On almost Blow-analytic equivalence

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NASH APPROXIMATION OF AN ANALYTIC DESINGULARIZATION

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Abstract. Approximation of real analytic functions by Nash functions is a classical topic in real geometry. In this paper, we focus on the Nash approximation of an analytic desingularization of a Nash function germ obtained by a sequence of blowings-up along smooth analytic centers. We apply the result to prove that Nash function germs that are analytically equivalent after analytic desingularizations are Nash equivalent after Nash desingularizations. Results are based on a precise Euclidean description of a sequence of blowings-up combined with Néron Desingularization.

The story of Nash manifolds and Nash maps begins with the fundamental paper [13] of J. Nash who realized any compact smooth manifold as a connected component of a real algebraic set. Nash manifolds, or Nash maps, are simply real analytic manifolds, or maps, with an additional semi-algebraic structure (i.e. described by finitely many equalities and inequalities of polynomial functions). In the further development of the theory, a crucial role has been played by approximation theorems, which state roughly speaking that real analytic solutions of a system of Nash equations may be approximated by Nash solutions, in a convenient topology ([1, 2, 16]).

Let f be a Nash function on a Nash manifold M. Then f is in particular a real analytic function on a real analytic manifold, and by Hironaka Desingularization Theorem [9] there exists a composition π of blowings-up along smooth analytic centers such that $f \circ \pi$ has only normal crossing singularities. We put our interest in this paper in approximating such a composition of blowings-up for which we can not apply classical Nash approximation theorems. Nevertheless, we prove as theorem 2.4 that each blowing-up along a smooth analytic center can be approximated by a blowing-up along a smooth Nash center in such a way that the normal crossing property of the modified function continues to hold.

Moreover we apply theorem 2.4 to deduce a Nash approximation theorem after desingularization (cf. theorem 1.4). More precisely for f and g Nash function germs on a compact semialgebraic set in a Nash manifold such that there exists analytic desingularizations π_f of f and π_g of g such that $f \circ \pi_f$ and $g \circ \pi_g$ are analytically equivalent, there exist Nash desingularizations $\widetilde{\pi}_f$ of f and $\widetilde{\pi}_g$ of g such that $f \circ \widetilde{\pi}_f$ and $g \circ \widetilde{\pi}_g$ are Nash equivalent. Note that here we do not only approximate the desingularizations but also the analytic diffeomorphism that realizes the equivalence between the modified germs. In the language of blow-analytic equivalence (cf. [11, 7, 8] and section 1), this result says that almost blow-analytically equivalent Nash function germs are almost blow-Nash equivalent. This question remains open for blow-analytic equivalence, namely if we impose moreover to the diffeomorphism that realizes the equivalence to induce a homeomorphism between the germs before desingularization.

The paper is organized as follows. We first recall in section 1 some definitions about blow-analytic equivalence before stating theorem 1.4. Then, in order to approximate a sequence of blowings-up along smooth centers, we focus in section 2.2 on a Euclidean realization of such a sequence in order to describe precisely its behavior under a perturbation of the defining ideal of the centers (cf. lemma 2.2). Combine with Néron Desingularization [17], this implies theorem 2.4. But this is not sufficient to prove theorem 1.4 since we need

2000 *Mathematics Subject Classification.* 26E05, 34C08, 58K20.

Key words and phrases. Real analytic functions, Nash functions, desingularization.

to approximate also the analytic diffeomorphism of the equivalence after the desingularization. To this aim, we need to generalize the Nash Approximation Theorem in [2] to a more general noncompact situation (cf. proposition 3.1). We obtain as a corollary that analytically equivalent Nash function germs on a compact semialgebraic set in a Nash manifold are Nash equivalent (cf. theorem 3.2). The last section is devoted to the proof of theorem 1.4.

In this paper a manifold means a manifold without boundary, analytic manifolds and maps mean real analytic ones unless otherwise specified, and id stands for the identity map.

1. Almost blow-analytic equivalence

Blow-analytic equivalence (cf. [11, 7, 8]) can be considered as a real counterpart of the topological equivalence between complex analytic germs of functions. Almost blow-analytic equivalence and almost blow-Nash equivalence are released version of the blow-analytic equivalence, for which cardinality results are known [5].

Definition 1.1. Let M be an analytic manifold and $f, g : M \longrightarrow \mathbf{R}$ be analytic functions on M. Then f and g are said to be *almost blow-analytically equivalent* if there exist two compositions of finite sequences of blowings-up along smooth analytic centers $\pi_f : N \longrightarrow M$ and $\pi_g : L \longrightarrow M$ and an analytic diffeomorphism $h : N \longrightarrow L$ so that $f \circ \pi_f = g \circ \pi_g \circ h$. In case there exist $\pi_f : N \to M$ and $\pi_g : L \to M$ with the above property and analytic diffeomorphisms $h : N \to L$ and $\tau : \mathbf{R} \to \mathbf{R}$ such that $\tau \circ f \circ \pi_f = g \circ \pi_g \circ h$, then f and g are called *almost blow-analytically R-L (=right-left) equivalent*. We define also the *almost blow-analytic (R-L) equivalence* of germs of analytic functions.

In this paper, we only treat the case where the images of the centers of the blowings-up of π_f and π_g are contained in their singular point sets $\operatorname{Sing} f$ and $\operatorname{Sing} g$, respectively, and the center C of each blowing-up is of codimension > 1 and normal crossing with the union D of the inverse images of the previous centers, i.e. there exists an analytic local coordinate system $(x_1, ..., x_n)$ at each point of C such that $C = \{x_1 = \cdots = x_k\}$ and $D = \{x_{i_1} \cdots x_{i_l} = 0\}$ for some $0 < k \in \mathbf{N}$ and $1 \leq i_1 < \cdots < i_l \leq n \in \mathbf{N}$, where $\mathbf{N} = \{0, 1, ...\}$.

Remark 1.2.

(1) Note that the cardinality of the set of classes of analytic functions on a compact analytic manifold, classified by almost blow-analytic equivalence, is of the continuum even if $\dim M = 0$, whereas that of almost blow-analytic R-L equivalence is countable [5].

(2) We do not know whether the almost blow-analytical (R-L) equivalence and the blow-analytical (R-L) equivalence give equivalence relations (see [7]). But this is the case if we admit blowings-up along non-smooth analytic center. Moreover, even in the case of germs of functions, almost blow-analytically equivalent function germs are not necessarily blow-analytically equivalent. We refer to [5] for these results.

Definition 1.3. A *semialgebraic* set is a subset of a Euclidean space which is described by finitely many equalities and inequalities of polynomial functions. A *Nash manifold* is a C^ω submanifold of a Euclidean space which is semialgebraic. A *Nash function* on a Nash manifold is a C^ω function with semialgebraic graph. A *Nash subset* is the zero set of a Nash function on a Nash manifold. Let M be a Nash manifold, $X \subset M$ be a semialgebraic subset and f, g be Nash function germs on X in M. Then f and g are said to be *almost blow-Nash equivalent* if there exist open semialgebraic neighborhoods U and V of X in M, two compositions of finite sequences of blowings-up along smooth Nash centers

$\pi_f : N \longrightarrow U$ and $\pi_g : L \longrightarrow V$ and a Nash diffeomorphism h from an open semialgebraic neighborhood of $\pi_f^{-1}(X)$ in N to one of $\pi_g^{-1}(X)$ in L so that f and g are supposed to be defined on U and V, respectively, $f \circ \pi_f = g \circ \pi_g \circ h$ and $h(\pi_f^{-1}(X)) = \pi_g^{-1}(X)$. We naturally define also *almost blow-Nash R-L equivalence*.

The aim of the paper is to prove the next result which is a natural counterpart of the classical Nash Approximation Theorem (cf. [1, 16]) in the case of almost blow-analytic equivalence.

Theorem 1.4. *Let M be a Nash manifold, $X \subset M$ be a compact semialgebraic subset such that $X = M$ or $X \subset \operatorname{Sing} f$, and f, g be Nash function germs on X in M. If f and g are almost blow-analytically (R-L) equivalent, then f and g are almost blow-Nash (respectively R-L) equivalent.*

Remark 1.5.

(1) Here the compactness assumption of X is necessary. Indeed, there exist a non-compact Nash manifold M and Nash functions f and g on M which are C^ω right equivalent but not almost blow-Nash equivalent as follows. Let N be a compact contractible Nash manifold with non-simply connected boundary of dimension $n > 3$ (e.g., see [12]). Set $M = (\operatorname{Int} N) \times (0, 1)$ and let $f : M \to (0, 1)$ denote the projection. Then M and f are of class Nash, and M is Nash diffeomorphic to $\mathbf{R}^{n+1}$ for the following reason. Smooth the corners of $N \times [0, 1]$. Then $N \times [0, 1]$ is a compact contractible Nash manifold with simply connected boundary of dimension > 4. Hence by the positive answers to Poincaré conjecture and Schönflies problem (Brown-Mazur Theorem) $N \times [0, 1]$ is C^∞ diffeomorphic to an $(n+1)$-ball. Hence by Theorem VI.2.2, [16] M is Nash diffeomorphic to an open $(n+1)$-ball. Let $g : M \to \mathbf{R}$ be a Nash function which is Nash right equivalent to the projection $\mathbf{R}^n \times (0, 1) \to (0, 1)$. Then f and g are C^ω right equivalent since $\operatorname{Int} N$ is C^ω diffeomorphic to $\mathbf{R}^n$, but they are not almost blow-Nash equivalent because if they are so then their levels are Nash diffeomorphic except for a finite number of values and hence $\operatorname{Int} N$ and $\mathbf{R}^n$ are Nash diffeomorphic, which contradicts Theorem VI 2.2, [16].

(2) The similar result concerning blow-Nash equivalence remains open. Namely we do not know whether blow-analytically equivalent Nash function germs on X in M are blow-Nash equivalent.

2. Nash approximation of an analytic desingularization

2.1. **Preliminaries on real analytic sheaf theory.** We recall the statements of the real analytic case of Cartan Theorems A and B, and Oka Theorem, in the refined version given in [5].

Let $\mathcal{O}$, $\mathcal{N}$ and $N(M)$ denote respectively the sheaves of analytic and Nash function germs on an analytic and Nash manifold and the ring of Nash functions on a Nash manifold M. We write $\mathcal{O}_M$ and $\mathcal{N}_M$ when we emphasize the domain M. For a function f on an analytic (Nash) manifold M, a subset X of M, a vector field v on M and for a sheaf of $\mathcal{O}$-($\mathcal{N}$-) modules $\mathcal{M}$ on M, let f_x, X_x, v_x and $\mathcal{M}_x$ denote the germs of f and X at a point x of M, the tangent vector assigned to x by v and the stalk of $\mathcal{M}$ at x, respectively. For a compact semialgebraic subset X of a Nash manifold M, let $\mathcal{N}(X)$ denote the germs of Nash functions on X in M with the topology of the inductive limit space of the topological spaces $N(U)$ with the compact-open C^∞ topology where U runs through the family of open semialgebraic neighborhoods of X in M.

Theorem 2.1. *Let $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}$-modules on an analytic manifold M.*

(1) (Cartan Theorem A) For any $x \in M$ we have $\mathcal{M}_x = H^0(M, \mathcal{M})\mathcal{O}_x$.

(2) *Assume moreover that $\mathcal{M}_x$ is generated by a uniform number of elements for any $x \in M$. Then $H^0(M, \mathcal{M})$ is finitely generated as a $H^0(M, \mathcal{O})$-module.*
(3) *(Cartan Theorem B) $H^1(M, \mathcal{M}) = 0$.*
(4) *Let $X \subset M$ be a global analytic set—the zero set of an analytic function. Let $\mathcal{I}$ be a coherent sheaf of $\mathcal{O}$-ideals on M such that any element of $\mathcal{I}$ vanishes on X. Then any $f \in H^0(M, \mathcal{O}/\mathcal{I})$ can be extended to some $F \in C^\omega(M)$, i.e. f is the image of F under the natural map $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}/\mathcal{I})$. If X is normal crossing, we can choose $\mathcal{I}$ to be the function germs vanishing on X. Then $H^0(M, \mathcal{O}/\mathcal{I})$ consists of functions on X whose germs at each point of X are extensible to analytic function germs on M.*
(5) *(Oka Theorem) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be coherent sheaves of $\mathcal{O}$-modules on M, and $h : \mathcal{M}_1 \to \mathcal{M}_2$ be an $\mathcal{O}$-homomorphism. Then* $\operatorname{Ker} h$ *is a coherent sheaf of $\mathcal{O}$-modules.*

2.2. **Euclidean realization of a sequence of blowings-up.** Let C be a smooth analytic subset of an analytic manifold U, and let $\pi : M \longrightarrow U$ denote the blowing-up of U along center C. In this section, we describe M as a smooth analytic subset of $U \times \mathbf{P}(k)$ for some $k \in \mathbf{N}$.

Let $\mathcal{I}$ denote the sheaf of $\mathcal{O}$-ideals defined by C. Since C is smooth, each stalk $\mathcal{I}_x$ is generated by $c = \operatorname{codim} C$ elements. Hence there exist a finite number of global generators $h_0, \ldots, h_k \in H^0(U, \mathcal{I})$ of $\mathcal{I}$ by theorem 2.1.(2). Define $\mathcal{A}$ to be the sheaf of relations of $h_0, \ldots, h_k$:

$$\mathcal{A} = \cup_{x \in U} \{(\mu_0, \ldots, \mu_k) \in \mathcal{O}_x^{k+1} : \sum_{i=0}^{k} \mu_i h_{ix} = 0\}.$$

Then $\mathcal{A}$ is coherent by theorem 2.1.(5), and each $\mathcal{A}_x$ is generated by $k - c + 1 + (c-1)!$ elements as follows. If $x_0 \notin C$ then $h_i(x_0) \neq 0$ for some i, say 0. On a small neighborhood of x_0, the map $\mathcal{O}^k \supset \mathcal{O}_x^k \ni (\mu_1, ..., \mu_k) \to (-\sum_{i=1}^k \mu_i h_{ix}/h_{0x}, \mu_1, ..., \mu_k) \in \mathcal{O}_x^{k+1} \subset \mathcal{O}^{k+1}$ is an isomorphism onto $\mathcal{A}$. Hence $\mathcal{A}_x$ is generated by k elements. If $x_0 \in C$, let x denote a point near x_0. In this case we can assume that $h_{0x}, ..., h_{c-1x}$ are regular function germs and generate $\mathcal{I}_x$. Then each h_{ix}, $c \leq i \leq k$, is of the form $\sum_{i=0}^{c-1} \phi_i h_{ix}$ for some $\phi_i \in \mathcal{O}_x$. Hence the projection image of $\mathcal{A}_x$ to the last $k-c+1$ factors of $\mathcal{O}_x^{k+1}$ is $\mathcal{O}_x^{k-c+1}$, and it suffices to see that $\mathcal{A}_x \cap \mathcal{O}_x^c \times \{0\} \times \cdots \times \{0\}$ is generated by $(c-1)!$ elements. We do this as follows: $\mathcal{A}_x \cap \mathcal{O}_x^c \times \{0\} \times \cdots \times \{0\}$ is generated by $(0, ..., 0, \overbrace{h_{j-1x}}^{i}, 0, ..., \overbrace{-h_{i-1x}}^{j}, 0, ..., 0)$ for $1 \leq i < j \leq c$. Therefore, $\mathcal{A}$ is generated by its global cross-sections $g_1 = (g_{1,0}, ..., g_{1,k}), ..., g_{k'} = (g_{k',0}, ..., g_{k',k}) \in C^\omega(U)^{k+1}$ for some $k' \in \mathbf{N}$. Moreover, it follows from these arguments that

(1) $\sum_{j=0}^k g_{i,j} h_j = 0,\ i = 1, ..., k'$,
(2) for each $x \in U - C$, the vectors $g_1(x), ..., g_{k'}(x)$ in $\mathbf{R}^{k+1}$ span a hyperplane and $(h_0(x), ..., h_k(x))$ in $\mathbf{R}^{k+1}$ is non-zero and orthogonal to the hyperplane,
(3) for each $x \in C$, the linear subspace $\{(s_0, ..., s_k) \in \mathbf{R}^{k+1} : \sum_{j=0}^k s_j g_{i,j}(x) = 0,\ i = 1, ..., k'\}$ of $\mathbf{R}^{k+1}$ is of dimension c.

Hence we can regard set-theoretically $M - \pi^{-1}(C)$ as

$$\{(x, t) \in (U - C) \times \mathbf{P}(k) : t_i h_j(x) = t_j h_i(x),\ i, j = 0, \ldots, k\}$$

by (2), hence M as

$$\{(x, t) \in U \times \mathbf{P}(k) : t_i h_j(x) = t_j h_i(x),\ i, j = 0, \ldots, k,\ \text{and} \sum_{j=0}^{k} t_j g_{i,j}(x) = 0,\ i = 1, ..., k'\}$$

by (3) and by $\sum_{j=0}^{k} t_j g_{i,j}(x) = 0$, $i = 1, ..., k'$, for $(x,t) \in (U - C) \times \mathbf{P}(k)$ with $t_i h_j(x) = t_j h_i(x)$, $i, j = 0..., k$, and π as the restriction to M of the projection $U \times \mathbf{P}(k) \to U$. When we identify M with the subset of $U \times \mathbf{P}(k)$, we say M is *realized* in $U \times \mathbf{P}(k)$.

Since we treat only finite sequences of blowings-up, we can imbed M into a Euclidean space. For that we imbed algebraically $\mathbf{P}(k)$ in $\mathbf{R}^{(k+1)^2}$ as in [1] by

$$(t_0 : \ldots : t_k) \mapsto (\frac{t_i t_j}{|t|^2}),$$

where $|t|^2 = \sum_{i=0}^{k} t_i^2$. It is known that $\mathbf{P}(k)$ is a non-singular algebraic subvariety in $\mathbf{R}^{(k+1)^2}$. We denote by $y_{i,j}$ the coordinates on $\mathbf{R}^{(k+1)^2}$ such that $y_{i,j} = t_i t_j / |t|^2$ on $\mathbf{P}(k)$. Let $\xi_1, \ldots, \xi_s$ be generators of the ideal of $\mathbf{R}[y_{i,j}]$ of functions vanishing on $\mathbf{P}(k)$. Set $l_{i,j,m}(x,y) = y_{i,j} h_m(x) - y_{m,i} h_j(x)$ for $i, j, m = 0, \ldots, k$. Define

$$N = \{(x,y) \in U \times \mathbf{R}^{(k+1)^2} : l_{i,j,m}(x,y) = 0,\ i, j, m = 0, \ldots, k,$$

$$\sum_{j=0}^{k} y_{j,m} g_{i,j}(x) = 0,\ i = 1, ..., k',\ m = 0, ..., k,\ \text{and } \xi_i(y) = 0,\ i = 1, ..., s\}.$$

Then $M = N$. Moreover the analytic sets of both sides coincide algebraically, i.e. the functions $l_{i,j,m}$, $\sum_{j=0}^{k} y_{j,m} g_{i,j}$, ξ_i generate $I(M)$—the ideal of $C^\omega(U \times \mathbf{R}^{(k+1)^2})$ of functions vanishing on M. Indeed, by theorem 2.1.(4) the problem is local. If $x \in U - C$, the claim locally at x is clear. Assume that $x \in C$, and let $(x_1, ..., x_n)$ denote a local coordinate system of U around x. As the claim does not depend on the choice of $\{g_i\}$, we can assume that $h_j = x_{j+1}$, $j = 0, ..., c-1$, $h_j = \sum_{i=0}^{c-1} \phi_{i,j} h_i$, $j = c, ..., k$, and $g_1 = (-\phi_{0,c}, ..., -\phi_{c-1,c}, 1, 0, ..., 0), ..., g_{k-c+1} = (-\phi_{0,k}, ..., -\phi_{c-1,k}, 0, ..., 0, 1)$, $g_{k-c+2} = (x_2, -x_1, 0, ...,0), ..., g_{k'} = (0, ...,0, x_c, -x_{c-1}, 0, ..., 0)$ for some C^ω functions $\phi_{i,j}$ on a neighborhood of x and for $k' = k - c + 1 + (c-1)!$. Then

$$(t_0 \cdots t_k) = (t_0 \cdots t_{c-1}) \begin{pmatrix} 1 & & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix},$$

$$\begin{pmatrix} t_0 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_{c-1} \end{pmatrix}.$$

Therefore $\sum_{j=0}^{k} t_j^2$ is equal to

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_{c-1} \end{pmatrix} (t_0 \cdots t_{c-1}) \begin{pmatrix} 1 & & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix}$$

whereas the matrix $(y_{i,j})_{i,j=0,...,k}$ equals

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} y_{0,0} & \cdots & y_{0,c-1} \\ \vdots & & \vdots \\ y_{c-1,0} & \cdots & y_{c-1,c-1} \end{pmatrix} \begin{pmatrix} 1 & & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix}.$$

Hence we can forget h_j and $y_{i,j} = y_{j,i},\ i = 0, ..., k,\ j = c, ..., k$, and we can replace N with its image under the projection $U \times \mathbf{R}^{(k+1)^2} \ni (x, y_{i,j}) \to (x, y_{i,j})_{i,j\le c-1} \in U \times \mathbf{R}^{c^2}$. Come back to the realization of M in $U \times \mathbf{P}(k)$. Then it becomes

$$\widetilde{M} = \{(x, y) \in U \times \mathbf{R}^{c^2} : l_{i,j,m}(x, y) = 0,\ i, j, m = 0, \dots, c-1,$$

$$\sum_{j=0}^{c-1} y_{j,m} g_{i,j}(x) = 0,\ i = k-c+2, ..., k', m = 0, ..., c-1, \text{ and } \xi'_i(y) = 0,\ i = 1, ..., s'\},$$

where ξ'_i are generators of $I(\mathbf{P}(c-1)) \subset \mathbf{R}[y_{i,j}]_{i,j\le c-1}$. Therefore it suffices to show that $l_{i,j,m}$, $\sum_{j'=0}^{c-1} y_{j',m} g_{i',j'}$, $\xi'_{i''}$, $i, j, m = 0, ..., c-1$, $i' = k-c+2, ..., k'$, $i'' = 1, ..., s'$, generate $I(\widetilde{M})$. However, by easy calculations we prove that $l_{i,j,m}$ and $\xi'_{i''}$ generate $I(\widetilde{M})$. (To realize M in $U \times \mathbf{P}(k)$ we need the equations $\sum_{j=0}^{k} t_j g_{i,j}(x) = 0,\ i = 1, ..., k-c+1$, which are equivalent to $t_c = t_0\phi_{0,c} + \cdots + t_{c-1}\phi_{c-1,c}, ..., t_k = t_0\phi_{0,k} + \cdots t_{c-1}\phi_{c-1,k}$.)

2.3. **Perturbation of a blowing-up.** When we perturb $h_i,\ i = 0, ..., k$, in the strong Whitney C^∞ topology, the common zero set $Z(h_i)$ of h_i's may become of smaller dimension than C and singular, where the strong Whitney C^∞ topology on $C^\infty(U)$ is defined to be the topology of the projective limit space of the topological spaces $C^\infty(U_k)$ with the C^∞ topology for all compact C^∞ submanifolds possibly with boundary U_k of U. (Note that Whitney Approximation Theorem in [18] holds also in this topology, and we call it Whitney Approximation Theorem.) However,

Lemma 2.2. *Let $\tilde{h}_i,\ i = 0, ..., k$, and $\tilde{g}_i = (\tilde{g}_{i,0}, ..., \tilde{g}_{i,k}),\ i = 1, ..., k'$, be C^ω functions on U and C^ω maps from U to $\mathbf{R}^{k+1}$ close to h_i and g_i, respectively, in the strong Whitney C^∞ topology. Assume that (1) $\sum_{j=0}^{k} \tilde{g}_{i,j}\tilde{h}_j = 0,\ i = 1, ..., k'$. Then*

- *$\tilde{C} = Z(\tilde{h}_i)$ is smooth and of the same dimension as C, $\tilde{h}_0, ..., \tilde{h}_k$ generate $I(Z(\tilde{h}_i))$ and $\tilde{g}_1, ..., \tilde{g}_{k'}$ are generators of the sheaf of relations $\tilde{\mathcal{A}}$ of $\tilde{h}_0, ..., \tilde{h}_k$.*
- *Let $\pi : M \to U$ and $\tilde{\pi} : \tilde{M} \to U$ denote the blowings-up along centers C and $\tilde{C}$, respectively. Let M and $\tilde{M}$ be realized in $U \times \mathbf{P}(k)$ as in section 2.2. Then there exist analytic diffeomorphisms τ of U and $\psi : M \to \tilde{M}$ close to id in the strong Whitney C^∞ topology such that $\tau(C) = \tilde{C}$ and $\tilde{\pi} \circ \psi = \tau \circ \pi$.*

Proof. The problem in the former half is local and clear around a point outside of C, and hence we assume that $h_j = x_{j+1},\ j = 0, ..., c-1$, for a local coordinate system $(x_1, ..., x_n)$, and $h_j = \sum_{i=0}^{c-1} \phi_{i,j} h_i,\ j = c, ..., k$, for some C^ω functions $\phi_{i,j}$ on U. Then $Z(\tilde{h}_0, ..., \tilde{h}_{c-1})$ is smooth and of the same dimension as C. Hence we need to see that $\tilde{h}_j,\ j = c, ..., k$, are contained in the ideal of $C^\omega(U)$ generated by $\tilde{h}_j,\ j = 0, ..., c-1$. Choose C^ω functions $\alpha_{i,j},\ i = 1, ..., k-c+1,\ j = 1, ..., k'$, on U so that

$$\begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_{k'} \end{pmatrix}$$

is of the form

$$\begin{pmatrix} -\phi_{0,1} & \cdots & -\phi_{c-1,1} & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ -\phi_{0,k-c+1} & \cdots & -\phi_{c-1,k-c+1} & 0 & & 1 \end{pmatrix}.$$

Set

$$\begin{pmatrix} \tilde{g}'_{1,0} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,0} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}.$$

Then $\begin{pmatrix} \tilde{g}'_{1,c} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,c} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix}$ is closed to $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$. Hence

$$\begin{pmatrix} \tilde{g}'_{1,c} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,c} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$$

is well-defined and of the form $\begin{pmatrix} \tilde{g}''_{1,0} & \cdots & \tilde{g}''_{1,c-1} & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \tilde{g}''_{k-c+1,0} & \cdots & \tilde{g}''_{k-c+1,c-1} & 0 & & 1 \end{pmatrix}$. Now (1) implies $\begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Therefore,

$$\begin{pmatrix} \tilde{g}''_{1,0} & \cdots & \tilde{g}''_{1,c-1} & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \tilde{g}''_{k-c+1,0} & \cdots & \tilde{g}''_{k-c+1,c-1} & 0 & & 1 \end{pmatrix} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and $\tilde{h}_j = -\sum_{i=0}^{c-1} \tilde{g}''_{j-c+1,i}\tilde{h}_i,\ j = c, ..., k$.

We need to see that $\tilde{g}_1, ..., \tilde{g}_{k'}$ are generators of $\tilde{\mathcal{A}}$. By (1) they are global cross-sections of $\tilde{\mathcal{A}}$. We postpone proving $\tilde{g}_1, ..., \tilde{g}_{k'}$ generate $\tilde{\mathcal{A}}$.

Next we prove the latter half of the lemma. We first define τ on C. The condition on $\tau|_C$ to be satisfied is $\tau(C) = \tilde{C}$. Let $U \subset \mathbf{R}^N$, let q denote the orthogonal projection of a tubular neighborhood of U in $\mathbf{R}^N$, and let $p : V \to C$ denote the proper orthogonal projection of a small closed tubular neighborhood of C in U. We require τ to satisfy, moreover, $p \circ \tau = \mathrm{id}$ on C. Then $\tau|_C$ is unique and the problem of finding $\tau|_C$ is local. Hence we assume that as above $h_j = x_{j+1}$, $j = 0, ..., c-1$, for a local coordinate system $(x_1, ..., x_n)$ at each point of C and $h_j = \sum_{i=0}^{c-1} \phi_{i,j}h_i$, $j = c, ..., k$, for some C^ω functions $\phi_{i,j}$. Then $\tau|_C$ is well-defined (cf. proof of lemma 3.5 in [5]), and $\tau|_C$ is an analytic embedding of C into U close to id in the strong Whitney C^∞ topology.

Secondly, we extend $\tau|_C$ to V by setting $\tau(x) = q(\tau \circ p(x) + x - p(x))$ for $x \in V$, which is close to id in the strong Whitney C^∞ topology. Moreover, using the extension we extend $\tau|_C$ to an analytic diffeomorphism τ of U close to id in the same topology by a partition of unity of class C^∞, theorem 2.1.(4) and by Whitney Approximation Theorem.

Lastly, we need to find ψ. Set $\tilde{\tilde{h}}_i = h_i \circ \tau^{-1}$ and $\tilde{\tilde{g}}_i = g_i \circ \tau^{-1}$ and define $\tilde{\tilde{M}}$ by $\tilde{\tilde{h}}_i$ and $\tilde{\tilde{g}}_i$ in $U \times \mathbf{P}(k)$ and $\tilde{\tilde{\pi}} : \tilde{\tilde{M}} \to U$. Then

$$\tilde{\tilde{M}} = \{(x,t) \in U \times \mathbf{P}(k) : t_i h_j \circ \tau^{-1}(x) = t_j h_i \circ \tau^{-1}(x),\ i, j = 0, \ldots, k, \text{ and}$$

$$\sum_{j=0}^{k} t_j g_{i,j} \circ \tau^{-1}(x) = 0,\ i = 1, ..., k'\},$$

and $\tilde{\tilde{\psi}} : M \ni (x,t) \to (\tau(x), t) \in \tilde{\tilde{M}}$ is an analytic diffeomorphism close to id in the strong Whitney C^∞ topology such that $\tilde{\tilde{\pi}} \circ \tilde{\tilde{\psi}} = \tau \circ \pi$. Hence we can replace h_i and g_i with $h_i \circ \tau^{-1}$ and $g_i \circ \tau^{-1}$, respectively. Thus we assume from the beginning that $Z(h_i) = Z(\tilde{h}_i)$. Set $h'_i = \tilde{h}_i - h_i$. Then there exist analytic functions $\chi_{i,j}$, $i, j = 0, ..., k$, on U close to 0 in the topology such that $h'_i = \sum_{j=0}^{k} \chi_{i,j} h_j$, $i = 0, ..., k$, which is proved, as before, by a partition of unity of class C^∞, theorem 2.1.(4) and by Whitney Approximation Theorem.

Set $A(x) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} \chi_{0,0} & \cdots & \chi_{k,0} \\ \vdots & & \vdots \\ \chi_{0,k} & \cdots & \chi_{k,k} \end{pmatrix}$ and define an analytic diffeomorphism ψ of $U \times \mathbf{P}(k)$ by $\psi(x,t) = (x, tA(x))$ for $(x,t) \in U \times \mathbf{P}(k)$. Then $(h_0, ..., h_k)A = (\tilde{h}_0, ..., \tilde{h}_k)$ on U, hence $\psi(M) = \tilde{M}$, $\tilde{\pi} \circ \psi = \pi$ on M and $\psi|_M$ is close to id in the topology, which proves the latter half.

It remains to show that $\tilde{g}_1, ..., \tilde{g}_{k'}$ generate $\tilde{\mathcal{A}}$, i.e. $\tilde{g}_1, ..., \tilde{g}_{k'}$ generate the $C^\omega(U)$-module $\tilde{X}$ defined by $\tilde{X} = \{\tilde{g} \in (C^\omega(U))^{k+1} : \tilde{g} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = 0\}$ by theorem 2.1.(3). As above we can assume that $Z(h_i) = Z(\tilde{h}_i)$. Moreover, we suppose that $\tilde{h}_i = h_i$ for any i for the following reason. For the above A we have ${}^tA \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix}$. Hence $\tilde{X} = \{\tilde{g} \in (C^\omega(U))^{k+1} : \tilde{g}^t A \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = 0\}$. Then it suffices to see that $\tilde{g}_1{}^tA, ..., \tilde{g}_{k'}{}^tA$ generate the $C^\omega(U)$-module $X = \{g \in (C^\omega(U))^{k+1} : g \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = 0\}$ because the map $(C^\omega(U))^{k+1} \ni \tilde{g} \to \tilde{g}^t A \in (C^\omega(U))^{k+1}$ is an isomorphism as $C^\omega(U)$-modules. Here $\tilde{g}_1{}^tA, ..., \tilde{g}_{k'}{}^tA$ are close to $g_1, ..., g_{k'}$ respectively. Therefore, replacing $\begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix}$ and $\begin{pmatrix} \tilde{g}_0 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$ with $\begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix}$ and $\begin{pmatrix} \tilde{g}_0 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} {}^tA$, respectively, we suppose from the beginning that $\tilde{h}_i = h_i$ for all i and $\tilde{\mathcal{A}} = \mathcal{A}$.

As above, the problem is local at each point of C and we assume that $h_j = x_{j+1}$, $j = 0, ..., c-1$, for a local coordinate system $(x_1, ..., x_n)$. Recall that

$$(*) \qquad \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,k'} \\ \vdots & & \vdots \\ \beta_{k-c+1,1} & \cdots & \beta_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots & \cdot & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \cdot & \cdots & \cdot & 0 & & 1 \end{pmatrix}$$

for some C^ω functions $\beta_{i,j}$ on U. Let r denote the restriction to $\mathcal{A}$ of the projection of $\mathcal{O}^{k+1}$ to the last $k - c + 1$ factors and $r_* : H^0(U, \mathcal{A}) \to (C^\omega(U))^{k-c+1}$ denote the induced map. Then $(*)$ implies that $r_*(\tilde{g}_1), ..., r_*(\tilde{g}_{k'})$ generate $r(\mathcal{A}) = \mathcal{O}^{k-c+1}$. Hence it suffices to see that $\mathcal{A} \cap \mathcal{O}^c \times \{0\} \times \cdots \times \{0\}$ is generated by $\{\sum_{i=1}^{k'} \beta_i \tilde{g}_i : \beta_i \in C^\omega(U),\ r_*(\sum_{i=1}^{k'} \beta_i \tilde{g}_i) = 0\}$.

Since $g_1, ..., g_{k'}$ generate $\mathcal{A}$, there exist C^ω functions $\gamma_{i,j},\ i = 1, ..., (c-1)!,\ j = 1, ..., k'$, such that

$$\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_{k'} \end{pmatrix} = \begin{pmatrix} x_2 & -x_1 0 \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots 0 x_c & -x_{c-1} & 0 & \cdots & 0 \end{pmatrix},$$

whose rows are global generators of $\mathcal{A} \cap \mathcal{O}^c \times \{0\} \times \cdots \times \{0\}$. Consider the matrix $\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$. Its (i,j) components, $i = 1, ..., (c-1)!,\ j = (c-1)! + 1, ..., k'$, are close to 0. Hence by $(*)$ there exist C^ω functions $\delta_{i,j},\ i = 1, ..., (c-1)!,\ j = 1, ..., k'$, close to 0 such that the (i,j) components, $i = 1, ..., (c-1)!,\ j = (c-1)! + 1, ..., k'$, of the matrix

$$\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} - \begin{pmatrix} \delta_{1,1} & \cdots & \delta_{1,k'} \\ \vdots & & \vdots \\ \delta_{(c-1)!,1} & \cdots & \delta_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$$

are 0. Replace $\gamma_{i,j}$ with $\tilde{\gamma}_{i,j} = \gamma_{i,j} - \delta_{i,j}$. Then the (i,j) components, $i = 1, ..., (c-1)!,\ j = (c-1)! + 1, ..., k'$, of the matrix $\begin{pmatrix} \tilde{\gamma}_{1,1} & \cdots & \tilde{\gamma}_{1,k'} \\ \vdots & & \vdots \\ \tilde{\gamma}_{(c-1)!,1} & \cdots & \tilde{\gamma}_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$ are 0, and each row is an approximation of the corresponding row of the matrix $\begin{pmatrix} x_2 & -x_1 0 \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots 0 x_c & -x_{c-1} & 0 & \cdots & 0 \end{pmatrix}$. Therefore, we can suppose from the beginning that $k = c-1,\ k' = (c-1)!$, $g_1 = (x_2, x_1, 0, ..., 0), ..., g_{k'} = (0, ..., 0, x_c, -x_{c-1})$.

Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_x$ for $x \in U$. For each $x \in C$, $\tilde{g}_{1x}, ..., \tilde{g}_{k'x}$ generate $\mathcal{A}_x$ if and only if $\tilde{g}_{1x}, ..., \tilde{g}_{k'x}$ and $\mathfrak{m}_x \mathcal{A}_x$ generate $\mathcal{A}_x$ by Nakayama lemma. On the other hand, the images of $g_{1x}, ..., g_{k'x}$ in the linear space $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$, $x \in C$, are a basis and hence $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$ is a linear space of dimension k'. Hence it suffices to see that the images of $\tilde{g}_{1x}, ..., \tilde{g}_{k'x}$ in $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$, $x \in C$, are linearly independent. Here $\mathfrak{m}_x \mathcal{A}_x = \mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x$ because clearly $\mathfrak{m}_x \mathcal{A}_x \subset \mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x$ and $\mathcal{A}_x / (\mathfrak{m}_x^2 \cap \mathcal{A}_x)\, (\approx\ (\mathfrak{m}_x^2 \mathcal{O}_x^c + \mathcal{A}_x) / \mathfrak{m}_x^2 \mathcal{O}_x^c)$ and $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$ are linear spaces of the same dimension. Now $\cup_{x \in U} \mathcal{O}_x^c / \mathfrak{m}_x^2 \mathcal{O}_x^c$ coincides with the space of 1-jets from U to $\mathbf{R}^c$. Hence for $x \in C$, the images of $\tilde{g}_{1x}, ..., \tilde{g}_{k'x}$ in $\mathcal{O}_x^c / \mathfrak{m}_x^2 \mathcal{O}_x^c$, i.e. in $\mathcal{A}_x / (\mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x)$ is linearly independent if $\tilde{g}_1, ..., \tilde{g}_{k'}$ are sufficiently close to $g_1, ..., g_{k'}$, respectively, in the Whitney C^1 topology because the images of $g_{1x}, ..., g_{k'x}$ are linearly independent.

□

Remark 2.3.

(1) In lemma 2.2, $\tau|_C$ is an embedding of C into U close to id in the strong Whitney C^∞ topology such that $\tau(C) = \tilde{C}$. Conversely, assume that there exist an analytic embedding τ_C of C into U close to id in the same topology. Then τ_C is extensible to an analytic diffeomorphism τ of U close to id in the topology. Define $\tilde{C}$, $\tilde{h}_i$ $\tilde{g}_i$ and $\tilde{\pi} : \tilde{M} \to \tilde{U}$ to be $\tau(C)$, $h_i \circ \tau^{-1}$, $g_i \circ \tau^{-1}$ and the blowing-up of U along center $\tilde{C}$, respectively. Realize M and $\tilde{M}$ in $U \times \mathbf{P}(k)$ as before. Then $\tilde{h}_i$ and $\tilde{g}_i$ are close to h_i and g_i respectively, $\sum_{j=0}^k \tilde{g}_{i,j} \tilde{h}_j = 0$, and hence by lemma 2.2 there exists an analytic diffeomorphism $\psi : M \to \tilde{M}$ close to id in the topology such that $\tilde{\pi} \circ \psi = \tau \circ \pi$.

When there exists τ_C such as above, we say $\tilde{C}$ is *close* to C in the strong Whitney C^∞ topology. Let $\psi : M_1 \to M_2$ and $\tilde{\psi} : \tilde{M}_1 \to \tilde{M}_2$ be analytic maps between analytic manifolds with $M_1 \subset \mathbf{R}^{n_1}$, $\tilde{M}_1 \subset \mathbf{R}^{n_1}$, $M_2 \subset \mathbf{R}^{n_2}$ and $\tilde{M}_2 \subset \mathbf{R}^{n_2}$. Assume that $\tilde{M}_1$ is close to M_1 in the topology through an analytic diffeomorphism $\tau : M_1 \to \tilde{M}_1$ close to id in the topology. Then we say $\tilde{\psi}$ is *close* to ψ in the topology if $\tilde{\psi}$ is so to $\psi \circ \tau$.

(2) The germ case of lemma 2.2 holds in the following sense. Let h_i, g_i, U and C be the same as above. Let X be a compact subset of U, and let $\tilde{h}_i$ and $\tilde{g}_i$ be C^ω functions and maps defined on an open neighborhood V of X in U close to $h_i|_V$ and $g_i|_V$, respectively, in the compact-open C^∞ topology with $\sum_{j=0}^{k} \tilde{g}_{i,j}\tilde{h}_j = 0$. Shrink V. Then the same statement as the former half of lemma 2.2 holds. For the latter half, let $\pi : M \to U$ and $\tilde{\pi} : \tilde{M} \to V$ denote the blowings-up along centers C and $\tilde{C} = Z(\tilde{h}_i)$. Let $M \subset U \times \mathbf{P}(k)$ and $\tilde{M} \subset V \times \mathbf{P}(k)$ be as above. Then there exist analytic imbed dings $\tau_- : V \to U$ and $\psi_- : \tilde{M} \to M$ close to id in the compact-open C^∞ topology such that $\tau_- \circ \tilde{\pi} = \pi \circ \psi_-$. In this case we say $\tilde{C}$ is *close* to C at X in the C^∞ topology, and define *closeness* of an analytic map to another one at a compact set.

2.4. **Nash approximation.** An analytic function with *only normal crossing singularities* at a point x of an analytic manifold is a function whose germ at x is of the form $\pm x^\alpha (= \pm \prod_{i=1}^{n} x_i^{\alpha_i}) + \text{const}$, $\alpha = (\alpha_1, ..., \alpha_n) \neq 0 \in \mathbf{N}^n$ for some local analytic coordinate system $(x_1, ..., x_n)$ at x. If the function has only normal crossing singularities everywhere we say the function has *only normal crossing singularities*. Remember that by Hironaka Desingularization Theorem an analytic function becomes one with only normal crossing singularities after a finite sequence of blowings-up along smooth centers. An analytic subset of an analytic manifold is called *normal crossing* if it is the zero set of an analytic function with only normal crossing singularities. This analytic function is called *defined by* the analytic set. It is not unique. However, the sheaf of $\mathcal{O}$-ideals *defined by* the analytic set is naturally defined and unique. We can naturally stratify a normal crossing analytic subset X into analytic manifolds X_i of dimension i. We call $\{X_i\}$ the *canonical stratification* of X.

Now we state and prove a Nash approximation theorem of an analytic desingularization of a Nash function. This result will be crucial for the proof of theorem 1.4.

Theorem 2.4. *Let f be a Nash function on a Nash manifold M. Let $M_m \xrightarrow{\pi_m} M_{m-1} \longrightarrow \cdots \xrightarrow{\pi_1} M_0 = M$ be a sequence of blowings-up along smooth analytic centers C_{m-1} in $M_{m-1}, ..., C_0$ in M_0, respectively, such that $f \circ \pi_1 \circ \cdots \circ \pi_m$ has only normal crossing singularities. Let X be a compact subset of M. Then there exist an open semialgebraic neighborhood U of X in M, a sequence of blowings-up $U_m \xrightarrow{\tau_m} U_{m-1} \longrightarrow \cdots \xrightarrow{\tau_1} U_0 = U$ along smooth Nash centers D_{m-1} in $U_{m-1}, ..., D_0$ in U_0, respectively, and an analytic embedding $\psi : U_m \to M_m$ such that $\psi(\tau_m^{-1}(D_{m-1})) \subset \pi_m^{-1}(C_{m-1}), ..., \psi((\tau_1 \circ \cdots \circ \tau_m)^{-1}(D_0)) \subset (\pi_1 \circ \cdots \circ \pi_m)^{-1}(C_0)$ and $f \circ \pi_1 \circ \cdots \circ \pi_m \circ \psi = f \circ \tau_1 \circ \cdots \circ \tau_m$. Let $M_1, ..., M_m$ be realized in $M \times \mathbf{P}(k_0), ..., M \times \mathbf{P}(k_0) \times \cdots \times \mathbf{P}(k_{m-1})$, respectively, for some $k_0, ..., k_{m-1} \in \mathbf{N}$. Then we can realize $U_1, ..., U_m$ in $U \times \mathbf{P}(k_0), ..., U \times \mathbf{P}(k_0) \times \cdots \times \mathbf{P}(k_{m-1})$, respectively, so that each pair $D_i \subset U_i$ and ψ are close to $C_i \subset M_i$ at $(\tau_1 \circ \cdots \circ \tau_{i-1})^{-1}(X)$ and to id at $(\tau_1 \circ \cdots \circ \tau_m)^{-1}(X)$, respectively, in the C^∞ topology.*

The proof of theorem 2.4 is the heart of the paper. It consists in a combination of algebra and topology, via a nested Néron Desingularization Theorem (see Theorem 11.4, [17]) and Nash Approximation Theorem. We proceed as follows. First we describe the analytic situation of the sequence of blowings-up in terms of ideals. Next we apply the nested type of Néron Desingularization Theorem and come down to a regular situation.

Then, in the regular situation, the classical Nash Approximation Theorem enables to realize the approximation. The idea comes from the proof of Theorem 1.1 in [2], where the usual Néron Desingularization Theorem is used.

Proof. Consider the blowing-up $\pi_1 : M_1 \to M_0$ along center C_0. Let

- $\mathcal{I}^0$ denote the sheaf of $\mathcal{O}$-ideals on M_0 defined by C_0,
- $h_0^0, ..., h_{k_0}^0$ its global generators,
- $\mathcal{A}^0 \subset \mathcal{O}_{M_0}^{k_0+1}$ the sheaf of relations of $h_0^0, ..., h_{k_0}^0$,
- $g_1^0 = (g_{1,0}^0, ..., g_{1,k_0}^0), ..., g_{k_0'}^0 = (g_{k_0',0}^0, ..., g_{k_0',k_0}^0)$ global generators of $\mathcal{A}^0$,
- $\xi_1^0, ..., \xi_{s_0}^0$ generators of the ideal $I(\mathbf{P}(k_0))$ of $\mathbf{R}[y_{i,j}^0]_{0\le i,j\le k_0}$ of functions vanishing on $\mathbf{P}(k_0)$ in $\mathbf{R}^{(k_0+1)^2}$.

Set $l_{i_1,i_2,i_3}^0(x,y^0) = y_{i_1,i_2}^0 h_{i_3}^0(x) - y_{i_3.i_1}^0 h_{i_2}^0(x)$ for $(x,y^0) \in M_0 \times \mathbf{R}^{(k_0+1)^2}$ and for $i_1, i_2, i_3 = 0, ..., k_0$. Then

$$\sum_{j=0}^{k_0} g_{i,j}^0(x) h_j^0(x) = 0 \quad \text{on } M_0 \text{ for } i = 1, ..., k_0', \tag{1}$$

$$M_1 = \{(x,y^0) \in M_0 \times \mathbf{R}^{(k_0+1)^2} : l_{i_1,i_2,i_3}^0(x,y^0) = 0,\ i_1, i_2, i_3 = 0, \dots, k_0,$$

$$\sum_{j=0}^{k_0} y_{j,i_1}^0 g_{i_2,j}^0(x) = 0,\ i_1 = 0, ..., k_0,\ i_2 = 1, ..., k_0', \text{ and } \xi_i^0(y^0) = 0,\ i = 1, ..., s_0\},$$

and both sides coincide algebraically, i.e. $I(M_1)\,(\subset C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2}))$ is generated by those $l_{i_1,i_2,i_3}^0, \sum_{j=0}^{k_0} y_{j,i_1}^0 g_{i_2,j}^0$ and ξ_i^0. Let $\{\alpha_i^1\}$ denote the generators. Note that π_1 is the restriction to M_1 of the projection $M_0 \times \mathbf{R}^{(k_0+1)^2} \to M_0$. Come back to definition 1.1. There we assume that $C_0 \subset \operatorname{Sing} f$, which is described as follows. Let $v_1, ..., v_n$ be Nash vector fields on M_0 which span the tangent space of M_0 at each point of M_0. Then we see, as previously, that $C_0 \subset \operatorname{Sing} f$ if and only if there exist C^ω functions $a_{i,j}^0$ on M_0, $i = 1, ..., n,\ j = 0, ..., k_0$, such that

$$v_i f = \sum_{j=0}^{k_0} a_{i.j}^0 h_j^0 \quad \text{on } M_0 \text{ for } i = 1, ..., n. \tag{2}$$

Let $\tilde{h}_0^0, ..., \tilde{h}_{k_0}^0$, $\tilde{g}_1^0, ..., \tilde{g}_{k_0'}^0$, $\tilde{a}_{i,j}^0$, $i = 1, ..., n$, $j = 0, ..., k_0$, be C^ω approximations of $h_0^0, ..., h_{k_0}^0$, $g_1^0, ..., g_{k_0'}^0$, $a_{i,j}^0$, $i = 1, ..., n$, $j = 0, ..., k_0$, respectively, in the strong Whitney C^∞ topology such that

($\tilde{1}$) $\sum_{j=0}^{k_0} \tilde{g}_{i,j}^0 \tilde{h}_j^0 = 0$ for $i = 1, ..., k_0'$ and

($\tilde{2}$) $v_i f = \sum_{j=0}^{k_0} \tilde{a}_{i,j}^0 \tilde{h}_j^0$ for $i = 1, ..., n$.

Then by lemma 2.2, the common zero set $\tilde{C}_0 = Z(\tilde{h}_j^0)$ is smooth and of the same dimension as C_0, $\tilde{g}_1^0, ..., \tilde{g}_{k_0'}^0$ are generators of the sheaf of relations $\tilde{\mathcal{A}}^0$ of $\tilde{h}_0^0, ..., \tilde{h}_{k_0}^0$, if we let $M_0 \times \mathbf{R}^{(k_0+1)^2} \supset M_0 \times \mathbf{P}(k_0) \supset \tilde{M}_1 \xrightarrow{\tilde{\pi}_1} M_0$ denote the blowing-up of M_0 along center $\tilde{C}_0$ defined by $\tilde{h}_0^0, ..., \tilde{h}_{k_0}^0$ then there exist analytic diffeomorphisms ψ_0 of M_0 and $\tilde{\psi}_0 : M_1 \to \tilde{M}_1$ close to id in the strong Whitney C^∞ topology such that $\psi_0(C_0) = \tilde{C}_0$ and $\tilde{\pi}_1 \circ \tilde{\psi}_0 = \psi_0 \circ \pi_1$. Hence $f \circ \tilde{\pi}_1 : \tilde{M}_1 \to \mathbf{R}$ is close to $f \circ \pi_1 : M_1 \to \mathbf{R}$ in the strong Whitney C^∞ topology because if we regard f as a function on $M_0 \times \mathbf{R}^{(k_0+1)^2}$ then $f \circ \tilde{\pi}_1 = f|_{\tilde{M}_1}$ and $f \circ \pi_1 = f|_{M_1}$. Moreover, $\tilde{C}_0 \subset \operatorname{Sing} f$ by ($\tilde{2}$) for $\tilde{h}_j^0$, and $I(\tilde{M}_1)$ is generated by $\tilde{l}_{i_1,i_2,i_3}^0(x,y^0)$ which

is defined by $\tilde{l}^0_{i_1,i_2,i_3}(x,y^0) = y^0_{i_1,i_2}\tilde{h}^0_{i_3}(x) - y^0_{i_3.i_1}\tilde{h}^0_{i_2}(x)$, $\sum_{j=0}^{k_0} y^0_{j,i_1}\tilde{g}^0_{i_2,j}(x)$ and $\xi^0_i(y^0)$ in $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2})$. Let $\tilde{\alpha}^1_i$ denote the generators corresponding to α^1_i.

Consider the second blowing-up $\pi_2 : M_2 \to M_1$ along C_1. In the same way as for the first blowing-up we define

- $\mathcal{I}^1 \subset \mathcal{O}_{M_1}$,
- $h^1_0, ..., h^1_{k_1} \in H^0(M_1, \mathcal{I}^1)$,
- $\mathcal{A}^1 \subset \mathcal{O}^{k_1+1}_{M_1}$,
- $g^1_1 = (g^1_{1,0}, ..., g^1_{1,k_1}), ..., g^1_{k'_1} = (g^1_{k'_1,0}, ..., g^1_{k'_1,k_1}) \in H^0(M_1, \mathcal{A}^1)$,
- $\xi^1_1, ..., \xi^1_{s_1} \in \mathbf{R}[y^1_{i,j}]_{0\le i,j\le k_1}$,
- $l^1_{i_1,i_2,i_3}(x,y^0,y^1) = y^1_{i_1,i_2}h^1_{i_3}(x,y^0) - y^1_{i_3.i_1}h^1_{i_2}(x,y^0)$ for $(x,y^0,y^1) \in M_1 \times \mathbf{R}^{(k_1+1)^2}$ and for $i_1, i_2, i_3 = 0, ..., k_1$,
- C^ω functions $a^1_{i,j}$ on M_1 for $i = 1, ..., n,\ j = 0, ..., k_1$

so that

$$\sum_{j=0}^{k_1} g^1_{i,j}(x,y^0)h^1_j(x,y^0) = 0 \quad \text{on } M_1 \text{ for } i = 1, ..., k'_1, \tag{1}$$

$$v_i f(x) = \sum_{j=0}^{k_1} a^1_{i.j}(x,y^0)h^1_j(x,y^0) \quad \text{on } M_1 \text{ for } i = 1, ..., n, \tag{2}$$

$$M_2 = \{(x,y^0,y^1) \in M_1 \times \mathbf{R}^{(k_1+1)^2} : l^1_{i_1,i_2,i_3}(x,y^0,y^1) = 0,\ i_1,i_2,i_3 = 0,\dots,k_1,$$

$$\sum_{j=0}^{k_1} y^1_{j,i_1}g^1_{i_2,j}(x,y^0) = 0,\ i_1 = 0, ..., k_1,\ i_2 = 1, ..., k'_1, \text{ and } \xi^1_i(y^1) = 0,\ i = 1, ..., s_1\},$$

where $I(M_2)$ in $C^\omega(M_1 \times \mathbf{R}^{(k_1+1)^2})$ is generated by those functions, denoted by $\{\alpha^2_i\}$, in the last braces, and π_2 is the restriction to M_2 of the projection $M_1 \times \mathbf{R}^{(k_1+1)^2} \to M_1$.

Here we require as another prescription of blowings-up that C_1 is normal crossing with $\pi_1^{-1}(C_0)$. For each $(x,y^0) \in C_1 \cap \pi^{-1}(C_0)$ there are two possible cases to consider: C_1 is transversal to $\pi_1^{-1}(C_0)$ at (x,y^0) or $C_{1(x,y^0)} \subset \pi_1^{-1}(C_0)_{(x,y^0)}$. Divide, if necessary, C_1 into two unions of its connected components so that on each union, only one case arises, and regard $\pi_1 : M_1 \to M_0$ as the composition of two blowings-up along centers one union and the inverse image of the other. Then we can assume that globally C_1 is transversal to $\pi_1^{-1}(C_0)$ or $C_1 \subset \pi_1^{-1}(C_0)$. The latter case occurs if and only if there exist C^ω functions $b^1_{j_0,j_1}$ on M_1, $j_0 = 0, ..., k_0,\ j_1 = 0, ..., k_1$, such that

$$h^0_{j_0}(x) = \sum_{j_1=0}^{k_1} b^1_{j_0,j_1}(x,y^0)h^1_{j_1}(x,y^0) \quad \text{on } M_1 \text{ for } j_0 = 0, ..., k_0. \tag{3}$$

We extend $h^1_j, g^1_{i,j}$, $a^1_{i,j}$ and $b^1_{j_0,j_1}$ (if exist) to analytic functions on $M_0 \times \mathbf{R}^{(k_0+1)^2}$. We use the same notation for the extensions because no confusion can arise. Then (1), (2) and (3) become

$$\sum_j g^1_{i,j}h^1_j = \sum_{i_1} \beta^1_{1,i_1,i}\alpha^1_{i_1} \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2}, \tag{1}$$

$$v_i f = \sum_j a^1_{i.j}h^1_j + \sum_{i_2} \beta^1_{2,i_2,i}\alpha^1_{i_2} \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2}, \tag{2}$$

$$h^0_{j_0} = \sum_{j_1} b^1_{j_0,j_1} h^1_{j_1} + \sum_i \beta^1_{3,i,j_0} \alpha^1_i \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \tag{3}$$

for some C^ω functions $\beta^i_{i_1,i_2,i_3}$ on $M_0 \times \mathbf{R}^{(k_0+1)^2}$. Now $Z(h^1_j) \cap M_1 = C_1$, and $I(M_2)$ in $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2})$ is generated by α^1_i and α^2_i, which we naturally regard as C^ω functions on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$.

For the second blowing-up, we consider again C^ω approximations $\tilde{h}^1_0, ..., \tilde{h}^1_{k_1}$, $\tilde{g}^1_1 = (\tilde{g}^1_{1,0}, ..., \tilde{g}^1_{1,k_1}), ..., \tilde{g}^1_{k'_1} = (\tilde{g}^1_{k'_1,0}, ..., \tilde{g}^1_{k'_1,k_1})$, $\tilde{a}^1_{i,j}$, $i = 1, ..., n$, $j = 0, ..., k_1$, $\tilde{b}^1_{j_0,j_1}$, $j_0 = 0, ..., k_0$, $j_1 = 0, ..., k_1$ and $\tilde{\beta}^1_{i_1,i_2,i_3}$ of $h^1_0, ..., h^1_{k_1}$, $g^1_1 = (g^1_{1,0}, ..., g^1_{1,k_1}), ..., g^1_{k'_1} = (g^1_{k'_1,0}, ..., g^1_{k'_1,k_1})$, $a^1_{i,j}$, $b^1_{j_0,j_1}$ (if exist) and $\beta^1_{i_1,i_2,i_3}$ on $M_0 \times \mathbf{R}^{(k_0+1)^2}$ in the strong Whitney C^∞ topology such that

($\tilde{1}$) $\sum_j \tilde{g}^1_{i,j} \tilde{h}^1_j = \sum_{i_1} \tilde{\beta}^1_{1,i_1,i} \tilde{\alpha}^1_{i_1}$ on $M_0 \times \mathbf{R}^{(k_0+1)^2}$ for $i = 1, ..., k'_1$,
($\tilde{2}$) $v_i f = \sum_j \tilde{a}^1_{i,j} \tilde{h}^1_j + \sum_{i_2} \tilde{\beta}^1_{2,i_2,i} \tilde{\alpha}^1_{i_2}$ on $M_0 \times \mathbf{R}^{(k_0+1)^2}$ for $i = 1, ..., n$,
($\tilde{3}$) $\tilde{h}^0_{j_0} = \sum_{j_1} \tilde{b}^1_{j_0,j_1} \tilde{h}^1_{j_1} + \sum_i \tilde{\beta}^1_{3,i,j_0} \tilde{\alpha}^1_i$ on $M_0 \times \mathbf{R}^{(k_0+1)^2}$ for $j_0 = 0, ..., k_0$.

Then $\tilde{C}_1 = Z(\tilde{h}^1_j) \cap \tilde{M}_1$ is smooth and of the same dimension as C_1. If $C_1 \subset \pi_1^{-1}(C_0)$, then $\tilde{C}_1$ is contained in $\pi_1^{-1}(\tilde{C}_0)$ by (3) and ($\tilde{3}$). If $C_1 \not\subset \pi_1^{-1}(C_0)$, i.e. if C_1 is transversal to $\pi_1^{-1}(C_0)$ in M_1, $\tilde{C}_1$ is transversal to $\tilde{\pi}_1^{-1}(\tilde{C}_0)$ in $\tilde{M}_1$ because the above diffeomorphism $\tilde{\psi}_0 : M_1 \to \tilde{M}_1$ is close to id in the strong Whitney C^∞ topology and carries $\pi_1^{-1}(C_0)$ to $\tilde{\pi}_1^{-1}(\tilde{C}_0)$ and because $\tilde{C}_1$ is close to C_1 in the strong Whitney C^∞ topology. Hence, in any case $\tilde{C}_1$ is normal crossing with $\tilde{\pi}_1^{-1}(\tilde{C}_0)$. It also follows from ($\tilde{2}$) that $\tilde{C}_1 \subset \operatorname{Sing} f = \tilde{\pi}_1^{-1}(\operatorname{Sing} f)$. Thus we can take the blowing-up $\tilde{\pi}_2 : \tilde{M}_2 \to \tilde{M}_1$ of $\tilde{M}_1$ along center $\tilde{C}_1$, and imbed $\tilde{M}_2$ by $\tilde{h}^1_0, ..., \tilde{h}^1_{k_1}$ into $\tilde{M}_1 \times \mathbf{P}(k_1) \subset \tilde{M}_1 \times \mathbf{R}^{(k_1+1)^2} \subset M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ so that $\tilde{\pi}_2$ is the restriction to $\tilde{M}_2$ of the projection $\tilde{M}_1 \times \mathbf{R}^{(k_1+1)^2} \to \tilde{M}_1$. Then there exist analytic diffeomorphisms $\psi_1 : M_1 \to \tilde{M}_1$ and $\tilde{\psi}_1 : M_2 \to \tilde{M}_2$ close to id in the strong Whitney C^∞ topology (ψ_1 is not necessarily equal to $\tilde{\psi}_0$) such that $\psi_1(C_1) = \tilde{C}_1$ and $\tilde{\pi}_2 \circ \tilde{\psi}_1 = \psi_1 \circ \pi_2$; $f \circ \tilde{\pi}_1 \circ \tilde{\pi}_2 : \tilde{M}_2 \to \mathbf{R}$ is close to $f \circ \pi_1 \circ \pi_2 : M_2 \to \mathbf{R}$ in the strong Whitney C^∞ topology; $I(\tilde{M}_2)$ is generated in $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2})$ by $\tilde{l}_{i_1,i_2,i_3}(x, y^0, y^1) = y^1_{i_1,i_2} \tilde{h}^1_{i_3}(x, y^0) - y^1_{i_3,i_1} \tilde{h}^1_{i_2}(x, y^0)$, $\sum_{j=0}^{k_1} y^1_{j,i_1} \tilde{g}^1_{i_2,j}(x, y^0)$, $\xi^1_i(y^1)$ and $\tilde{\alpha}^1_i$. Let $\tilde{\alpha}^2_i$ denote the former generators, and let $\tilde{\alpha}^1_i$ be naturally extended to $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$.

Note that there exists a C^ω diffeomorphism from M_2 to $\tilde{M}_2$ close to id in the strong Whitney C^∞ topology and carrying $\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$ to $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$ for the following reason. First by lemma 3.6 in [5], we have a C^ω diffeomorphism from M_1 to $\tilde{M}_1$ close to id in the strong Whitney C^∞ topology and carrying $C_1 \cup \pi_1^{-1}(C_0)$ to $\tilde{C}_1 \cup \tilde{\pi}_1^{-1}(\tilde{C}_0)$. Hence we can assume that $C_1 \cup \pi_1^{-1}(C_0) = \tilde{C}_1 \cup \tilde{\pi}_1^{-1}(\tilde{C}_0)$. Then in the same way as in the proof of lemma 2.2 we construct a C^ω diffeomorphism $\eta : M_2 \to \tilde{M}_2$ close to id in the strong Whitney C^∞ topology such that $\tilde{\pi}_2 \circ \eta = \pi_2$ and hence $\eta(\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)) = \eta(\pi_2^{-1}(C_1 \cup \pi_1^{-1}(C_0))) = \tilde{\pi}_2^{-1}(C_1 \cup \pi_1^{-1}(C_0)) = \tilde{\pi}_2^{-1}(\tilde{C}) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$.

We repeat inductively the same arguments on each blowing-up. Then the condition (3) becomes a little complicated because the union of the inverse images of the previous centers is not necessarily smooth. Let us consider the center C_2 of the blowing-up $\pi_3 : M_3 \to M_2$. We describe the condition that C_2 is normal crossing with $A = \pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$ as follows. Let h^2_j, $g^2_j = (g^2_{j,0}, ..., g^2_{j,k_2})$, ξ^2_i, $a^2_{i,j}$, $\beta^2_{i_1,i_2,i_3}$ on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ and their C^ω approximations $\tilde{h}^2_j$, $\tilde{g}^2_j = (\tilde{g}^2_{j,0}, ..., \tilde{g}^2_{j,k_2})$, $\tilde{\xi}^2_i$, $\tilde{a}^2_{i,j}$, $\tilde{\beta}^2_{i_1,i_2,i_3}$ in the strong Whitney C^∞ topology be given for for the center C_2 so that the corresponding equalities to (1), ($\tilde{1}$), (2) and ($\tilde{2}$) hold. Set $Y = \overline{A - \pi_2^{-1}(C_1)}$. Then Y is a smooth analytic set of codimension

1 in M_2, if $C_1 \not\subset \pi_1^{-1}(C_0)$ then $Y = A$, if $C_1 \subset \pi_1^{-1}(C_0)$ then $\pi_2^{-1}(C_1) \subset (\pi_1 \circ \pi_2)^{-1}(C_0)$ and $Y \cup \pi_2^{-1}(C_1)$ is a decomposition of $(\pi_1 \circ \pi_2)^{-1}(C_0)$ to smooth analytic sets, it follows from the normal crossing property of C_2 with A that C_2 is normal crossing with $\pi_2^{-1}(C_1)$ and with Y (the converse is not necessarily correct), and $I(Y)$ in $C^\omega(M_2)$ coincides with

$$\{f \in C^\omega(M_2) : fh_i^1 = \sum_{j=0}^{k_0} c_{i,j} h_j^0 \text{ for some } c_{i,j} \in C^\omega(M_2),\ i = 0, ..., k_1\}.$$

Let $\chi_j^{0,2},\ j = 1, ..., t_2$, be generators of $I(Y)$. Then there exist C^ω functions $c_{j_0,j_1,j}^{0,2}$ on $M_2,\ j_0 = 0, ..., k_0,\ j_1 = 1, ..., k_1,\ j = 1, ..., t_2$, such that

$$\chi_j^{0,2}(x, y^0, y^1) h_{j_1}^1(x, y^0) = \sum_{j_0=0}^{k_0} c_{j_0,j_1,j}^{0,2}(x, y^0, y^1) h_{j_0}^0(x) \quad \text{on } M_2, \tag{4}$$

and as in the case of the second blowing-up, dividing C_2 we can assume that C_2 is transversal to $\pi_2^{-1}(C_1)$ or contained in $\pi_2^{-1}(C_1)$ and that C_2 is transversal to Y or contained in Y. If $C_2 \subset \pi_2^{-1}(C_1)$ then there exist C^ω functions b_{j_1,j_2}^2 on $M_2,\ j_1 = 0, ..., k_1,\ j_2 = 0, ..., k_2$, such that

$$h_{j_1}^1(x, y^0) = \sum_{j_2=0}^{k_2} b_{j_1,j_2}^2(x, y^0, y^1) h_{j_2}^2(x, y^0, y^1) \quad \text{on } M_2, \tag{3}$$

and if $C_2 \subset Y$ then there exist C^ω functions $d_{j,j_2}^{0,2}$ on $M_2,\ j = 1, ..., t_2,\ j_2 = 0, ..., k_2$, such that

$$\chi_j^{0,2}(x, y^0, y^1) = \sum_{j_2=0}^{k_2} d_{j,j_2}^{0,2}(x, y^0, y^1) h_{j_2}^2(x, y^0, y^1) \quad \text{on } M_2. \tag{5}$$

As before we assume that $\chi_j^{0,2}$, b_{j_1,j_2}^2, $c_{j_1,j_2}^{0,2}$, $d_{j,j_2}^{0,2}$ are defined on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$. Then there exist C^ω functions γ_{i_1,i_2,i_3}^i and $\gamma_{i_1,i_2,i_3,i_4}^i$ on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ such that

$$h_{j_1}^1 = \sum_{j_2} b_{j_1,j_2}^2 h_{j_2}^2 + \sum_i \gamma_{1,i,j_1}^1 \alpha_i^1 + \sum_i \gamma_{1,i,j_1}^2 \alpha_i^2, \tag{3}$$

$$\chi_j^{0,2} h_{j_1}^1 = \sum_{j_0} c_{j_0,j_1,j}^{0,2} h_{j_0}^0 + \sum_i \gamma_{2,i,j,j_1}^1 \alpha_i^1 + \sum_i \gamma_{2,i,j,j_1}^2 \alpha_i^2, \tag{4}$$

$$\chi_j^{0,2} = \sum_{j_2} d_{j,j_2}^{0,2} h_{j_2}^2 + \sum_i \gamma_{3,i,j}^1 \alpha_i^1 + \sum_i \gamma_{3,i,j}^2 \alpha_i^2 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}. \tag{5}$$

We need to consider also C^ω approximations $\tilde\chi_j^{0,2}$, $\tilde b_{j_1,j_2}^2$, $\tilde c_{j_1,j_2,j}^{0,2}$, $\tilde d_{j,j_2}^{0,2}$, $\tilde\gamma_{i_1,i_2,i_3}^i$ and $\tilde\gamma_{i_1,i_2,i_3,i_4}^i$ of $\chi_j^{0,2}$, b_{j_1,j_2}^2 (if exist), $c_{j_1,j_2,j}^{0,2}$, $d_{j,j_2}^{0,2}$ (if exist), γ_{i_1,i_2,i_3}^i (if exist) and $\gamma_{i_1,i_2,i_3,i_4}^i$ on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ in the strong Whitney C^∞ topology such that

($\tilde 3$) $\tilde h_{j_1}^1 = \sum_{j_2=0}^{k_2} \tilde b_{j_1,j_2}^2 \tilde h_{j_2}^2 + \sum_i \tilde\gamma_{1,i,j_1}^1 \tilde\alpha_i^1 + \sum_i \tilde\gamma_{1,i,j_1}^2 \tilde\alpha_i^2$,

($\tilde 4$) $\tilde\chi_j^{0,2} \tilde h_{j_1}^1 = \sum_{j_0} \tilde c_{j_0,j_1,j}^{0,2} \tilde h_{j_0}^0 + \sum_i \tilde\gamma_{2,i,j,j_1}^1 \tilde\alpha_i^1 + \sum_i \tilde\gamma_{2,i,j,j_1}^2 \tilde\alpha_i^2$,

($\tilde 5$) $\tilde\chi_j^{0,2} = \sum_{j_2} \tilde d_{j,j_2}^{0,2} \tilde h_{j_2}^2 + \sum_i \tilde\gamma_{3,i,j}^1 \tilde\alpha_i^1 + \sum_i \tilde\gamma_{3,i,j}^2 \tilde\alpha_i^2$ on $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$.

Set $\tilde Y = Z(\tilde\chi_j) \cap \tilde M_2$. Then

$$\tilde Y = \overline{(\tilde\pi_1 \circ \tilde\pi_2)^{-1}(\tilde C_0) - \tilde\pi_2^{-1}(\tilde C_1)}$$

because $\tilde Y$ contains the right hand side by ($\tilde 4$) and because the converse inclusion does from the facts that Y and the right hand side are smooth and of codimension 1 in M_2 and

in $\tilde{M}_2$, respectively, and that χ_j are generators of $I(Y)$ in $C^\omega(M_2)$. Hence $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \pi_2)^{-1}(\tilde{C}_0)$, which is normal crossing, is the union of the smooth analytic sets $\tilde{\pi}_3^{-1}(\tilde{C}_1)$ and $\tilde{Y}$. Moreover, $\tilde{C}_2$ is normal crossing with $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \pi_2)^{-1}(\tilde{C}_0)$ for the following four reasons.

If C_2 is transversal to $\pi_2^{-1}(C_1)$ or to Y, so is $\tilde{C}_2$ to $\tilde{\pi}_2^{-1}(\tilde{C}_1)$ or to $\tilde{Y}$, respectively, by the same reason as before. If $C_2 \subset \pi_2^{-1}(C_1)$, then there exist C^ω $b^2_{j_1,j_2}$ with (3) on M_2, hence $\tilde{h}^1_{j_1} = \sum_{j_2=0}^{k_2} \tilde{b}^2_{j_1,j_2} \tilde{h}^2_{j_2}$ on $\tilde{M}_2$ and $\tilde{C}_2 \subset \tilde{\pi}_2^{-1}(\tilde{C}_1)$. In the same way we see that if $C_2 \subset Y$ then $\tilde{C}_2 \subset \tilde{Y}$. The fourth reason is that C_2 is normal crossing with $\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$.

By these four properties we can find also a C^ω diffeomorphism from M_2 to $\tilde{M}_2$ close to id in the strong Whitney C^∞ topology and carrying C_2, $\pi_2^{-1}(C_1)$ and $(\pi_1 \circ \pi_2)^{-1}(C_0)$ to $\tilde{C}_2$, $\tilde{\pi}_2^{-1}(\tilde{C}_1)$ and $(\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$, respectively.

Let $1 < m' (< m) \in \mathbf{N}$. As above we inductively imbed $M_{m'}$ into $M_{m'-1} \times \mathbf{R}^{(k_{m'-1}+1)^2} (\subset M_0 \times \mathbf{R}^{(k_0+1)^2} \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2})$ and obtain a finite number of C^ω functions on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$, namely $h^{m'}_j$, $a^{m'}_{i,j}$, $\xi^{m'}_i$, $\chi^{m'',m'}_j$, $c^{m'',m'}_{j_0,j_1,j}$, $d^{m'',m'}_{j,j_1}$, $\alpha^{m'}_i$, $\beta^{m'',m'}_{i_1,i_2,i_3}$, $\beta^{m''',m'',m'}_{i_1,i_2,i_3,i_4}$, $\beta^{m''',m'',m'}_{i_1,i_2,i_3}$ for $m'' (< m'), m''' (\leq m') \in \mathbf{N}$ and a finite number of C^ω maps from $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ to $\mathbf{R}^{(k_{m'}+1)^2}$, namely $g^{m'}_j = (g^{m'}_{j,0}, ..., g^{m'}_{j,k_{m'}})$ such that the following conditions are satisfied:

- the blowing-up $\pi_{m'} : M_{m'} \to M_{m'-1}$ is the restriction to $M_{m'}$ of the projection $M_{m'-1} \times \mathbf{R}^{(k_{m'-1}+1)^2} \to M_{m'}$;
- $\{h^{m'}_j : j\}$ are generators of $I(C_{m'})$ in $C^\omega(M_{m'})$;
- $\{\xi^{m'}_i(y^{m'}) : i\}$ are generators of $I(\mathbf{P}(k_{m'}))$ in $\mathbf{R}[y^{m'}_{i,j}]_{0 \leq i,j \leq k_{m'}}$ $(\subset C^\omega(\mathbf{R}^{(k_{m'}+1)^2}))$;
- $\{g^{m'}_j : j\}$ are generators of the sheaf of relations of $h^{m'}_0, ..., h^{m'}_{k_{m'}}$ on $M_{m'}$;
- $\{\chi^{m'',m'}_j : j\}$ are generators of $I(Y_{m'',m'})$ in $C^\omega(M_{m'})$, where $Y_{m'-1,m'} = \pi^{-1}_{m'}(C_{m'-1})$ and

$$Y_{m'',m'} = \overline{(\pi_{m''+1} \circ \cdots \circ \pi_{m'})^{-1}(C_{m''}) - (\pi_{m''+2} \circ \cdots \circ \pi_{m'})^{-1}(C_{m''+1}) - \cdots - \pi^{-1}_{m'}(C_{m'-1})}$$

  for $m'' < m' - 1$; $\{\alpha^{m'}_i : i\} = \{y^{m'-1}_{i_1,i_2} h^{m'-1}_{i_3} - y^{m'-1}_{i_3,i_1} h^{m'-1}_{i_2}, \sum_j y^{m'-1}_{j,i_1} g^{m'-1}_{i_2,j}, \xi^{m'-1}_i : i_1, i_2, i_3, i\}$;
- $\{\alpha^{m'''}_i : m''' \leq m', i\}$ are generators of $I(M_{m'})$ in $C^\omega(M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2})$, where we naturally regard $h^{m'-1}_i$, $g^{m'-1}_{i,j}$, $\xi^{m'-1}_i$ and $\alpha^{m'''}_i$ as functions on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1})^2}$;
- (1) $\sum_j g^{m'}_{i,j} h^{m'}_j = \sum_{\substack{m''' \leq m' \\ i_1}} \beta^{m''',m'}_{1,i_1,i} \alpha^{m'''}_{i_1}$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$;
- (2) $v_i f = \sum_j a^{m'}_{i,j} h^{m'}_j + \sum_{\substack{m''' \leq m' \\ i_2}} \beta^{m''',m'}_{2,i_2,i} \alpha^{m'''}_{i_2}$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$;
- (4) $\chi^{m'',m'}_j h^{m'-1}_{j_1} = \sum_{j_0} c^{m'',m'}_{j_0,j_1,j} \chi^{m'',m'-1}_{j_0} + \sum_{\substack{m''' \leq m' \\ i}} \beta^{m''',m'',m'}_{4,i,j,j_1} \alpha^{m'''}_i$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ for $m'' < m' - 1$;
- (5) $\chi^{m'',m'}_j = \sum_{j_1} d^{m'',m'}_{j,j_1} h^{m'}_{j_1} + \sum_{\substack{m''' \leq m' \\ i}} \beta^{m''',m'',m'}_{5,i,j} \alpha^{m'''}_i$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ for $m'' < m' - 1$ if $C_{m'} \subset Y_{m'',m'}$;
- (6) $\chi^{m'-1,m'}_j = h^{m'-1}_j$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$.

(The condition (3) is included in (5) and (6).) Here $d^{m'',m'}_{j,j_1}$ and $\beta^{m''',m'',m'}_{5,i,j}$ exist only if $C_{m'} \subset Y_{m'',m'}$ and we assume that if $C_{m'} \not\subset Y_{m'',m'}$ then $C_{m'}$ is transversal to $Y_{m'',m'}$ in $M_{m'}$. Note that $\cup_{m''<m'} Y_{m'',m'}$ is a decomposition of $\pi^{-1}_{m'}(C_{m'-1}) \cup \cdots \cup (\pi_1 \circ \cdots \circ \pi_{m'})^{-1}(C_0)$ into smooth analytic sets.

Assume that there exist inductively also a blowing-up $M_0\times\cdots\times\mathbf{R}^{(k_{m'-1}+1)^2}\supset\tilde{M}_{m'}\xrightarrow{\tilde{\pi}_{m'}}\tilde{M}_{m'-1}$ along center $\tilde{C}_{m'-1}$ close to $M_0\times\cdots\times\mathbf{R}^{(k_{m'-1}+1)^2}\supset M_{m'}\xrightarrow{\pi_{m'}}M_{m'-1}$ in the strong Whitney C^∞ topology and C^ω approximations

$$\tilde{h}_j^{m'},\ \tilde{a}_{i,j}^{m'},\tilde{\chi}_j^{m'',m'},\ \tilde{c}_{j_0,j_1,j}^{m'',m'},\ \tilde{d}_{j,j_1}^{m'',m'},\ \tilde{\alpha}_i^{m'},\tilde{\beta}_{i_1,i_2,i_3}^{m'',m'},\tilde{\beta}_{i_1,i_2,i_3,i_4}^{m''',m'',m'},\tilde{\beta}_{i_1,i_2,i_3}^{m''',m'',m'},\tilde{g}_j^{m'}=(\tilde{g}_{j,0}^{m'},...,\tilde{g}_{j,k_{m'}}^{m'})$$

of

$$h_j^{m'},\ a_{i,j}^{m'},\ \chi_j^{m'',m'},\ c_{j_0,j_1,j}^{m'',m'},\ d_{j,j_1}^{m'',m'},\ \alpha_i^{m'},\beta_{i_1,i_2,i_3}^{m'',m'},\beta_{i_1,i_2,i_3,i_4}^{m''',m'',m'},\beta_{i_1,i_2,i_3}^{m''',m'',m'},g_j^{m'}=(g_{j,0}^{m'},...,g_{j,k_{m'}}^{m'})$$

on $M_0\times\cdots\times\mathbf{R}^{(k_{m'-1}+1)^2}$ in the strong Whitney C^∞ topology such that $\tilde{\pi}_{m'}$ is the restriction to $\tilde{M}_{m'}$ of the projection $M_0\times\cdots\times\mathbf{R}^{(k_{m'-1}+1)^2}\to M_0\times\cdots\times\mathbf{R}^{(k_{m'-2}+1)^2}$,

$$\{\tilde{\alpha}_i^{m'}:i\}=\{y_{i_1,i_2}^{m'-1}\tilde{h}_{i_3}^{m'-1}-y_{i_3,i_1}^{m'-1}\tilde{h}_{i_2}^{m'-1},\ \sum_j y_{j,i_1}^{m'-1}\tilde{g}_{i_2,j}^{m'-1},\ \xi_i^{m'-1}:i_1,i_2,i_3,i\}$$

and the corresponding conditions $(\tilde{1})$, $(\tilde{2})$, $(\tilde{4})$, $(\tilde{5})$ (if $C_{m'}\subset Y_{m'',m'}$) and $(\tilde{6})$ to (1), (2), (4), (5) (if $C_{m'}\subset Y_{m'',m'}$) and (6) are satisfied. Set $\tilde{C}_{m'}=Z(\tilde{h}_j^{m'})\cap\tilde{M}_{m'}$, $\tilde{Y}_{m'-1,m'}=\tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1})$ and

$$\tilde{Y}_{m'',m'}=\overline{(\tilde{\pi}_{m'-1}\circ\cdots\circ\tilde{\pi}_{m'})^{-1}(\tilde{C}_{m''})-\tilde{(}\pi_{m''}\circ\cdots\circ\tilde{\pi}_{m'})^{-1}(\tilde{C}_{m''+1})-\cdots-\tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1})}$$

for $m''<m'-1$. Then, as before:

- $I(\tilde{M}_{m'})$ in $C^\omega(M_0\times\cdots\times\mathbf{R}^{(k_{m'-1}+1)^2})$ is generated by $\{\tilde{\alpha}_i^{m''}:m''\le m',\,i\}$;
- $\tilde{C}_{m'}$ is smooth and of the same dimension as $C_{m'}$;
- $I(\tilde{C}_{m'})$ in $C^\omega(\tilde{M}_{m'})$ is generated by $\{\tilde{h}_j^{m'}:j\}$;
- $\{\tilde{g}_j^{m'}:j\}$ are generators of the sheaf of relations of $\tilde{h}_0^{m'},...,\tilde{h}_{k_{m'}}^{m'}$ on $\tilde{M}_{m'}$;
- $I(\tilde{Y}_{m'',m'})$ in $C^\omega(\tilde{M}_{m'})$ for each $m''<m'$ is generated by $\{\tilde{\chi}_j^{m'',m'}:j\}$ by $(\tilde{4})$;
- $\tilde{C}_{m'}\subset\tilde{Y}_{m'',m'}$ if and only if $C_{m'}\subset Y_{m'',m'}$;
- if $C_{m'}\not\subset Y_{m'',m'}$ then $\tilde{C}_{m'}$ is transversal to $\tilde{Y}_{m'',m'}$ in $\tilde{M}_{m'}$;
- $\cup_{m''<m'}\tilde{Y}_{m'',m'}$ is a decomposition of $\tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1})\cup\cdots\cup(\tilde{\pi}_1\circ\cdots\circ\tilde{\pi}_{m'})^{-1}(\tilde{C}_0)$ into smooth analytic sets;
- $\tilde{C}_{m'}$ is normal crossing with this set;
- there exists a C^ω diffeomorphism from $M_{m'}$ to $\tilde{M}_{m'}$ close to id in the strong Whitney C^∞ topology and carrying $C_{m'},...,(\pi_1\circ\cdots\circ\pi_{m'})^{-1}(C_0)$ to $\tilde{C}_{m'},...,(\tilde{\pi}_1\circ\cdots\circ\tilde{\pi}_{m'})^{-1}(\tilde{C}_0)$, respectively;
- $f\circ\tilde{\pi}_1\circ\cdots\circ\tilde{\pi}_{m'}$ is close to $f\circ\pi_1\circ\cdots\circ\pi_{m'}$ in the strong Whitney C^∞ topology.

Finally, as above, we imbed M_m and $\tilde{M}_m$ into $M_0\times\mathbf{R}^{(k_0+1)^2}\times\cdots\times\mathbf{R}^{(k_{m-1}+1)^2}$ by $h_0^{m-1},...,h_{k_{m-1}}^{m-1}$ and $\tilde{h}_0^{m-1},...,\tilde{h}_{k_{m-1}}^{m-1}$, respectively, define $\alpha_i^m,\tilde{\alpha}_i^m,Y_{m',m}$ and $\tilde{Y}_{m',m}$ for $0\le m'<m$, and let $\{\chi_j^{m',m}:j\}$ and $\{\tilde{\chi}_j^{m',m}:j\}$ be finitely many C^ω functions on $M_0\times\cdots\times\mathbf{R}^{(k_{m-1}+1)^2}$ which are generators of $I(Y_{m',m})$ in $C^\omega(M_m)$ and of $I(\tilde{Y}_{m',m})$ in $C^\omega(\tilde{M}_m)$, respectively, for each $m'<m$ such that each $\tilde{\chi}_j^{m',m}$ is close to $\chi_j^{m',m}$ in the strong Whitney C^∞ topology. Then there exists a C^ω diffeomorphism $\psi_m:M_m\to\tilde{M}_m$ close to id in the strong Whitney C^∞ topology carrying $\pi_m^{-1}(C_{m-1}),...,(\pi_1\circ\cdots\circ\pi_m)^{-1}(C_0)$ to $\tilde{\pi}_m^{-1}(\tilde{C}_{m-1}),...,(\tilde{\pi}_1\circ\cdots\circ\tilde{\pi}_m)^{-1}(\tilde{C}_0)$, respectively. Set $F=f\circ\pi_1\circ\cdots\circ\pi_m$ and $\tilde{F}=f\circ\tilde{\pi}_1\circ\cdots\circ\tilde{\pi}_m$. Then F has only normal crossing singularities. We require ψ_m to carry, moreover, $\operatorname{Sing}F$ to $\operatorname{Sing}\tilde{F}$. That is possible if $\tilde{F}$ has only normal crossing singularities by the same reason as before.

We will describe a condition for $\tilde{F}$ to have only normal crossing singularities. As the problem in the theorem is local around the compact subset X of M, we assume that M_m is covered by a finite number of good open subsets in the following sense. We have the

disjoint union B of finitely many closed balls B_i in the Euclidean space of same dimension as M, a C^ω immersion $\rho = (\rho_{-1}, ..., \rho_{m-1}) : B \to M_0 \times \mathbf{R}^{(k_0+1)^2} \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$, finitely many C^ω functions $\delta_{i,j}$ on each B_i regular at $\delta_{i,j}^{-1}(0)$ and $\theta_{i,j} > 0 \in \mathbf{N}$ such that $\Im\rho \subset M_m$, $\rho(\mathrm{Int}\, B) \supset X$, for each i $\rho|_{B_i}$ is an embedding, $F \circ \rho|_{\mathrm{Int}\, B_i}$ has only normal crossing singularities with unique singular value z_{0i}, and

$$F \circ \rho|_{B_i} = \prod_j \delta_{i,j}^{\theta_{i,j}} + z_{0i}.$$

Here the condition $\Im\rho \subset M_m$ and the last condition are equivalent to

$$f \circ \rho_{-1}|_{B_i} = \prod_j \delta_{i,j}^{\theta_{i,j}} + z_{0i} \tag{7}$$

and there exist C^ω functions $\nu_{i,j}^{m'}$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$ such that for each $\alpha_i^{m'}$ with $m' \leq m$

$$\alpha_i^{m'}(x, y^0, ..., y^{m'-1}) = \nu_{i,-1}^{m'}(x, y^0, ..., y^{m-1}, z)(x - \rho_{-1}(z)) + \tag{8}$$

$$\sum_{j=0}^{m-1} \nu_{i,j}^{m'}(x, y^0, ..., y^{m-1}, z)(y^j - \rho_j(z)) \text{ on } M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$$

because $x - \rho_{-1}(z)$, $y^j - \rho_j(z)$, $j = 0, ..., m-1$, generate the ideal of $C^\omega(M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B)$ defined by the graph of ρ—$\{(\rho(z), z) : z \in B\}$. Conversely, existence of such ρ, $\delta_{i,j}$, $\theta_{i,j}$ and $\nu_{i,j}^{m'}$ implies the normal crossing property of F. Note

$$\{z_{0i}\} = F(\mathrm{Sing}\, F|_{(\pi_1 \circ \cdots \circ \pi_m)^{-1}(U)}) = f(\mathrm{Sing}\, f|_U)$$

for an open neighborhood U of X in M. (Assume that $U = M$ for simplicity of notation.) Hence a condition for $\tilde{F}$ to have only normal crossing singularities is that there exist C^ω approximations $\tilde{\rho} = (\tilde{\rho}_{-1}, ..., \tilde{\rho}_{m-1}) : B \to M_0 \times \mathbf{R}^{(k_0+1)^2} \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$ of ρ, $\tilde{\delta}_{i,j}$ of $\delta_{i,j}$ and $\tilde{\nu}_{i,j}^{m'}$ of $\nu_{i,j}^{m'}$ in the strong Whitney C^∞ topology such that

$$f \circ \tilde{\rho}_{-1}|_{B_i} = \prod_j \tilde{\delta}_{i,j}^{\theta_{i,j}} + z_{0i}, \tag{$\tilde{7}$}$$

$$\tilde{\alpha}_i{}^{m'}(x, y^0, ..., y^{m'-1}) = \tilde{\nu}_{i,-1}^{m'}(x, y^0, ..., y^{m-1}, z)(x - \tilde{\rho}_{-1}(z)) + \tag{$\tilde{8}$}$$

$$\sum_{j=0}^{m-1} \tilde{\nu}_{i,j}^{m'}(x, y^0, ..., y^{m-1}, z)(y^j - \tilde{\rho}_j(z)) \quad \text{on } M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B.$$

However, we cannot find directly the approximations by proposition 3.1 below. Indeed, we need additional arguments as follows. Extend trivially ρ to $\rho = (\rho_{-1}, ..., \rho_{m-1}) : M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \to M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$ and $\delta_{i,j}$ to $\delta_{i,j} : M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i \to \mathbf{R}$. Then (7) holds on $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$. Approximate these extended ρ and $\delta_{i,j}$ by a C^ω map $\tilde{\rho} = (\tilde{\rho}_{-1}, ..., \tilde{\rho}_{m-1}) : M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \to M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$ and C^ω functions $\tilde{\delta}_{i,j}$ on $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$, respectively, so that ($\tilde{7}$) and ($\tilde{8}$) hold on $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$ and $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$, respectively. Regard locally M_0 as a Euclidean space, and consider the map $\tilde{P} : M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \ni (x, y^0, ..., y^{m-1}, z) \to (x - \tilde{\rho}_{-1}(x, ..., z), ..., y^{m-1} - \tilde{\rho}_{m-1}(x, ..., z)) \in M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$. As $\tilde{P}$ is close to the map : $M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \ni (x, y^0, ..., y^{m-1}, z) \to (x - \rho_{-1}(x, ..., z), ..., y^{m-1} - \rho_{m-1}(x, ..., z)) \in M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$, the Jacobian matrix

$\left|\frac{D(\tilde{P})}{D(x,...,y^{m-1})}\right|$ vanishes nowhere. Hence by the implicit function theorem we have an analytic map $\hat{\rho} = (\hat{\rho}_{-1}, ..., \hat{\rho}_{m-1}) : B \to M_0 \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$ such that $\tilde{\rho}(\hat{\rho}(z), z)) = \hat{\rho}(z)$ and $\hat{\rho}$ is close to ρ in the strong Whitney topology. Then $\hat{\rho}$ is a C^ω immersion,

$$(\hat{7}) \qquad f \circ \hat{\rho}_{-1}(z) = f \circ \tilde{\rho}_{-1}(\hat{\rho}(z), z) = \prod_j \tilde{\delta}_{i,j}^{\theta_{i,j}}(\hat{\rho}(z), z) + z_{0i} \quad \text{for } z \in B_i,$$

$$(\hat{8}) \qquad \tilde{\alpha}_i^{m'} \circ \hat{\rho}(z) = \tilde{\nu}_{i,-1}^{m'}(\hat{\rho}(z), z)(\hat{\rho}_{-1}(z) - \tilde{\rho}_{-1}(\hat{\rho}_{-1}(z), z)) + \sum_{j=0}^{m-1} \tilde{\nu}_{i,j}^{m'}(\hat{\rho}(z), z)(\hat{\rho}_j(z) - \tilde{\rho}_j(\hat{\rho}(z), z)) = 0 \quad \text{for } z \in B.$$

By $(\hat{8})$, $\Im\hat{\rho} \subset \tilde{M}_m$, hence $\rho(\operatorname{Int} B) \supset X$, and by $(\hat{7})$, $\tilde{F}$ has only normal crossing singularities because $\tilde{\delta}_{i,j}(\hat{\rho}(z), z)$ is close to $\delta_{i,j}(z)$ in the strong Whitney C^∞ topology. Note that if $\tilde{\rho}$ is of class Nash, so is $\hat{\rho}$.

Under the conditions $(\tilde{7})$ and $(\tilde{8})$, F and $\tilde{F}$ are C^ω right equivalent through a C^ω diffeomorphism close to id in the strong Whitney C^∞ topology for the following reason. Since F and $\tilde{F}$ have only normal crossing singularities, and since $f \circ \rho_{-1}$ and $f \circ \hat{\rho}_{-1}$ are C^ω right equivalent by (7) and $(\hat{7})$, we can modify ψ_m to carry $\operatorname{Sing} F$ to $\operatorname{Sing} \tilde{F}$ (cf. step 1 of the proof of theorem 3.1 in [5]). Replacing $\tilde{F}$ with $\tilde{F} \circ \psi_m$, we assume that $\tilde{M}_m = M_m$, $\tilde{\pi}_m^{-1}(\tilde{C}_{m-1}) = \pi_m^{-1}(C_{m-1}), ..., (\tilde{\pi}_1 \circ \cdots \circ \tilde{\pi}_m)^{-1}(\tilde{C}_0) = (\pi_1 \circ \cdots \circ \pi_m)^{-1}(C_0)$ and $\operatorname{Sing} F = \operatorname{Sing} \tilde{F}$. Let κ be a Nash function on $\mathbf{R}$ with zero set $\{z_{0i}\}$ and regular there. Then $\kappa \circ F$ and $\kappa \circ \tilde{F}$ satisfy the assumption of lemma 3.10 in [5]:

- they have the same sign at each point of M, only normal crossing singularities at $(\kappa \circ F)^{-1}(0) = F^{-1}(F(\operatorname{Sing} F))$ and the same multiplicity at each point of $(\kappa \circ F)^{-1}(0)$,
- the C^ω function on M_m, defined to be $\kappa \circ \tilde{F}/\kappa \circ F$ on $M_m - (\kappa \circ F)^{-1}(0)$, is close to 1 in the strong Whitney C^∞ topology. Indeed, the map $:C^\omega(M_m) \ni \phi \to \phi \cdot (\kappa \circ F) \in \kappa \circ FC^\omega(M_m)$ is open in the strong Whitney C^∞ topology (cf. remark in step 1 of the proof of lemma 3.1,(i) in [5]), $\kappa \circ \tilde{F}$ is contained in $\kappa \circ FC^\omega(M_m)$ and close to $\kappa \circ F$ and hence there exist $\phi \in C^\omega(M_m)$ close to 1 such that $\phi \cdot (\kappa \circ F) = \kappa \circ \tilde{F}$.

Therefore there exists a C^ω diffeomorphism ψ'_m of M_m close to id in the strong Whitney C^∞ topology such that $\psi'_m((\kappa \circ F)^{-1}(0)) = (\kappa \circ F)^{-1}(0)$ and $F - \tilde{F} \circ \psi'_m$ is r-flat at $(\kappa \circ F)^{-1}(0)$ for a large integer r. Then by lemma 3.11,(i) in [5], F and $\tilde{F}$ are C^ω right equivalent through a C^ω diffeomorphism close to id in the strong Whitney C^∞ topology.

Consider the case of germ on X. Enlarging X if necessary we assume that X is semialgebraic. Set $X_0 = X$. Let $h_j^{m'}$, $g_j^{m'}$, $a_{i,j}^{m'}$, , .., $\nu_{i,j}^{m'}$ be the same as above. Let $\tilde{h}_j^0$, $\tilde{g}_j^0$, $\tilde{a}_{i,j}^0$ be defined not on M_0 but on an open neighborhood U_0 of X_0 in M_0 close to h_j^0, g_j^0, $a_{i,j}^0$, respectively, at X_0 in the C^∞ topology so that $(\tilde{1})$ and $(\tilde{2})$ hold on U_0. Shrink U_0 if necessary. Then by remark 2.3.(2) of lemma 2.2 we have the blowing-up $U_0 \times \mathbf{R}^{(k_0+1)^2} \supset U_1 \xrightarrow{\tau_1} U_0$ along center $D_0 = Z(\tilde{h}_j^0)$ defined by $\tilde{h}_0^0, ..., \tilde{h}_{k_0}^0$ and analytic imbed dings ψ_0 of U_0 into M_0 and $\tilde{\psi}_0$ of U_1 into M_1 close to id at X_0 and at $\tau_1(X)$, respectively, such that $\psi_0(D_0) \subset C_0$ and $\psi_0 \circ \tau_1 = \pi_1 \circ \tilde{\psi}_0$.

Next let $\tilde{h}_j^1$, $\tilde{g}_j^1$, $\tilde{a}_{i,j}^1$, $\tilde{b}_{j_0,j_1}^1$, $\tilde{\beta}_{i_1,i_2,i_3}^1$ be defined on an open neighborhood of $X_0 \times X_1$ in $M_0 \times \mathbf{R}^{(k_0+1)^2}$ close to $h_j^1, ..., \beta_{i_1,i_2,i_3}^1$, respectively, at $X_0 \times X_1$ in the C^∞ topology such that $(\tilde{1})$, $(\tilde{2})$ and $(\tilde{3})$ hold on the neighborhood, where $\tilde{\alpha}_i^1$ are defined as in the global case and X_1 denotes a large ball in $\mathbf{R}^{(k_0+1)^2}$ with center 0 such that $\pi_1^{-1}(X_0)$ and $\tau_1^{-1}(X_0)$ are contained in $X_0 \times \operatorname{Int} X_1$. Shrink U_0 and the neighborhood of $X_0 \times X_1$ so that U_1 and $M_1 \cap U_0 \times \mathbf{R}^{(k_0+1)^2}$ are closed subsets of the neighborhood, which is possible because π_1

and τ_1 are proper. Then there exist the blowing-up $U_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2} \supset U_2 \xrightarrow{\tau_2} U_1$ along center $D_1 = Z(\tilde{h}_j^1) \cap U_1$ defined by $\tilde{h}_0^1, ..., \tilde{h}_{k_1}^1$ and analytic imbed dings ψ_1 of U_1 into M_1 and $\tilde{\psi}_1$ of U_2 into M_2 close to id at $\tau_1^{-1}(X_0)$ and at $(\tau_1 \circ \tau_2)^{-1}(X_0)$, respectively, such that $\psi_1(D_1) \subset C_1$ and $\psi_1 \circ \tau_2 = \pi_2 \circ \tilde{\psi}_1$.

Let $1 < m' < m$, $m'' < m'$ and $m''' \leq m'$. By induction, let $\tilde{h}_j^{m'}$, $\tilde{g}_j^{m'}$, $\tilde{a}_{i,j}^{m'}$, $\tilde{\chi}_j^{m'',m'}$, $\tilde{c}_{j_0,j_1,j}^{m'',m'}$, $\tilde{d}_{j,j_1}^{m'',m'}$, $\tilde{\beta}_{i_1,i_2,i_3}^{m'',m'}$, $\tilde{\beta}_{i_1,i_2,i_3,i_4}^{m''',m'',m'}$, $\tilde{\beta}_{i_1,i_2,i_3}^{m''',m'',m'}$ be defined on an open neighborhood of $X_0 \times \cdots \times X_{m'}$ in $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \cdots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ close to $h_j^{m'}$, $g_j^{m'}$, ..., respectively, at $X_0 \times \cdots \times X_{m'}$ in the C^∞ topology such that $(\tilde{1})$, $(\tilde{2})$, $(\tilde{4})$, $(\tilde{5})$, $(\tilde{6})$ hold on the neighborhood, where $\tilde{\alpha}_i^{m'}$ are given as in the global case and X_i denotes a large ball in $\mathbf{R}^{(k_{i-1}+1)^2}$ with center 0 for $i = 2, ..., m'$. For $m' (\leq m) \in \mathbf{N}$, let $\tilde{\alpha}_i^m$ and $\tilde{\chi}_j^{m',m}$ be defined on an open neighborhood of $X_0 \times \cdots \times X_m$ close to α_i^m and $\chi_j^{m',m}$, respectively, at $X_0 \times \cdots \times X_m$ as before, and $\tilde{\rho}_i$, $\tilde{\delta}_{i,j}$, $\tilde{\nu}_{i,j}^{m'}$ on an open neighborhood of $X_0 \times \cdots \times X_m \times B$ close to ρ_i, $\delta_{i,j}$, $\nu_{i,j}^{m'}$, respectively, at $X_0 \times \cdots \times X_m \times B$ so that $(\tilde{7})$ and $(\tilde{8})$ hold on the neighborhood.

Then we obtain a sequence of blowings-up $U_m \xrightarrow{\tau_m} U_{m-1} \longrightarrow \cdots \xrightarrow{\tau_1} U_0$ along smooth analytic centers $D_{m-1} = Z(\tilde{h}_j^{m-1}) \cap U_{m-1}$ in $U_{m-1}, ..., D_0 = Z(\tilde{h}_j^0)$ in U_0, respectively, and an analytic embedding $\psi : U_m \to M_m$ such that $\psi(\tau_m^{-1}(D_{m-1})) \subset \pi_m^{-1}(C_{m-1}), ..., \psi((\tau_1 \circ \cdots \circ \tau_m)^{-1}(D_0)) \subset (\pi_1 \circ \cdots \circ \pi_m)^{-1}(C_0)$, $f \circ \pi_1 \circ \cdots \circ \pi_m \circ \psi = f \circ \tau_1 \circ \cdots \circ \tau_m$, $U_1, ..., U_m$ are realized in $U_0 \times \mathbf{P}(k_0) \subset U_0 \times \mathbf{R}^{(k_0+1)^2}, ..., U_0 \times \mathbf{P}(k_0) \times \cdots \times \mathbf{P}(k_{m-1}) \subset U_0 \times \mathbf{R}^{(k_0+1)^2} \times \cdots \times \mathbf{R}^{(k_{m-1}+1)^2}$, respectively, and each pair $D_i \subset U_i$ and ψ are close to $C_i \subset M_i$ at $(\tau_1 \circ \cdots \circ \tau_{i-1})^{-1}(X_0)$ and to id at $(\tau_1 \circ \cdots \circ \tau_m)^{-1}(X_0)$, respectively, in the C^∞ topology.

Thus it remains only to find the approximations $\tilde{h}_j^0$, $\tilde{g}_j^0$, ... of class Nash. This is a consequence of proposition 3.1 below.

□

3. Nested Nash approximation

3.1. **Nash approximation of an analytic diffeomorphism.** In view of theorem 3.2 and the proof of theorem 1.4, we need to be able to make a Nash approximation of analytic solutions of a system of Nash equations. Next proposition is a nested version of the Nash Approximation Theorem established in [2].

Proposition 3.1. *Let $M_1, ..., M_m$ be Nash manifolds, $X_1 \subset M_1, ..., X_m \subset M_m$ compact semialgebraic subsets, and let $l_1, ..., l_m, n_1, ..., n_m \in \mathbf{N}$. Let $F_i \in \mathcal{N}(X_1 \times \cdots \times X_i \times \mathbf{R}^{l_1} \times \cdots \times \mathbf{R}^{l_i})^{n_i}$ and $f_i \in \mathcal{O}(X_1 \times \cdots \times X_i)^{l_i}$ for $i = 1, ..., m$ such that*

$$F_i(x_1, ..., x_i, f_1(x_1), ..., f_i(x_1, ..., x_i)) = 0$$

as elements of $\mathcal{O}(X_1 \times \cdots \times X_i)^{n_i}$. Then there exist $\tilde{f}_i \in \mathcal{N}(X_1 \times \cdots \times X_i)^{l_i}$ close to f_i in the C^∞ topology for $i = 1, ..., m$ such that $F_i(x_1, ..., x_i, \tilde{f}_1(x_1), ..., \tilde{f}_i(x_1, ..., x_i)) = 0$ in $\mathcal{N}(X_1 \times \cdots \times X_i)^{n_i}$.

Proof. The proof is inspired by Nested Smoothing Theorem 11.4, [17] by Teissier and its proof. The proof for $m = 1$ coincide with Theorem 1.1, [2] if M_1 is compact and if $X_1 = M_1$, and we can prove the proposition for $m = 1$ in the same way even if M_1 is non-compact.

Regard each F_i as n_i elements of $\mathcal{N}(X_1 \times \cdots \times X_i \times \mathbf{R}^{l_1} \times \cdots \times \mathbf{R}^{l_i})$. We can assume that M_i and X_i are all connected and that F_i are polynomial functions in the variables $(y_1, ..., y_i) \in \mathbf{R}^{l_1} \times \cdots \times \mathbf{R}^{l_i}$ with coefficients in $\mathcal{N}(X_1 \times \cdots \times X_i)$ for the same reason as in the proof of Theorem 1.1, [2]. Let $\mathcal{N}(X_1 \times \cdots \times X_i)[y_1, ..., y_i]$ denote the ring of polynomials in the variables $(y_1, ..., y_i) \in \mathbf{R}^{l_1} \times \cdots \times \mathbf{R}^{l_i}$ with coefficients in $\mathcal{N}(X_1 \times \cdots \times X_i)$ and $(F_1, ..., F_i)$ the ideal of $\mathcal{N}(X_1 \times \cdots \times X_i)[y_1, ..., y_i]$ generated by $F_1, ..., F_i$.

Consider a commutative diagram of homomorphisms between rings:

$$\begin{array}{ccccc}
\mathcal{N}(X_1) & \xrightarrow{\phi_1} & \frac{\mathcal{N}(X_1)[y_1]}{(F_1)} & \xrightarrow{\psi_1} & \mathcal{O}(X_1) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle r_1} \\
\mathcal{N}(X_1\times X_2) & \xrightarrow{\phi_2} & \frac{\mathcal{N}(X_1\times X_2)[y_1,y_2]}{(F_1,F_2)} & \xrightarrow{\psi_2} & \mathcal{O}(X_1\times X_2) \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle r_2} \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle p_{m-1}} & & \downarrow{\scriptstyle q_{m-1}} & & \downarrow{\scriptstyle r_{m-1}} \\
\mathcal{N}(X_1\times\cdots\times X_m) & \xrightarrow{\phi_m} & \frac{\mathcal{N}(X_1\times\cdots\times X_m)[y_1,\ldots,y_m]}{(F_1,\ldots,F_m)} & \xrightarrow{\psi_m} & \mathcal{O}(X_1\times\cdots\times X_m),
\end{array}$$

where for each i, ϕ_i, p_i, q_i and r_i are naturally defined, $\psi_i = \mathrm{id}$ on $\mathcal{N}(X_1 \times \cdots \times X_i)$ and $\psi_i(y_j)$ is defined to be f_j as an element of $\mathcal{O}(X_1 \times \cdots \times X_i)$ for each $j \leq i$. Then it suffices to find homomorphisms $\tilde{\psi}_1 : \mathcal{N}(X_1)[y_1]/(F_1) \to \mathcal{N}(X_1), ..., \tilde{\psi}_m : \mathcal{N}(X_1 \times \cdots \times X_m)[y_1, ..., y_m]/(F_1, ..., F_m) \to \mathcal{N}(X_1 \times \cdots \times X_m)$ such that $\tilde{\psi}_1 \circ \phi_1 = \mathrm{id}, ..., \tilde{\psi}_m \circ \phi_m = \mathrm{id}$, $\tilde{\psi}_1(y_1), ..., \tilde{\psi}_m(y_m)$ are close to $f_1, ..., f_m$, respectively, and $r_i \circ \tilde{\psi}_i = \tilde{\psi}_{i+1} \circ q_i$ for $0 < i < m$. For that we only need to decide the values $\tilde{\psi}_1(y_1), ..., \tilde{\psi}_m(y_m)$ because $\tilde{\psi}_i(y_k) = \tilde{\psi}_k(y_k)$ as elements of $\mathcal{N}(X_1 \times \cdots \times X_i)$ for $k < i$ by the equality $r_j \circ \tilde{\psi}_j = \tilde{\psi}_{j+1} \circ q_j$. By [3], [6] and [14] we know that $\mathcal{O}(X_1 \times \cdots \times X_i)$ and $N(M_1 \times \cdots \times M_i)$ are Noetherian, and the proofs in [3] and [14] work for the Noetherian property of $\mathcal{N}(X_1 \times \cdots \times X_i)$. Hence all the rings in the diagram are Noetherian. Therefore, we assume that ψ_i are injective enlarging F_i if necessary.

We will find $k_i \in \mathbf{N}$, finite subsets G_i of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]$, $z_j \in \mathbf{R}^{k_j}$, and a commutative diagram of homomorphisms between rings:

$$\begin{array}{ccccc}
\frac{\mathcal{N}(X_1)[y_1]}{(F_1)} & \xrightarrow{\rho_1} & \frac{\mathcal{N}(X_1)[z_1]}{(G_1)} & \xrightarrow{\xi_1} & \mathcal{O}(X_1) \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle r_1} \\
\frac{\mathcal{N}(X_1\times X_2)[y_1,y_2]}{(F_1,F_2)} & \xrightarrow{\rho_2} & \frac{\mathcal{N}(X_1\times X_2)[z_1,z_2]}{(G_1,G_2)} & \xrightarrow{\xi_2} & \mathcal{O}(X_1\times X_2) \\
\downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle r_2} \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle q_{m-1}} & & \downarrow{\scriptstyle s_{m-1}} & & \downarrow{\scriptstyle r_{m-1}} \\
\frac{\mathcal{N}(X_1\times\cdots\times X_m)[y_1,\ldots,y_m]}{(F_1,\ldots,F_m)} & \xrightarrow{\rho_m} & \frac{\mathcal{N}(X_1\times\cdots\times X_m)[z_1,\ldots,z_m]}{(G_1,\ldots,G_m)} & \xrightarrow{\xi_m} & \mathcal{O}(X_1\times\cdots\times X_m)
\end{array}$$

such that for each i, s_i is naturally defined, $\xi_i \circ \rho_i = \psi_i$, $\rho_i = \xi_i = \mathrm{id}$ on $\mathcal{N}(X_1 \times \cdots \times X_i)$, the zero set Z_i of $(G_1, ..., G_i)$ is the germ on $X_1 \times \cdots \times X_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$ of a Nash submanifold of $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$ and $(G_1, ..., G_i)$ is the ideal of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]$ of function germs vanishing on Z_i. Note that then the restriction π_i to Z_i of the projection $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i} \to M_1 \times \cdots \times M_i$ is submersive because $\mathcal{N}(X_1 \times \cdots \times X_i) \subset \mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]/(G_1, ..., G_i)$ and $\xi_i|_{\mathcal{N}(X_1 \times \cdots \times X_i)} = \mathrm{id}$, that $\xi_i(z_1, ..., z_i)$ is an analytic cross-section of π_i and that when we regard locally M_j as Euclidean spaces the rank of the Jacobian matrix $\frac{D(G_1,...,G_i)}{D(x_1,...,x_i,z_1,...,z_i)}$ equals the codimension of Z_i in $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$ at each point of Z_i.

As in the proof of Theorem 1.1, [2] there exist $k_i \in \mathbf{N}$, finite subsets G_i of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_i]$, $z_i \in \mathbf{R}^{k_i}$ and homomorphisms of $\mathcal{N}(X_1 \times \cdots \times X_i)$-algebras

$$\frac{\mathcal{N}(X_1 \times \cdots \times X_i)[y_1, ..., y_i]}{(F_1, ..., F_i)} \xrightarrow{\rho_i'} \frac{\mathcal{N}(X_1 \times \cdots \times X_i)[z_i]}{(G_i)} \xrightarrow{\xi_i'} \mathcal{O}(X_1 \times \cdots \times X_i)$$

such that for each i, $\xi_i' \circ \rho_i' = \psi_i$, the zero set Z_i' of (G_i) is the germ on $X_1 \times \cdots \times X_i \times \mathbf{R}^{k_i}$ of a Nash submanifold of $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_i}$ and (G_i) is the ideal of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_i]$ of function germs vanishing on Z_i'. Then as above the restriction π_i' to Z_i' of the projection $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_i} \to M_1 \times \cdots \times M_i$ is submersive and $\xi_i'(z_i)$ is an analytic cross-section of π_i'. Define ρ_i to be the composition of ρ_i' with the canonical homomorphism $\mathcal{N}(X_1 \times \cdots \times X_i)[z_i]/(G_i) \to \mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]/(G_1, ..., G_i)$ and ξ_i by $\xi_i(z_i) = \xi_i'(z_i)$ and $\xi_i(z_j) = r_{i-1} \circ \cdots \circ r_j \circ \xi_j'(z_j)$. Then the conditions on G_i, ρ_i and ξ_i are satisfied. Indeed, first the zero set Z_i of $(G_1, ..., G_i)$ in $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$ is the fibre product of the submersions $(\pi_1', \mathrm{id}) : Z_1' \times M_2 \times \cdots \times M_i \to M_1 \times \cdots \times M_i$, $(\pi_2', \mathrm{id}) : Z_2' \times M_3 \times \cdots \times M_i \to M_1 \times \cdots \times M_i, ..., \pi_i' : Z_i' \to M_1 \times \cdots \times M_i$ and hence the germ on $X_1 \times \cdots \times X_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$ of some Nash submanifold of $M_1 \times \cdots \times M_i \times \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_i}$. Next, add some finite subset of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]$ to G_i whose elements vanish on Z_i, if necessary. Then $(G_1, ..., G_i)$ is the ideal of $\mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]$ of function germs vanishing on Z_i. Thus we obtain the required diagram.

For the construction of $\tilde{\psi}_i$'s it suffices to find homomorphisms of $\mathcal{N}(X_1 \times \cdots \times X_i)$-algebras $\tilde{\xi}_i : \mathcal{N}(X_1 \times \cdots \times X_i)[z_1, ..., z_i]/(G_1, ..., G_i) \to \mathcal{N}(X_1 \times \cdots \times X_i)$ so that $\tilde{\xi}_i(z_i) \in \mathcal{N}(X_1 \times \cdots \times X_i)^{k_i}$ are close to $\xi_i(z_i) \in \mathcal{O}(X_1 \times \cdots \times X_i)^{k_i}$ in the C^∞ topology because if we define $\tilde{\xi}_i$ by $\tilde{\xi}_i(z_j) = r_{i-1} \circ \cdots r_j \circ \tilde{\xi}_j(z_j)$ for $j < i$ then $\tilde{\psi}_i = \tilde{\xi}_i \circ \rho_i$ fulfill the requirements. By induction on m we assume that $\tilde{\xi}_1, ..., \tilde{\xi}_{m-1}$ are given. Then as before we only need to decide $\tilde{\xi}_m(z_m) \in \mathcal{N}(X_1 \times \cdots \times X_m)^{k_m}$ close to $\xi_m(z_m) \in \mathcal{O}(X_1 \times \cdots \times X_m)^{k_m}$ in the C^∞ topology so that $G_m(x_1, ..., x_m, \tilde{\xi}_m(z_m)) = \{0\}$ as a subset of $\mathcal{N}(X_1 \times \cdots \times X_m)$, i.e. $\tilde{\xi}_m(z_m)$ is a Nash cross-section of π_m'. (Here the elements of G_m may be of the variables $x_1, ..., x_m, z_1, ..., z_m$. However, we can remove some elements from G_m so that they are all in the variables $x_1, ..., x_m, z_m$ by the above arguments.) Let $U \subset U'$ be small open semialgebraic neighborhoods of $X_1 \times \cdots \times X_m$ in $M_1 \times \cdots \times M_m$ such that $\overline{U}$ is compact and contained in U', Z_m' is the germ on $X_1 \times \cdots \times X_m \times \mathbf{R}^{k_m}$ of a closed Nash submanifold Z' of $U' \times \mathbf{R}^{k_m}$, π_m' is the germ on $X_1 \times \cdots \times X_m \times \mathbf{R}^{k_m}$ of a surjective submersion $\pi' : Z' \to U'$ and $\xi_m(z_m)$ is the germ on $X_1 \times \cdots \times X_m$ of an analytic cross-section $\xi : \overline{U} \to Z'$ of π'. Let η be a Nash approximation of $\xi|_{\overline{U}} : \overline{U} \to Z'$ in the C^∞ topology (Nash Approximation Theorem), which is an embedding but not necessarily a cross-section of $\pi'|_{\pi'^{-1}(\overline{U})}$. Let $\tilde{\xi}_m(z_m)$ be the germ of $\eta \circ (\pi_m' \circ \eta)^{-1}$ on $X_1 \times \cdots \times X_m$. Then $\tilde{\xi}_m(z_m)$ is a Nash cross-section of π_m' close to $\xi_m(z_m)$ in the C^∞ topology. Thus we complete the proof.

□

As a corollary of proposition 3.1 we obtain the following Nash approximation theorem, which generalizes that proved in [2] in the case where $X = M$ and M is compact.

Theorem 3.2. *Let M be a Nash manifold, $X \subset M$ be a compact semialgebraic subset, and f, g be Nash function germs on X in M. If f and g are analytically right equivalent, then f and g are Nash right equivalent. The diffeomorphism of Nash right equivalence can be chosen to be close to the given one of analytic right equivalence in the C^∞ topology.*

Here we naturally define analytic or Nash right equivalence of two analytic or Nash function germs, respectively, on X in M. We note only that the diffeomorphism germ of equivalence is X-preserving.

For the proof we introduce some notions. Let X be a semialgebraic subset of a Nash manifold M. We consider the germs of sets on X in M. For a germ A on X of a subset of M, let $\overline{A}^X$ or A^{-X} denote the *Nash closure* of A in M, i.e. the smallest Nash set germ in M containing A. In the case where A is a subset of M also, $\overline{A}^X$ stands for the Nash closure of the germ of A on X in M. We define by induction a sequence of Nash set germs M_i in M as follows. Let M_1 be the germ $\overline{X}^X$ and assume that $M_1, ..., M_{k-1}$ are given for $k\,(>1) \in \mathbf{N}$. Then, set

$$M_k = [\overline{(M_{k-1} - X)} \cap \overline{(M_{k-1} \cap X)}]^X.$$

We call $\{M_i\}$ the *canonical Nash germ decomposition* of X. Then $\{M_i\}$ is a decreasing sequence of Nash set germs, for each i the set $X \cap M_i - M_{i+1}$ is a union of some connected components of $M_i - M_{i+1}$ and $\{M_i\}$ is canonical in the following sense. Let $\{M_i'\}$ be another decreasing sequence of Nash set germs such that for each i the set $X \cap M_i' - M_{i+1}'$ is a union of some connected components of $M_i' - M_{i+1}'$, which is called a *Nash germ decomposition of* X. Assume that $\{M_i'\}$ is distinct from $\{M_i\}$. Then $M_1' = M_1, ..., M_{k-1}' = M_{k-1}$ and $M_k' \supsetneq M_k$ for some k.

A subset Y of an analytic manifold N is called *global semianalytic* if Y is described by finitely many equalities and inequalities of global analytic functions on N. Let Y be a relatively compact and global semianalytic subset of N. Then we can define the *global analytic closure* $\overline{Z}^Y$ of the germ on Y of a subset Z of N (or of the germ Z on Y of a subset of N) and a (the *canonical) global analytic germ decomposition* of Y in the same way. Indeed, for a global semianalytic set Z in N, $\dim Z = \dim \overline{Z}^Y$ for the reason explained below, moreover if Z is, moreover, relatively compact then $\overline{Z}$ is global semianalytic by [15] and finally, a global analytic set is global semianalytic (and moreover is the zero set of one global analytic function, cf. proof of corollary 2.2 in [5]). To prove that $\dim Z = \dim \overline{Z}^Y$ we can assume that Z is a global semianalytic set of the form $\{x \in N : f(x) = 0, f_1 > 0, ..., f_k(x) > 0\}$ for some analytic functions $f, f_1, ..., f_k$ on N dividing Z if necessary, and it suffices to prove that the global analytic closure $\overline{Z}^N$ of Z is of the same dimension as Z. Let $x_0 \in Z$ where the germ of $\overline{Z}^N$ is of dimension $\dim \overline{Z}^N$, which exists since $Z \cap \mathrm{Reg}\, \overline{Z}^N \neq \emptyset$. Then $f_1 > 0, ..., f_k > 0$ on a neighborhood of x_0 in $\overline{Z}^N$. Hence Z contains the neighborhood and is of dimension $\dim \overline{Z}^N$. (We do not know whether the canonical global analytic germ decomposition of Y does exist if Y is a non-relatively compact global semianalytic set.)

Remark 3.3.

(1) Let $N \supset Y$ and $N' \supset Y'$ be analytic manifolds and respective relatively compact and global semianalytic subsets and $\phi : N \to N'$ an analytic diffeomorphism such that $\phi(Y) = Y'$. Then ϕ carries the canonical global analytic germ decomposition of Y to the canonical global analytic germ decomposition of Y', which is clear.

(2) Let $M \supset X$ be a Nash manifold and a semialgebraic subset. Then the canonical global analytic germ decomposition of X is well-defined and coincides with the canonical Nash germ decomposition of X because the global analytic closure of a semialgebraic set equals its Nash closure (cf. remark (vi) in section 2.2 of [5]).

Proof of theorem 3.2. Let $M \subset \mathbf{R}^n$, set $M_0 = \overline{M}^X$, and let $\{M_i : i = 1, 2, ...\}$ be the canonical Nash germ decomposition of X. Let $\mathcal{O}(X)$ and $\mathcal{N}(X)$ denote respectively the germs of analytic and Nash functions on X in $\mathbf{R}^n$ but not in M. Let $\{\phi_{i,j} : j\}$ for each $i = 0, 1, ...$ be finitely many generators of the ideal of $\mathcal{N}(X)$ defined by M_i. Extend f and g to elements $\hat{f}$ and $\hat{g}$ of $\mathcal{N}(X)$, respectively. Then we have $\pi = (\pi_1, ..., \pi_n) \in \mathcal{O}(X)^n$ such that $\pi|_M$ is the germ on X of a C^ω diffeomorphism between neighborhoods of X in M

and $f \circ \pi = g$ on M. Hence there exist $\alpha_j \in \mathcal{O}(X)$ such that

$$\hat{f} \circ \pi = \hat{g} + \sum_j \alpha_j \phi_{0,j}. \tag{1}$$

By remarks 3.3.(1) and 3.3.(2), π is M_i-preserving. Hence there exist $\beta_{i,j,j'} \in \mathcal{O}(X)$ such that for each $\phi_{i,j}$

$$\phi_{i,j} \circ \pi = \sum_{j'} \beta_{i,j,j'} \phi_{i,j'}. \tag{2}$$

Apply proposition 3.1 to (1) and (2). Then there exist $\tilde{\pi} \in \mathcal{N}(X)^n$, $\tilde{\alpha}_j \in \mathcal{N}(X)$ and $\tilde{\beta}_{i,j,j'} \in \mathcal{N}(X)$ close to π, α_j and $\beta_{i,j,j'}$, respectively, in the C^∞ topology such that

$$\hat{f} \circ \tilde{\pi} = \hat{g} + \sum_j \tilde{\alpha}_j \phi_{0,j}, \tag{$\widetilde{1}$}$$

$$\phi_{i,j} \circ \tilde{\pi} = \sum_{j'} \tilde{\beta}_{i,j,j'} \phi_{i,j'}. \tag{$\widetilde{2}$}$$

Since $\tilde{\pi}$ is an approximation of π, ($\widetilde{2}$) implies that $\tilde{\pi}|_M$ is the germ on X of a Nash diffeomorphism between open semialgebraic neighborhoods of X in M. Hence by ($\widetilde{1}$), $f \circ (\tilde{\pi}|_M) = g$, and the theorem is proved.

□

Consider the plural case of $\{X\}$. Let X and X_j, $j = 1, ..., k$, be semialgebraic subsets of a Nash manifold M. We define the *canonical Nash germ decomposition* $\{M_i\}$ of $\{X; X_j\}$ as follows. Set $X_0 = \cup_{j=1}^k X_j$ and $M_1 = \overline{X_0}^X$, and assume by induction that we have defined M_i for some $i > 0$. Then we set

$$M_{i+1} = (\cup_{j=1}^k [\overline{(M_i - X_j)} \cap \overline{(M_i \cap X_j)}]\,)^{-X}.$$

The same properties as in the single case hold. To be precise, $\{M_i\}$ is a decreasing sequence of Nash set germs on X, for each i and $j > 0$ $X_j \cap M_i - M_{i+1}$ is a union of some connected components of $M_i - M_{i+1}$, and $\{M_i\}$ is canonical in the same sense as in the single case. We define also a *Nash germ decomposition* of $\{X; X_j\}$ and a (the *canonical) global analytic germ decomposition* of a finite family of relatively compact global semianalytic sets in an analytic manifold in the same way. Then remark 3.3.(1) and 3.3.(2) hold also in the plural case.

Using these notions and remarks in the same way as above we can refine theorem 3.2 as follows.

Remark 3.4. In theorem 3.2, let A_i and B_i be a finite number of semialgebraic subsets of M such that the diffeomorphism germ of analytical right equivalence carries the germ on X of each A_i to the one of B_i. Then the diffeomorphism germ of Nash right equivalence is chosen to have the same property.

In particular, if we set $f = g =$ constant then we have the following statement.

Let M and X be the same as in theorem 3.2, and let C_i and D_i be finitely many semialgebraic subsets of M. Assume that there exists the germ π on X of an analytic diffeomorphism between neighborhoods of X in M which carries the germ on X of each C_i to the one of D_i and such that $\pi(X) = X$. Then π is approximated by the germ on X of a Nash diffeomorphism between semialgebraic neighborhoods of X in M in the C^∞ topology keeping the properties of π.

3.2. **Proof of theorem 1.4.**

3.2.1. *Proof of theorem 1.4 in the case where $X = M$ and M is compact.* Assume that f and g are almost blow-analytically equivalent. Let $\pi_f : N \to M$, $\pi_g : L \to M$ and $h : N \to L$ be two compositions of finite sequences of blowings-up along smooth analytic centers and an analytic diffeomorphism, respectively, such that $f \circ \pi_f = g \circ \pi_g \circ h$. By Hironaka Desingularization Theorem [9] there exists a sequence of blowings-up $N_m \xrightarrow{\tau_{f,m}} N_{m-1} \longrightarrow \cdots \xrightarrow{\tau_{f,1}} N_0 = N$ along smooth analytic centers C_{m-1} in $N_{m-1}, ..., C_0$ in N_0, respectively, such that $f \circ \pi_f \circ \tau_{f,1} \circ \cdots \circ \tau_{f,m}$ has only normal crossing singularities. Since $h : N \to L$ is an analytic diffeomorphism, we have a sequence of blowings-up $L_m \xrightarrow{\tau_{g,m}} L_{m-1} \longrightarrow \cdots \xrightarrow{\tau_{g,1}} L_0 = L$ along smooth analytic centers D_{m-1} in $L_{m-1}, ..., D_0$ in L_0, respectively, and analytic diffeomorphisms $h_m : N_m \to L_m, ..., h_0 = h : N_0 \to L_0$ such that $h_{m-1}(C_{m-1}) = D_{m-1}, ..., h_0(C_0) = D_0$, and the following diagram is commutative :

$$\begin{array}{ccccccc} N_m & \xrightarrow{\tau_{f,m}} & N_{m-1} & \longrightarrow \cdots \xrightarrow{\tau_{f,1}} & N_0 \\ \downarrow{\scriptstyle h_m} & & \downarrow{\scriptstyle h_{m-1}} & & \downarrow{\scriptstyle h_0} \\ L_m & \xrightarrow{\tau_{g,m}} & L_{m-1} & \longrightarrow \cdots \xrightarrow{\tau_{g,1}} & L_0. \end{array}$$

Hence, replacing π_f, π_g and h with $\pi_f \circ \tau_{f,1} \circ \cdots \circ \tau_{f,m}$, $\pi_g \circ \tau_{g,1} \circ \cdots \circ \tau_{g,m}$ and h_m, respectively, we can assume from the beginning that $f \circ \pi_f$ and $g \circ \pi_g$ have only normal crossing singularities. Then by theorem 2.4 there exist compositions of finite sequences of blowings-up along smooth Nash centers $\tilde{\pi}_f : \tilde{N} \to M$ and $\tilde{\pi}_g : \tilde{L} \to M$ and analytic diffeomorphisms $\tau_f : \tilde{N} \to N$ and $\tau_g : \tilde{L} \to L$ such that $f \circ \pi_f \circ \tau_f = f \circ \tilde{\pi}_f$ and $g \circ \pi_g \circ \tau_g = g \circ \tilde{\tau}_g$. Hence $f \circ \tilde{\pi}_f \circ \tau_f^{-1} = g \circ \tilde{\pi}_g \circ \tau_g^{-1} \circ h$, and $f \circ \tilde{\pi}_f$ and $g \circ \tilde{\pi}$ are analytically right equivalent. Then by theorem 3.2 they are Nash right equivalent, i.e. f and g are almost blow-Nash equivalent.

Next we prove that if f and g are almost blow-analytically R-L equivalent then they are almost blow-Nash R-L equivalent. For that it suffices to prove that two analytically R-L equivalent Nash functions ϕ and ψ with only normal crossing singularities are Nash R-L equivalent. Let π and τ be analytic diffeomorphisms of M and $\mathbf{R}$, respectively, such that $\tau \circ \phi = \psi \circ \pi$. Then $\pi(\mathrm{Sing}\,\phi) = \mathrm{Sing}\,\psi$, $\tau(\phi(\mathrm{Sing}\,\phi)) = \psi(\mathrm{Sing}\,\psi)$ and $\pi(\phi^{-1}(\phi(\mathrm{Sing}\,\phi))) = \psi^{-1}(\psi(\mathrm{Sing}\,\psi))$. By remark 3.4 we have a Nash diffeomorphism π_0 of M close to π in the C^∞ topology such that $\pi_0(\mathrm{Sing}\,\phi) = \mathrm{Sing}\,\psi$ and $\pi_0(\phi^{-1}(\phi(\mathrm{Sing}\,\phi))) = \psi^{-1}(\psi(\mathrm{Sing}\,\psi))$, and since $\phi(\mathrm{Sing}\,\phi)$ is a finite set, we have a Nash diffeomorphism τ_0 of $\mathbf{R}$ close to τ in the compact-open C^∞ topology such that $\tau_0 = \tau$ on $\phi(\mathrm{Sing}\,\phi)$. Replace ψ with $\tau_0^{-1} \circ \psi \circ \pi_0$. Then we can assume from the beginning that $\mathrm{Sing}\,\phi = \mathrm{Sing}\,\psi$, $\phi(\mathrm{Sing}\,\phi) = \psi(\mathrm{Sing}\,\psi)$, $\phi^{-1}(\phi(\mathrm{Sing}\,\phi)) = \psi^{-1}(\psi(\mathrm{Sing}\,\psi))$, and π and τ are close to id in the C^∞ topology and in the compact-open C^∞ topology, respectively. Hence for each $z_0 \in \phi(\mathrm{Sing}\,\phi)$, $\phi - z_0$ and $\psi - z_0$ have the same sign at each point of M and the same multiplicity at each point of $\phi^{-1}(z_0)$. Let ρ be a Nash function on $\mathbf{R}$ with zero set $\phi(\mathrm{Sing}\,\phi)$ and regular there. Then $\rho \circ \phi$ and $\rho \circ \psi$ satisfy the conditions in lemma 3.10, [5]—$(\rho \circ \phi)^{-1}(0) = (\rho \circ \psi)^{-1}(0)$ $(= \phi^{-1}(\phi(\mathrm{Sing}\,\phi)))$, $\rho \circ \phi$ and $\rho \circ \psi$ have the same sign at each point of M, only normal crossing singularities at $(\rho \circ \phi)^{-1}(0)$ and the same multiplicity at each point of $(\rho \circ \phi)^{-1}(0)$, and the natural extension to M of the function $\rho \circ \psi / \rho \circ \phi$ defined on $M - (\rho \circ \phi)^{-1}(0)$ is close to 1 in the C^∞ topology. Hence by lemma 3.10 in [5] there exists a Nash diffeomorphism π_1 of M close to id in the C^∞ topology such that $\pi_1(\phi^{-1}(\phi(\mathrm{Sing}\,\phi))) = \phi^{-1}(\phi(\mathrm{Sing}\,\phi))$ and $\phi - \psi \circ \pi_1$ is l-flat at $\phi^{-1}(\phi(\mathrm{Sing}\,\phi))$ for a large integer l. Replace, once more, ψ with $\psi \circ \pi_1$. Then we can assume, moreover, that $\phi - \psi$ is l-flat at $\phi^{-1}(\phi(\mathrm{Sing}\,\phi))$ and close to 0 in the C^∞ topology. Hence by proposition 3.11,(i) in [5], ϕ and ψ are analytically right equivalent and then by theorem 3.2 they are Nash right equivalent.

3.2.2. *Proof of theorem 1.4 in the case of* $X \subset \mathrm{Sing}\, f$. Assume that f and g are Nash functions defined on open semialgebraic neighborhoods U and V, respectively, of X in M, and let $\pi_f : N \to U$, $\pi_g : L \to V$ and $h : N' \to L'$ be two compositions of finite sequences of blowings-up along smooth analytic centers and an analytic diffeomorphism from an open neighborhood N' of $\pi_f^{-1}(X)$ in N to one L' of $\pi_g^{-1}(X)$ in L, respectively, such that $f \circ \pi_f = g \circ \pi_g \circ h$ and $h(\pi_f^{-1}(X)) = \pi_g^{-1}(X)$. When we proceed as in the proof in the case of $X = M$ we can replace $\pi_f : N \to U$, $\pi_g : L \to V$ and $h : N' \to L'$ by Nash $\tilde{\pi}_f : \tilde{N} \to \tilde{U}$, $\tilde{\pi}_g : \tilde{L} \to \tilde{V}$ and $h : \tilde{N}' \to \tilde{L}'$, respectively, so that $f \circ \tilde{\pi}_f = g \circ \tilde{\pi}_g \circ \tilde{h}$. However, we cannot expect the equality $\tilde{h}(\tilde{\pi}_f^{-1}(X)) = \tilde{\pi}_g^{-1}(X)$. For the equality we need to modify π_f and π_g.

As in the construction of the canonical Nash germ decomposition we have a decreasing sequence of Nash sets X_i, $i = 1, 2, ...$, in U such that X_1 is the Nash closure of X in U and for each i the set $X_i \cap X - X_{i+1}$ is a union of some connected components of $X_i - X_{i+1}$. Set $X_{f,i} = \pi_f^{-1}(X_i)$. Then $\{X_{f,i}\}$ is a decreasing sequence of global analytic sets in N, $\pi_f^{-1}(X) \subset X_{f,1}$, and for each i the set $\pi_f^{-1}(X) \cap X_{f,i} - X_{f,i+1}$ is a union of some connected components of $X_{f,i} - X_{f,i+1}$. By Hironaka Desingularization Theorem we reduce the problem, in the same way as in the case of $X = M$, to the case where $X_{f,i}$ are normal crossing, $f \circ \pi_f$ has only normal crossing singularities, and hence $\pi_f^{-1}(X)$ is a union of some connected components of strata of the canonical stratification of $\mathrm{Sing}(f \circ \pi_f)$. Shrink N', V, L and L' so that $L' = L$ if necessary. Then $\pi_g^{-1}(X)$ and $g \circ \pi_g$ have these properties of $\pi_f^{-1}(X)$ and $f \circ \pi_f$. We call these properties $(*)$.

Let N and L be realized in $U \times \mathbf{P}(k_{f,0}) \times \cdots \times \mathbf{P}(k_{f,m'-1})$ and in $V \times \mathbf{P}(k_{g,0}) \times \cdots \times \mathbf{P}(k_{g,m''-1})$, respectively, as in theorem 2.4. Then by theorem 2.4 there exist compositions of finite sequences of blowings-up along smooth Nash centers $\tilde{\pi}_f : \tilde{N} \to \tilde{U}$ and $\tilde{\pi}_g : \tilde{L} \to \tilde{V}$ and analytic embeddings $h_f : \tilde{N} \to N'$ and $h_g : \tilde{L} \to L'$ such that

- $\tilde{U}$ and $\tilde{V}$ are open semialgebraic neighborhoods of X in U and V, respectively,
- $f \circ \pi_f \circ h_f = f \circ \tilde{\pi}_f$, $g \circ \pi_g \circ h_g = g \circ \tilde{\pi}_g$, $\tilde{N}$ and $\tilde{L}$ are realized in $\tilde{U} \times \mathbf{P}(k_{f,0}) \times \cdots \times \mathbf{P}(k_{f,m'-1})$ and in $\tilde{V} \times \mathbf{P}(k_{g,0}) \times \cdots \times \mathbf{P}(k_{g,m''-1})$, respectively,
- $\tilde{N}$ and $\tilde{L}$ are close to N and L at $\tilde{\pi}_f^{-1}(X)$ and $\tilde{\pi}_g^{-1}(X)$, respectively, in the C^∞ topology,
- $(**)$ h_f and h_g are close to id at $\tilde{\pi}_f^{-1}(X)$ and $\tilde{\pi}_g^{-1}(X)$, respectively, in the C^∞ topology.

Hence

$$f \circ \tilde{\pi}_f \circ h_f^{-1} = g \circ \tilde{\pi}_g \circ h_g^{-1} \circ h \quad \text{on } \Im h_f \cap h^{-1}(\Im h_g).$$

Clearly $h_f(\mathrm{Sing}(f \circ \tilde{\pi}_f)) \subset \mathrm{Sing}(f \circ \pi_f)$ and $h_g(\mathrm{Sing}(g \circ \tilde{\pi}_g)) \subset \mathrm{Sing}(g \circ \pi_g)$. It follows from $(*)$ and $(**)$ that $\tilde{\pi}_f^{-1}(X)$ and $\tilde{\pi}_g^{-1}(X)$ are unions of some connected components of strata of the canonical stratifications of $\mathrm{Sing}(f \circ \tilde{\pi}_f)$ and $\mathrm{Sing}(g \circ \tilde{\pi}_g)$, respectively, and hence

$$h_f(\tilde{\pi}_f^{-1}(X)) = \pi_f^{-1}(X) \quad \text{and} \quad h_g(\tilde{\pi}_g^{-1}(X)) = \pi_g^{-1}(X).$$

Therefore, the germs of $f \circ \tilde{\pi}_f$ on $\tilde{\pi}_f^{-1}(X)$ and of $g \circ \tilde{\pi}_g$ on $\tilde{\pi}_g^{-1}(X)$ are analytically right equivalent. On the other hand, by remark 3.4, the germs of $\tilde{N}$ on $\tilde{\pi}_f^{-1}(X)$ and of $\tilde{L}$ on $\tilde{\pi}_g^{-1}(X)$ are Nash diffeomorphic. Hence we can regard them as the same Nash set germ. Then by theorem 3.2 and remark 3.4, the germs of $f \circ \tilde{\pi}_f$ on $\tilde{\pi}_f^{-1}(X)$ and of $g \circ \tilde{\pi}_g$ on $\tilde{\pi}_g^{-1}(X)$ are Nash right equivalent. Thus the germs of f and g on X are almost blow-Nash equivalent.

Finally, the case of the R-L equivalences runs in the same way as that of $X = M$.

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