HAL open science

Some examples of 'second order elliptic integrable systems associated to a 4-symmetric space'

Frédéric Hélein

To cite this version:

Frédéric Hélein. Some examples of 'second order elliptic integrable systems associated to a 4-symmetric space'. London Mathematical Society Durham symposium : Methods of Integrable Systems in Geometry, Aug 2006, Durham, United Kingdom. hal-00346771

HAL Id: hal-00346771

https://hal.science/hal-00346771v1

Submitted on 12 Dec 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some examples of 'second order elliptic integrable systems associated to a 4-symmetric space'

Frédéric Hélein, Université Paris 7

Saturday 11 August 2006

London Mathematical Society Durham symposium
Methods of Integrable Systems in Geometry
11–21 August 2006

1 Hamiltonian Stationary Lagrangian (HSL) surfaces

1.1 A variational problem in $\mathbb{R}^4$

$\mathbb{R}^4$ has the canonical Euclidean structure $\langle\cdot,\cdot\rangle$ and the symplectic form $\omega := dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. An immersed surface $\Sigma \subset \mathbb{R}^4$ is

(i) **Lagrangian** iff $\omega_{|\Sigma} = 0$

(ii) **Hamiltonian Stationary Lagrangian** (HSL) iff $\omega_{|\Sigma} = 0$ and Σ is a critical point of the area functional $\mathcal{A}$ with respect to all *Hamiltonian vector fields* ξ_h s.t.:

- $\exists h \in \mathcal{C}_c^\infty(\mathbb{R}^4)$, $\xi_h \lrcorner\, \omega + dh = 0$
- equivalentely, if J is the complex structure s.t. $\omega = \langle J\cdot,\cdot\rangle$, $\xi_h = J\nabla h$.

It means that $\delta\mathcal{A}_\Sigma(\xi_h) = 0$, $\forall h \in \mathcal{C}_c^\infty(\mathbb{R}^4)$.

What is the Euler equation ?
The Gauss map is:

$$\begin{array}{rcccl} \gamma : \Sigma & \longrightarrow & Gr_{Lag}(\mathbb{R}^4) & \subset & Gr_2(\mathbb{R}^4) \\ & & S^1 \times S^2 & \subset & S^2 \times S^2 \end{array}$$

Denote $\gamma = (\rho_\Sigma, \sigma_\Sigma)$ the two components of γ. For a Lagrangian immersion $\rho_\Sigma \simeq e^{i\beta}$. Then the *mean curvature vector* is

$$\vec{H} = J\nabla\beta.$$

Lemma 1.1 *Σ is HSL iff*

$$\begin{cases} \omega_{|\Sigma} = 0 \\ \Delta_\Sigma \beta = 0. \end{cases}$$

Remark: Σ is special Lagrangian iff $\begin{cases} \omega_{|\Sigma} = 0 \\ \beta = \text{Constant.} \end{cases} \iff \begin{cases} \omega_{|\Sigma} = 0 \\ \Sigma \text{ is minimal} \end{cases}$.

An analytic study was done by R. Schoen and J. Wolfson [6] (in a 4-dimensional Calabi–Yau manifold).

1.2 It is a completely integrable system (F.H.–P. Romon [1, 2])

Let $\Omega \subset \mathbb{C}$ be an open subset and $X : \Omega \longrightarrow \mathbb{R}^4$ a (local) conformal parametrization of Σ. Set

$$\rho_X := \rho_\Sigma \circ X,$$

the *left Gauss map.*

Idea: to lift the pair (X, ρ_X) to a map $F : \Omega \longrightarrow \mathfrak{G}$, where $\mathfrak{G}$ is a local symmetry group of the problem. The more naive choice is $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$, the group of isometries of $\mathbb{R}^4$. Then

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X),$$

where $R : \Omega \longrightarrow SO(4)$ encodes $\rho_X \simeq e^{i\beta}$. (Alternatively one can choose $\mathfrak{G} = U(2) \ltimes \mathbb{C}^2$, with the identification $\mathbb{C}^2 \simeq (\mathbb{R}^4, J)$ and $U(2)$: subgroup of $SO(4)$. Then the way $R \in U(2)$ encodes β is simply through the relation $\det_{\mathbb{C}} R = e^{i\beta}$).

In all cases there exists an automorphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$ s.t. $\tau^4 = Id$. This automorphism acts on the Lie algebra $\mathfrak{g}$ and can be diagonalized with the eigenvalues i, 1, $-i$ and -1. Hence the vector space decomposition

$$\begin{array}{rccccccc} \mathfrak{g}^{\mathbb{C}} = & \mathfrak{g}_{-1} & \oplus & \mathfrak{g}_0^{\mathbb{C}} & \oplus & \mathfrak{g}_1 & \oplus & \mathfrak{g}_2^{\mathbb{C}} \\ \text{eigenvalues:} & -i & & 1 & & i & & -1 \end{array}$$

Then consider the (pull-back of the) Maurer–Cartan form

$$\alpha := F^{-1} dF$$

and split $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ according to this decomposition. Then do the further splitting $\alpha_2 = \alpha_2' + \alpha_2''$, where $\alpha_2' = \alpha(\frac{\partial}{\partial z})dz$ and $\alpha_2'' = \overline{\alpha_2'}$. And consider the family of deformations

$$\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2'', \quad \lambda \in \mathbb{C}^*.$$

Then:

Theorem 1.1 *(i) X is Lagrangian iff $\alpha''_{-1} = 0$*

(ii) X is HSL iff $\alpha''_{-1} = 0$ and, $\forall \lambda \in \mathbb{C}^*$, $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$.

Using this characterisation one see easily that HSL surfaces are solutions of a completely integrable system.

Note that analogous formulations work for HSL surfaces in $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1)) = U(3)/U(2) \times U(1)$, $\mathbb{C}D^2 = SU(2,1)/S(U(2) \times U(1))$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}D^1 \times \mathbb{C}D^1$ [3].

2 Generalizations in $\mathbb{R}^4$ (after I. Khemar [4])

Again $\mathbb{R}^4$ is endowed with its canonical Euclidean structure. We will also use an identification of $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. We recall that this allows to represent rotations $R \in SO(4)$ by a pair $(p,q) \in S^3 \times S^3 \subset \mathbb{H} \times \mathbb{H}$ of unit quaternions such that $\forall z \in \mathbb{H}$, $R(z) = pz\overline{q}$. In other words, denoting by $L_p : z \longmapsto pz$ and $R_{\overline{q}} : z \longmapsto z\overline{q}$, we have $R = L_p R_{\overline{q}} = R_{\overline{q}} L_p$. The pair (p,q) is unique up to sign, hence the identification $SO(4) \simeq S^3 \times S^3/\{\pm\}$.

Moreover we can also precise the identification $Gr_2(\mathbb{R}^4) \simeq S^2 \times S^2$. Let

$$Stiefel_2(\mathbb{H}) := \{(e_1, e_2) \in \mathbb{H} \times \mathbb{H} |\, |e_1| = |e_2| = 1, \langle e_1, e_2 \rangle = 0\}.$$

Observe that $\forall (e_1, e_2) \in Stiefel_2(\mathbb{H})$, $e_2\overline{e_1}$ (resp. $\overline{e_1}e_2$) is unitary (because e_1 and e_2 are so) and imaginary (because $\langle e_1, e_2 \rangle = 0$). Hence this defines two maps

$$\begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1, e_2) & \longmapsto & e_2\overline{e_1} \end{array}, \qquad \begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1, e_2) & \longmapsto & \overline{e_1}e_2 \end{array}.$$

These maps factor through the natural map $P : (e_1, e_2) \longmapsto \mathrm{Span}\{e_1, e_2\}$ from $Stiefel_2(\mathbb{H})$ to the oriented Grassmannian $Gr_2(\mathbb{H})$: let

$$\begin{array}{c} \rho : Gr_2(\mathbb{H}) \longrightarrow S^2 \\ \text{s. t. } \rho \circ P(e_1, e_2) = e_2\overline{e_1} \end{array}, \qquad \begin{array}{c} \sigma : Gr_2(\mathbb{H}) \longrightarrow S^2 \\ \text{s. t. } \sigma \circ P(e_1, e_2) = \overline{e_1}e_2. \end{array}$$

Then $(\rho, \sigma) : Gr_2(\mathbb{H}) \longrightarrow S^2 \times S^2$ is a diffeomorphism.

2.1 Immersions of a surface in $\mathbb{H}$ with a harmonic 'left Gauss map'

Let $X : \Omega \longrightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega \longrightarrow S^2$ its *left Gauss map*, i.e. $\forall z \in \Omega$, $\rho_X(z)$ is the image of $\mathrm{Span}(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z))$ by ρ. It is characterised by

$$\frac{\partial X}{\partial y} = \rho_X \frac{\partial X}{\partial x} \quad \Longleftrightarrow \quad i\frac{\partial X}{\partial z} = \rho_X \frac{\partial X}{\partial z}.$$

(In the second equation the i on the l.h.s. is the complex structure on $\Omega \subset \mathbb{C}$, whereas the ρ_X on the r.h.s. denotes the left multiplication in $\mathbb{H}$.)

Remark: instead of viewing ρ_X as the left component of the Gauss map in $Gr_2(\mathbb{H}) \simeq S^2 \times S^2$, an alternative interpretation is that ρ_X is a map into the 'left' connected component of the manifold of compatible complex structures $\mathcal{J}_{\mathbb{H}} \simeq S^2 \cup S^2$ on $\mathbb{H}$ (cf. the work of F. Burstall).

Idea: to lift the pair (X, ρ_X) by a framing $F : \Omega \longrightarrow \mathfrak{G}$, $\mathfrak{G}$ is a subgroup of $SO(4) \ltimes \mathbb{R}^4$.
How ? We fix some constant imaginary unit vector $u \in S^2 \subset \mathrm{Im}\mathbb{H}$.

- *First method:* we lift X **and** its full Gauss map $T_X\Sigma \simeq (\rho_X, \sigma_X)$: we let (e_1, e_2) be any moving frame which is an orthonormal basis of $T_{X(z)}\Sigma$ (e.g. $e_1 = \frac{\partial X}{\partial x}/|\frac{\partial X}{\partial x}|$, $e_2 = \frac{\partial X}{\partial x}/|\frac{\partial X}{\partial y}|$) and we choose $F = (R, X)$ s.t. R satisfies:

  $$R(1) = e_1, \quad R(u) = e_2.$$

  Decompose $R = L_p R_{\overline{q}}$, then

  $$R(1) = p\overline{q}, R(u) = pu\overline{q}, \text{ so that } \quad \rho_X = e_2\overline{e_1} = pu\overline{p}.$$

  Note: In this case we must choose $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$ (which acts transitively on $Stiefel_2(\mathbb{H})$).

- *Second method:* we lift **only** X and ρ_X. Then it means that we choose $F = (R, X)$, where $R = L_p R_{\overline{q}}$ is s.t.

  $$\rho_X = pu\overline{p}.$$

  Hence the choice of q is not relevant. In other words introducing the *(left) Hopf fibration*

  $$\begin{array}{rccc} \mathcal{H}^u_L : & SO(4) & \longrightarrow & S^2 \\ & L_p R_{\overline{q}} & \longmapsto & pu\overline{p}, \end{array}$$

  we choose the lift $F = (R, X)$ in such a way that $\mathcal{H}^u_L \circ R = \rho_X$.
  We observe that in this case one may choose $q = 1$ and assume that $R \in \{L_p | p \in S^3\} \simeq Spin3$, i.e. work with $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$. The restriction of $\mathcal{H}^u_L$ to $Spin3$ (viewed as a subgroup of $SO(4)$) is just the Hopf fibration $\mathcal{H}^u : S^3 \longrightarrow S^2$.

Actually the second point of view is more general and leads to a simpler theory.

Now let $\tau : (R, X) \longmapsto (L_u R L_u^{-1}, -L_u X)$, a 4th order automorphism of $\mathfrak{G}$ (i.e. $\tau^4 = Id$). It induces a 4th order automorphism on its Lie algebra $\mathfrak{g}$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^{\mathbb{C}}$$

be its associated eigenspace decomposition. Split the Maurer–Cartan form $\alpha = F^{-1}dF$ according to this decomposition: $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ and let

$$\beta_{\lambda^2} := \lambda^{-2}\alpha_2' + \alpha_0 + \lambda^2\alpha_2'',$$

$$\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2'' = \beta_{\lambda^2} + \lambda^{-1}\alpha_{-1} + \lambda\alpha_1.$$

Then:

Lemma 2.1 *If $X : \Omega \longrightarrow \mathbb{R}^4$ is a conformal immersion and if $R : \Omega \longrightarrow SO(4)$ is an arbitrary smooth map, then*

$$\mathcal{H}_L^u \circ R = \rho_X \quad \Longleftrightarrow \quad \alpha_{-1}'' = 0.$$

In other words $F = (R, X) : \Omega \longrightarrow SO(4) \ltimes \mathbb{R}^4$ lifts (X, ρ_X) iff $\alpha_{-1}'' = 0$.

Remark: α_1 is the complex conjugate of α_{-1}, so that $\alpha_{-1}'' = 0$ iff $\alpha_1' = 0$.

Lemma 2.2 *We have:*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] + (\lambda^{-3} - \lambda)[\alpha_2' \wedge \alpha_{-1}''] + (\lambda^3 - \lambda^{-1})[\alpha_2'' \wedge \alpha_1']. \quad (1)$$

Hence in particular, if F lifts (X, ρ_X), then $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}]$.

In order to interpret (1) we further observe that

(i) $\mathfrak{G}^\tau$, the fixed subset of $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of $\mathfrak{G}$ with Lie algebra $\mathfrak{g}_0$

(ii) $\mathfrak{G}^{\tau^2} = \{(R, 0) \in \mathfrak{G}\}$, the fixed subset of $\tau^2 : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of $\mathfrak{G}$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_2$,

with the inclusions

$$\mathfrak{G}^\tau \subset \mathfrak{G}^{\tau^2} \subset \mathfrak{G}.$$

Moreover $\mathfrak{G}/\mathfrak{G}^{\tau^2} \simeq \mathbb{H}$ and $\mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2$ and the projection map

$$\begin{array}{ccc} \mathfrak{G}^{\tau^2} & \longrightarrow & \mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2 \\ R \simeq (R, 0) & \longmapsto & R \bmod \mathfrak{G}^\tau \end{array}$$

coincides with the Hopf fibration $\mathcal{H}_L^u$. Hence, by applying the standard theory of harmonic maps into symmetric spaces, we deduce that:

$$d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] = 0 \quad \Longleftrightarrow \quad \mathcal{H}_L^u \circ R : \Omega \longrightarrow S^2 \text{ is harmonic.}$$

Putting Lemmas 2.1 and 2.2 and these observations together we conclude with the following:

Theorem 2.1 *Let $X : \Omega \longrightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega \longrightarrow S^2$ its left Gauss map. Let $F = (R, X) : \Omega \longrightarrow \mathfrak{G}$ be any smooth map. Then*

(i) $\mathcal{H}_L^u \circ R = \rho_X$ (i.e. F is a lift of (X, ρ_X)) iff $\alpha''_{-1} = 0$

(ii) If so, i.e. if F is a lift of (X, ρ_X), then ρ_X is harmonic iff

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

2.2 Examples

2.2.1 HSL surfaces revisited

Let us introduce again the symplectic form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. Note that $\omega = \omega_1 := \langle L_i \cdot, \cdot \rangle$. Let us introduce also $\omega_2 := \langle L_j \cdot, \cdot \rangle = dx^1 \wedge dx^3 + dx^4 \wedge dx^2$ and $\omega_3 := \langle L_k \cdot, \cdot \rangle = dx^1 \wedge dx^4 + dx^2 \wedge dx^4$. Then

$$e_2 \overline{e_1} = \rho(e_1, e_2) = i\omega_1(e_1, e_2) + j\omega_2(e_1, e_2) + k\omega_3(e_1, e_2).$$

So X is a conformal *Lagrangian* immersion iff $X^*\omega_1 = 0$, i.e. iff ρ_X takes values in

$$S^1 = \{j \cos\beta + k \sin\beta = e^{i\beta} j \,|\, \beta \in \mathbb{R}\}.$$

Hence a lift of (X, ρ_X) is characterized by

$$pu\overline{p} = \mathcal{H}_L^u \circ R = \rho_X = e^{i\beta} j.$$

A convenient choice for u is to assume that $u \perp i$, e.g. $u = j$. In that case

$$\{p \in S^3 \,|\, pu\overline{p} = e^{i\beta} j\} = \{e^{i\beta/2} e^{j\theta} \,|\, \theta \in \mathbb{R}\}$$

and the simplest choices are $p = \pm e^{i\beta/2}$.

With this choice:

- if we start with the group $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$, our lift satisfies $R = L_{e^{i\beta/2}} R_{\overline{q}}$, i.e. we can reduce $SO(4) \ltimes \mathbb{R}^4$ to $U(2) \ltimes \mathbb{C}^2$
- if we start with the group $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$, our lift satisfies $R = L_{e^{i\beta/2}}$, i.e. we can reduce $Spin3 \ltimes \mathbb{R}^4$ to $U(1) \ltimes \mathbb{C}^2$ (cf. spinor lifts, related to the Konopelchenko–Taimanov representation formula).

2.2.2 Constant mean curvature surfaces in $\mathbb{R}^3$

Consider an immersed surface Σ in $\mathbb{H}$ with a harmonic left Gauss map. If we assume further that this surface is contained in $\mathrm{Im}\mathbb{H}$, then any orthonormal basis (e_1, e_2) of $T_{X(z)}\Sigma$ is composed of imaginary vectors. Hence

$$\rho_X = e_2 \overline{e_1} = -\overline{e_1} e_2 = -\sigma_X,$$

so that ρ_X is harmonic iff σ_X is so. Actually ρ_X is nothing but the Gauss map of Σ in $\mathrm{Im}\mathbb{H} \simeq \mathbb{R}^3$. Hence by Ruh–Vilms theorem we know that Σ is a *constant mean curvature surface* in $\mathbb{R}^3$. Conversely any constant mean curvature surface in $\mathbb{R}^3$ arises that way.

2.3 Other generalizations in dimension 4

This theory can be generalized to surfaces in S^4 or $\mathbb{C}P^2$: then (X, ρ_X) is replaced by a lift of the immersion X in the four dimensional manifold into the twistor bundle of complex structures. The condition of ρ_X being harmonic is replaced by the fact this lift is vertically harmonic (the fiber being the set of (left) compatible complex structures, diffeomorphic to S^2). This follows from independant works by F. Burstall and I. Khemar.

3 A generalization for surfaces in $\mathbb{R}^8$ (I. Khemar [4])

The following theory is based on the identification of $\mathbb{R}^8$ with octonions $\mathbb{O}$. Again the map

$$\begin{array}{ccc} Stiefel_2(\mathbb{O}) & \longrightarrow & S^6 \\ (e_1, e_2) & \longmapsto & e_2\overline{e_1}, \end{array}$$

where $S^6 \in \mathrm{Im}\mathbb{O} \subset \mathbb{O}$, can be factorized through the map $P : Stiefel_2(\mathbb{O}) \longrightarrow Gr_2(\mathbb{O})$, $(e_1, e_2) \longmapsto \mathrm{Span}\{e_1, e_2\}$ by introducing

$$\begin{array}{c} \rho : Gr_2(\mathbb{O}) \longrightarrow S^6 \\ \text{s.t. } \rho \circ P(e_1, e_2) = e_2\overline{e_1}. \end{array}$$

Let Σ be an immersed surface in $\mathbb{O}$ we say that Σ *is ρ-harmonic iff the composition of the Gauss map $\Sigma \longrightarrow Gr_2(\mathbb{O})$ with ρ is harmonic.*

This theory is completely similar with the theory of surfaces in quaternions $\mathbb{H}$ which used the group $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$, where $Spin3$ can be seen as the subgroup of $SO(4)$ generated by L_i, L_j and L_k and the induced representation of $Spin3$ was the spinor representation $\mathbb{H}$. Here we will use $\mathfrak{G} = Spin7 \ltimes \mathbb{O}$, where $Spin7$ can be identified with the subgroup of $SO(8)$ generated by $\{L_v |\, v \in S^6 \subset \mathrm{Im}\mathbb{O}\}$ and the induced representation on $\mathbb{R}^8$ coincides with the spinor representation of $Spin7$ on $\mathbb{O}$. A difference however is that $Spin7$ is "bigger" than $Spin3$ and in particular acts transitively on $Stiefel_2(\mathbb{O})$ (with isotropy $SU(3)$) and $Gr_2(\mathbb{O})$ (with isotropy G_2), whereas $Spin3$ do not act transitively on $Gr_2(\mathbb{H})$. After fixing an imaginary unit octonion $u \in \mathbb{O}$, a 'Hopf' fibration

$$\begin{array}{cccl} \mathcal{H}^u : & Spin7 & \longrightarrow & S^6 \\ & p & \longmapsto & \mathcal{H}^u(p), \text{ s.t. } pL_u p^{-1} = L_{\mathcal{H}^u(p)} \end{array}$$

can be defined.

Now let $X : \mathbb{C} \supset \Omega \longrightarrow \mathbb{O}$ be a conformal immersion and denote $\rho_X := \rho \circ T_X\Sigma$ the composition of the Gauss map $T_X\Sigma$ of X with ρ. After having fixed $u \in S^6 \subset \mathbb{O}$ we let

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X) : \Omega \longrightarrow Spin7 \ltimes \mathbb{O},$$

be a smooth map. We say that F lifts (X, ρ_X) iff $\mathcal{H}^u \circ R = \rho_X$. Using the 4th order automorphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$ defined by

$$\tau(R, X) = (L_u R L_u^{-1}, -L_u X),$$

we can characterized among all maps $F = (R, X)$ those which lift ρ_X by the condition $\alpha''_{-1} = 0$ (after a decomposition of the Maurer–Cartan form $\alpha := F^{-1}dF$ along the eigenspaces of the action of τ on the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$). Then the ρ-harmonic immersions satisfy a zero curvature equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ similar to the previous case.

Again ρ_X can be interpreted as a map into the manifold $\mathcal{J}_{\mathbb{O}}$ of compatible complex structures on $\mathbb{O}$, because of the relation $\rho_X \frac{\partial X}{\partial z} = i\frac{\partial X}{\partial z}$. However the embedding $S^6 \subset \mathcal{J}_{\mathbb{O}}$ is much less clear than the inclusion $S^2 \subset \mathcal{J}_{\mathbb{H}}$ that we used previously: we recall indeed that $\mathcal{J}_{\mathbb{H}} \simeq S^2_L \cup S^2_R$ and hence that our S^2 was just the (left) connected component of $\mathcal{J}_{\mathbb{H}}$. However $\mathcal{J}_{\mathbb{O}} \simeq SO(8)/U(4)$ is 12 dimensional, so that our S^6 is now a particular submanifold of $\mathcal{J}_{\mathbb{O}}$. Hence a twistor intepretation of the theory in $\mathbb{O}$ seems less clear.

4 Towards a supersymmetric interpretation

Observation : the coefficients of α_{-1} and α_1 actually behave like spinors (they turn half less than those of α_2 when λ run over S^1 and they satisfy a kind of Dirac equation). This motivates the following results by I. Khemar [5].

4.1 Superharmonic maps into a symmetric space

For simplicity we restrict ourself to maps into the sphere $S^n \subset \mathbb{R}^{n+1}$. It can be seen as a system of PDE's on a map $u : \Omega \longrightarrow S^n$ (where $\Omega \subset \mathbb{C}$) and *odd* sections ψ_1, ψ_2 of u^*TS^n. This system is

$$\left\{ \begin{array}{rcl} \nabla_{\overline{z}} \dfrac{\partial u}{\partial z} & = & \dfrac{1}{4}\left(\psi \langle \psi, \dfrac{\partial u}{\partial \overline{z}} \rangle - \overline{\psi} \langle \overline{\psi}, \dfrac{\partial u}{\partial z} \rangle \right) \\ \nabla_{\overline{z}} \psi & = & \dfrac{1}{4} \langle \overline{\psi}, \psi \rangle \overline{\psi}, \end{array} \right. \tag{2}$$

where $\psi = \psi_1 - i\psi_2$. By "odd" we mean that the components ψ_1 and ψ_2 are anticommuting (Grassmann) variables. An alternative elegant reformulation of this system can be obtained by adding the extra field $F : \Omega \longrightarrow \mathbb{R}^{n+1}$, which satisfies the 0th order PDE's

$$F = \frac{1}{2i} \langle \psi, \overline{\psi} \rangle u \tag{3}$$

and by setting

$$\Phi := u + \theta^1 \psi_1 + \theta^2 \psi_2 + \theta^1 \theta^2 F,$$

where θ^1 and θ^2 are anticommuting coordinates, so that $(x, y, \theta^1, \theta^2)$ forms a complete system of coordinates on the *superplane* $\mathbb{R}^{2|2}$. Then (2) and (3) are equivalent to

$$\overline{D} D\Phi + \langle \overline{D}\Phi, D\Phi \rangle \Phi = 0, \tag{4}$$

where $D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}$, $\overline{D} = \frac{\partial}{\partial\overline{\theta}} - \overline{\theta}\frac{\partial}{\partial\overline{z}}$.

Actually, from (2) and (3) to (4), we have used the fact that u, ψ_1, ψ_2 and F are the components (supermultiplet) of a single map Φ from $\mathbb{R}^{2|2}$ to $S^n \subset \mathbb{R}^{n+1}$, which satisfies the superharmonic map equation (4).

Now we lift Φ to a framing supermap $\mathcal{F} : \mathbb{R}^{2|2} \longrightarrow SO(n+1)$ such that the composition of $\mathcal{F}$ with the projection $SO(n+1) \longrightarrow SO(n+1)/SO(n) \simeq S^n$ is Φ. Set $\alpha := \mathcal{F}^{-1}d\mathcal{F}$ and decompose $\alpha = \alpha_0 + \alpha_1$, according to the splitting of the Lie algebra $so(n+1)$ by the Cartan involution.

Before giving a characterization of the superharmonic equation, it is useful to present a technical result concerning the exterior calculus of 1-forms on $\mathbb{R}^{2|2}$.

Lemma 4.1 *For a 1-form α on $\mathbb{R}^{2|2}$ with coefficients in a Lie algegra $\mathfrak{g}$, we have the equivalence*

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \quad \Longleftrightarrow \quad \overline{D}\alpha(D) + D\alpha(\overline{D}) + [\alpha(\overline{D}), \alpha(D)] = 0.$$

Remark: $\Lambda^1(\mathbb{R}^{2|2})^*$ is spanned by $(d\theta, d\overline{\theta}, dz+(d\theta)\theta, d\overline{z}+(d\overline{\theta})\overline{\theta})$, the dual basis of $(D, \overline{D}, \frac{\partial}{\partial z}, \frac{\partial}{\partial\overline{z}})$. Hence in particular $\Lambda^2(\mathbb{R}^{2|2})^*$ is 6 dimensional. So the expansion of the l.h.s. of $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ leads to 6 equations which are a priori independant. The content of this lemma is that these 6 terms vanish as soon as one of these 6 coefficients (namely the coefficient of $d\theta \wedge d\overline{\theta}$) vanishes.

Now the supermap $\mathcal{F}$ is superharmonic iff

$$\overline{D}\alpha_1(D) + [\alpha_0(\overline{D}), \alpha_1(D)] = 0.$$

We hence deduce:

Theorem 4.1 *$\mathcal{F}$ is superharmonic iff*

$$\forall\lambda \in \mathbb{C}^*, \quad \overline{D}\alpha(D)_\lambda + D\alpha(\overline{D})_\lambda + [\alpha(\overline{D})_\lambda, \alpha(D)_\lambda] = 0,$$

where $\alpha(D)_\lambda := \alpha_0(D) + \lambda^{-1}\alpha_1(D)$ and $\alpha(\overline{D})_\lambda := \alpha_0(\overline{D}) + \lambda\alpha_1(\overline{D})$.

It results that this problem has the structure of a completely integrable system (F. O'Dea, I. Khemar). In particular the DPW algorithm for harmonic maps works.

The DPW potential is a $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$-valued holomorphic 1-form μ on $\mathbb{R}^{2|2}$ s.t.

$$\mu(D) = \mu_0(D) + \theta\mu_\theta(D) = \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

One integrates the equation

$$Dg = g\mu(D)$$

to get a holomorphic map $g = g_0 + \theta g_\theta : \mathbb{R}^{2|2} \longrightarrow \Lambda\mathfrak{G}^{\mathbb{C}}_\tau$. This implies in particular that

$$g_0^{-1}\frac{\partial g_0}{\partial z} = -\left((\mu_0(D))^2 + \mu_\theta(D)\right) = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

Similarly, if $\mathcal{F} = \mathcal{F}_0 + \theta\mathcal{F}_\theta + \overline{\theta}\mathcal{F}_{\overline{\theta}} + \theta\overline{\theta}\mathcal{F}_{\theta\overline{\theta}}$, it turns out that $\mathcal{F}_0^{-1}d\mathcal{F}_0 = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \lambda(\dot{)} + \lambda^2(\cdot)$. Hence we recover (for $\mathcal{F}_0$) something similar to a second order elliptic integrable system.

4.2 Superprimitive maps [5]

More precisely we can recover a second order elliptic integrable system close to the HSL surface theory in $\mathbb{R}^4$ by looking at *superprimitive maps* from $\mathbb{R}^{2|2}$ to the 4-symmetric space $SU(3)/SU(2)$: if $\Phi : \mathbb{R}^{2|2} \longrightarrow SU(3)/SU(2)$ is a superprimitive map then the first component u in the decomposition $\Phi = u + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F$ is a conformal HSL immersion (with the restriction that the Lagrangian angle β is equal to a *real* constant plus a harmonic non constant *nilpotent* function).

References

[1] F. Hélein, P. Romon, *Hamiltonian stationary Lagrangian surfaces in* $\mathbb{C}^2$, Comm. in Analysis and Geometry, Vol. 10, N. 1 (2002), 79–126.

[2] F. Hélein, P. Romon, *Weierstrass representation of Lagrangian surfaces in four dimensional spaces using spinors and quaternions*, Comment. Math. Helv., 75 (2000), 668–680.

[3] F. Hélein, P. Romon, *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces*, in *Differential Geomety and Integrable Systems*, M. Guest, R. Miyaoka, Y. Ohnita, ed., AMS, 2002.

[4] I. Khemar, *Surfaces isotropes de* $\mathbb{O}$ *et systèmes intégrables*, arXiv: math.DG/0511258.

[5] I. Khemar, *Supersymmetric harmonic maps into symmetric spaces*, arXiv: math.DG/0511258.

[6] R. Schoen, J. Wolfson, *Minimizing volume among Lagrangian submanifolds*, Proc. Sympos. Pure Math., 65, Amer. Math. Soc. (1999).

[7] C.-L. Terng, *Geometries and symmetries of soliton equations and integrable elliptic systems*, arXiv: math.DG/0212372.