_v f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \\ f_{t=0} = f_0 \end{cases}$$

where $f(t,x,v)$ is the density of ions, with $t \in \mathbb{R}^+, x \in \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d, v \in \mathbb{R}^d$ (usually $d = 2$ or 3), meaning that $f(t,x,v)dxdv$ gives the number of ions in the infinitesimal volume $[x, x+dx] \times [v, v+dv]$ at time t (note that in this model, electrons are for the moment neglected).

1.1.3 The gyrokinetic approximation

It is important from a numerical point of view to establish the asymptotic equation when $|B|$ tends to infinity. Indeed, we expect the asymptotic equation to be "easier" to handle: only one time and space scale, perhaps less variables in the phase space to deal with... The derivation of such equations is usually referred to in the mathematic literature as "gyrokinetic approximation".

Rigorous justifications of these derivations with various time and space observation scales have only appeared at the end of the nineties. We refer for instance to the works of Brenier ([4]), Frénod and Sonnendrücker ([10], [11]), Frénod, Raviart and Sonnendrücker ([9]), Golse and Saint-Raymond ([14],[15]), Saint-Raymond ([18],[19]).

The classical "guiding center approximation" corresponds to the following scaling for the Vlasov-Poisson system (for now on and until the end of the paper, B is a constant vector, say for instance $B = e_z$):

$$\begin{cases} \partial_t f_\epsilon + v.\nabla_x f_\epsilon + (E_\epsilon + \frac{v \wedge B}{\epsilon}).\nabla_v f_\epsilon = 0 \\ E_\epsilon = -\nabla_x V_\epsilon \\ -\Delta_x V_\epsilon = \int f_\epsilon dv \\ f_{\epsilon,t=0} = f_0 \end{cases} \tag{1.6}$$

The articles [10] and [14] show that when $\epsilon \to 0$, this leads to a monokinetic equation in the direction of B:

$$\begin{cases} \partial_t f + v_\parallel.\nabla_x f + E_\parallel.\nabla_v f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \\ f_{t=0} = f_0 \end{cases} \tag{1.7}$$

Notice that the electric drift does not appear; this was expected since we have seen in the formal analysis that this drift was of higher order in ϵ than the other effects. This shows in particular that this approximation is not sufficient for the numerical simulation of tokamaks. In order to make this drift appear, there exists to our knowledge two main possibilities:

1. one consists in restricting to a $2D$ problem in the plane orthogonal to B ([14]),
2. the other consists in rescaling the orthogonal scales in order to get both transport and electric drift to the same order ([11]).

This work directly follows the articles [11] and [9] where the authors considered the "finite Larmor radius approximation". This corresponds to say that the spatial observation scale in the plane orthogonal to B is chosen smaller than the one in the parallel direction, more precisely with the same order as the Larmor radius r_L, so that one can expect the electric drift to appear in the asymptotic equation. In some sense, having such a scaling allows the electric field to significantly vary across a Larmor radius. Moreover, in that case, the positions of the particles are no longer assimilated to the position of their "guiding center" and we will have to perform an average over one fast oscillation period (the so-called gyroaverage) in order to get a sort of averaged number density.

1.2 Scaling and existing results

The system we are going to study is based on the "finite Larmor radius scaling" and takes into account the quasineutrality of the plasma.

1.2.1 The (refined) mathematical model

We refer to [11] for a complete discussion on the scaling. Let us recall briefly how it works: let $L_\parallel$ be the characteristic length in the direction of the magnetic field and $L_\perp$ be the characteristic length in the perpendicular plane. We consider that $L_\parallel \sim 1$ and $L_\perp \sim \epsilon$ and define the dimensionless variables $x'_\parallel = L_\parallel x_\parallel$ and $x'_\perp = L_\perp x_\perp$. In the same fashion we also define the dimensionless variables t' and v' with characteristic time and velocities with the same order as $L_\parallel$ and introduce the new number density f' defined by $\bar{f} f'(t', x', v') = f(t, x, v)$ (and we define likewise the new electric field and potential $\bar{E}E'(t', x', v') = f(t, x, v)$ and $\bar{V}V'(t', x', v') = V(t, x, v)$). We consider the scaling $\bar{f}, \bar{E} \sim 1$ and $\bar{V} \sim \epsilon$. At last, we introduce the Debye length of the plasma λ_D, which appears in the Poisson equation. In order to take into account the quasineutrality of the plasma, we take from now on $\lambda_D^2 \sim \epsilon$. The dimensionless system (1.6) becomes:

$$\begin{cases} \partial_{t'} f'_\epsilon + \frac{v'_\perp}{\epsilon}.\nabla_{x'} f'_\epsilon + v'_\parallel.\nabla_{x'} f'_\epsilon + (E'_\epsilon + \frac{v' \wedge B}{\epsilon}).\nabla_{v'} f'_\epsilon = 0 \\ E'_\epsilon = (-\nabla_{x'_\perp} V'_\epsilon, -\epsilon \nabla_{x'_\parallel} V'_\epsilon) \\ -\epsilon \Delta_{x'_\parallel} V'_\epsilon - \frac{1}{\epsilon} \Delta_{x'_\perp} V'_\epsilon = \frac{1}{\epsilon^2} \int f'_\epsilon dv \\ f'_{\epsilon, t'=0} = f'_0 \end{cases} \tag{1.8}$$

with the notation $\Delta_{x'_\parallel} = \partial_{x'_\parallel}$ and $\Delta_{x'_\perp} = \Delta - \Delta_{x'_\parallel}$. Frow now on, for the sake of simplicity, we forget the primes. Since we want to take into account the electrons, we will rather consider the following Poisson equation:

$$-\epsilon \Delta_{x_\parallel} V_\epsilon - \frac{1}{\epsilon} \Delta_{x_\perp} V_\epsilon = \frac{1}{\epsilon^2} \left(n^i_\epsilon - n^e_\epsilon \right) \tag{1.9}$$

where $n^i_\epsilon = \int f_\epsilon dv$ the density of ions and n^e_ϵ the density of electrons. We make the assumption that we deal with a fixed background of electrons, so that, because of global neutrality,

$$n^\epsilon_e = \int f_0 dv dx$$

The Poisson equation can now be written:

$$-\epsilon^2 \Delta_{x_\parallel} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_0 dv dx \tag{1.10}$$

the problem being posed for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$ (with $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ equipped with the restriction of the Lebesgue measure to $[0, 1]^3$).

1.2.2 State of the art about the Finite Larmor Radius Approximation

Using homogenization arguments, Frénod and Sonnendrücker established the convergence in some weak sense of sequences of solutions $(f_\epsilon)_{\epsilon \geq 0}$ of similar systems, in two cases, namely in some pseudo $2D$ case (assuming that nothing depends on $x_\parallel$ and $v_\parallel$) and in a $3D$ case when the electric field is external. The main tool used to establish the convergence is the "2-scale convergence" introduced by Nguetseng [17] and Allaire [2] that we will recall later on.

1. The 3D case:

Assume that we deal with an external electric field $E_\epsilon = E \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^3)$:

$$\begin{cases} \partial_t f_\epsilon + v_\parallel . \nabla_x f_\epsilon + \frac{v_\perp}{\epsilon} . \nabla_x f_\epsilon + \left(E + \frac{v \wedge e_z}{\epsilon} \right) . \nabla_v f_\epsilon = 0 \\ f_{t=0} = f_0 \end{cases}$$

Frénod and Sonnendrücker proved the following theorem:

Theorem 1.1. *For each ϵ let f_ϵ be the unique solution of the scaled Vlasov equation in $L^\infty_t(L^1_{x,v} \cap L^2_{x,v})$. Then the following convergence holds as ϵ tends to 0:*

$$f_\epsilon \rightharpoonup f \text{ weak-* } L^\infty_t(L^2_{x,v}) \tag{1.11}$$

where $f \in L^\infty_t(L^2_{x,v})$ is the unique solution to:

$$\begin{aligned} \partial_t f + v_\parallel . \nabla_x f + \frac{1}{2\pi} \left(\int_0^{2\pi} \mathcal{R}(\tau) E(t, x + \mathcal{R}(-\tau)v) d\tau \right) . \nabla_x f \\ + \frac{1}{2\pi} \left(\int_0^{2\pi} R(\tau) E(t, x + \mathcal{R}(-\tau)v) d\tau \right) . \nabla_v f = 0 \end{aligned}$$

$$f_{|t=0} = \frac{1}{2\pi} \left(\int_0^{2\pi} f_0(x + \mathcal{R}(\tau)v, R(\tau)v) d\tau \right)$$

denoting by R and $\mathcal{R}$ the linear operators defined by:

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = (-R(-\pi/2) + R(-\pi/2 + \tau))$$

2. The pseudo $2D$ case:

   The Vlasov-Poisson system considered for this case is the following $2D$ system:

$$\partial_t f_\epsilon + \frac{v}{\epsilon} . \nabla_x f_\epsilon + \left(E_\epsilon + \frac{v^\perp}{\epsilon} \right) . \nabla_v f_\epsilon = 0 \tag{1.12}$$

$$f_{\epsilon|t=0} = f_0 \tag{1.13}$$

$$E_\epsilon = -\nabla V_\epsilon, -\Delta_x V_\epsilon = \rho_\epsilon \tag{1.14}$$

$$\rho_\epsilon = \int f_\epsilon dv \tag{1.15}$$

   If $v = (v_x, v_y)$, $v^\perp$ is defined by $(v_y, -v_x)$.

   From now on, we will deal with global weak solutions of Vlasov-Poisson systems in the sense of Arsenev ([3]).

   Assuming here that $f_0 \geq 0, f_0 \in L^1_{x,v} \cap L^p_{x,v}$ (for some $p > 2$) and that the initial kinetic energy is bounded:

$$0 < \int f_0(1 + |v|^2) dv < \infty$$

   Frénod and Sonnendrücker proved the following theorem (we voluntarily write an unprecise meta-version of the result)

Theorem 1.2. *For each ϵ, let (f_ϵ, E_ϵ) be a solution in the sense of Arsenev to (1.12)-(1.15).*

Then, up to a subsequence, in a certain sense, f_ϵ 2-scale converges to F and E_ϵ 2-scale converges to $\mathcal{E}$. Moreover, there exists a function G such that :

$$F(t,\tau,x,v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v) \tag{1.16}$$

and $(G, \mathcal{E})$ satisfies :

$$\partial_t G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0$$

$$G_{|t=0} = \frac{1}{2\pi} f_0$$

$$\mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int_v G(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv$$

Note that the authors actually developped a generic framework that allows them to deal with different scalings and to give a precise approximation at any order. We do not wish to do so in our study.

1.3 A bit of homogenization theory and some useful definitions

Let us now precisely state the "2-scale" convergence tools used in this paper.

Definition 1.1. *Let X be a separable Banach space, X' be its topological dual space and $(.,.)$ the duality bracket between X' and X. For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L^{q'}_\alpha(\mathbb{R}; X')$) the space of α-periodic continuous (respectively $L^{q'}$) functions on $\mathbb{R}$ with values in X. Let $q \in [1; \infty[$.*

Given a sequence (u_ϵ) of functions belonging to the space $L^{q'}(0,t;X')$ and a function $U^0(t,\theta) \in L^{q'}(0,T;L^{q'}_\alpha(\mathbb{R};X'))$ we say that

$$u_\epsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))$ we have:

$$\lim_{\epsilon\to 0}\int_0^T \left(u_\epsilon(t), \Psi\left(t, \frac{t}{\epsilon}\right)dt\right) = \frac{1}{\alpha}\int_0^T\int_0^\alpha \left(U^0(t,\theta), \Psi(t,\theta)\right)d\theta dt \tag{1.17}$$

Theorem 1.3. *Given a sequence (u_ϵ) bounded in $L^{q'}(0,t;X')$, there exists for all $\alpha > 0$ a function $U^0_\alpha \in L^{q'}(0,T;L^{q'}_\alpha(\mathbb{R};X')$ such that up to a subsequence,*

$$u_\epsilon \text{ 2-scale converges to } U^0_\alpha$$

The profile U^0_α is called the α-periodic two scale limit of u_ϵ and the link between U^0_α and the weak- limit u of u_ϵ is given by:*

$$\frac{1}{\alpha}\int_0^\alpha U^0 d\theta = u \tag{1.18}$$

We also introduce some notations:

Definition 1.2. *We define for all $p \in [1;\infty]$ the space $L^p_{x,v}$:=$L^p_x(\mathbb{T}^d,(L^p_v(\mathbb{R}^d)))$.*

In the same fashion, we define the spaces $L^p_{t,x}$, $L^p_{t,x,v}$...

Let $L^p_{2\pi,\tau}$ be the space of 2π-periodic functions of τ which are in L^p_τ.

Let $L^p_{x,loc}$ be the space of functions f such that for all infinitely differentiable cut-off functions $\varphi \in \mathcal{C}^\infty_c$, φf belongs to L^p_x. We will say that a sequence (f_ϵ) is uniformly bounded in $L^p_{x,loc}$ if for each compact set K, the sequence of the restrictions to K is uniformly bounded in L^p_x with respect to ϵ (but this bound can depend on K).

We will also use the same notations for Sobolev spaces $W^{s,p}$ ($s \in \mathbb{R}$).

1.4 Statement of the result

In this paper we prove that the 2-scale convergence established in the previous $2D$ case is also true in the real $3D$ framework. The difficulty comes from the fact that there is no uniform full elliptic regularity for the electric field because of the factor ϵ^2 in front of $\Delta_{x_\parallel}$ in the Poisson equation:

$$-\epsilon^2\Delta_{x_\parallel}V_\epsilon - \Delta_{x_\perp}V_\epsilon = \int f_\epsilon dv$$

In particular there is no a priori regularity on $x_\parallel$ and therefore no strong compactness. Nevertheless, we actually prove that due to the particular form of the asymptotic equation, the moments of the solution with respect to $v_\parallel$ are more regular in $x_\parallel$ than the solution itself. We can then easily pass to the weak limit.

Notice that this result is in the same spirit as the proof of the weak stability of the Vlasov-Maxwell system by DiPerna and Lions ([7]), where the authors have regularity on moments, by opposition to the proof of the weak stability of the Vlasov-Poisson system by Arsenev ([3]), where the author has compactness on the electric field. Actually our result is a sort of an hybrid one, since we get on one hand regularity with respect to $x_\perp$ by elliptic regularity and in the other hand regularity with respect to $x_\parallel$ by averaging.

We assume here that the initial data f_0 satisfies the following conditions: $f_0 \geq 0, f_0 \in L^1_{x,v} \cap L^p_{x,v}$ (for some $p > 7/2$) and the initial kinetic energy is bounded:

$$0 < \int f_0|v|^2 dv < \infty$$

Proposition 1.1. *For each ϵ, let (f_ϵ, E_ϵ) in $L^\infty_t(L^1_{x,v} \cap L^p_{x,v}) \times L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,\frac{7}{5}}_{x_\perp}))$ be a solution in the sense of Arsenev to (1.8)-(1.10). Then up to a subsequence we have the following convergence as ϵ tends to 0:*

$$f_\epsilon \quad \textit{2-scale converges to} \quad F \in L^\infty_t(L^\infty_{2\pi,\tau}(L^1_{x,v} \cap L^p_{x,v})) \tag{1.19}$$

$$E_\epsilon \quad \textit{2-scale converges to} \quad \mathcal{E} \in L^\infty_t(L^\infty_{2\pi,\tau}(L^{7/5}_{x_\parallel}(W^{1,\frac{7}{5}}_{x_\perp}))) \tag{1.20}$$

Moreover, there exists a function $G \in L^\infty_t(L^1_{x,v} \cap L^p_{x,v})$ such that:

$$F(t,\tau,x,v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v) \tag{1.21}$$

and $(G,\mathcal{E})$ is solution to:

$$\partial_t G + v_\parallel.\nabla_x G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0$$

$$G_{|t=0} = \frac{1}{2\pi} f_0$$

$$\mathcal{E} = (-\nabla_\perp V, 0), \quad -\Delta_\perp V = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dvdx$$

denoting by R and $\mathcal{R}$ the linear operators defined by:

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = (R(-\pi/2) - R(-\pi/2 + \tau))$$

As it has been said, for the proof of this proposition, we will first prove a lemma which gives the regularity of moments in $v_\parallel$ of the solution. For this, we use an averaging lemma. Then the proof is very similar to the proof in the $2D$ case, but we will give it again for the sake of completeness.

Remark: This proposition implies that for a given non-negative initial data $G_{|t=0} = G_0$ in $L^1_{x,v} \cap L^p_{x,v}$ (with $p > \frac{7}{2}$) and satisfying the energy bound, the asymptotic system admits at least one global weak solution $G \in L^\infty_t(L^1_{x,v} \cap L^p_{x,v})$. With the additional assumptions on the inital data:

$$G_0 \in W^{1,1}_{x,v},$$
$$\|(1 + |v|^4)G_0\|_{L^\infty_{x,v}} < \infty,$$
$$\|(1 + |v|^4)DG_0\|_{L^\infty_{x,v}} < \infty.$$

we are able to prove the uniqueness of the solution, using the same ideas than Degond in [6] (and also used afterwards by Saint-Raymond in a gyrokinetic context ([18])).

2 A priori uniform estimates for the scaled Vlasov-Poisson system

2.1 Conservation of L^p norms and energy for the scaled system

In this section we give a priori estimates which are very classical for the Vlasov-Poisson system. (see for example [10], [11], [14] for recent references in the gyrokinetic literature). In order to recall how one can get them, we will give some formal computations. In order to have rigorous proofs we should deal with smooth and compactly supported functions, especially with a sequence $(f^n_\epsilon)_{n\geq 0}$ of solutions of regularized Vlasov-Poisson equations (that is the way one can clasically build a global weak solution in the sense of Arsenev ([3])). Then one can justify the estimations by passing to the limit.

First, as usual for such Vlasov equations, L^p norms are conserved:

Lemma 2.1. *For all $1 \leq p \leq \infty$,*

$$\forall t \geq 0, \|f(t)\|_{L^p_{x,v}} = \|f(0)\|_{L^p_{x,v}} \tag{2.1}$$

Moreover, $f(0) \geq 0$ if and only if $\forall t \geq 0, f(t) \geq 0$ (referred to as the maximum principle)

That precisely means that if $f(0) \in L^p_{x,v}$, then $f \in L^\infty_t(L^p_{x,v})$.

Let us now compute the energy for the scaled system:

Lemma 2.2. *We have:*

$$\mathcal{E}_\epsilon(t) = \left(\int f_\epsilon |v|^2 dvdx + \epsilon \int |\nabla_{x_\perp} V_\epsilon|^2 dx + \epsilon^3 \int |\nabla_{x_\parallel} V_\epsilon|^2 dx \right) \leq \mathcal{E}_\epsilon(0) \tag{2.2}$$

In particular there exists $C > 0$ independant of ϵ such that:

$$\int f_\epsilon |v|^2 dvdx \leq C \tag{2.3}$$

Formal proof. We multiply the scaled Vlasov equation by $|v|^2$ and integrate with respect to x and v.

$$\int \partial_t f_\epsilon |v|^2 dvdx + \int E_\epsilon . \nabla_v f_\epsilon |v|^2 dvdx = \frac{d}{dt} \left(\int f_\epsilon |v|^2 dvdx \right) - 2 \int E_\epsilon(x).v f_\epsilon dvdx = 0$$

We then integrate the Vlasov equation with respect to v. We get the so called conservation of charge:

$$\frac{d}{dt} \left(\int f dv \right) + \nabla_{x_\parallel} . \left(\int f v_\parallel dv \right) + \frac{\nabla_{x_\perp}}{\epsilon} . \left(\int f v_\perp dv \right) = 0 \tag{2.4}$$

Therefore, we have:

$$\begin{aligned} \int E_\epsilon(x).v f_\epsilon dvdx &= -\int (\nabla_{x_\perp} V_\epsilon, \epsilon \nabla_{x_\perp} V_\epsilon).v f_\epsilon dvdx \\ &= \int V_\epsilon \left(\nabla_{x_\perp} .(f_\epsilon v_\perp) + \epsilon \nabla_{x_\parallel} .(f_\epsilon v_\parallel) \right) dvdx \\ &= -\epsilon \int V_\epsilon \partial_t f_\epsilon dvdx \end{aligned}$$

Finally, using the Poisson equation, we get:

$$\begin{aligned} -\epsilon \int V_\epsilon \partial_t f_\epsilon dvdx &= -\epsilon \int V_\epsilon \partial_t \left(-\epsilon^2 \Delta_{x_\parallel} V_\epsilon - \Delta_{x_\perp} V_\epsilon \right) dx \\ &= -\epsilon \left(\int \nabla_{x_\perp} V_\epsilon \partial_t \nabla_{x_\perp} V_\epsilon dx + \epsilon^2 \int \nabla_{x_\parallel} \partial_t V_\epsilon \nabla_{x_\parallel} V_\epsilon dx \right) \\ &= -\epsilon \frac{1}{2} \frac{d}{dt} \left(\int |\nabla_{x_\perp} V_\epsilon|^2 dx + \epsilon^2 \int |\nabla_{x_\parallel} V_\epsilon|^2 dx \right) \end{aligned}$$

Thus it comes:

$$\frac{d}{dt} \left(\int f_\epsilon |v|^2 dvdx + \epsilon \int |\nabla_{x_\perp} V_\epsilon|^2 dx + \epsilon^3 \int |\nabla_{x_\parallel} V_\epsilon|^2 dx \right) = 0 \tag{2.5}$$

□

2.2 Regularity of the electric field

Let us recall a classical lemma obtained by a standard real interpolation argument (see [10] for instance):

Lemma 2.3. *Let $f(x,v)$ be a mesurable positive function on $\mathbb{R}^3 \times \mathbb{R}^3$. Then:*

$$\int \left(\int f(x,v) dv \right)^{7/5} dx \leq C \|f\|_{L^2_{x,v}}^{4/5} \left(\int |v|^2 f dxdv \right)^{3/5} \tag{2.6}$$

Proof. For any $R > 0$, we can write the following decomposition:

$$\begin{aligned}\int f_\epsilon dv &= \int_{|v|\leq R} f dv + \int_{|v|>R} f dv \\ &\leq CR^{3/2}\|f\|_{L^2_v} + \frac{1}{R^2}\int |v|^2 f dv\end{aligned}$$

Then we can take R such that $R^{3/2}\|f\|_{L^2_v} = \frac{1}{R^2}\int |v|^2 f dv$ so that we get:

$$\int f_\epsilon dv \leq C\left(\int f^2 dv\right)^{2/7}\left(\int |v|^2 f dv\right)^{3/7} \tag{2.7}$$

We then raise the quantities to the power $7/5$, integrate with respect to x and use Hölder's inequality which gives the estimate.

□

By conservation of the L^2 norm and the uniform bound on the kinetic energy, this entails that:

$$\rho_\epsilon \in L^\infty_t(L^{7/5}_x) \tag{2.8}$$

and the norm is bounded uniformly with respect to ϵ.

We now use the Poisson equation to compute the regularity of the electric field. Let us recall that:

$$E_\epsilon = \left(-\epsilon\nabla_{x_\parallel} V_\epsilon, -\nabla_{x_\perp} V_\epsilon\right)$$
$$-\epsilon^2\Delta_{x_\parallel} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \rho_\epsilon - \int \rho_\epsilon dx$$

Lemma 2.4. *With the above notations and assumptions:*
E_ϵ is uniformly bounded with respect to ϵ in $L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,7/5}_{x_\perp}))$

Proof. Let $\epsilon > 0$ and $t > 0$ be fixed. For the sake of simplicity we write V instead of V_ϵ and E instead of E_ϵ.

For any function $f(x_\parallel, x_\perp)$, define the rescaled function $\hat{f}(z, x_\perp)$ by

$$\hat{f}\left(\frac{x_\parallel}{\epsilon}, x_\perp\right) = \epsilon^{\frac{5}{7}} f(x_\parallel, x_\perp)$$

so that:

$$\|\hat{f}\,(z, x_\perp)\,\|_{L^{7/5}_z} = \|f(x_\parallel, x_\perp)\|_{L^{7/5}_{x_\parallel}} \tag{2.9}$$

The Poisson equation becomes:

$$-\Delta_z \hat{V} - \Delta_{x_\perp}\hat{V} = \hat{\rho} - \epsilon^{\frac{5}{7}}\int \rho_0 dx$$

and the scaled electric field is given by:

$$\hat{E} = \left(-\nabla_z \hat{V}, -\nabla_{x_\perp}\hat{V}\right)$$

Since $\rho_\epsilon(t, ., .)$ is uniformly bounded in $L^{7/5}_x$, standard results of elliptic regularity on the torus $\mathbb{T}^3$ show that there exists $C > 0$ independant of ϵ such that:

$$\|\hat{V}\|_{W^{2,7/5}_{z,x_\perp}} \leq C \left\| \hat{\rho} - \epsilon^{\frac{5}{7}} \int \rho_0 dx \right\|_{L^{7/5}_{z,x_\perp}}$$

Thanks to (2.9) we get:

$$\|\hat{V}\|_{W^{2,7/5}_{z,x_\perp}} \leq C \left\| \rho - \int \rho_0 dx \right\|_{L^\infty_t(L^{7/5}_x)} \leq C_0$$

with C_0 independant of ϵ.
Consequently, we have:

$$\|\hat{E}\|_{L^{7/5}_z(W^{1,7/5}_{x_\perp})} \leq \|\hat{E}\|_{W^{1,7/5}_z(W^{1,7/5}_{x_\perp})} \leq C_0$$

Finally from (2.9) we get

$$\|E_\epsilon\|_{L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,7/5}_{x_\perp}))} \leq C_0$$

□

We can see as expected that the regularity of the electric field with respect to the $x_\parallel$ variable is not sufficient to get some strong compactness.

Remark: If we work on the whole space $\mathbb{R}^3$ instead of $\mathbb{T}^3$, we only get homogeneous estimates for V_ϵ and we have not been able to deal with such anisotropic estimates in the following of the paper. Roughly speaking, if V is a solution of the Poisson equation $-\Delta V = \rho$ with $\rho \in L^{7/5}(\mathbb{R}^3)$, we can only say that $V \in \dot{W}^{2,7/5}(\mathbb{R}^3)$ (the homogeneous Sobolev space) and not $W^{2,7/5}$.

3 Proof of Proposition 1.1

Proof. The beginning of the proof is identical to the one given in [11]. For the sake of completeness we recall here the main arguments and refer to [11] for the details.

Step 1: Deriving the constraint equation

First of all, since (f_ϵ) is bounded in $L^\infty_t(L^1_{x,v} \cap L^p_{x,v})$, Theorem 1.3 shows that for all $\alpha > 0$:

$$f_\epsilon \text{ 2-scale converges to } F_\alpha \in L^\infty(0,T; L^\infty_\alpha(\mathbb{R}; L^p_{x,v}))$$

Let $\Psi(t,\tau,x,v)$ be an α-periodic oscillating test function in τ and define:

$$\Psi^\epsilon \equiv \Psi(t, \frac{t}{\epsilon}, x, v)$$

We start by writing the weak formulation of the scaled Vlasov equation against Ψ^ϵ. Since

$$\nabla_{x_\parallel}.v_\parallel = \nabla_{x_\perp}.v_\perp = \mathrm{div}_v \left(E_\epsilon + \frac{v \wedge e_z}{\epsilon} \right) = 0,$$

we get the following equation:

$$\int f_\epsilon \left((\partial_t \Psi)^\epsilon + \frac{1}{\epsilon}(\partial_\tau \Psi)^\epsilon + v_\parallel.(\nabla_x \Psi)^\epsilon + \frac{v_\perp}{\epsilon}.(\nabla_x \Psi)^\epsilon + \left(E_\epsilon + \frac{v \wedge e_z}{\epsilon} \right).(\nabla_v \Psi)^\epsilon \right) dt dx dv$$
$$= - \int f_0 \Psi(x, v0, 0) dx dv$$

Multiply then by ϵ and pass to the (2-scale) limit. We get the so called constraint equation for the α-periodic profile F_α:

$$\partial_\tau F_\alpha + v_\perp.\nabla_x F_\alpha + v \wedge e_z.\nabla_v F_\alpha = 0, \tag{3.1}$$

which means that F_α is constant along the characteristics:

$$\frac{dV}{d\tau} = V \wedge e_z \tag{3.2}$$

$$\frac{dX}{d\tau} = V_\perp \tag{3.3}$$

A straightforward calculation therefore shows that there exists $F^0_\alpha \in L^\infty(0,T,L^p_{x,v})$ such that:

$$F_\alpha(t,\tau,x,v) = F^0_\alpha(t, x + \mathcal{R}(\tau)v, R(\tau)v) \tag{3.4}$$

with:

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = (-R(-\pi/2) + R(-\pi/2+\tau))$$

i.e. $\mathcal{R}(\tau) = \begin{bmatrix} \sin\tau & \cos\tau - 1 & 0 \\ 1-\cos\tau & \sin\tau & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since R and $\mathcal{R}$ are 2π-periodic, we will consider the 2π profile: indeed if α and 2π were incommensurable, F_α could not depend on τ and consequently we would have no information on the oscillations.

Step 2: Filtering the essential oscillation

We now look for the equation satisfied by $F^0_{2\pi} := G$; we introduce the filtered function g_ϵ:

$$g_\epsilon(t,x,v) = f_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v, R(-t/\epsilon)v) \tag{3.5}$$

(meaning that we have removed the oscillations)

We easily compute the equation satisfied by g_ϵ:

$$\begin{aligned} \partial_t g_\epsilon + v_\parallel.\nabla_x g_\epsilon + \mathcal{R}(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_x g_\epsilon \\ + R(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_v g_\epsilon = 0 \end{aligned} \tag{3.6}$$

Remark: Note here that g_ϵ 2-scale converges to G, and since it does not depend on τ, it also weakly converges to G.

Step 3: Getting some regularity on moments

From now on, the goal is to get some compactness for the moments of g_ϵ with respect to $v_\parallel$. The main tool is the following averaging lemma proved by Bézard in [5], which is a refined version of the fundamental result of DiPerna, Lions and Meyer ([8]):

Theorem 3.1. *Let $1 < p \leq 2$. Let $f, g \in L^p(dt \otimes dx \otimes dv)$ be solutions of the following transport equation*

$$\partial_t f + v.\nabla_x f = (I - \Delta_{t,x})^{\tau/2}(I - \Delta_v)^{m/2} g \tag{3.7}$$

with $m \in \mathbb{R}^+, \tau \in [0,1[$. *Then* $\forall \Psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\rho_\Psi(t,x) = \int f(t,x,v)\Psi(v)dv \in W^{s,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)$ *where*

$$s = \frac{1-\tau}{(1+m)p'} \tag{3.8}$$

Moreover,

$$\|\rho_\Psi\|_{W^{s,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C\left(\|f\|_{L^p(dt\otimes dx\otimes dv)} + \|(I-\Delta_{t,x})^{\tau/2}(I-\Delta_v)^{m/2}g\|_{W^{-\tau,p}_{t,x}(W^{-m,p}_v)}\right) \tag{3.9}$$

(C is a positive constant independant of f and g)

Averaging lemmas are an important feature of transport equations: since the transport equation (3.7) is hyperbolic, one can obviously not expect the solution f to be more regular than the right hand side or the inital data. Nevertheless, if one considers the averaged quantity in velocity ρ_Ψ, one can actually notice a gain of regularity. This phenomenon was first observed independently by Golse, Perthame and Sentis ([13]) and Agoshkov ([1]) then was formulated in a precise way for the first time by Golse, Lions, Perthame and Sentis (see [12]); it is referred to as "velocity averaging". There exists many refined versions of these results and numerous interesting applications in kinetic theory, but we shall not dwell on that. We simply point out that this tool has been successfully applied to Vlasov equations, for instance to prove the existence of global weak solutions to the Vlasov-Maxwell as it has been done by DiPerna and Lions ([7]).

Remarks :

1. The fundamental averaging lemmas proved by DiPerna, Lions and Meyer ([8]), which lead to a Besov estimate instead of a Sobolev estimate, would have been sufficient for our purpose, since we do not need the optimal regularity.

2. These lemmas have been proved for functions with values in $\mathbb{R}$ but the proof is the same with functions with values in more complicated Banach spaces. Indeed one can consider "abstract" Banach norms instead of the usual norm in $\mathbb{R}$ and the proof is then formally identical.

Equipped with this tool, we can now prove that moments in $v_\parallel$ are more regular with respect to t and $x_\parallel$ than the solution itself.

Proposition 3.1. *For each* $\epsilon > 0$, *let* g_ϵ *a uniformly bounded function in* $L^1_{x,v} \cap L^p_{x,v}$ *(with* $p > 7/2$*) be some solution of:*

$$\begin{aligned}\partial_t g_\epsilon + v_\parallel.\nabla_x g_\epsilon + \mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_x g_\epsilon \\ + R(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_v g_\epsilon = 0\end{aligned}$$

with E_ϵ *uniformly bounded in* $L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,7/5}_{x_\perp}))$.

Let $\Psi \in \mathcal{D}(\mathbb{R})$. *Define*

$$\eta_\epsilon(t,x,v_\perp) = \int g_\epsilon(t,x,v)\Psi(v_\parallel)dv_\parallel$$

Then,

$$\eta_\epsilon \text{ is uniformly bounded in } W^{s,\gamma}_{t,loc}(W^{s,\gamma}_{x_\parallel,loc}(W^{-1,\gamma}_{x_\perp,v_\perp,loc})) \tag{3.10}$$

for $\gamma \in]1;2[$ *such that,* $\frac{1}{\gamma} = \frac{5}{7} + \frac{1}{p}$ *and some* $s \in]0;1[$ *(depending on* γ*)*

Proof. • The first step is to localize the equation. Let K be the cartesian product of compact sets:

$$K = [0,T] \times K_{x_\parallel} \times K_{x_\perp} \times K_{v_\parallel} \times K_{v_\perp}$$

We now consider some positive smooth function $\Phi(t, x_\parallel, x_\perp, v_\parallel, v_\perp)$ which is $\mathcal{C}_c^\infty$ and which satisfies the condition:

$$\Phi \equiv 0 \text{ outside } K \tag{3.11}$$

The equation satisfied by $g_\epsilon \Phi$ is the following one:

$$\begin{aligned}\partial_t(g_\epsilon\Phi) + v_\parallel.\nabla_x(g_\epsilon\Phi) = &-\underbrace{\mathcal{R}(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_x(g_\epsilon\Phi)}_{(1)}\\ &-\underbrace{R(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_v(g_\epsilon\Phi)}_{(2)} - \partial_t(\Phi)g_\epsilon - v_\parallel.\nabla_x(\Phi)g_\epsilon\\ &-\mathcal{R}(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_x(\Phi)g_\epsilon - R(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_v(\Phi)g_\epsilon\end{aligned}$$

The idea is now to consider this equation as a kinetic equation with respect to the variables $(t, x_\parallel, v_\parallel)$ and with values in an abstract Banach space (which will be $W^{-1,\gamma}_{x_\perp,v_\perp}$). We then only study the first two members of the right-hand side (noticing that the other terms have more regularity than these ones).

From now on, for the sake of simplicity and readability, we will write L^p and $W^{s,p}$ norms without always specifying that they are taken on the compact support of Φ.

- **Estimate on the first term (1)**

Since E_ϵ does not depend on v, we have:

$$E_\epsilon \in L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,\infty}_v(W^{1,7/5}_{x_\perp})))$$

In particular if we restrain ourselves to compact supports:

$$E_\epsilon \in L^{7/5}_t\left([0,T], L^{7/5}_{x_\parallel}(K_{x_\parallel}, W^{1,7/5}_{v_\parallel}(K_{v_\parallel}, W^{1,7/5}_{v_\perp}(K_{v_\perp}, W^{1,7/5}_{x_\perp}(K_{x_\perp}))))\right)$$

The second point is that the differential operator

$$\mathcal{R}(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(t/\epsilon)v).\nabla_x$$

involves only derivatives with respect to the $x_\perp$ variable and not in the parallel direction (this remark is fundamental for using an averaging lemma). Product laws in Sobolev spaces give then:

$$\begin{aligned}&\|\mathcal{R}(t/\epsilon)E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v).\nabla_x(g_\epsilon\Phi)\|_{W^{-1,\gamma}_{x_\perp,v_\perp}(K_{x_\perp}\times K_{v_\perp})} \leq\\ &\|E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v)\|_{W^{1,7/5}_{x_\perp,v_\perp}(K_{x_\perp}\times K_{v_\perp})}\|\mathcal{R}(t/\epsilon).\nabla_x(g_\epsilon\Phi)\|_{W^{-1,p}_{x_\perp,v_\perp}(K_{x_\perp}\times K_{v_\perp})}\end{aligned}$$

where $\frac{1}{\gamma} = \frac{5}{7} + \frac{1}{p}$.

Remark: The regularity of (1) with respect to $v_\perp$ is not optimal (since it involves no derivative in $v_\perp$ for g_ϵ). Nevertheless we are interested in the regularity of the whole right hand side, and we will see that the term (2) has this regularity in $v_\perp$.

The change of variables $(x,v) \mapsto (x+\mathcal{R}(s)v, R(s)v)$ has unit Jacobian for all $s \in \mathbb{R}$ so that:

$$\|E_\epsilon(t, x+\mathcal{R}(-t/\epsilon)v)\|_{W^{1,7/5}_{x_\perp,v_\perp}(K_{x_\perp}\times K_{v_\perp})} = \|E_\epsilon(t,x)\|_{W^{1,7/5}_{x_\perp,v_\perp}(J_{x_\perp}\times J_{v_\perp})}$$

where $J_{x_\perp}$ and $J_{v_\perp}$ are the compact images of $K_{x_\perp}$ and $K_{v_\perp}$ by the change of variables. Moreover, one gets:

$$\|\mathcal{R}(t/\epsilon).\nabla_x(g_\epsilon\Phi)\|_{W^{-1,p}_{x_\perp,v_\perp}} \leq C\|g_\epsilon\Phi\|_{L^p_{x_\perp,v_\perp}}$$

Finally we have after integrating in $t, x_\parallel, v_\parallel$:

$$\begin{aligned}\|\mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_x(g_\epsilon\Phi)\|_{L^\gamma_{t,x_\parallel,v_\parallel}(W^{-1,\gamma}_{x_\perp,v_\perp})}\\ \leq C\|E_\epsilon(t,x)\|_{L^{7/5}_{t,x_\parallel,v_\parallel}(W^{1,7/5}_{x_\perp,v_\perp})}\|g_\epsilon\Phi\|_{L^p_{t,x_\parallel,v_\parallel}(L^p_{x_\perp,v_\perp})}\end{aligned}$$

and C is a constant independant of ϵ.

- **Estimate on the second term (2)**

  By the same method one gets:

$$\begin{aligned}\|R(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_v(g_\epsilon\Phi)\|_{L^\gamma_{t,x_\parallel}(W^{-1,\gamma}_{v_\parallel}(W^{-1,\gamma}_{x_\perp,v_\perp}))}\\ \leq C\|E_\epsilon(t,x)\|_{L^{7/5}_{t,x_\parallel}(W^{1,7/5}_{x_\perp,v})}\|g_\epsilon\Phi\|_{L^p_{t,x_\parallel}(L^p_{x_\perp,v})}\end{aligned}$$

  Finally we see that the right hand side is uniformly bounded in:

$$L^\gamma_{t,x_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{v_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}}))$$

- **Regularity of the moments**

  By theorem 3.1 (and the second remark given afterwards), for all $\Psi \in \mathcal{C}^\infty_c$, the moment:

$$\eta_\epsilon(t,x,v_\perp) = \int g_\epsilon(t,x,v)\Psi(v_\parallel)dv_\parallel$$

  is then uniformly bounded in the space $W^{s,\gamma}_{t,x_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}})$ with $s>0$ given by $s=\frac{1}{2\gamma'}$.

□

We can now prove that the sequence of moments η_ϵ is compact in a space of distributions which is the dual of some space where the sequence (E_ϵ) is uniformly bounded.

Corollary 3.1. *There exists $\theta \in]0,1[$ and $\eta \in W^{s\theta,7/2}_{t,loc}(W^{s\theta,7/2}_{x_\parallel,loc}(W^{-\theta,7/2}_{x_\perp,v_\perp,loc}))$ such that for all $\xi>0$, up to a subsequence:*

$$\|\eta_\epsilon - \eta\|_{L^{7/2}_{t,loc}(L^{7/2}_{x_\parallel,loc}(W^{-\theta-\xi,7/2}_{x_\perp,v_\perp,loc}))} \to 0 \tag{3.12}$$

(the "norm" means that this convergence holds when restricting to any compact set)

Proof. By the initial assumption on the initial data, there exists $q > 7/2$ such that $f_0 \in L^q_{x,v}$; thanks to the a priori L^p estimate, we get $g_\epsilon \in L^\infty_t(L^q_{x,v})$. Define γ by:

$$\frac{1}{\gamma} = \frac{5}{7} + \frac{1}{q}$$

The previous lemma shows that for some $s > 0$:

$$\eta_\epsilon \in W^{s,\gamma}_{t,\mathrm{loc}}(W^{s,\gamma}_{x_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}}))$$

Since $g_\epsilon \in L^q_{t,\mathrm{loc}}(L^q_{x,v})$ and Ψ has compact support, we get by Hölder's inequality:

$$\eta_\epsilon \in L^q_{t,\mathrm{loc}}(L^q_{x_\parallel}(L^q_{x_\perp,v_\perp}))$$

Since $\frac{1}{\gamma} > \frac{5}{7} > \frac{2}{7}$ and $\frac{1}{q} < \frac{2}{7}$, there exists $\theta \in]0,1[$ such that

$$\frac{2}{7} = \frac{1-\theta}{q} + \frac{\theta}{\gamma}$$

By interpolation we deduce that:

$$\eta_\epsilon \in W^{s\theta,7/2}_{t,\mathrm{loc}}(W^{s\theta,7/2}_{x_\parallel,\mathrm{loc}}(W^{-\theta,7/2}_{x_\perp,v_\perp,\mathrm{loc}}))$$

We then used the following refined interpolation result proved by Simon in [20], which is, roughly speaking, an anisotropic adaptation of the classical Riesz-Fréchet-Kolmogorov criterion for compactness in L^p:

Theorem 3.2. *Let $1 \leq p \leq \infty$ and $s > 0$. Let $T > 0$ and B_1, B_2 be two Banach Spaces with B_1 compactly embedded in B_2. Let F be a bounded set of $W^{s,p}_t([0,T], B_1)$.*

Then F is relatively compact in $L^p_t([0,T], B_2)$.

This entails, thanks to the Sobolev's embeddings, that the sequence (η_ϵ) is relatively compact in $L^{7/2}_{t,\mathrm{loc}}(L^{7/2}_{x_\parallel,\mathrm{loc}}(W^{-\theta-\xi,7/2}_{x_\perp,v_\perp,\mathrm{loc}}))$, for all $\xi > 0$.

□

Remark: By uniqueness of the limit in the sense of distributions, we get:

$$\eta = \int G\Psi(v_\parallel) dv_\parallel$$

From now on, we consider ξ such that $\theta + \xi < 1$, which is of course possible since $\theta < 1$.

Step 4: Passing to the weak limit

We will first need a technical lemma which is obtained directly from the 2-scale convergence of E_ϵ.

Lemma 3.1.

$$\mathcal{R}(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \text{ 2-scale converges to } \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau \quad (3.13)$$

$$R(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \text{ 2-scale converges to } \frac{1}{2\pi}\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau \quad (3.14)$$

Proof. E_ϵ is uniformly bounded in $L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,7/5}_{x_\perp})$, so there exists $\mathcal{E} \in L^\infty_t(L^{7/5}_{x_\parallel}(W^{1,7/5}_{x_\perp})$ such that E_ϵ 2 scale converge to $\mathcal{E}$.

We take $\Psi(t,\tau,x)$ a 2π-periodic w.r.t. τ test function and use the 2 scale convergence of E_ϵ:

$$\begin{aligned}
&\int \mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\Psi(t,t/\epsilon,x)dtdx \\
&= \int E_\epsilon(t,x).{}^t\mathcal{R}(t/\epsilon)\Psi(t,t/\epsilon,x-\mathcal{R}(-t/\epsilon)v)dtdx \\
&\to \int\int_0^{2\pi} \mathcal{E}(t,\tau,x).{}^t\mathcal{R}(\tau)\Psi(t,\tau,x-\mathcal{R}(-\tau)v)dtd\tau dx \\
&= \int\int_0^{2\pi} \mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v).{}^t\mathcal{R}(\tau)\Psi(t,\tau,x)dtd\tau dx
\end{aligned}$$

The proof is the same for $R(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v)$.

□

Now, we can write the weak formulation of the kinetic equation (3.6) against a smooth test function of the form $\Phi(t,x,v_\perp)\Psi(v_\parallel)$such that the support of this test function is included in in the cartesian product of compact sets $[0,T]\times K_{x_\parallel}\times K_{x_\perp}\times K_{v_\parallel}\times K_{v_\perp}$. If we can pass to the limit for such test functions, then by density it will be also the case for all test functions.

Noticing that $\mathrm{div}_x\, v_\parallel = 0$ and that

$$\mathrm{div}_x\left(\mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v)\right) + \mathrm{div}_v\left(R(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v)\right) = 0$$

we get:

$$\begin{aligned}
\int \Big(\partial_t(\Phi(t,x,v_\perp)\Psi(v_\parallel)) + v_\parallel.\nabla_x(\Phi\Psi) + \mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_x(\Phi\Psi) \\
+R(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_v(\Phi\Psi)\Big)g_\epsilon dtdx_\perp dx_\parallel dv_\perp dv v_\parallel \\
= -\int u_0\Phi(0,x,v_\perp)\Psi(v_\parallel)dxdv
\end{aligned}$$

We can easily take the weak limit in the linear part $\partial_t g_\epsilon + v_\parallel.\nabla_x g_\epsilon$.

Consider now the "non linear" term:

$$\begin{aligned}
&\int \mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_x\Phi(t,x,v_\perp)g_\epsilon\Psi(v_\parallel)dtdx_\perp dx_\parallel dv_\perp dv_\parallel = \\
&\int \mathcal{R}(t/\epsilon)E_\epsilon(t,x+\mathcal{R}(-t/\epsilon)v).\nabla_x\Phi(t,x,v_\perp)\left(\int g_\epsilon\Psi(v_\parallel)dv_\parallel\right)dtdx_\perp dx_\parallel dv_\perp
\end{aligned}$$

The convergence of this term can be established by the strong/weak convergence principle. Nevertheless, we have to carefully use this technique to get the result so that we

wish to explicitly evaluate the difference:

$$\begin{aligned}
&\Big| \int \mathcal{R}(t/\epsilon) E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v).\nabla_x \Phi(t, x, v_\perp)\eta_\epsilon dt dx_\perp dx_\| dv_\perp \\
&\qquad\qquad - \int \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v)d\tau.\nabla_x \Phi \eta dt dx dv_\perp \Big| \\
\leq &\Big| \int \Big(\mathcal{R}(t/\epsilon) E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) - \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v)d\tau \Big).\nabla_x \Phi \eta dt dx dv_\perp \Big| \\
&+ \Big| \int \mathcal{R}(t/\epsilon) E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v).\nabla_x \Phi(t, x, v_\perp)\left(\eta_\epsilon - \eta\right) dt dx_\perp dx_\| dv_\perp \Big|
\end{aligned}$$

The first term of the right hand side converges to zero because of the 2-scale convergence of E_ϵ (lemma 3.1). We can control the second term by:

$$C\|E_\epsilon.\nabla_x \Phi\|_{L_t^{7/5}(L_{x_\|}^{7/5}(W_{x_\perp, v_\perp}^{\theta+\xi, 7/5}))} \|\eta_\epsilon - \eta\|_{L_t^{7/2}(L_{x_\|}^{7/2}(W_{x_\perp, v_\perp}^{-\theta-\xi, 7/2}))} \tag{3.15}$$

(these norms are actually taken on the compact support of Φ but we do not write it for the sake of simplicity)

Using the fact that E_ϵ is uniformly bounded in $L_{t,\mathrm{loc}}^{7/5}(L_{x_\|}^{7/5}(W_{x_\perp}^{\theta+\xi,7/5}))$ (this is an easy consequence of lemma 2.4) and that thanks to corollary 3.1,

$$\|\eta_\epsilon - \eta\|_{L_t^{7/2}([0,T], L_{x_\|}^{7/2}(K_{x_\|}, W_{x_\perp, v_\perp}^{-\theta-\xi, 7/2}(K_{x_\perp} \times K_{v_\perp})))} \to 0$$

We can deduce that

$$\left| \int \mathcal{R}(t/\epsilon) E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v).\nabla_x \Phi(t, x, v_\perp)\left(\eta_\epsilon - \eta\right) dt dx_\perp dx_\| dv_\perp \right| \to 0$$

The proof is of course the same for the other non linear term:

$$R(t/\epsilon) E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v).\nabla_v g_\epsilon$$

To conclude let us compute the asymptotic equation satisfied by the 2 scale limit of V_ϵ denoted by V. We take $\Psi(t, \tau, x)$ a 2π-periodic w.r.t. a τ test function. We write the weak formulation of the Poisson equation:

$$\begin{aligned}
&\epsilon^2 \int \nabla_{x_\|} V_\epsilon \nabla_{x_\|} \Psi(t, t/\epsilon, x) dt dx + \int \nabla_{x_\perp} V_\epsilon \nabla_{x_\perp} \Psi(t, t/\epsilon, x) dt dx \\
= &\int f_\epsilon(t, x, v)\Psi(t, t/\epsilon, x) dt dv dx - \int \left(- \int f_0 dv dx \right) \Psi(t, t/\epsilon, x) dt dv dx
\end{aligned}$$

We then pass to the 2 scale limit:

$$\begin{aligned}
0 \quad + \quad & \int\int_0^{2\pi} \nabla_{x_\perp} V(t, \tau, x) \nabla_{x_\perp} \Psi(t, \tau, x) d\tau dt dx = \int\int_0^{2\pi} F(t, \tau, x, v)\Psi(t, \tau, x) dt dv dx \\
= \quad & \int\int_0^{2\pi} G(t, \tau, x + \mathcal{R}(\tau)v, R(\tau)v)\Psi(t, \tau, x) d\tau dv dx \\
- \quad & \int\int_0^{2\pi} \left(\int f_0 dv dx \right) \Psi(t, \tau, x) d\tau dv dx
\end{aligned}$$

From which we get the "Poisson" equation given in Proposition 1.1:

$$-\Delta_\perp V = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dvdx$$

Moreover since $E_\epsilon = (-\nabla_{x_\perp} V_\epsilon, -\epsilon\nabla_{x_\parallel} V_\epsilon)$, we easily get if we pass to the 2 scale limit:

$$\mathcal{E} = (-\nabla_{x_\perp} V, 0)$$

□

4 Closing comments

Finallty we can that see as in [11] that the drift involving the electric field in the asymptotic "kinetic" equation corresponds to the electric drift that we mentioned in the introduction and which was expected to appear. Nevertheless, we point out that the parallel component of the electric field disappears in the end, which seems to be quite an unpleasant feature of this scaling.

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