KINETIC EQUATIONS WITH MAXWELL BOUNDARY CONDITION

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Abstract

We prove global stability results of *DiPerna-Lions* renormalized solutions for the initial boundary value problem associated to some kinetic equations, from which existence results classically follow. The (possibly nonlinear) boundary conditions are completely or partially diffuse, which include the so-called Maxwell boundary conditions, and we prove that it is realized (it is not relaxed!). We are able to deal with Boltzmann, Vlasov-Poisson and Fokker-Planck type models. The proofs use some trace theorems of the kind previously introduced by the author for the Vlasov equations, new results concerning weak-weak convergence (the renormalized convergence and the biting L^1 weak convergence), as well as the Darrozès-Guiraud information in a crucial way.

Équations cinétiques avec conditions aux limites de Maxwell

Résumé - Nous montrons la stabilité des solutions renormalisées au sens de *DiPerna-Lions* pour des équations cinétiques avec conditions initiale et aux limites. La condition aux limites (qui peut être non linéaire) est partiellement diffuse et est réalisée (c'est-à-dire qu'elle n'est pas relaxée!). Les techniques que nous introduisont sont illustrées sur l'équation de Fokker-Planck-Boltzmann et le système de Vlasov-Poisson-Fokker-Planck ainsi que pour des conditions aux limites linéaires sur l'équation de Boltzmann et le système de Vlasov-Poisson. Les démonstrations utilisent des théorèmes de trace du type de ceux introduits par l'auteur pour les équations de Vlasov, des résultats d'Analyse Fonctionnelle sur les convergences faible-faible (la convergence renormalisée et la convergence au sens du *Biting Lemma*), ainsi que l'information de Darroès-Guiraud d'une manière essentielle.

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Contents

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1 Introduction and main results

Let Ω be an open and bounded subset of $\mathbb{R}^N$ and set $\mathcal{O} = \Omega \times \mathbb{R}^N$. We consider a gas confined in $\Omega \subset \mathbb{R}^N$. The state of the gas is given by the distribution function $f = f(t,x,v) \geq 0$ of particles, which at time $t \geq 0$ and at position $x \in \Omega$, move with the velocity $v \in \mathbb{R}^N$. The evolution of f is governed by a kinetic equation written in the domain $(0,\infty) \times \mathcal{O}$ and it is complemented with a boundary condition that we describe now.

We assume that the boundary $\partial\Omega$ is sufficiently smooth; the regularity that we need is that there exists a vector field $n \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ such that $n(x)$ coincides with the outward unit normal vector at $x \in \partial\Omega$. We then define $\Sigma^x_\pm := \{v \in \mathbb{R}^N; \pm\, v \cdot n(x) > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$ as well as $\Sigma = \partial\Omega \times \mathbb{R}^N$ and

$$\Sigma_\pm = \{(x,v) \in \Sigma; \pm n(x) \cdot v > 0\} = \{(x,v);\, x \in \partial\Omega,\, v \in \Sigma^x_\pm\}.$$

We also denote by $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$ and by $d\lambda_k$ the measure on $(0,\infty) \times \Sigma$ defined by $d\lambda_k = |n(x) \cdot v|^k \, dv d\sigma_x dt$, $k = 1$ or 2.

The boundary condition takes into account how the particles are reflected by the wall and thus takes the form of a balance between the values of the trace γf of f on the outgoing and incoming velocities subsets of the boundary:

$$(\gamma_- f)(t,x,v) = \mathcal{R}_x(\gamma_+ f(t,x,.))(v) \quad \text{on} \quad (0,\infty) \times \Sigma_-, \tag{1.1}$$

where $\gamma_\pm f := \mathbf{1}_{(0,\infty)\times\Sigma_\pm} \gamma f$. In order to describe the interaction between particles and wall by the mean of the reflection operator $\mathcal{R}$, J.-C. Maxwell [54] proposed in 1879 the following phenomenological law by splitting the reflection operator into a local reflection operator and a diffuse (or Maxwell) reflection operator:

$$\mathcal{R} = (1-\alpha)\, L + \alpha\, D. \tag{1.2}$$

Here $\alpha \in [0,1]$ is a constant, called the *accommodation coefficient*. The local reflection operator L is defined by

$$(L_x\, \phi)\,(v) = \phi(R_x\, v),$$

with $R_x\, v = -v$ (inverse reflection) or $R_x\, v = v - 2\,(v \cdot n(x))\, n(x)$ (specular reflection). The diffuse reflection operator $D = (D_x)_{x\in\partial\Omega}$ according to the Maxwellian profile M with temperature (of the wall) $\Theta > 0$ is defined at the boundary point $x \in \partial\Omega$ for any measurable function ϕ on Σ^x_+ by

$$(D_x\, \phi)(v) = M(v)\ \tilde{\phi}(x),$$

where the normalized Maxwellian M is

$$M(v) = (2\,\pi)^{\frac{1-N}{2}}\, \Theta^{-\frac{N+1}{2}}\, e^{-\frac{|v|^2}{2\,\Theta}}, \tag{1.3}$$

and the out-coming flux of mass of particles $\tilde{\phi}(x)$ is

$$\tilde{\phi}(x) = \int_{v'\cdot n(x)>0} \phi(v')\, v'\cdot n(x)\, dv' = \int_{\Sigma_+^x} \frac{\phi}{M}\, d\mu_x. \tag{1.4}$$

It is worth mentioning that $d\mu_x(v) := M(v)\,|n(x)\cdot v|\,dv$ is a probability measure because of the normalization condition (1.3). Moreover, there holds for any measurable function ϕ on Σ_+^x

$$\int_{\Sigma_-^x} \mathcal{R}_x\phi\,|n(x)\cdot v|\,dv = \int_{\Sigma_-^x} L_x\phi\,|n(x)\cdot v|\,dv = \int_{\Sigma_-^x} D_x\phi\,|n(x)\cdot v|\,dv = \int_{\Sigma_+^x} \phi\, n(x)\cdot v\, dv, \tag{1.5}$$

which means that all the particles which reach the boundary are reflected (no particle goes out of the domain nor enters in the domain).

The reflexion law (1.2) was the only model for the gas/surface interaction that appeared in the literature before the late 1960s. In order to describe with more accuracy the interaction between molecules and wall, other models have been proposed in [26], [27], [51] where the reflexion operator $\mathcal{R}$ is a general integral operator satisfying the so-called non-negativity, normalization and reciprocity conditions, see [30] and Remark 6.4. We do not know if our analysis can be adapted to such a general kerenel. However, the boundary condition can be generalized in an other direction, see [31], [12], and we will sometimes assume that the following non linear boundary condition holds

$$\mathcal{R}\,\phi = (1-\tilde{\alpha})\,L\,\phi + \tilde{\alpha}\,D\,\phi, \qquad \tilde{\alpha} = \alpha(\tilde{\phi}), \tag{1.6}$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function which satisfies $0 < \bar{\alpha} \le \alpha(s) \le 1$ for any $s \in \mathbb{R}_+$.

In the domain, the evolution of f is governed by a kinetic equation

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \mathcal{I}(f) \quad \text{ in } \quad (0,\infty)\times\mathcal{O}, \tag{1.7}$$

where $\mathcal{I}(f)$ models the interactions of particles each one with each other and with the environment: typically, it may be a combination of the quadratic Boltzmann collision operator (describing the collision interactions of particles by binary elastic shocks), the Vlasov-Poisson operator (describing the fact that particles interact by the way of the two-body long range Coulomb force) or the Fokker-Planck operator (which takes into account the fact that particles are submitted to an heat bath). More precisely, for the nonlinear boundary condition (1.6) we are able to deal with Fokker-Planck type equations, in particular the Fokker-Planck-Boltzmann equation (FPB in short) and the Vlasov-Poisson-Fokker-Planck system (VPFP in short), while for a constant accommodation coefficient we are able to deal with Vlasov type equations such as the Boltzmann equation and the Vlasov-Poisson system (VP in short). We refer to section 6 where these models are presented. It is worth mentioning that we are not able to deal with the Vlasov-Maxwell system.

Finally, we complement these equations with a given initial condition

$$f(0,.) = f_{in} \ge 0 \ \text{ on } \mathcal{O}, \tag{1.8}$$

which satisfies the natural physical bound of finite mass, energy and entropy

$$\iint_{\mathcal{O}} f_{in}\,(1+|v|^2+|\log f_{in}|)\,dxdv =: C_0 < \infty. \tag{1.9}$$

We begin with a general existence result that we state deliberately in an imprecise way and we refer to section 6 (and Theorem 6.2) for a more precise statement.

Theorem 1.1 *Let consider the initial boundary value problem (1.1)-(1.7)-(1.8) associated to the FPB equation or the VPFP system with possibly mass flux depending constant accommodation coefficient (1.6) or the boundary value problem associated to the Boltzmann equation or the VP system with constant accommodation coefficient (1.2). For any nonnegative initial datum f_{in} with*

finite mass, energy and entropy ((1.9) holds) there exists at least one (renomalized) solution $f \in C([0,\infty); L^1(\mathcal{O}))$ *with finite mass, energy and entropy to the kinetic equation (1.7) associated to the initial datum* f_{in} *and such that the trace function* γf *exactly fulfills the boundary condition (1.1) (it is not relaxed!).*

The Boltzmann equation and the FPB equation for initial data satisfying the natural bound (1.9) was first studied by R. DiPerna and P.-L. Lions [35, 37, 38] who proved stability and existence results for renomalized global solutions in the case of the entire space ($\Omega = \mathbb{R}^N$). Afterwards, the corresponding boundary value problem with reflection boundary conditions (1.1) and constant accommodation coefficient has been extensively studied in the case of the Boltzmann model [47], [4], [5], [6], [7], [28], [43], [48], [29], [56]. It has been proved, in the partial absorption case $\gamma_- f = \theta \mathcal{R}\gamma_+ f$ with $\theta \in [0,1)$ and in the completely local reflection case (i.e. (1.1) holds with $\alpha \equiv 0$), that there exists a global renormalized solution. But in the most interesting physical case (when $\theta \equiv 1$ and $\alpha \in (0,1]$), it has only been proved in [6] that the boundary condition (1.1) holds in the relaxed form

$$\gamma_- f \geq \mathcal{R}(\gamma_+ f) \quad \text{ on } (0,\infty) \times \Sigma_- \tag{1.10}$$

instead of (1.1). Also, the Boltzmann equation with nonlinear boundary conditions has been treated in the setting of a strong but non global solution framework in [44].

With regard to existence results for the initial value problem for the VPFP system set in the whole space, we refer to [14], [15], [16], [19], [22], [23], [34], [59], [66], [24], [61] as well as [20] for physical motivations. The initial boundary value problem has been addressed in [13], [21]. We also refer to [3], [11], [46], [58], [68] for the initial boundary value problem for the VP system and to [58] for the corresponding stationary problem. We emphasize that in all these works only local reflection or prescribed incoming data are treated, and to our knowledge, there is no result concerning the diffuse boundary condition for the VP system or for the VPFP system.

We also mention that there is a great deal of information for the boundary value problem in an abstract setting in [67], [45] with possibly non linear boundary conditions [10], [57].

In short, the present work improves the already known existence results for kinetic equations with diffusive boundary reflection into two directions.

• On the one hand, we prove that (1.1) is fulfilled, while only its relaxed form (1.10) was previously established.

• On the other hand, we are able to consider a large class of kinetic models (including Vlasov-Poisson term) while only Boltzmann equation (or linear equations) had been previously tackled.

• Finally, we are able to handle some nonlinear boundary condition in the case of Fokker-Planck type equation.

We do not present the proof of Theorem 1.1 (nor the proof of its accurate version Theorem 6.2) because it classically follows from a sequential stability or sequential compactness result that we present below and a standard (but tedious) approximation procedure, see for instance [56] or the above quoted references. We deliberately state again the sequential stability result in an imprecise way, referring to section 6 for a more accurate version.

Theorem 1.2 *Let consider the initial boundary value problem (1.1)-(1.7)-(1.8) associated to the FPB equation or the VPFP system with possibly mass flux depending constant accommodation coefficient (1.6) or the boundary value problem associated to the Boltzmann equation or the VP system with constant accommodation coefficient (1.2). Let then* (f_n) *be a sequence of (renormalized) solutions to that equations and let assume that* (f_n) *and the trace sequence* (γf_n) *satisfy the natural physical a priori bounds (to be specified for each model). If* $f_n(0,.)$ *converges to* f_{in} *weakly in* $L^1(\mathcal{O})$ *then, up to the extraction of a subsequence,* f^n *converges (at least) weakly in* $L^1([0,T] \times \mathcal{O})$ *for all* $T \in (0,\infty)$ *to a (renormalized) solution* f *to the kinetic equation (1.7) with initial value* f_{in}*. Furthermore, for any* $\varepsilon > 0$ *and* $T > 0$ *there exists a measurable set* $A \subset (0,T) \times \partial\Omega$ *such that* $meas\,((0,T) \times \partial\Omega \setminus A) < \varepsilon$ *and*

$$\gamma_+ f_n \rightharpoonup \gamma_+ f \quad \textit{weakly in} \quad L^1(A \times \mathbb{R}^N, d\lambda_1), \tag{1.11}$$

the convergence being strong in the case of the Fokker-Planck type equations. As a consequence we can pass to the limit in the reflection boundary condition (1.1)-(1.2) (and (1.1)-(1.6) in the case of the Fokker-Planck type equations), so that the reflection boundary condition (1.1) is fulfilled.

Let us briefly explain the main steps and difficulties in the proof of the stability result.

• The first step consists in collecting the physical estimates available on the solution f to the equation (1.1)-(1.7)-(1.8) and on its trace γf. In the interior of the domain the a priori bounds satisfied by f strongly depend on the model considered but they are the same than those available in the case of the entire space. In general, for the trace, we are only able to prove that

$$\forall\, T \qquad \int_0^T \int_{\partial\Omega} \mathcal{E}\big(\frac{\gamma_+ f}{M}\big)\, d\sigma_x dt \leq C_T, \tag{1.12}$$

with C_T only depending on C_0 and T, where the functional $\mathcal{E} = \mathcal{E}_x$ is the *Darrozès-Guiraud information* defined by

$$\mathcal{E}\big(\phi\big) := \int_{\Sigma_+^x} h(\phi)\, d\mu_x - h\left(\int_{\Sigma_+^x} \phi\, d\mu_x\right), \qquad h(s) = s\, \log s, \tag{1.13}$$

and where we recall that $d\mu_x(v) := M(v)\, |n(x)\cdot v|\, dv$ is a probability measure on Σ_+^x so that $\mathcal{E}(\phi) \geq 0$ thanks to the Jensen inequality. Let us emphasize that additionally to the *a priori* bound of the *Darozès-Guiraud information* (1.12), we can prove an L^1 *a priori* bound in the case of the Boltzmann equation (and the FPB equation) and only an $L^{1/2}$ *a priori* (but also *a posteriori*) bound in the case of the VP system (and the FPVP system): in both cases, we do not have any *a priori* information on the trace which guaranties uniform local equi-integrability on the trace of a sequence of solutions. The main difficulty is thus the lack of a good *a priori* bound on the trace.

• The next step consists of specify the sense of the equations. The physical a priori estimates on f make possible to give a sense to (1.7) in a renormalized sense as introduced by DiPerna and Lions. Then, what sense gives to the trace γf of f? That so-called trace problem has been studied in [9], [32], [2], [64], [45], [18] for the Vlasov equation with a Lipschitz force field and extended to the Vlasov-Fokker-Planck equation in [21]. In the case of the VP and the VPFP systems, the *a priori* estimate on the force field does not guarantee Lipschitz regularity but only Sobolev regularity. A trace theory has been developed in [55, 56] for the (possibly renormalized) solutions of the Vlasov equation with a force field in Sobolev space that we extend here to the solutions of the Vlasov-Fokker-Planck equation. The trace of a solution is here defined by a Green formula written on the renormalized equation.

• In a last step, we have to pass to the limit in a sequence of solutions which satisfy the "natural physic bounds". For the equation satisfied by f in the interior of the domain, the proofs have been done yet by DiPerna-Lions [34, 35, 37] and Lions [52], and nothing has to be changed. The main difficulty solved here is to handle with the boundary condition. Let us emphasize that using only the L^1 boundedness information (as it is available for the Boltzmann equation for instance) on a sequence of the trace of solutions to a kinetic equation satisfying the boundary condition (1.1) it is only possible to prove the relaxed boundary condition (1.10), see [6, 43].

• In this paper, we prove some L^1-weak (L^1-strong in the case of FP models) convergence in the velocity variable for the sequence $(\gamma_+ f_n)$ (as stated in Theorem 1.2) which is strong enough to conclude. Our proof is based on the use of notions of weak-weak convergences, namely the renormalized convergence (r-convergence) and the biting L^1-weak convergence (b-convergence). We say weak-weak convergences in order to express the fact that they are extremely weak sense of convergence: weaker, for instance, to the L^1-weak convergence and to the a.e. convergence. On the one hand, thanks to the trace theory, we prove that the sequence of trace functions (γf_n) r-converges to γf (as well as a.e. for FP models). Next, thanks to some additional L^1 a priori bounds, or because the r-convergence is *almost equivalent to* the b-convergence when the limit

function belongs to L^0, we deduce that $\widetilde{\gamma_+ f_n}$ b-converges to $\widetilde{\gamma_+ f}$. Finally, that information and the boundedness of the Darrozès-Guiraud information leads to (1.11).

Let us now briefly outline the contents of the paper. In section 2, we consider the free transport equation for which we apply the above strategy. We present for this very simple case the differents tools (renomalized and biting L^1-weak covergence, trace theory and Darrozès-Guiraud information), we state a first velocity L^1-weak compactness result and then we prove the corresponding version of the stability Theorem 1.2. In Section 3, we develop the notion of renormalized convergence in a more general framework and we prove some more accurate version of biting L^1-weak covergence and velocity L^1-weak compactness. In Section 4, we present the trace theory for the Vlasov-Fokker-Planck equation with Sobolev regularity on the force field. In Section 5, puting together the results from Section 3 and Section 4, we establish the renormalized and the almost everywhere convergence of trace functions sequences. In Section 6 we present the models and we establish the main stablity (up to the boundary) results. Finally, in the appendix, we come back to the notion of renormalized convergence for which we give several relevant examples and counterexamples.

2 An illuminating example: the free transport equation.

In this section we assume that f is governed by the free transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0 \quad \text{in } (0,\infty) \times \mathcal{O}, \tag{2.1}$$

complemented with the initial condition (1.8) and the boundary reflection condition (1.1) with constant restitution coefficient $\alpha \in (0,1]$. Our aim is to adapt the *DiPerna-Lions stability theory* to that simple boundary value problem. We follow the strategy expounded in the introduction. We first collect the a priori bounds satisfied by a solution to the boundary value problem (2.1)-(1.1)-(1.8) with initial datum satisfying (1.9). We next present some general functional analysis tools which roughly speaking makes possible to deduce L^1 weak convergence in the v variable from L^1 boundedness and *Darrozès-Guiraux information* boundedness. We finally state and prove the stability result associated to the boundary value problem (2.1)-(1.1)-(1.8).

Remark 2.1 *It is worth mentioning that the proof of the corresponding stability result for the Boltzmann equation is essentially the same as for the free transport equation. We refer to section 6 where that model is handled. However, the reader who is only interested by the Boltzmann model may easily adapt the proof below with the arguments introduced in [55] (it will be more elementary than the proof presented in Section 3 to Section 6 which is made in order to also deal with a Vlasov-Poisson term and/or with a Fokker-Planck term).*

2.1 A priori bounds.

Lemma 2.2 *For any nonnegative initial datum f_{in} such that (1.9) holds and any time $T \in (0,\infty)$ there exists a constant C_T (only depending on C_0 and T) such that any solution f to the initial boundary value problem (2.1)-(1.1)-(1.8) satisfies (at least formally)*

$$\sup_{[0,T]} \iint_{\mathcal{O}} f\,(1+|v|^2+|\log\, f|)\,dxdv + \alpha \int_0^T \int_{\partial\Omega} \mathcal{E}\left(\frac{\gamma_+ f}{M}\right) d\sigma_x dt \leq C_T, \tag{2.2}$$

where $\mathcal{E}$ is defined in (1.13), and

$$\alpha \int_0^T \iint_{\Sigma} \gamma f\,(1+|v|^2)\,|n(x)\cdot v|\,dvd\sigma_x dt \leq C_T. \tag{2.3}$$

Proof of Lemma 2.2. We consider a solution f of (1.1)-(2.1)-(1.8), which is sufficiently regular and decreasing at the infinity in such a way that all integrations (by parts) that we shall perform are allowed.
Step 1. Mass conservation. Integrating the free transport equation (2.1) over x, v, using the Green formula and the fact that the mass flux on the boundary vanishes thanks to (1.5), we straightforwardly obtain that mass is conserved

$$\forall\, t \geq 0 \qquad \iint_{\mathcal{O}} f(t,.)\, dvdx = \iint_{\mathcal{O}} f_{in}\, dvdx.$$

Step 2. Relative entropy. Multiplying the free transport equation (2.1) by $h'(f/M)$, with $h(s) = s \log s$, and integrating it over x, v, we have

$$\frac{d}{dt} \iint_{\mathcal{O}} h(f/M)\, M\, dvdx = \iint_{\Sigma} h(\gamma f/M)\, M\, v \cdot n(x)\; dvd\sigma_x. \tag{2.4}$$

The Darrozès-Guiraud inequality states that the entropy boundary flux at the right hand side of equation (2.4) is nonnegative. That is a straightforward consequence of the Jensen inequality taking advantage that $d\mu_x(v) = M\,|v \cdot n(x)|\; dv$ is a probability measure. We present now the proof of an accurate version of the Darrozès-Guiraud inequality which make precise how much that term is nonnegative. From the boundary reflection condition (1.1), the convexity of h and the expression (1.2) of the reflection operator, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} h(\gamma f/M)\, d\mu_x(v) &= \int_{\Sigma_x^+} h(\gamma_+ f/M)\, d\mu_x(v) - \int_{\Sigma_x^-} h(\mathcal{R}\gamma_+ f/M)\, d\mu_x(v) \\
&\geq \alpha \left\{ \int_{\Sigma_x^+} h(\gamma_+ f/M)\, d\mu_x(v) - \int_{\Sigma_x^-} h(D\gamma_+ f/M)\, d\mu_x(v) \right\} \\
&+(1-\alpha) \left\{ \int_{\Sigma_x^+} h(\gamma_+ f/M)\, d\mu_x(v) - \int_{\Sigma_x^-} h(L\gamma_+ f/M)\, d\mu_x(v) \right\} \\
&\geq \alpha \left\{ \int_{\Sigma_x^+} h(\gamma_+ f/M)\, d\mu_x(v) - h(\widetilde{\gamma_+ f}) \right\} = \alpha\, \mathcal{E}_x(\frac{\gamma_+ f}{M}),
\end{aligned} \tag{2.5}$$

where we have performed the change of variables $L_x : v \mapsto R_x v$ in the second term with $jac\, L_x = 1$, so that this term vanishes, and where the Darrozès-Guirraud information functional $\mathcal{E}_x$ is defined in (1.13), see also (1.4). Gathering (2.4) and (2.5), we get

$$\frac{d}{dt} \iint_{\mathcal{O}} h(f/M)\, M\, dvdx + \alpha \int_{\partial\Omega} \mathcal{E}_x(\gamma_+ f)\, d\sigma_x \leq 0.$$

Finally, using the elementary estimates, that one can find in [53] for instance,

$$\int_{\mathbb{R}^N} f\, \big(\frac{|v|^2}{4\,\Theta} + |\log f|\big)\, dv \leq C_M + \int_{\mathbb{R}^N} h(f/M)\, M\, dv, \tag{2.6}$$

and

$$\int_{\mathbb{R}^N} h(f_{in}/M)\, M\, dv \leq \int_{\mathbb{R}^N} f_{in}\, \big(\frac{|v|^2}{4\,\Theta} + |\log f_{in}|\big)\, dv + C_M, \tag{2.7}$$

we obtain that (2.2) holds.

Step 3. Additional L^1 estimates. For the sake of completeness we sketch the proof of the L^1 a priori bound (2.3) yet established in [6, 56]. We multiply the free transport equation (2.1) by $n(x) \cdot v$ and we integrate it over all the variables, to get

$$\int_0^T \iint_{\Sigma} \gamma f\, (n(x) \cdot v)^2\, dvd\sigma_x dt = \Big[\iint_{\mathcal{O}} f\, n(x) \cdot v\, dvdx \Big]_T^0 + \int_0^T \iint_{\mathcal{O}} f\, v \cdot \nabla_x n(x) v\, dvdxdt,$$

so that, thanks to (2.2) and because $n \in W^{1,\infty}(\Omega)$,

$$\int_0^T \iint_\Sigma \gamma f \, (n(x)\cdot v)^2 \, dv d\sigma_x dt \leq C_T. \tag{2.8}$$

We then remark that there exists $C_1 = C_1(\Theta) \in (0,\infty)$ such that

$$\widetilde{\gamma_+ f} = C_1 \int_{\Sigma_-^x} M(v) \, \widetilde{\gamma_+ f} \, (n(x)\cdot v)^2 \, dv = C_1 \int_{\Sigma_-^x} \gamma_- f \, (n(x)\cdot v)^2 \, dv, \tag{2.9}$$

and there exists $C_2 = C_2(\Theta) \in (0,\infty)$ such that

$$\int_{\Sigma_-^x} \gamma_- f \, (1+|v|)^2 \, |n(x)\cdot v| \, dv = \int_{\Sigma_-^x} M(v) \, \widetilde{\gamma_+ f} \, (1+|v|^2) \, |n(x)\cdot v| \, dv \leq C_2 \, \widetilde{\gamma_+ f}. \tag{2.10}$$

Finally, we come back to the equation (2.1) that we multiply by $|v|^2$ and that we integrate in all variables. We obtain

$$\begin{aligned} \iint_{\mathcal{O}} f(T,.) \, |v|^2 \, dvdx \quad &+ \quad \alpha \int_0^T \iint_{\Sigma_+} \gamma_+ f \, |v|^2 \, n(x)\cdot v \, dv d\sigma_x dt \\ &= \quad \iint_{\mathcal{O}} f_{in} \, |v|^2 \, dvdx + \alpha \int_0^T \iint_{\Sigma_-} \gamma_- f \, |v|^2 \, |n(x)\cdot v| \, dv d\sigma_x dt. \end{aligned} \tag{2.11}$$

Estimate (2.3) follows gathering (2.9), (2.8) and (2.10), (2.11). □

2.2 Biting L^1-weak convergence and L^1-weak compactness in the velocity variable.

In this section we present some functional analysis results which make possible to gain L^1-weak convergence in the v variable from L^1 boundedness and boundedness of the *Darrozès-Guiraud information*. For that purpose, we introduce a first notion of weak-weak convergence, namely the biting L^1-weak convergence. It seems to have been introduced by Kadec and Pelzyński [50] and rediscovered and developed in a L^1 and bounded measure framework by Chacon and Rosenthal in the end of the 1970's, see [41], [17]. Let us first recall the definition of the biting L^1-weak convergence that we extend to a "L framework".

In the sequel Y stands for a closed and σ-compact topological space, i.e. $Y = \cup_k Y_k$ where (Y_k) is an increasing sequence of compact sets, that we provide with its σ-ring of Borel sets and with ν a locally finite Borel measure. We denote by $L(Y)$ the space of all measurable functions $\phi : Y \to \bar{\mathbb{R}}$ and by $L^0(Y)$ the subset of all measurable and almost everywhere finite functions. In order to simplify the presentation, we will be only concerned with nonnegative functions of L and L^0. Thus, in this section, we also denote by L and L^0 the cone of nonnegative functions in these spaces, and we do not specify it anymore.

Definition 2.3 *We say that a sequence (ψ_n) of $L(Y)$ converges in the biting L^1-weak sense to $\psi \in L(Y)$, in brief (ψ_n) b-converges to ψ, and we write $\psi_n \xrightarrow{b} \psi$, if for every $k \in \mathbb{N}$ we can find $A_k \subset Y_k$ in such a way that (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$, $\psi_n \in L^1(A_k)$ for all n large enough and $\psi_n \rightharpoonup \psi$ weakly in $L^1(A_k)$. In particular, that implies $\psi \in L^0(Y)$.*

The fundamental result concerning the biting L^1-weak convergence is the so-called Biting Lemma that we recall now. We refer to [25], [8], [17], [41] and [50] for a proof of this Lemma. We also refer to [1] and [33] for other developments related to the biting L^1-weak convergence. Extension of this theory to multi-valued functions has been done by Balder, Castaing, Valadier and others; we refer to [60] for precise references.

Theorem 2.4 (Biting Lemma). *Let (ψ_n) be a bounded sequence of $L^1(Y)$. There exists $\psi \in L^1(Y)$ and a subsequence $(\psi_{n'})$ such that $(\psi_{n'})$ b-converges to ψ and $\|\psi\|_{L^1} \le \liminf \|\psi_{n'}\|_{L^1}$.*

Our first result is a kind of intermediate result between the Biting Lemma and the Dunford-Pettis Lemma. More precisely, we prove a L^1-weak compactness in the v variable for sequences (ϕ_n) which are bounded in L^1 and such that the associated *Darrozès-Guiraud information* is uniformly (in n) bounded. It is based on the Biting Lemma, the Dunford-Pettis Lemma and a convexity argument.

Theorem 2.5 *Let consider $j : \mathbb{R}_+ \to \mathbb{R}$ a convex function of class $C^2(0,\infty)$ such that $j(s)/s \to +\infty$ when $s \nearrow +\infty$ and such that the application J from $(\mathbb{R}_+)^2$ to $\mathbb{R}$ defined by $J(s,t) = (j(t) - j(s))\,(t-s)$ is convex, ω a non negative function of $\mathbb{R}^N$ such that $\omega(v) \to \infty$ when $|v| \to \infty$ and, for any $y \in Y$, a probability measure μ_y on $\mathbb{R}^N$. Assume that (ϕ_n) is a sequence of non negative measurable functions on $Y \times \mathbb{R}^N$ such that*

$$\int_Y \int_{\mathbb{R}^N} \left[\phi_n(y,v)\,(1+\omega(v)) + \mathcal{E}(\phi_n(y,.))\right] d\mu_y(v)\, d\nu(y) \le C_1 < \infty, \tag{2.12}$$

where $\mathcal{E} = \mathcal{E}_{j,y}$ is the non negative Jensen information functional defined by

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^N} j(\phi)\, d\mu_y - j\Big(\int_{\mathbb{R}^N} \phi\, d\mu_y\Big) \qquad \text{if} \quad 0 \le \phi \in L^1(\mathbb{R}^N, d\mu_y).$$

Then, there exists $\phi \in L^1(Y \times \mathbb{R}^N)$ and a subsequence $(\phi_{n'})$ such that for every $k \in \mathbb{N}$ we can find $A_k \subset Y_k$ in such a way that (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$ and

$$\phi_{n'} \rightharpoonup \phi \quad \text{weakly in} \quad L^1(A_k \times \mathbb{R}^N; d\nu\, d\mu).$$

Furthermore, $\mathcal{E}$ is a convex and weakly L^1 l.s.c. functional, and thus

$$\int_Y \int_{\mathbb{R}^N} \left[\phi(y,v)\,(1+\omega(v)) + \mathcal{E}(\phi(y,.))\right] d\mu_y(v)\, d\nu(y) \le C_1. \tag{2.13}$$

Proof of Theorem 2.5. From bound (2.12) and the Biting Lemma we know that there exists a subsequence n' such that for every $k \in \mathbb{N}$ we can find a Borel set $A = A_k \subset Y_k$ with $\nu(Y_k \backslash A) < 1/k$ such that

$$\int_{\mathbb{R}^N} \phi_{n'}\, d\mu_y(v) \quad \text{weakly converges in} \quad L^1(A). \tag{2.14}$$

Thanks to the Dunford Pettis Lemma and (2.14) there is a convex function $\Phi = \Phi_k$ such that $\Phi(s)/s \to \infty$ when $s \to \infty$ and

$$\int_A \Phi\Big(\int_{\mathbb{R}^N} \phi_{n'}\, d\mu_y(v)\Big)\, d\nu(y) \le C_2 = C_2(k) < \infty.$$

Furthermore, we can assume that $\Phi(0) = 0$, $\Phi' = a_m$ in $[m, m+1]$ with $j'(s_0) \le a_m \nearrow +\infty$, where $s_0 \in \mathbb{N}^\star$ is such that $j(s_0) \ge 0$ and $j'(s_0) \ge 0$.

Then we define $\Psi = \Psi_k$ by $\Psi(s) = j(s)$ for $s \in [0, s_0]$ and by induction on $m \in \mathbb{N}$, we consider t_m such that $j'(t_m) = a_m - \Psi'(s_m) + j'(s_m)$ and we set $s_{m+1} = [t_m] + 1$, $\Psi'' := j''$ on $[s_m, t_m]$ and $\Psi'' := 0$ on $[t_m, s_{m+1}]$ so that $t_m \ge s_m \ge m$ and $\Psi'(s_{m+1}) \ge a_m \ge \Psi'(s_m)$. Therefore, we have built a convex function Ψ such that the function $s \mapsto j(s) - \Psi(s)$ is convex, $\Psi(s)/s \nearrow \infty$ since $\Psi'(s) \nearrow \infty$, and $\Psi \le \Phi$ since $\Psi' \le \Phi'$, so that

$$\int_A \Psi\Big(\int_{\mathbb{R}^N} \phi_{n'}\, d\mu\Big)\, d\nu \le C_2. \tag{2.15}$$

The Jensen inequality, written for the function $s \mapsto j(s) - \Psi(s)$, gives

$$\int_{\mathbb{R}^N} \Psi(\phi_{n'})\, d\mu - \Psi\big(\int_{\mathbb{R}^N} \phi_{n'}\, d\mu\big) \le \mathcal{E}(\phi_{n'}),$$

and combining it with (2.12) and (2.15) we get

$$\iint_{A\times\mathbb{R}^N} \Psi(\phi_{n'})\, d\mu\, d\nu \le C_1 + C_2,$$

and thus

$$\begin{aligned} \iint_{A\times\mathbb{R}^N} \Psi^+(\phi_{n'})\, d\mu_y\, d\nu \qquad &\le C_1 + C_2 + \iint_{A\times\mathbb{R}^N} \Psi^-(\phi_{n'})\, d\mu_y\, d\nu \\ &\le C_3(k) := C_1 + C_2 + \nu(A)\, \sup j^- < \infty. \end{aligned} \tag{2.16}$$

Thanks to estimates (2.12), (2.16) and the Dunford-Pettis Lemma we get that $(\phi_{n'})$ falls in a relatively weakly compact set of $L^1(A_k \times \mathbb{R}^N)$ for any $k \in \mathbb{N}$. We conclude, by a diagonal process, that there is a function $\phi \in L^1(Y \times \mathbb{R}^N)$ and a subsequence $(\phi_{n''})$ which converges to ϕ in the sense stated in Theorem 2.5.

In order to prove that $\mathcal{E}$ is a convex functional, we begin by assuming that $j \in C^1(\mathbb{R}_+, \mathbb{R})$, so that $\mathcal{E}$ is Gâteaux differentiable. By definition of the G-differential

$$\begin{aligned} \nabla\mathcal{E}(\phi)\cdot\psi \quad &:= \quad \lim_{t\to 0} \frac{\mathcal{E}(\phi + t\,\psi) - \mathcal{E}(\phi)}{t} \\ &= \quad \int_{\mathbb{R}^N} j'(\phi)\,\psi\, d\mu - j'\big(\int_{\mathbb{R}^N} \phi\, d\mu\big) \int_{\mathbb{R}^N} \psi\, d\mu, \end{aligned}$$

for any $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$. Therefore, by the Jensen inequality, we have

$$\langle\, \nabla\mathcal{E}(\psi) - \nabla\mathcal{E}(\phi), \psi - \phi\,\rangle \;= \int_{\mathbb{R}^N} J(\phi,\psi)\, d\mu - J\Big(\int_{\mathbb{R}^N} \phi\, d\mu, \int_{\mathbb{R}^N} \psi\, d\mu\Big) \ge 0,$$

so that $\nabla\mathcal{E}$ is monotone and thus $\mathcal{E}$ is convex on $L^\infty(\mathbb{R}^N)$: for any $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$ and any $t \in (0,1)$

$$\mathcal{E}(\phi + (1-t)\,\psi) \le t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi). \tag{2.17}$$

When $j \notin C^1(\mathbb{R}_+, \mathbb{R})$ we define, for any $\varepsilon > 0$, the function $j_\varepsilon(s) = j(s+\varepsilon) - j(\varepsilon)$ which belongs to $C^1(\mathbb{R}_+, \mathbb{R})$, and the above computations for the associated functional $\mathcal{E}_\varepsilon$ are correct, so that inequality (2.17) holds for $\mathcal{E}$ replaced by $\mathcal{E}_\varepsilon$. Then, writing inequality (2.17) for $\mathcal{E}_\varepsilon$ and fixed $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$, $t \in (0,1)$ and passing to the limit $\varepsilon \to 0$ we obtain that $\mathcal{E}$ is convex on $L^\infty(\mathbb{R}^N)$. Now let $0 \le \phi, \psi \in L^1(\mathbb{R}^N)$, $t \in (0,1)$. If $j(\phi)$ or $j(\psi) \notin L^1(\mathbb{R}^N)$ then $t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi) = +\infty$ and the convex inequality (2.17) obviously holds. In the other case, we have $j(\phi)$, $j(\psi) \in L^1(\mathbb{R}^N)$, we can choose two sequences $0 \le (\phi_n)$, (ψ_n) of $L^\infty(\mathbb{R}^N)$ such that $\phi_n \nearrow \phi$ and $\psi_n \nearrow \psi$ a.e.. Passing to the limit $\varepsilon \to 0$ in the convex inequality (2.17) written for ϕ_ε and ψ_ε we get, by the Lebesgue convergence dominated Theorem and the Fatou Lemma,

$$\begin{aligned} \int_{\mathbb{R}^N} j(t\,\phi + (1-t)\,\psi) \quad &\le \quad \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} j(t\,\phi_\varepsilon + (1-t)\,\psi_\varepsilon) \\ &\le \quad t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi) + j\Big(\int_{\mathbb{R}^N} t\,\phi + (1-t)\,\psi\Big), \end{aligned}$$

which exactly means that $\mathcal{E}$ is a convex functional in $L^1(\mathbb{R}^N)$. Finally, if $0 \le \phi$, $\psi \in L^1(Y \times \mathbb{R}^N)$ and $t \in (0,1)$, then $\phi(y,.)$, $\psi(y,.) \in L^1(\mathbb{R}^N)$ for almost every $y \in Y$ and, integrating the convex inequality (2.17), we obtain that the functional

$$0 \le \phi \in L^1(Y \times \mathbb{R}^N) \mapsto \mathcal{F}(\phi) = \int_Y \mathcal{E}(\phi)\, d\nu$$

is convex. Furthermore, by Fatou Lemma, $\mathcal{F}$ is l.s.c. for the strong convergence in L^1, for the weak $\sigma(L^1, L^\infty)$ convergence and for the biting L^1-weak convergence, so that (2.13) holds. □

We introduce a second kind of weak-weak convergence, namely the renormalized convergence, which is the very natural notion of convergence when we deal with sequences of trace functions, as we will see below. We now present the definition (in a simplified case) and a first elementary result that we will use in the next subsection. More about the renormalized convergence is presented in section 3.

Definition 2.6 *Let define the sequence (T_M) by setting $T_M(s) := s \wedge M = \min(s, M)\ \forall\, s,\ M \geq 0$. We say that a sequence (ϕ_n) of $L(Y)$ converges in the renormalized sense (or r-converges) if there exists a sequence $(\bar{T}_M)$ of $L^\infty(Y)$ such that*

$$T_M(\phi_n) \rightharpoonup \bar{T}_M \quad \sigma(L^\infty(Y), L^1(Y))\ \star \qquad \text{and} \qquad \bar{T}_M \nearrow \phi \qquad \text{a.e. in } Y.$$

Lemma 2.7 *For any sequence (ϕ_n) of $L(Y)$ and $\phi \in L^0(Y)$ such that $\phi_n \overset{b}{\rightharpoonup} \phi$ in the biting L^1-weak sense there exists a subsequence $(\phi_{n'})$ such that $\phi_{n'} \overset{r}{\rightharpoonup} \phi$ in the renormalized sense.*

Proof of Lemma 2.7. We follow the proof of [8] where that result is established in a L^1 framework. By assumption, for any $k \in \mathbb{N}$, there exists a Borel set A_k such that $\nu\,(Y_k \backslash A_k) < 1/k$ and $\phi_n \rightharpoonup \phi$ weakly in $L^1(A_k)$. Thanks to Dunford-Pettis Lemma, there is a function $\delta_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\delta_k(M) \to 0$ when $M \to +\infty$ and

$$\int_{A_k} \phi_n\, \mathbf{1}_{\{\phi_n \geq M\}}\, dy \leq \delta_k(M) \quad \forall\, n, M, k \in \mathbb{N}^*. \tag{2.18}$$

Moreover, there exists a subsequence $(\phi_{n'})$ of (ϕ_n) and a sequence $(\bar{T}_M)$ of $L^\infty(Y)$ such that for any $M \in \mathbb{N}$ there holds

$$T_M(\phi_{n'}) \rightharpoonup \bar{T}_M \quad \sigma(L^\infty(Y), L^1(Y)).$$

We obviously have that $(\bar{T}_M)$ is an increasing sequence in $L^\infty(Y)$ and $\bar{T}_M \leq \phi$ a.e. because that is true on any A_k. Observe that

$$0 \leq \phi_n - T_M(\phi_n) \leq (\phi_n - M)\, \mathbf{1}_{\phi_n \geq M} \qquad \text{a.e. in } Y. \tag{2.19}$$

Gathering (2.18) and (2.19) we get

$$\begin{aligned} \int_{A_k} |\phi - \bar{T}_M|\, d\nu &= \lim_{n' \to \infty} \int_{A_k} (\phi_{n'} - \bar{T}_M(\phi_{n'}))\, [\operatorname{sign}(\phi - \bar{T}_M)]\, d\nu \\ &\leq \liminf_{n' \to \infty} \int_{A_k} \phi_{n'}\, \mathbf{1}_{\{\phi_{n'} \geq M\}}\, dy \leq \delta_k(M). \end{aligned}$$

That proves $\bar{T}_M \to \phi$ a.e. in Y when $M \to \infty$, and then $\phi_{n'} \overset{r}{\rightharpoonup} \phi$. □

2.3 The trace theorem and the stability result.

Let us recall the following trace theorem which makes precise the sense of the trace of a solution.

Theorem 2.8 [56] *Let $g \in L^\infty(0, T; L^1(\mathcal{O}))$ satisfy*

$$\Lambda\, g := \partial_t g + v \cdot \nabla_x g = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathcal{O}).$$

There exists $\gamma g \in L^1_{loc}((0, T) \times \Sigma; d\lambda_2\, dt)$ and $g_0 \in L^1(\mathcal{O})$ which satisfy the renormalized Green formula

$$\int_0^T \iint_{\mathcal{O}} \beta(g)\, \Lambda\phi\, dvdxdt = \int_0^T \iint_{\Sigma} \beta(\gamma\, g)\, \phi\; n(x) \cdot v\; dvd\sigma_x dt - \iint_{\mathcal{O}} \beta(g_0)\, \phi\, dxdv, \tag{2.20}$$

for all $\beta \in W^{1,\infty}(\mathbb{R})$ and all test functions $\phi \in \mathcal{D}([0, T) \times \bar{\mathcal{O}})$, as well as for all $\beta \in W^{1,\infty}_{loc}(\mathbb{R})$, with $\beta' \in L^\infty(\mathbb{R})$, and all test functions $\phi \in \mathcal{D}([0, T) \times \bar{\mathcal{O}})$ such that $\phi = 0$ on $[0, T) \times \Sigma_0$.

We may then state our first main result.

Theorem 2.9 *Let $f_n \in L^\infty(0,\infty;L^1(\mathcal{O}))$ be a sequence of solutions to the initial boundary value problem (2.1)-(1.1)-(1.8) such that both (f_n) and the trace sequence (γf_n) satisfy the associated natural a priori bounds: for any $T>0$ there is a constant C_T*

$$\sup_{[0,T]} \iint_{\mathcal{O}} f_n \left(1+|v|^2+|\log f_n|\right) dvdx \le C_T \tag{2.21}$$

and

$$\int_0^T \iint_\Sigma \gamma f_n \,(1+|v|^2)\,|n(x)\cdot v|\, dvd\sigma_x dt + \int_0^T \int_{\partial\Omega} \mathcal{E}(\gamma_+ f_n)\, d\sigma_x dt \le \alpha^{-1}\, C_T. \tag{2.22}$$

On the one hand, there exists $f \in L^\infty(0,\infty;L^1(\mathcal{O}))$ satisfying (2.2) and $f_{in} \in L^1(\mathcal{O})$ satisfying (1.9) such that, up to the extraction of subsequences,

$$f_n \rightharpoonup f \quad \sigma(L^1,L^\infty), \qquad f_n(0,.) \rightharpoonup f_{in} \quad \sigma(L^1,L^\infty), \tag{2.23}$$

and f is a solution to the free transport equation (2.1)-(1.8) with initial datum f_{in}.

On the other hand, there exists $\eta_\pm \in L^1(0,T)\times\Sigma_\pm, d\lambda_1)$ for all $T\in(0,\infty)$, which furthermore satisfies (2.2)-(2.3) (with $\gamma_\pm f$ replaced by $\eta_\pm$), such that, up to the extraction of a subsequence, for any $T,\varepsilon>0$ there exists a measurable set $A\subset(0,T)\times\partial\Omega$ such that meas$((0,T)\times\partial\Omega\setminus A) < \varepsilon$ and

$$\gamma_\pm f_n \rightharpoonup \eta_\pm \quad \textit{weakly in} \quad L^1(A\times\mathbb{R}^N, d\lambda_1). \tag{2.24}$$

As a consequence, $\gamma_\pm f = \eta_\pm$ and the reflection boundary condition (1.1) holds.

Proof of Theorem 2.9. First, from (2.21) and the Dunford-Pettis lemma we deduce (2.23). Then, thanks to Lemma 2.7, extracting again a subsequence if necessary, we deduce that $f_n \xrightarrow{r} f$ or more precisely, there exists two sequences $(\bar{T}_M)$ and $(\bar{T}^0_M)$ such that

$$T_M(f_n) \rightharpoonup \bar{T}_M \quad \sigma(L^\infty,L^1)\star \quad \text{and} \quad \bar{T}_M \nearrow f \quad \text{a.e.}, \tag{2.25}$$

$$T_M(f_n(0,.)) \rightharpoonup \bar{T}^0_M \quad \sigma(L^\infty,L^1)\star \quad \text{and} \quad \bar{T}^0_M \nearrow f_{in} \quad \text{a.e..} \tag{2.26}$$

Next, from (2.22) and theorem 2.5 (with $\phi_n = \gamma_+ f_n/M$, $j(s)=s\log s$, $\omega(v)=|v|^2$, $d\nu(y)=d\sigma_x dt$, $d\mu_y(v) = |n(x)\cdot v|\, M(v)\, dv$) we deduce that (2.24) holds. Then, thanks to Lemma 2.7 again, extacting a subsequence if necessary, we deduce that $\gamma f_n \xrightarrow{r} \eta$ or more precisely, there exists a sequence $(\bar{\gamma}_M)$ such that

$$T_M(\gamma f_n) \rightharpoonup \bar{\gamma}_M \quad \sigma(L^\infty,L^1)\star \quad \text{and} \quad \bar{\gamma}_M \nearrow \eta \quad \text{a.e..} \tag{2.27}$$

One the one hand, starting from the Green renomalized formula (2.20) for the free transport equation

$$\int_0^T \iint_{\mathcal{O}} T_M(f_n)\,\Lambda\varphi\, dvdxdt = \int_0^T \iint_\Sigma T_M(\gamma\, f_n)\,\varphi\; n(x)\cdot v\; dvd\sigma_x dt - \iint_{\mathcal{O}} T_M(f_n(0,.))\,\varphi\, dxdv,$$

for any $\varphi\in\mathcal{D}([0,T)\times\bar{\mathcal{O}})$, and passing twice two the limit, first when $n\to\infty$, next when $M\to\infty$, we get

$$\int_0^T \iint_{\mathcal{O}} f\,\Lambda\varphi\, dvdxdt = \int_0^T \iint_\Sigma \eta\,\varphi\; n(x)\cdot v\; dvd\sigma_x dt - \iint_{\mathcal{O}} f_{in}\,\varphi\, dxdv.$$

In other words, f is a solution to the free transport equation and the trace Theorem 2.8 implies that $\eta=\gamma f$. On the other hand, thanks to (2.24), we may clearly pass to the limit $n\to\infty$ in the L^1-weak sense in the boundary equation

$$\gamma_- f_n = \mathcal{R}(\gamma_+ f_n) \quad \text{on} \quad A\times\mathbb{R}^N,$$

for any set A given by (2.24), and we get $\eta_- = \mathcal{R}(\eta_+)$. We conclude that the boundary condition (1.1) is well fulfilled. □

3 On the convergence in the renormalized sense.

3.1 Basic properties.

We present the main basic properties concerning the notion of convergence in the renormalized sense. More about renormalized convergence is set out in the appendix section. In that section the frameworks and notations are the same as those of subsection 2.2, and again, we only deal with nonnegative functions of $L = L(Y)$, but we do not specify it anymore.

Definition 3.1 *We say that α is a renormalizing function if $\alpha \in C_b(\mathbb{R})$ is increasing and $0 \leq \alpha(s) \leq s$ for any $s \geq 0$. We say that (α_M) is a renormalizing sequence if α_M is a renormalizing function for any $M \in \mathbb{N}$ and $\alpha_M(s) \nearrow s$ for all $s \geq 0$ when $M \nearrow \infty$. Given any renormalizing sequence (α_M), we say that (ϕ_n) (α_M)-renormalized converges to ϕ (or we just say that (ϕ_n) r-converges to ϕ) if there exists a sequence $(\bar{\alpha}_M)$ of $L^\infty(Y)$ such that*

$$\alpha_M(\phi_n) \rightharpoonup \bar{\alpha}_M \quad \sigma(L^\infty(Y), L^1(Y)) \star \quad \text{and} \quad \bar{\alpha}_M \nearrow \phi \quad \text{a.e. in } Y.$$

Notice that the renormalized convergence as defined in definition 2.6 is nothing but the (T_M)-renormalized convergence.

Proposition 3.2 *1. The (α_M)-renormalized limit in the definition 3.1 does not depend on the renormalizing sequence (α_M), but only on the sequence (ϕ_n). In other words, given two renormalizing sequences (α_M) and (β_M), if (ϕ_n) (α_M)-renormalized converges to ϕ^α and (β_M)-renormalized converges to ϕ^β then $\phi^\alpha = \phi^\beta$.*

2. For any sequence (ϕ_n) of L there exists a subsequence $(\phi_{n'})$ of (ϕ_n) and a function $\phi \in L$ such that $(\phi_{n'})$ (α_M)-renormalized converges to ϕ for any renormalizing sequence (α_M).

3. A sequence (ϕ_n) which converges to ϕ a.e. or strongly in L^p, $p \in [1, \infty]$, also r-converges to ϕ. From a sequence (ϕ_n) which converges to ϕ weakly in L^p, $p \in [1, \infty]$, or in the biting L^1-weak sense, we may extract a subsequence $(\phi_{n'})$ which r-converges to ϕ.

Remark 3.3 *1. The definition of the (α_M)-renormalized convergence with $\alpha_M \neq T_M$ is important in order to obtain the renormalized convergence of the trace functions sequence in Theorem 5.2. Indeed, T_M is not smooth enough in order to be taken as a renormalizing function for the VFP equation and we have to introduce the "smooth" renormalizing functions $\alpha := \Phi_{M,\theta}$.*

2. Because of Proposition 3.2 we will often make the abuse of language by not specifying the renormalizing sequence (α_M) used to define the (α_M)-renormalized convergence and by saying that (ϕ_n) r-converges (to ϕ) when it is only a subsequence of (ϕ_n) which r-converges (to ϕ).

3. Let notice that in general we can not exclude that the limit $\phi \equiv +\infty$, since for instance the sequence (ϕ_n) defined by $\phi_n = \phi \equiv \infty$ belongs to L and r-converges to ϕ.

Proof of the Proposition 3.2. Step 0. We first claim that for any sequence (ϕ_n) of L and any renormalizing sequence (α_M) there exists a subsequence $(\phi_{n'})$ of (ϕ_n) and $\phi \in L$ such that $(\phi_{n'})$ (α_M)-renormalized converges to ϕ. Indeed, for any M we can find a subsequence $(n_k^M)_k$ and $\bar{\alpha}_M \in L^\infty$ such that $\alpha_M(\phi_{n_k^M}) \rightharpoonup \bar{\alpha}_M$ weakly in L^∞. By a diagonal process we can obtain a unique subsequence (n') such that the above weak convergence holds for any $M \in \mathbb{N}$. Furthermore, since (α_M) is increasing, we get that $(\bar{\alpha}_M)$ is an increasing sequence of non negative measurable functions, so that it converges to a limit $\phi \in L$.

Step 1. Let assume that for a renormalizing sequence (α_K) we have $\alpha_K(\phi_n) \rightharpoonup \bar{\alpha}_K \nearrow \psi$. Thanks to step 0, there exits a sub-sequence $(\phi_{n'})$, a sequence $\bar{T}_M \in L^\infty$ and a function $\phi \in L$ such that $T_M(\phi_{n'}) \rightharpoonup \bar{T}_M \nearrow \phi$. It is clear that $\forall K, M \in \mathbb{N}\ \forall \varepsilon > 0$ there is $k_{M,\varepsilon}, m_K \in \mathbb{N}$ such that $\alpha_K \leq T_{m_K}$ and $T_M \leq \alpha_{k_{M,\varepsilon}} + \varepsilon$. Therefore, writing that $\alpha_K(\phi_n) \leq T_{m_K}(\phi_n)$ and $T_M(\phi_n) \leq \alpha_{k_{M,\varepsilon}}(\phi_n) + \varepsilon$, and passing to the limit $n \to +\infty$, we get

$$\bar{\alpha}_K \leq \bar{T}_{m_K} \leq \phi \quad \text{and} \quad \bar{T}_M \leq \bar{\alpha}_{k_{M,\varepsilon}} + \varepsilon \leq \psi + \varepsilon.$$

Then passing to the limit $M, K \nearrow \infty$ we obtain that $\psi \le \phi \le \psi + \varepsilon$ for any $\varepsilon > 0$, and finally passing to the limit $\varepsilon \to 0$ we conclude that $\psi = \phi$.

Step 2. Let remark that the class of renormalizing functions is separable for the uniform norm of $C(\mathbb{R}_+)$. For instance, the family $\mathcal{A} = \{\alpha^k\}$ of functions α such that

$$0 \le \alpha(s) \le s \quad \text{and} \quad \alpha'(s) = \sum_{j=1}^{J} \theta_j \, \mathbf{1}_{[a_j, a_{j+1}[}(s), \quad a_j, \, \theta_j \in \mathbb{Q}_+$$

is countable and dense. By a diagonal process and thanks to step 0, we can find a subsequence $(\phi_{n'})$ in such a way that for any $\alpha \in \mathcal{A}$ there exists $\bar{\alpha} \in L^\infty$ such that $\alpha(\phi_{n'}) \rightharpoonup \bar{\alpha}$. Let fix now (β_M) a renormalizing sequence. On one hand, for any M there exists a sequence (α_k) of $\mathcal{A}$ such that $\alpha^k \le \beta_M \le \alpha^k + 1/k$ for any $k \in \mathbb{N}$ and $\alpha_k \nearrow \beta_M$. We yet know that $\alpha_k(\phi_{n'}) \rightharpoonup \bar{\alpha}_k$. Since $(\bar{\alpha}_k)$ is not decreasing, it converges a.e., and we set $\beta^*_M = \lim \bar{\alpha}_k$. On the other hand, thanks to Step 0, there exists a subsequence $(\phi_{n''})$ and a function $\bar{\beta}_M$ such that $\beta_M(\phi_{n''}) \rightharpoonup \bar{\beta}_M$. That implies $\bar{\alpha}_k \le \bar{\beta}_M \le \bar{\alpha}_k + 1/k$. Passing to the limit $k \to \infty$, we get $\bar{\beta}_M = \beta^*_M$. Therefore, by uniqueness of the limit, it is the all sequence $\beta_M(\phi_{n'})$ which converges to β^*_M. Finally, thanks to the usual monotony argument we deduce that $\phi_{n'}$ converges in the (β_M)-renormalized sense and its limit is necessary ϕ thanks to Step 1.

Step 3. If $\phi_n \to \phi$ a.e. or strongly in L^p then clearly $\alpha_M(\phi_n) \rightharpoonup \alpha_M(\phi)$ L^∞-weak and $\alpha_M(\phi) \nearrow \phi$ for any renormalizing sequence (α_M), so that $\phi_n \xrightarrow{r} \phi$. If (ϕ_n) converges weakly in L^p, $p \in [1, \infty]$, then it obviously converges in the biting L^1-weak sense and we may apply Lemma 2.7. □

Let now define the limit superior and the limit inferior in the renormalized sense.

Definition 3.4 *Let (ϕ_n) be a sequence of L. Consider I the set of all the increasing applications $\imath : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(\phi_{\imath(k)})_{k \ge 0}$ of $(\phi_n)_{n \ge 0}$ converges in the renormalized sense and note $\phi_\imath = r\text{-}lim\, \phi_{\imath(k)}$. Thanks to the Proposition 3.2.2, we know that I is not empty. We defined the limit superior and the limit inferior of (ϕ_n) in the renormalized sense by*

$$r\text{-}limsup\, \phi_n := \sup_{\imath \in I} \phi_\imath \qquad \text{and} \qquad r\text{-}liminf\, \phi_n := \inf_{\imath \in I} \phi_\imath.$$

It is clear that if $r\text{-}limsup\, \phi_n = r\text{-}liminf\, \phi_n$ then (ϕ_n) r-converges (up to the extraction of a subsequence).

Proposition 3.5 *1. If $\phi_n \xrightarrow{r} \phi$, $\psi_n \xrightarrow{r} \psi$ and $\lambda_n \to \lambda$ in $\mathbb{R}^\star_+$ then $\phi_n + \lambda\, \psi_n \xrightarrow{r} \phi + \lambda\, \psi$.*

2. Let $\phi_n \xrightarrow{r} \phi$ and β be a nonnegative and concave function then $\beta(\phi) \ge r\text{-}limsup\, \beta(\phi_n)$.

3. Let β be a strictly concave function, and (ϕ_n) be a sequence such that $\phi_n \xrightarrow{r} \phi$ and $\beta(\phi) \le r\text{-}liminf\, \beta(\phi_n)$ then, up to the extraction a subsequence, $\phi_n \to \phi$ a.e. in Y.

4. Let $\phi_n \xrightarrow{r} \phi$ and S be a bounded and nonnegative operator of L^1 then $S\, \phi \le r\text{-}liminf\, S\, \phi_n$.

Proof of the Proposition 3.5. Step 1. From the elementary inequality

$$\forall\, a, b, M \ge 0 \qquad M \wedge (a + b) \le M \wedge a + M \wedge b \le (2\, M) \wedge (a + b),$$

we deduce

$$\text{w-lim}\, [M \wedge (\phi_n + \psi_n)] \le \text{w-lim}\, [M \wedge \phi_n] + \text{w-lim}\, [M \wedge \psi_n] \le \text{w-lim}\, [(2\, M) \wedge (\phi_n + \psi_n)]$$

so that $r\text{-}lim\, (\phi_n + \psi_n) = \phi + \psi$. Next, from the elementary identity

$$\forall\, a, b, M \ge 0 \qquad (a\, b) \wedge M = a\, \big(b \wedge (M/a)\big)$$

and because for any $\varepsilon > 0$ there holds $0 < \lambda - \varepsilon \le \lambda_n \le \lambda + \varepsilon$ for n large enough, we have

$$(\lambda - \varepsilon) \left[\phi_n \wedge \frac{M}{\lambda - \varepsilon}\right] \le (\lambda_n\, \phi_n) \wedge M \le (\lambda + \varepsilon) \left[\phi_n \wedge \frac{M}{\lambda + \varepsilon}\right].$$

We deduce that for a subsequence $(\lambda_{n'}\,\phi_{n'})$

$$\forall\,\varepsilon>0\qquad (\lambda-\varepsilon)\,\bar T_{\frac{M}{\lambda-\varepsilon}}\le \underset{n'\to\infty}{\text{w-lim}}\, T_M(\lambda_{n'}\,\phi_{n'})\le (\lambda+\varepsilon)\,\bar T_{\frac{M}{\lambda+\varepsilon}},$$

so that, passing to the limit $\varepsilon\to 0$ and using that $T_{M/(\lambda+\varepsilon)}\le T_{M/(\lambda-\varepsilon)}$,

$$\lambda\,\bar T_M=\underset{n'\to\infty}{\text{w-lim}}\, T_M(\lambda_{n'}\,\phi_{n'}).$$

Passing to the limit $M\to\infty$, we conclude that $r\text{-}lim\,(\lambda_n\,\phi_n)=\lambda\,\phi$.

Step 2. We know that

$$\beta(s)=\inf_{\ell\ge\beta}\ell(s),$$

where the inf is taken over all real values affine functions $\ell(t)=a\,t+b$ which satisfy $a,b\ge 0$ and $\beta(t)\le\ell(t)$ for any $t\ge 0$. Furthermore, for any ℓ and M, there clearly exists K_M such that

$$T_M(\ell(s))\le\ell(T_K(s))\qquad\text{and}\qquad \ell(T_M(s))\le T_K(\ell(s))\qquad\text{for all}\qquad K\ge K_M,\ s\ge 0.$$

We deduce that for any $\ell\ge\beta$, we have

$$T_M(\beta(\phi_n))\le\ell(T_K(\phi_n)).$$

Therefore, we get

$$\limsup_n T_M(\beta(\phi_n))\le\ell(\lim_n T_K(\phi_n))\le\ell(\phi)$$

and finally

$$\limsup_n T_M(\beta(\phi_n))\le\beta(\phi)\qquad\text{for any } M,$$

which exactly means that $r\text{-}limsup\,\beta(\phi_n)\le\beta(\phi)$.

Step 3. For any subsequence (n') such that $\beta(\phi_{n'})$, $\beta(\phi_{n'}/2+\phi/2))$ and $\beta(\phi_{n'}/2+\phi/2))-\beta(\phi_{n'})/2-\beta(\phi)/2\ge 0$ converge in the renormalized sense, we have

$$r\text{-}lim\,\big[\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi_{n'})}{2}-\frac{\beta(\phi)}{2}\big]+\frac{\beta(\phi)}{2}+r\text{-}lim\,\frac{\beta(\phi_{n'})}{2}=r\text{-}lim\,\beta\big(\frac{\phi_{n'}+\phi}{2}\big),$$

thanks to step 1. As a consequence, we get

$$\begin{aligned}0&\le\ r\text{-}lim\,\big[\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi_{n'})}{2}-\frac{\beta(\phi)}{2}\big]\\&=\ r\text{-}lim\,\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi)}{2}-r\text{-}lim\,\frac{\beta(\phi_{n'})}{2}\\&\le\ \beta(\phi)-\frac{\beta(\phi)}{2}-\frac{\beta(\phi)}{2}=0,\end{aligned}$$

thanks to step 2 and because $\phi_{n'}/2+\phi/2\overset{r}{\rightharpoonup}\phi$. Therefore, for any k, we have

$$\begin{aligned}0&\le\ \lim_{n\to\infty}\int_{Y_k}T_1\Big(\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi_{n'})}{2}-\frac{\beta(\phi)}{2}\Big)\,d\nu\\&\le\ \int_{Y_k}r\text{-}limsup\,\big[\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi_{n'})}{2}-\frac{\beta(\phi)}{2}\big]\,d\nu=0,\end{aligned}$$

so that, up to extraction a subsequence,

$$\beta\big(\frac{\phi_{n'}+\phi}{2}\big)-\frac{\beta(\phi_{n'})}{2}-\frac{\beta(\phi)}{2}\to 0\quad\text{a.e. on } Y\qquad\text{and}\qquad\phi_{n'}\to\phi\quad\text{a.e. on } Y.$$

Step 4. Fix $\chi \in C_c(Y)$, the space of continuous functions on Y with compact support, such that $0 \le \chi \le 1$. Since $T_M(\phi_n)\,\chi \rightharpoonup \bar{T}_M\,\chi$ weakly in L^1, we have

$$S(T_M(\phi_n)\,\chi) \rightharpoonup S(\bar{T}_M\,\chi) \quad \text{weakly in } L^1. \tag{3.1}$$

We deduce, using $T_K(S(T_M(\phi_n)\,\chi)) \le T_K(S(\phi_n))$ and Proposition 3.2.3 that

$$S(\bar{T}_M\,\chi) = \underset{n\to\infty}{r\text{-}liminf}\ S(T_M(\phi_n)\,\chi) \le \underset{n\to\infty}{r\text{-}liminf}\ S(\phi_n).$$

We conclude letting $\chi \nearrow 1$ and $M \to +\infty$. □

3.2 From renormalized convergence to weak convergence.

We give now a kind of extension of the Biting Lemma in the L^0 framework.

Definition 3.6 *We say that a sequence (ψ_n) is asymptotically bounded in $L^0(Y)$ if for any $k \in \mathbb{N}$ there exists $\delta_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\delta_k(M) \searrow 0$ when $M \nearrow +\infty$ and for any M there is $n_{k,M}$ such that*

$$meas\,\{y \in Y_k,\ \psi_n(y) \ge M\} \le \delta_k(M) \qquad \forall k \in \mathbb{N},\ \forall n \ge n_{k,M}. \tag{3.2}$$

Theorem 3.7 *Let (ψ_n) be a sequence of $L^0(Y)$ which r-converges to ψ with $\psi \in L^0(Y)$. Then (ψ_n) is asymptotically bounded in $L^0(Y)$ and there exists a subsequence $(\psi_{n'})$ which b-converges to ψ.*

Remark 3.8 *In the L^1 framework, J. Ball & F. Murat [8] have already proved that the biting L^1-weak convergence implies, up to the extraction of a subsequence, the convergence in the renormalized sense, as it has been recalled and extended to the L^0 framework in Lemma 2.7. As a consequence, combining Ball & Murat's result with Theorem 3.7, we get the equivalence between the biting L^1-weak convergence and the renormalized convergence. More precisely, considering a sequence (ψ_n) of $L(Y)$, it is equivalent to say that, up to the extraction of a subsequence,*

$$\psi_n \overset{b}{\rightharpoonup} \psi \quad \textit{in the biting } L^1\textit{-weak sense (so that } \psi \in L^0(Y)\textit{)}, \tag{3.3}$$

$$\psi_n \overset{r}{\rightharpoonup} \psi \quad \textit{in the renormalized sense and } \psi \in L^0(Y). \tag{3.4}$$

Furthermore, in both cases, the full sequence (ψ_n) is asymptotically bounded in L^0. Again, we refer to the appendix where some complements about r-convergence and b-convergence are given.

Proof of Theorem 3.7. Step 1. Proof of the asymptotic boundedness in L^0. We argue by contradiction. For an arbitrary $\varepsilon > 0$ we know that there exists $B \subset Y_k$ such that $\nu(Y_k \backslash B) < \varepsilon/2$ and $\psi \in L^1(B)$. If there is no $m \in \mathbb{N}$ such that $\text{meas}\,\{y \in B,\ \psi_n(y) \ge m\} < \varepsilon/2$ for all n large enough, this means that there exists an increasing sequence (n_m) such that

$$\text{meas}\,\{y \in B,\ \psi_{n_m}(y) \ge m\} \ge \varepsilon/2 \qquad \forall m \ge 0.$$

Therefore, for any $\ell \in \mathbb{N}$ and any $m \ge \ell$ we have

$$\int_B T_\ell(\psi_{n_m}) \ge \ell\,\text{meas}\,\{y \in B,\ \psi_{n_m}(y) \ge \ell\} \ge \ell\,\text{meas}\,\{y \in B,\ \psi_{n_m}(y) \ge m\} \ge \ell\,\frac{\varepsilon}{2},$$

and passing to the limit $m \to \infty$, we get

$$\int_B \psi \ge \int_B \underset{m\to\infty}{\text{w-lim}}\ T_\ell(\psi_{n_m}) \ge \ell\,\frac{\varepsilon}{2} \qquad \forall \ell \ge 0.$$

Letting $\ell \nearrow \infty$ we get a contradiction with the fact that $\psi \in L^1(B)$. As a conclusion, we have proved that for any $\varepsilon > 0$ there exists m_ε and n_ε such that $\text{meas}\,\{y \in Y_k,\ \psi_n \ge m_\varepsilon\} < \varepsilon$ for any $n \ge n_\varepsilon$, and (3.2) easily follows.

Step 2. Proof of the convergence in the biting L^1-weak sense. As in Step 1, for any $k \in \mathbb{N}$ we can choose B such that $\nu(Y_k \backslash B) < 1/3k$ and $\psi \in L^1(B)$. Setting $\displaystyle\int_B \psi \, dy = C_0$, we construct a sequence (n_ℓ) such that

$$\int_B T_\ell(\psi_{n_\ell}) \, dy \leq C_0 + \frac{1}{\ell}. \tag{3.5}$$

From (3.5), Theorem 2.4 (Biting Lemma) and Lemma 2.7, we may extract a subsequence, still denoted by (ψ_{n_ℓ}), which b-converges and r-converges to a limit denoted by $\psi^* \in L^1(B)$. On the one hand, for any $M \in \mathbb{N}$ we have $T_M(\psi_{n_\ell}) \leq T_\ell(\psi_{n_\ell})$ for $\ell \geq M$ so that, passing to the limit $\ell \to \infty$, we get w-$\lim T_M(\psi_{n_\ell}) \leq \psi^*$ and thus $\psi \leq \psi^*$. On the other hand, from Theorem 2.4 (Biting Lemma) again, we have $\|\psi^*\|_{L^1} \leq \liminf \|T_\ell(\psi_{n_\ell})\|_{L^1} \leq C_0 = \|\psi\|_{L^1}$. Gathering these two inequalities, we have proved

$$T_\ell(\psi_{n_\ell}) \rightharpoonup \psi \quad \text{ weakly in } L^1(B).$$

Furthermore, since (ψ_n) is asymptotically bounded in $L^0(Y)$ we have, up to the extraction of a subsequence again,

$$\text{meas}\{\psi_{n_\ell} \neq T_\ell(\psi_{n_\ell})\} = \text{meas}\{\psi_{n_\ell} > \ell\} \leq \delta_k(\ell) \underset{\ell \to \infty}{\longrightarrow} 0.$$

Therefore, we can choose an other subsequence, still noted (ψ_{n_ℓ}), such that $Z_L := \{\forall \ell \geq L \ / \ \psi_{n_\ell} \neq T_\ell(\psi_{n_\ell})\}$ satisfies

$$\text{meas}(Z_L) \leq \sum_{\ell \geq L} \text{meas}\{\psi_{n_\ell} > \ell\} \underset{L \to \infty}{\longrightarrow} 0.$$

Finally, choosing L large enough such that meas $(Z_L) < 1/3k$ and setting $A_k := B \cap Z_L^c$, we have $|Y_k \backslash A| < 1/k$, $\psi_{n_\ell} \in L^1(A)$ for all $\ell \geq L$ and

$$\psi_{n_\ell} = T_\ell(\psi_{n_\ell}) \rightharpoonup \psi \quad \text{ weakly in } L^1(A).$$

We conclude thanks to a diagonal process. □

A simple but fundamental consequence of Theorem 2.5 and Theorem 3.7 is the following.

Theorem 3.9 *Consider a function $m : \mathbb{R}^N \to \mathbb{R}$ and a family of measures $d\varpi_y$ on $\mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^N} m(v) \, d\varpi_y(v) = 1, \quad \int_{\mathbb{R}^N} m(v)^{1/4} \, d\varpi_y(v) \leq C_4 \ \forall y \quad \textit{and} \quad m(0) \geq m(v) \underset{|v| \to \infty}{\longrightarrow} 0.$$

Let (ϕ_n) be a sequence of $L^0(Y \times \mathbb{R}^N)$ which satisfies

$$\int_Y \mathcal{E}\Big(\frac{\phi_n(y,.)}{m(.)}\Big) \, d\nu(y) \leq C_1 < \infty,$$

with $\mathcal{E}$ just like in Theorem 2.5 with $d\mu_y(v) = m(v) \, d\varpi_y(v)$, and assume that

$$\psi_n(y) := \int_{\mathbb{R}^N} \phi_n(y,v) \, d\varpi_y(v) \overset{r}{\rightharpoonup} \psi \quad \textit{with} \quad \psi \in L^0(Y). \tag{3.6}$$

Then, there exists $\phi \in L^1(Y \times \mathbb{R}^N, d\nu d\varpi)$ and a subsequence $(\phi_{n'})$ such that for every $k \in \mathbb{N}$ we can find $A_k \subset Y_k$ in such a way that (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$ and

$$\phi_{n'} \rightharpoonup \phi \quad \textit{weakly in} \quad L^1(A_k \times \mathbb{R}^N, d\nu d\varpi).$$

As a consequence $\psi = \displaystyle\int_{\mathbb{R}^N} \phi \, d\varpi$ and $\mathcal{E}(\phi/m) \in L^1(Y)$.

Proof of Theorem 3.9. From (3.6), Theorem 3.7 and Definition 2.3 we know that there exists a subsequence $(\psi_{n'})$ such that for every $k \in \mathbb{N}$ we can find $A = A_k \subset Y_k$ satisfying (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$ and

$$\psi_{n'} \quad \text{is weakly compact in} \quad L^1(A).$$

Next, we come back to estimate (2.16) in the proof of Theorem 2.5, which written with the new notation, becomes

$$\iint_{A\times\mathbb{R}^N} \phi_{n'}\, \Xi\big(\frac{\phi_{n'}}{m(v)}\big)\, d\varpi_y d\nu \le C_3, \tag{3.7}$$

where we have set $\Xi(s) := \Psi^+(s)/s$. Of course, we can assume without loss of generality that Ξ is not decreasing, $\Xi(s) \nearrow \infty$ when $s \nearrow \infty$ and $\Xi(s) \le s^{1/2}$. From (3.7) we deduce

$$\iint_{A\times\mathbb{R}^N} \phi_{n'}\, \Xi\big(\frac{\phi_{n'}}{m(0)}\big)\, d\varpi_y\, d\nu \le C_3, \tag{3.8}$$

as well as

$$\begin{aligned} \iint_{A\times\mathbb{R}^N} \phi_{n'}\, \Xi(m(v)^{-1/2})\, d\varpi_y\, d\nu \le \\ \le \iint_{A\times\mathbb{R}^N} \phi_{n'}\, \Xi(m(v)^{-1/2}) \left(\mathbf{1}_{\{\phi_{n'}\le m(v)^{1/2}\}} + \mathbf{1}_{\{\phi_{n'}\ge m(v)^{1/2}\}}\right) d\varpi_y\, d\nu \\ \le \iint_{A\times\mathbb{R}^N} m(v)^{1/4}\, d\varpi_y\, d\nu + \iint_{A\times\mathbb{R}^N} \phi_{n'}\, \Xi\big(\frac{\phi_{n'}}{m(v)}\big)\, d\varpi_y\, d\nu \le C_4\, |Y_k| + C_3. \end{aligned} \tag{3.9}$$

Gathering (3.8) and (3.9), we deduce by the Dunford-Pettis Lemma that $(\phi_{n'})$ belongs to a weak compact set of $L^1(A\times\mathbb{R}^N, d\nu d\varpi)$, and we conclude as in the end of the proof of Theorem 2.5. □

4 Trace theorems for solutions of the Vlasov-Fokker-Planck equation.

4.1 Statement of the trace theorems

In this section we recall the trace results established in [55], [56] for the Vlasov equation (which corresponds to the case $\nu = 0$ in the Theorem below) and we extend them to the VFP equation. Given a vector field $E = E(t,x,v)$, a source term $G = G(t,x,v)$, a constant $\nu \ge 0$ and a solution $g = g(t,x,v)$ to the Vlasov-Fokker-Planck equation

$$\Lambda_E\, g = \frac{\partial g}{\partial t} + v\cdot\nabla_x g + E\cdot\nabla_v g - \nu\,\Delta_v g = G \ \text{ in } (0,T)\times\mathcal{O}, \tag{4.1}$$

we show that g has a trace γg on the boundary $(0,T)\times\Sigma$ and a trace $\gamma_t g$ on the section $\{t\}\times\mathcal{O}$ for any $t \in [0,T]$. These trace functions are defined thanks to a Green formula. We write indifferently $\gamma_t g = g(t,.)$.

The meaning of equation (4.1) is of two kinds. In the first case, we assume that $g \in L^\infty(0,T; L^p_{loc}(\bar{\mathcal{O}}))$, with $p \in [1,\infty]$, is a solution of (4.1) in the sense of distributions, i.e.,

$$\int_0^T \iint_{\mathcal{O}} (g\,\Lambda_E^\star\phi + G\,\phi)\, dvdxdt = 0, \tag{4.2}$$

for all test functions $\phi \in \mathcal{D}((0,T)\times\mathcal{O})$, where we have set

$$\Lambda_E^\star\,\phi = \frac{\partial\phi}{\partial t} + v\cdot\nabla_x\phi + E\cdot\nabla_v\phi + \nu\,\Delta_v\phi + (\mathrm{div}_v E)\,\phi.$$

In this case we assume

$$E \in L^1\big(0,T;W^{1,p'}_{loc}(\bar{\mathcal{O}})\big), \quad \mathrm{div}_v E \in L^1\big(0,T;L^{p'}_{loc}(\bar{\mathcal{O}})\big), \quad G \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}), \tag{4.3}$$

where $p' \in [1,\infty]$ stands for the conjugate exponent of p, given by $1/p + 1/p' = 1$, and we make one of the two additional hypothesis

$$\nu \int_0^T \int_{\mathcal{O}_R} |\nabla_v g|^2 \, dvdxdt \le C_{T,R} \tag{4.4}$$

or

$$\nu \int_0^T \int_{\mathcal{O}_R} |\nabla_v g|^2 \, \mathbf{1}_{\{M\le|g|\le M+1\}} \, dvdxdt \le C_{T,R} \quad \forall M \ge 0. \tag{4.5}$$

Remark 4.1 *The bound (4.5) is the natural bound that appears when we consider, for example, the initial value problem with initial datum $g_0 \in L^p(\mathcal{O})$ when $\Omega = \mathbb{R}^N$ or when Ω is an open subset of $\mathbb{R}^N$ and specular reflections are imposed at the boundary.*

In the second case, we assume that g is a renormalized solution of (4.1). In order to make precise the meaning of such a solution, we must introduce some notations. We denote by $\mathcal{B}_1$ the class of all functions $\beta \in W^{2,\infty}(\mathbb{R})$ such that β' has a compact support and by $\mathcal{B}_2$ the class of all functions $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' has a compact support. Remark that for every $u \in L(Y)$ and $\beta \in \mathcal{B}_1$ one has $\beta(u) \in L^\infty(Y)$. We shall write $g \in C([0,T];L(\mathcal{O}))$ if $\beta(g) \in C([0,T];L^1_{loc}(\bar{\mathcal{O}}))$ for every $\beta \in \mathcal{B}_1$.

We say that $g \in L((0,T)\times\mathcal{O})$ is a renormalized solution of (4.1) if for all $\beta \in \mathcal{B}_1$ we have

$$E \in L^1\big(0,T;W^{1,1}_{loc}(\bar{\mathcal{O}})\big), \ \beta'(g)\,G \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}), \ \nu\,\beta''(g)\,|\nabla_v\, g|^2 \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}), \tag{4.6}$$

and $\beta(g)$ is solution of

$$\Lambda_E\,\beta(g) = \beta'(g)\,G - \nu\,\beta''(g)\,|\nabla_v\, g|^2 \ \text{ in } \mathcal{D}'((0,T)\times\mathcal{O}). \tag{4.7}$$

We can now state the trace theorems for the Vlasov-Fokker-Planck equation (4.1).

Theorem 4.2 (The case $p = \infty$). *Let $g \in L^\infty([0,T]\times\mathcal{O})$ be a solution of equation (4.2)-(4.3)-(4.4). There exists γg defined on $(0,T)\times\Sigma$ and for every $t \in [0,T]$ there exists $\gamma_t g \in L^\infty(\mathcal{O})$ such that*

$$\gamma_t g \in C([0,T];L^a_{loc}(\bar{\mathcal{O}})) \quad \forall a \in [1,\infty) \quad \text{and} \quad \gamma\, g \in L^\infty((0,T)\times\Sigma), \tag{4.8}$$

and the following Green formula

$$\begin{aligned}\int_{t_0}^{t_1} \iint_{\mathcal{O}} \big(\beta(g)\,\Lambda_E^\star\phi + (\beta'(g)\,G - \nu\,\beta''(g)\,|\nabla_v\, g|^2)\,\phi\big)\,dvdxdt = \\ = \Big[\iint_{\mathcal{O}} \beta(g(t,.))\,\phi\,dxdv\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1}\iint_{\Sigma} \beta(\gamma\, g)\,\phi\ n(x)\cdot v\ dvd\sigma_x dt,\end{aligned} \tag{4.9}$$

holds for all t_0, $t_1 \in [0,T]$, all $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ and all test function $\phi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$.

Remark 4.3 *A fundamental point, which is a consequence of the Green formula (4.9), is the possibility of renormalizing the trace function, i.e.*

$$\gamma\,\beta(g) = \beta(\gamma\, g) \tag{4.10}$$

for all $\beta \in W^{2,\infty}(\mathbb{R})$. More generally, (4.10) holds as soon as $\gamma\,\beta(g)$ is defined. This is the property that will allow us to define the trace of a renormalized solution.

Theorem 4.4 (The case $p \in [1,\infty)$). *Let $g \in L^\infty(0,T;L^p_{loc}(\bar{\mathcal{O}}))$ be a solution of equation (4.2)-(4.3)-(4.5). There exists γg defined on $(0,T)\times\Sigma$ and for every $t\in[0,T]$ there exists $\gamma_t g \in L^p(\mathcal{O})$ such that*

$$\gamma_t g \in C([0,T];L^1_{loc}(\mathcal{O})) \quad \text{and} \quad \gamma\, g \in L^1_{loc}\big([0,T]\times\Sigma, d\lambda_2\big), \tag{4.11}$$

and satisfy the Green formula (4.9) for every t_0, $t_1\in[0,T]$, every $\beta\in\mathcal{B}_1$ and every test functions $\phi\in\mathcal{D}([0,T]\times\bar{\mathcal{O}})$, and also for every t_0, $t_1\in[0,T]$, every $\beta\in\mathcal{B}_2$ and every test functions $\phi\in\mathcal{D}_0([0,T]\times\bar{\mathcal{O}})$, the space of functions $\phi\in\mathcal{D}([0,T]\times\bar{\mathcal{O}})$ such that $\phi=0$ on $(0,T)\times\Sigma_0$.

Theorem 4.5 (The renormalized case). *Let $g\in L((0,T)\times\mathcal{O})$ satisfy the bound condition (4.6) and the equation (4.7). Then $\gamma g\in L([0,T]\times\Sigma)$ and for every $t\in[0,T]$ there exists $\gamma_t g\in C([0,T];L(\mathcal{O}))$ which satisfy the Green formula (4.9) for all t_0, $t_1\in[0,T]$, all $\beta\in\mathcal{B}_1$ and all test functions $\phi\in\mathcal{D}([0,T]\times\bar{\mathcal{O}})$. Furthermore, if (4.7) makes sense for at least one function β such that $\beta(s)\nearrow\infty$ when $s\nearrow\infty$, then of course $\gamma_t g\in L^0(\mathcal{O})$ for any $t\in[0,T]$ and $\gamma g\in L^0([0,T]\times\Sigma)$.*

4.2 Proof of the trace theorems

We begin with some notations. For a given real $R>0$, we define $B_R=\{y\in R^N \ / \ |y|<R\}$, $\Omega_R=\Omega\cap B_R$, $\mathcal{O}_R=\Omega_R\times B_R$ and $\Sigma_R=(\partial\Omega\cap B_R)\times B_R$. We also denote by $L^{a,b}_R$ the space $L^a(0,T;L^b(\mathcal{O}_R))$ or $L^a(0,T;L^b(\Omega_R))$, and $L^{a,b}_{loc}$ the space $L^a(0,T;L^b_{loc}(\bar{\mathcal{O}}))$ or $L^a(0,T;L^b_{loc}(\bar{\Omega}))$.

Proof of Theorem 4.2. First step: a priori bounds. In this step we assume that g is a solution of (4.1) and is "smooth". Precisely, $g\in W^{1,1}\big(0,T;W^{1,\infty}(\Omega;W^{2,\infty}(\mathbb{R}^N))\big)$, in such a way that the Green formula (4.9) holds. The trace γg in (4.9) is defined thanks to the usual trace theorem in the Sobolev spaces. We shall prove two *a priori* bounds on g. Let define $\beta\in W^{2,\infty}_{loc}(\mathbb{R})$ by

$$\beta(s)=\begin{cases}|s|-1/2 & \text{if } |s|\geq 1\\ s^2/2 & \text{if } |s|\leq 1\end{cases} \text{ so that } \beta'(s)=\begin{cases}1 & \text{if } s\geq 1\\ s & \text{if } |s|\leq 1\\ -1 & \text{if } s\leq -1\end{cases} \text{ and } \beta''(s)=\begin{cases}0 & \text{if } |s|\geq 1\\ 1 & \text{if } |s|\leq 1\end{cases},$$

and thus $\beta\in\mathcal{B}_1$. Fix $R>0$ and consider $\chi\in\mathcal{D}(\bar{\mathcal{O}})$ such that $0\leq\chi\leq 1$, $\chi=1$ on $\mathcal{O}_R$ and $\operatorname{supp}\chi\subset\bar{\mathcal{O}}_{R+1}$. We set $\phi=\chi\, n(x)\cdot v$. The Green formula (4.9) gives

$$\int_0^T\!\!\iint_\Sigma \beta(\gamma\, g)\,\chi\,(n(x)\cdot v)^2\,dvd\sigma_x dt = -\Big[\iint_{\mathcal{O}}\beta(g(t,.))\,\phi\,dxdv\Big]_0^T + \int_0^T\!\!\iint_{\mathcal{O}}\big(\beta(g)\,\Lambda_E^\star\phi+(\beta'(g)\,G-\nu\,\beta''(g)\,|\nabla_v\, g|^2)\,\phi\big)\,dvdxdt.$$

We deduce from it a first *a priori* bound: there are some constants γ_R and C_R such that

$$\begin{aligned}\gamma_R\int_0^T\!\!\iint_{\Sigma_R}|\gamma\, g|\,(n(x)\cdot v)^2\,dvd\sigma_x dt &\leq \int_0^T\!\!\iint_{\Sigma_R}\beta(\gamma\, g)\,(n(x)\cdot v)^2\,dvd\sigma_x dt\\ &\leq C_R\int_0^T\!\!\iint_{\mathcal{O}_{R+1}}\big(g^2\,(1+|E|)+|G|+\nu\,|\nabla_v\, g|^2\big)\,dvdxdt\\ &\quad + C_R\iint_{\mathcal{O}_{R+1}}\big(g^2(0,.)+g^2(T,.)\big)\,dxdv,\end{aligned} \tag{4.12}$$

where we have used the fact that for $u\in L^\infty(Y_R)$ with $Y_R=\mathcal{O}_R$ or Σ_R there holds

$$\gamma_R\int_{Y_R}|u| \ \leq\ \int_{Y_R}\beta(u) \ \leq\ \gamma_R^{-1}\int_{Y_R}u^2.$$

Let $K\subset\mathcal{O}$ be a compact set and consider $\phi\in\mathcal{D}(\mathcal{O})$ such that $0\leq\phi\leq 1$, $\phi=1$ on K and $R>0$ such that $\operatorname{supp}\phi\subset\mathcal{O}_R$. We fix $t_0\in[0,T]$. The Green formula (4.9) implies

$$\iint_{\mathcal{O}}\beta(g(t_1,.))\,\phi\,dxdv=\iint_{\mathcal{O}}\beta(g(t_0,.))\,\phi\,dxdv + \int_{t_0}^{t_1}\!\!\iint_{\mathcal{O}}\big(\beta(g)\,\Lambda_E^\star\phi+(\beta'(g)\,G-\nu\,\beta''(g)\,|\nabla_v\, g|^2)\,\phi\big)\,dvdxdt, \tag{4.13}$$

and we get a second *a priori* bound

$$\gamma_R \iint_K |g|(t_1,.)\,dxdv \leq C_R \iint_{\mathcal{O}_R} g^2(t_0,.)\,dxdv + C_R \int_0^T \iint_{\mathcal{O}_R} \left(g^2\,(1+|E|) + |G| + \nu\,|\nabla_v\, g|^2\right) dvdxdt. \tag{4.14}$$

Second step: regularization and passing to the limit. Let us now consider a function g which satisfies the assumptions of Theorem 4.2. We define the mollifier ρ_k by

$$\rho_k(z) = k^N\,\rho(k\,z) \geq 0, \quad k \in \mathbb{N}^\star, \quad \rho \in \mathcal{D}(\mathbb{R}^N), \quad \mathrm{supp}\,\rho \subset B_1, \quad \int_{\mathbb{R}^N} \rho(z)\,dz = 1,$$

and we introduce the regularized functions $g_k = g \star_{x,k} \rho_k *_v \rho_k$, where $*$ stands for the usual convolution and $\star_{x,k}$ for the convolution-translation defined by

$$(u \star_{x,k} h_k)(x) = [\tau_{2\,n(x)/k}(u * h_k)](x) = \int_{\mathbb{R}^N} u(y)\,h_k(x - \frac{2}{k}\,n(x) - y)\,dy,$$

for all $u \in L^1_{loc}(\bar\Omega)$ and $h_k \in L^1(\mathbb{R}^N)$ with $\mathrm{supp}\,h_k \subset B_{1/k}$.

Lemma 4.6 *With this notation one has* $g_k \in W^{1,1}\big(0,T;W^{1,\infty}(\Omega;W^{2,\infty}(\mathbb{R}^N))\big)$ *and*

$$\Lambda_E g_k = G_k \quad \textit{in} \quad \mathcal{D}'((0,T)\times\mathcal{O}),$$

with $G_k \in L^1_{loc}((0,T)\times\bar{\mathcal{O}})$ *for all* $k \in \mathbb{N}$. *Moreover, the sequences* (g_k) *and* (G_k) *satisfy*

$$\begin{cases} (g_k) \textit{ is bounded in } L^\infty((0,T)\times\mathcal{O}), \quad g_k \longrightarrow g \quad \textit{a.e. in } (0,T)\times\mathcal{O}, \\ \nabla_v g_k \longrightarrow \nabla_v g \quad \textit{in } L^2_{loc}([0,T]\times\bar{\mathcal{O}}) \quad \textit{and} \quad G_k \longrightarrow G \quad \textit{in } L^1_{loc}([0,T]\times\bar{\mathcal{O}}). \end{cases} \tag{4.15}$$

The proof of Lemma 4.6 is similar to the proof of [55, Lemma 1] and of [36, Lemma II.1] to which we refer.

From Lemma 4.6 we have that for all $k,\ell \in \mathbb{N}^\star$ the difference $g_k - g_\ell$ belongs to $W^{1,1}\big(0,T;W^{1,\infty}(\Omega;W^{2,\infty}(\mathbb{R}^N))\big)$ and is a solution of

$$\Lambda_E(g_k - g_\ell) = G_k - G_\ell \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathcal{O}).$$

We know, thanks to (4.15), that $g_k(t,.)$ converges to $g(t,.)$ in $L^2_{loc}(\bar{\mathcal{O}})$ for a.e. $t \in [0,T]$; we fix t_0 such that $g_k(t_0,.) \to g(t_0,.)$. Moreover, up to a choice for the continuous representation of g_k, we can assume that $g_k \in C([0,T],L^1_{loc}(\bar{\mathcal{O}}))$. Therefore, the estimate (4.13) applied to $g_k - g_\ell$ in t_0 and the convergence (4.15) imply that for all compact sets $K \subset \mathcal{O}$ we have

$$\sup_{t\in[0,T]} \|(g_k - g_\ell)(t,.)\|_{L^1(K)} \underset{k,\ell\to+\infty}{\longrightarrow} 0. \tag{4.16}$$

We deduce from this, that there exists, for any time $t \in [0,T]$, a function $\gamma_t g$ such that $g_k(t,.)$ converges to $\gamma_t g$ in $C([0,T];L^1_{loc}(\mathcal{O}))$; in particular,

$$g(t,x,v) = \gamma_t g(x,v) \text{ for a.e. } (t,x,v) \in (0,T)\times\mathcal{O}.$$

Thus, we also have $g_k(t,.) = (\gamma_t\,g) \star_{x,k} \rho_k *_v \rho_k$ a.e. in $(0,T)\times\mathcal{O}$, and since these two functions are continuous, the equality holds for all $(t,x,v) \in [0,T]\times\bar{\mathcal{O}}$ and $k \in \mathbb{N}^\star$, so that $g_k(t,.) \to \gamma_t g$ in $L^2_{loc}(\bar{\mathcal{O}})$ for all $t \in [0,T]$.

Using now the estimate (4.12), applied to $g_k - g_\ell$, and the convergence (4.15) and (4.16) we get that

$$\int_0^T \iint_{\Sigma_R} |\gamma g_k - \gamma g_\ell| \, (n(x)\cdot v)^2 \, dv d\sigma_x dt \underset{k,\ell\to+\infty}{\longrightarrow} 0,$$

for all $R > 0$. We deduce that there exists a function $\gamma g \in L^1_{loc}([0,T]\times\Sigma, (n(x)\cdot v)^2\, dv d\sigma_x dt)$, which is the limit of γg_k in this space. Moreover, since $\|\gamma g_k\|_{L^\infty} \le \|g_k\|_{L^\infty}$ is bounded, we have $\gamma g \in L^\infty((0,T)\times\mathcal{O})$.

Finally, we obtain the Green formula (4.9) writing it first for g_k and then passing to the limit $k\to\infty$ thanks to the convergence previously obtained. Uniqueness of the trace function follows from the Green formula. □

Proof of Theorem 4.5. The proof is based on Theorem 4.2 and on an monotony argument. This is exactly the same as the one presented in [56] in the case of Vlasov equation. Let $(\beta_M)_{M\ge1}$ be a sequence of odd functions of $\mathcal{B}_1$ such that

$$\beta_M(s) = \begin{cases} s & \text{if } s\in[0,M] \\ M+1/2 & \text{if } s\ge M+1, \end{cases}$$

and $|\beta_M(s)| \le |s|$ for all $s\in\mathbb{R}$. The function $\alpha_M(s) := \beta_M(\beta_{M+1}^{-1}(s))$, with the convention $\alpha_M(s) = M+1/2$ if $s \ge M+3/2$, is well defined, odd and also belongs to $\mathcal{B}_1$. We will construct the trace function γg as the limit of $(\gamma\beta_M(g))$ when $M\to\infty$, that one being defined thanks to Theorem 4.2. Indeed, the condition (4.5) implies that

$$\nabla_v g\, \mathbf{1}_{|g|\le M+1} \in L^2_{loc}([0,T]\times\bar{\mathcal{O}}),$$

and then $\nabla_v\beta_M(g) = \beta_M'(g)\,\nabla_v g \in L^2_{loc}([0,T]\times\bar{\mathcal{O}})$ in such a way that $\beta_M(g)$ satisfies the assumption on Theorem 4.2. We define $\Gamma_M^{(\pm)} = \{(t,x,v)\in(0,T)\times\Sigma, \pm\gamma\beta_M(g)(t,x,v) > 0\}$ and $\Gamma_M^{(0)} = \{(t,x,v)\in(0,T)\times\Sigma, \gamma\beta_M(g)(t,x,v) = 0\}$. Thanks to the definition of α_M and the renormalization property (4.10) of the trace, one has $\gamma\,\beta_M(g) = \gamma\,\alpha_M(\beta_{M+1}(g)) = \alpha_M(\gamma\,\beta_{M+1}(g))$. We deduce that, up to a set of measure zero,

$$\Gamma_M^{(+)} = \Gamma_1^{(+)}, \quad \Gamma_M^{(-)} = \Gamma_1^{(-)} \quad \text{and} \quad \Gamma_M^{(0)} = \Gamma_1^{(0)} \quad \text{for all } M\ge1.$$

Therefore the sequence $(\gamma\,\beta_M(g))_{M\ge1}$ is increasing on $\Gamma_1^{(+)}$ and decreasing on $\Gamma_1^{(-)}$. This implies that $\gamma\,\beta_M(g)$ converges a.e. to a limit denoted by γg which belongs to $L([0,T]\times\Sigma)$. Obviously, if (4.7) holds for one function β such that $\beta(s)\nearrow+\infty$ when $s\nearrow\pm\infty$, then $\beta(\gamma g)\in L^1((0,T)\times\Sigma, d\lambda_2)$ and $\gamma g\in L^0((0,T)\times\Sigma)$. In order to establish the Green formula (4.9) we fix $\beta\in\mathcal{B}_1$ and $\phi\in\mathcal{D}((0,T]\times\bar{\mathcal{O}})$. We write the Green formula for the function $\beta(\beta_M(g))$, and using the fact that $\gamma\big[\beta\circ\beta_M(g)\big] = \beta(\gamma\beta_M(g))$, we find

$$\int_0^T\iint_{\mathcal{O}} \big(\beta\circ\beta_M(g)\,(\frac{\partial\phi}{\partial t} + v\cdot\nabla_x\phi + E\cdot\nabla_v\phi) + (\beta\circ\beta_M)'(g)\,G\,\phi\big)\,dvdxdt =$$
$$= \int_0^T\iint_\Sigma \beta(\gamma\,\beta_M(g))\,\phi\,n(x)\cdot v\,dvd\sigma_x ds.$$

We get (4.9) by letting $M\to\infty$ and noticing that $\beta\circ\beta_M(s)\to\beta(s)$ for all $s\in\mathbb{R}$. □

Remark 4.7 *Theorem 4.4 is now a quite simple consequence of Theorem 4.5 using the a priori bounds stated in the proof of Theorem 4.2. Let emphasize that with the additional assumption (4.4) in hands, it is possible to give a direct proof of Theorem 4.4 (following the proof of Theorem 4.2) instead of passing through the renormalization step. See [55] for details.*

Proof of Theorem 4.4. For all $\beta\in\mathcal{B}_1$ it is clear that $\beta(g)\in L^\infty$, $\nabla_v\beta(g)\in L^2$ and that $\beta(g)$ is solution of (4.7) using Lemma 5.5 below (we just have to multiply equation (5.20), in the case

$\mu \equiv 0$, by $\beta'(g_k)$ and pass to the limit $k \to \infty$). Thanks to Theorem 4.5, we already know that g has a trace $\gamma_t g \in L(\mathcal{O})$ and $\gamma g \in L((0,T)\times\mathcal{O})$ which satisfies the Green formula (4.9) for all $\beta \in \mathcal{B}_1$ and $\phi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$. We just have to prove that γg and $\gamma_t g$ belong to the appropriate space. On one hand, for all $\beta \in \mathcal{B}_1$ such that $|\beta(s)| \le |s|$ one has

$$\|\beta(\gamma_t g)\|_{L^p_R} \le \sup_k \sup_{[0,T]} \|\beta(g_k(t,.))\|_{L^p_R} \le \sup_{[0,T]} \|g_k(t,.)\|_{L^p_R} \le \|g\|_{L^{\infty,p}_R},$$

and thus, choosing $\beta = \beta_M$, defined in the proof of Theorem 4.5, one gets, passing to the limit $M \to \infty$,

$$\sup_{[0,T]} \|\gamma_t g\|_{L^p_R} \le \|g\|_{L^{\infty,p}_R} < \infty.$$

In the same way and using (4.12), we show that

$$\|\gamma g\|_{L^1([0,T]\times\Sigma_R, d\lambda_2)} < \infty.$$

We still have to prove that $\gamma_t g \in C([0,T], L^1_{loc}(\bar{\mathcal{O}}))$, which is an immediate consequence of the following Lemma. □

Lemma 4.8 *Let (u_n) be a bounded sequence of $L^1_{loc}(\mathcal{O})$ such that $\beta(u_n) \rightharpoonup \beta(u)$ in $\big(C_c(\mathcal{O})\big)'$ for all $\beta \in \mathcal{B}_2$. Then $u_n \to u$ in $L^1_{loc}(\mathcal{O})$.*

Proof of Lemma 4.8. We fix $j : \mathbb{R} \to \mathbb{R}$ a nonnegative function of class C^2, strictly convex on the interval $[-M, M]$ and such that $j''(t) = 0$ for all $t \notin [-M, M]$; in particular $j \in \mathcal{B}_2$. We also consider $\chi \in C_c(\mathcal{O})$ such that $0 \le \chi \le 1$. By assumption

$$\int_{\mathcal{O}} j(u_n)\,\chi \to \int_{\mathcal{O}} j(u)\,\chi \tag{4.17}$$

and by convexity of j one also has

$$\liminf_{n\to\infty} \int_{\mathcal{O}} j\big(\frac{u_n+u}{2}\big)\,\chi \ge \int_{\mathcal{O}} j(u)\,\chi \quad \text{since} \quad \frac{u_n+u}{2} \rightharpoonup u \ \text{ in } \ \big(C_c(\mathcal{O})\big)'. \tag{4.18}$$

Remarking that

$$\frac{1}{2}\,j(t) + \frac{1}{2}\,j(s) - j\big(\frac{t+s}{2}\big) \ge 0 \qquad \forall t,\, s \in \mathbb{R}, \tag{4.19}$$

we deduce from (4.17) and (4.18) that

$$\int_{\mathcal{O}} \big[\frac{1}{2}\,j(u_n) + \frac{1}{2}\,j(u) - j\big(\frac{u_n+u}{2}\big)\big]\,\chi \to 0. \tag{4.20}$$

From the fact that in (4.19) the inequality is strict whenever $t,\, s \in [-M, M]$ and $t \ne s$, we obtain from (4.20) that there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e. on $\operatorname{supp}\chi \cap [\,|u| < M]$. The preceding argument being valuable for arbitrary M and χ, we obtain, by a diagonal process, a subsequence of (u_n), still denoted by (u_{n_k}), such that $u_{n_k} \to u$ a.e. in $\mathcal{O}$.

We now set $j_\pm(s) = s_\pm$. We first remark that we can write $j_\pm = j_{\pm,1} + j_{\pm,2}$ with $j_{\pm,1} \in \mathcal{B}_2$ and $j_{\pm,2} \in W^{2,\infty}(\mathbb{R})$ in such a way that

$$\int_{\mathcal{O}} j_\pm(u_{n_k})\,\chi \to \int_{\mathcal{O}} j_\pm(u)\,\chi.$$

On the other hand, the elementary inequality $\big|\, b - |a-b| \,\big| \le a \quad \forall a,\, b \ge 0$ and the dominated convergence Theorem imply $j_\pm(u_{n_k}) - |j_\pm(u_{n_k}) - j_\pm(u)| \to j_\pm(u)$ in $L^1_{loc}(\mathcal{O})$. It follows that

$$\limsup_{k\to\infty} \int_{\mathcal{O}} \big|\, j_\pm(u_{n_k}) - j_\pm(u) \,\big|\,\chi = \int_{\mathcal{O}} j_\pm(u)\,\chi - \lim_{k\to\infty} \int_{\mathcal{O}} j_\pm(u_{n_k})\,\chi = 0.$$

We conclude that $u_{n_k} = j_+(u_{n_k}) - j_-(u_{n_k}) \to j_+(u) - j_-(u) = u$ strongly in $L^1_{loc}(\mathcal{O})$ and that, in fact, it is the entire sequence (u_n) which converges. □

5 Renormalized convergence for trace functions sequence.

We present now a quite general stability result in both the interior and up to the boundary for a sequence of renormalized solutions to the Vlasov-Fokker-Planck equation on a domain. This will be a key argument in the proof of Theorem 1.2. In some sense, this result says that renormalized convergence, as well as the a.e. convergence, can be propagated from the interior to the boundary. Notice that it is not clear that a similar result holds for the L^1-weak convergence.

Theorem 5.1 *Consider three sequences (g_n), (E_n) and (G_n), with $G_n = G_n^+ - G_n^-$, $G_n^\pm \geq 0$, which satisfy the convergence assumptions*

$$g_n \rightharpoonup g \text{ weakly in } L^\infty(0,T:L^1(\mathcal{O})), \tag{5.1}$$

$$E_n \to E \text{ strongly in } L^1((0,T)\times\Omega), \text{uniformaly bounded in } L^1(0,T;W^{1,1}(\Omega)), \tag{5.2}$$

$$\begin{aligned} &\alpha_M'(g_n)\, G_n^\pm \rightharpoonup \bar G_M^\pm \text{ weakly in } L^1((0,T)\times\mathcal{O}_R), \\ &\quad \text{with } \bar G_M^\pm \nearrow G^\pm \quad \text{a.e. with } \beta(g)\, G^\pm \in L^1((0,T)\times\mathcal{O}_R), \end{aligned} \tag{5.3}$$

for all $\beta \in \mathcal{B}_3$ the class of functions of $W^{1,\infty}_{loc}(\mathbb{R})$ such that $|\beta'(s)|\,(1+s)^{-1} \in L^\infty(\mathbb{R})$, all renormalizing sequence (α_M) of $\mathcal{B}_3$ as well as the renormalized Vlasov equation

$$\Lambda_{E_n}\,\beta(g_n) = \beta'(g_n)\, G_n \quad \text{ in } \mathcal{D}'((0,T)\times\mathcal{O}), \tag{5.4}$$

for which clearly each term makes sense thanks to (5.1)–(5.3). Then $g \in L^\infty(0,T;L^1(\mathcal{O}))$ is a solution of

$$\Lambda_E\,\beta(g) = \beta'(g)\, G \quad \text{ in } \mathcal{D}'((0,T)\times\mathcal{O}), \quad G = G^+ - G^-, \tag{5.5}$$

for all $\beta \in \mathcal{B}_3$. Furthermore, the traces γg_n and γg defined thanks to the Theorem 4.5 satisfy

$$\gamma g_n \overset{r}{\rightharpoonup} \gamma g \quad \text{in the renormalized sense.} \tag{5.6}$$

Proof of Theorem 5.1. The proof is esentially the same as Step 2 in the proof of [56, Proposition 5] and as the proof of Theorem 2.9. Nevertheless, for the sake of completeness, we sketch the main arguments.

Step 1. Up to the extaction of a subsequence, we have $g_n \overset{r}{\rightharpoonup} g$ thanks to (5.1) and Lemma 2.7, and there exists $\eta \in L((0,T)\times\Sigma)$ such that $\gamma g_n \overset{r}{\rightharpoonup} \eta$ thanks to Proposition 3.2. More precisely, there exists two sequences $(\bar\alpha_M)$ and $(\bar\gamma_M)$ and such that

$$\alpha_M(g_n) \rightharpoonup \bar\alpha_M \quad \sigma(L^\infty,L^1)\star \quad \text{and} \quad \bar\alpha_M \nearrow g \quad \text{a.e.}, \tag{5.7}$$

$$\alpha_M(\gamma g_n) \rightharpoonup \bar\gamma_M \quad \sigma(L^\infty,L^1)\star \quad \text{and} \quad \bar\gamma_M \nearrow \eta \quad \text{a.e.}. \tag{5.8}$$

The Green formula (4.9) associated to the equation (5.4) with $\beta = \alpha_M$ implies

$$\int_0^T \iint_{\mathcal{O}} \big(\alpha_M(g_n)\,\Lambda_E^*\varphi + \alpha_M'(g_n)\,G_n\varphi\big)\,dvdxdt = \int_0^T \iint_\Sigma \alpha_M(\gamma\, g_n)\,\varphi\; n(x)\cdot v\; dvd\sigma_x dt,$$

for any $\varphi \in \mathcal{D}((0,T)\times\bar{\mathcal{O}})$. Passing to the limit $M\to\infty$ with the help of (5.7), (5.2), (5.3) in the above identity, we obtain

$$\Lambda_E\,\bar\alpha_M = \bar G_M \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathcal{O}) \tag{5.9}$$

and $\gamma\bar\alpha_M = \bar\gamma_M$ thanks to the trace Theorem 4.2 and the convergence (5.8).

Step 2. For a given function $\beta \in \mathcal{B}_3 \cap L^\infty$, we write the renormalized Green formula (4.9) associated to the equation (5.9) as

$$\int_0^T \iint_{\mathcal{O}} \big(\beta(\bar\alpha_M)\,\Lambda_E^*\varphi + \beta'(\bar\alpha_M)\,\bar G_M\varphi\big)\,dvdxdt = \int_0^T \iint_\Sigma \beta(\gamma_M)\,\varphi\; n(x)\cdot v\; dvd\sigma_x dt, \tag{5.10}$$

for any $\varphi \in \mathcal{D}((0,T)\times\bar{\mathcal{O}})$. Using that $(\bar\alpha_M)$, $(G_M^\pm)$ and $(\bar\gamma_M)$ are a.e. increasing sequences we have

$$\beta(\bar\alpha_M) \nearrow \beta(g), \quad \beta'(\bar\alpha_M)\,\bar G_M^\pm \nearrow \beta'(g)\,G^\pm \quad \text{in} \quad L^1((0,T)\times\mathcal{O}) \tag{5.11}$$

as well as

$$\beta(\bar\gamma_M) \nearrow \beta(\eta) \text{ a.e. and uniformaly bounded in } L^\infty((0,T)\times\mathcal{O}). \tag{5.12}$$

Passing to the limit in (5.10) with the help of (5.11) and (5.12) we obtain

$$\int_0^T \iint_{\mathcal{O}} \big(\beta(g)\,\Lambda_E^*\varphi + \beta'(g)\,G\,\varphi\big)\,dvdxdt = \int_0^T \iint_\Sigma \beta(\eta)\,\varphi\; n(x)\cdot v\; dvd\sigma_x dt,$$

which precisely means that $\eta = \gamma g$. We conclude by gathering that information with (5.8). □

Theorem 5.2 *Consider three sequences (g_n), (E_n) and (G_n) which satisfy, for all $\beta \in \mathcal{B}_4$ the class of functions of $W^{2,\infty}_{loc}(\mathbb{R})$ such that $|\beta'(s)|\,(1+s)^{-1} \in L^\infty(\mathbb{R})$ and $|\beta''(s)|\,(1+s)^{-2} \in L^\infty(\mathbb{R})$,*

$$g_n \to g \text{ strongly in } L^1((0,T)\times\mathcal{O}) \text{ and is uniformly bounded in } L^\infty(0,T;L^1(\mathcal{O})), \tag{5.13}$$

$$E_n \rightharpoonup E \quad \text{weakly in } L^1(0,T;W^{1,1}_{loc}(\bar{\mathcal{O}})), \tag{5.14}$$

$$\beta'(g_n)\,G_n \rightharpoonup \beta'(g)\,G \quad \text{weakly in } L^1((0,T)\times\mathcal{O}_R) \quad \forall \beta\in\mathcal{B}_4,\ \forall R\geq 0, \tag{5.15}$$

$$\int_0^T \int_{\mathcal{O}} \frac{|\nabla_v g_n|^2}{1+g_n}\,dvdxdt \leq C_T, \tag{5.16}$$

as well as the renormalized Vlasov-Fokker-Planck equation

$$\Lambda_{E_n}\,\beta(g_n) = \beta'(g_n)\,G_n - \nu\,\beta''(g_n)\,|\nabla_v g_n|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}), \tag{5.17}$$

for which clearly each term makes sense thanks to (5.13)–(5.16). Then $g \in L^\infty(0,T;L^1(\mathcal{O}))$ is a solution of

$$\Lambda_E\,\beta(g) = \beta'(g)\,G - \nu\,\beta''(g)\,|\nabla_v g|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}) \tag{5.18}$$

for all $\beta \in \mathcal{B}_4$. Furthermore, the traces γg_n and γg defined thanks to the Theorem 4.5 satisfy

$$\gamma g_n \overset{r}{\rightharpoonup} \gamma g \quad \text{in the renormalized sense, and} \quad \gamma_+ g_n \to \gamma_+ g \quad \text{a.e.} \tag{5.19}$$

We shall need the following auxiliary results in the proof of Theorem 5.2.

Lemma 5.3 *Let (u_n) be a bounded sequence of $L^2(Y)$ such that $u_n \rightharpoonup u$ weakly in $L^2(Y)$. Then, there exists $\mu \in (C_c(Y))'$, a nonnegative measure, such that, up to the extraction of a subsequence,*

$$|u_n|^2 \rightharpoonup |u|^2 + \mu \qquad \text{weakly in} \quad (C_c(Y))'.$$

Lemma 5.4 *For any $\theta \in (0,1)$ and $M \in (0,\infty)$ we set*

$$\Phi(s) = \Phi_{M,\theta}(s) := \begin{cases} 1/\theta\,(e^{\theta\,s}-1) & \text{if } s \leq M \\ (s-M)\,e^{\theta\,M} + 1/\theta\,(e^{\theta\,M}-1) & \text{if } s \geq M, \end{cases}$$

and $\beta(s) := \beta_1(s) = \log(1+s)$. Then

$$\begin{cases} \Phi'(s) \geq 1, \qquad \Phi\circ\beta(s) \nearrow s \text{ when } M\nearrow\infty,\ \theta\nearrow 1, \\ \text{and} \qquad 0 \leq -(\Phi\circ\beta)''(s) \leq \dfrac{1-\theta+e^{(\theta-1)\,M}}{1+s} \end{cases} \qquad \forall s \geq 0.$$

Lemma 5.5 *Let $g \in L^\infty(0,T;L^p_{loc}(\mathcal{O}))$ be a solution to the Vlasov-Fokker-Planck equation*

$$\Lambda_E\, g = G + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}), \tag{5.20}$$

with $E \in L^1(0,T;W^{1,p'}_{loc}(\mathcal{O}))$, $G \in L^1_{loc}((0,T)\times\mathcal{O}))$ and $\mu \in \mathcal{D}'((0,T)\times\mathcal{O})$, $\mu \geq 0$. For a given mollifer ρ_k in $\mathbb{R}^N$, we set

$$g_k := g *_t \rho_k *_x \rho_k *_v \rho_k \quad \text{and} \quad \mu_k := \mu *_t \rho_k *_x \rho_k *_v \rho_k.$$

Then g_k satisfies the Vlasov-Fokker-Planck equation

$$\Lambda_E\, g_k = G_k + \mu_k \quad \text{in all compact set of } (0,T)\times\mathcal{O},$$

with $G_k \to G$ strongly in $L^1_{loc}([0,T]\times\mathcal{O}))$.

The proof of Lemma 5.3 is classical, the one of Lemma 5.4 is elementary, and we refer to [35] for the proof of Lemma 5.5.

Proof of the Theorem 5.2. Step 1: Proof of (5.18). This step is inspired from [35] and it is clear from the theory of renormalized solution [36] that it is enough to prove (5.18) only for $\beta(s) := \log(1+s)$. With the notation $h_n := \beta(g_n)$ and $h = \beta(g)$ we have $\nabla_v h_n = \sqrt{-\beta''(g_n)}\,\nabla_v g_n \rightharpoonup \sqrt{-\beta''(g)}\,\nabla_v g = \nabla_v h$ weakly in $L^2((0,T)\times\mathcal{O})$ so that, thanks to Lemma 5.3, there is a bounded measure $\mu \geq 0$ such that, up to the extraction of a subsequence, $|\nabla_v h_n|^2 \rightharpoonup |\nabla_v h|^2 + \mu$ weakly in $\mathcal{D}'([0,T]\times\bar{\mathcal{O}})$. Passing to the limit $n \to \infty$ in (5.17) we get

$$\Lambda_E\, \beta(g) = \beta'(g)\, G - \beta''(g)\, |\nabla_v g|^2 + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}).$$

We just point out that

$$E_n\, \beta(g_n) \rightharpoonup E\, \beta(g) \quad \text{weakly in } L^1((0,T)\times\mathcal{O}),$$

since $\beta(g_n) \to \beta(g)$ strongly in $L^2(0,T;L^p(\mathcal{O}))$ for all $p < \infty$ and $E_n \rightharpoonup E$ weakly in $L^2(0,T;L^q(\mathcal{O}))$ for every $q \in [1, N/(N-1))$. We prove now that $\mu = 0$ in $(0,T)\times\mathcal{O}$.

With the notations introduced in Lemma 5.4 and Lemma 5.5 we have

$$\Lambda_E\, \Phi(h_k) = \Phi'(h_k)\left(\beta'(g)\, G - \beta''(g)\, |\nabla_v g|^2\right) *_{t,x,v} \rho_k - \Phi''(h_k)\, |\nabla_v h_k|^2 + \Phi'(h_k)\, \mu_k.$$

Using that $\Phi' \geq 1$ (thanks to Lemma 5.4) and passing to the limit $k \to \infty$ (thanks to Lemma 5.5), we get

$$\Lambda_E\, (\Phi\circ\beta)(g) \geq \Phi'(\beta(g))\, \beta'(g)\, G - (\Phi'(\beta(g))\, \beta''(g) + \Phi''(\beta(g))\, (\beta'(g))^2)\, |\nabla_v g|^2 + \mu$$

and then

$$\Lambda_E\, (\Phi\circ\beta)(g) - (\Phi\circ\beta)'(g)\, G \geq (\Phi\circ\beta)''(g)\, |\nabla_v g|^2 + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}). \tag{5.21}$$

In order to have an estimate of the left hand side we come back to equation (5.17), and we write

$$\Lambda_{E_n}\, \Phi\circ\beta(g_n) = (\Phi\circ\beta)'(g_n)\, G_n - (\Phi\circ\beta)''(g_n)\, |\nabla_v g_n|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O})$$

since $\Phi\circ\beta \in \mathcal{B}_4$. Then, for all $\chi \in \mathcal{D}((0,T)\times\mathcal{O}$ such that $0 \leq \chi \leq 1$ we have (thanks to Lemma 5.4)

$$\begin{aligned}
\Big|\int_0^T\!\!\int_{\mathcal{O}} \left(\Phi\circ\beta(g_n)\, \Lambda_{E_n}\, \chi + (\Phi\circ\beta)'(g_n)\, G_n\, \chi\right) dvdxdt\Big| &= \\
&= -\int_0^T\!\!\int_{\mathcal{O}} (\Phi\circ\beta)''(g_n)\, |\nabla_v g_n|^2\, \chi\, dvdxdt \\
&\leq [1 - \theta + e^{(\theta-1)\,M}] \int_0^T\!\!\int_{\mathcal{O}} \frac{|\nabla_v g_n|^2}{1+g_n}\, dvdxdt.
\end{aligned}$$

Passing to the limit $n \to \infty$ we get, thanks to (5.16),

$$\Big| \int_0^T \int_{\mathcal{O}} \big(\Phi \circ \beta(g)\, \Lambda_E\, \chi + (\Phi \circ \beta)'(g)\, G\, \chi \big)\, dvdxdt \Big| \leq [1 - \theta + e^{(\theta-1)\,M}]\, C_T.$$

Then, coming back to (5.21), we have (thanks to Lemma 5.4 again)

$$\begin{aligned} \int_0^T \int_{\mathcal{O}} \chi\, d\mu &\leq -\int_0^T \int_{\mathcal{O}} \big(\Phi \circ \beta(g)\, \Lambda_E\, \chi + (\Phi \circ \beta)'(g)\, G\, \chi + (\Phi \circ \beta)''(g)\, |\nabla_v g|^2 \big)\, dvdxdt \\ &\leq 2\, [1 - \theta + e^{(\theta-1)\,M}]\, C_T \qquad \forall \theta \in [0,1],\ M > 0, \end{aligned}$$

and letting $M \to \infty$ and then $\theta \to 1$ we obtain $\mu = 0$ on $\operatorname{supp} \chi$, which is precisely saying that $\mu = 0$ in $(0,T) \times \mathcal{O}$.

Step 2: Proof of (5.19). We fix $\phi \in \mathcal{D}((0,T) \times \bar{\mathcal{O}})$ such that $0 \leq \phi \leq 1$. By definition of γg_n we have

$$\begin{aligned} &\Big| \int_0^T \iint_\Sigma \Phi \circ \beta(\gamma\, g_n)\, \phi\ n(x) \cdot v\ dvd\sigma_x dt \\ &\qquad - \int_0^T \int_{\mathcal{O}} \big(\Phi \circ \beta(g_n)\, \Lambda_{E_n}\, \chi + (\Phi \circ \beta)'(g_n)\, G_n\, \chi \big)\, dvdxdt \Big| = \\ &= \int_0^T \int_{\mathcal{O}} (\Phi \circ \beta)''(g_n)\, |\nabla_v g_n|^2\, \chi\, dvdxdt \leq [1 - \theta + e^{(\theta-1)\,M}] \int_0^T \int_{\mathcal{O}} \frac{|\nabla_v g_n|^2}{1 + g_n}\, dvdxdt. \end{aligned}$$

We note $\overline{\Phi \circ \beta}$ the L^1-weak limit of $\Phi \circ \beta(\gamma\, g_n)$. Passing to the limit $n \to \infty$ we get

$$\begin{aligned} &\Big| \int_0^T \iint_\Sigma \overline{\Phi \circ \beta}\, \phi\ n(x) \cdot v\ dvd\sigma_x dt - \int_0^T \int_{\mathcal{O}} \big(\Phi \circ \beta(g)\, \Lambda_E\, \chi + (\Phi \circ \beta)'(g)\, G\, \chi \big)\, dvdxdt \Big| \leq \\ &\qquad \leq [1 - \theta + e^{(\theta-1)\,M}]\, C_T, \end{aligned}$$

and thus

$$\begin{aligned} &\Big| \int_0^T \iint_\Sigma \overline{\Phi \circ \beta}\, \phi\ n(x) \cdot v\ dvd\sigma_x dt - \int_0^T \int_{\mathcal{O}} \big[\Phi \circ \beta(g)\, \Lambda_E\, \chi \\ &\qquad + \big((\Phi \circ \beta)'(g)\, G - (\Phi \circ \beta)''(g)\, |\nabla_v g|^2 \big)\, \chi \big]\, dvdxdt \Big| \leq 2\, [1 - \theta + e^{(\theta-1)\,M}]\, C_T. \end{aligned}$$

Once again, by definition of γg, we obtain

$$\Big| \int_0^T \iint_\Sigma \big(\overline{\Phi \circ \beta} - \Phi \circ \beta(\gamma\, g) \big)\, \phi\ n(x) \cdot v\ dvd\sigma_x dt \Big| \leq 2\, [1 - \theta + e^{(\theta-1)\,M}]\, C_T \underset{M \to \infty, \theta \to 1}{\longrightarrow} 0,$$

and $\overline{\Phi \circ \beta} \nearrow r\text{-}lim\, \gamma g_n$ since $\Phi \circ \beta(s) \nearrow s$ when $M \nearrow \infty$, $\theta \searrow 1$, so that $\gamma g = r\text{-}lim\, \gamma g_n$.

In order to prove the a.e. convergence we only have to show, thanks to Proposition 3.5.3, that, up to the extraction of a subsequence,

$$r\text{-}liminf\, \beta(\gamma_+ g_n) \geq \beta(\gamma_+ g). \tag{5.22}$$

Using Lemma 5.3 and the first step, we can pass to the limit in (5.17), up to the extraction of a subsequence, and we get

$$\begin{aligned} \int_0^T \iint_\Sigma \overline{\beta}\, \phi\ n(x) \cdot v\ dvd\sigma_x dt &= \int_0^T \int_{\mathcal{O}} \big(\beta(g)\, \Lambda_E\, \chi + (\beta'(g)\, G + \beta''(g)\, |\nabla_v g|^2 + \mu)\, \chi \big)\, dvdxdt \\ &= \int_0^T \iint_\Sigma (\beta(\gamma g)\ n(x) \cdot v + \mu)\, \phi\ dvd\sigma_x dt, \end{aligned}$$

where $\overline{\beta} = \text{w-lim}\, \beta(\gamma g_n)$ is the weak limit in $L^1((0,T)\times\Sigma)$ of $\beta(\gamma g_n)$. We deduce that $\overline{\beta}\ n(x)\cdot v = \beta(\gamma g)\ n(x)\cdot v + \mu$ on $(0,T)\times\Sigma$, and in particular

$$\overline{\beta} \geq \beta(\gamma_+ g) \quad \text{ on } \quad (0,T)\times\Sigma_+.$$

Since $r\text{-}liminf\, \beta(\gamma_+ g_n) = \bar{\beta}$, that ends the proof of (5.22). □

6 Boltzmann, Vlasov-Poisson and Fokker-Planck equations

In this section we derive the *a priori* physical bound, then make precise the exact meaning of renormalized solution we deal with and finally state and present a proof of the corresponding stability results. In order to do not repeat many times the exposition, we consider the full Vlasov-Poisson-Fokker-Planck-Boltzmann system (VPFPB in short)

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f - \text{div}_v\left((\nabla_x V_f + \lambda\, v)\, f\right) - \nu\,\Delta_v f = Q(f,f) \ \text{ in } \ (0,\infty)\times\mathcal{O}, \tag{6.1}$$

where $\nu \geq 0$, $\lambda \in \mathbb{R}$, $Q(f,f)$ stands for the bilinear Boltzmann collision operator and V_f is given by the mean of the Poisson equation

$$-\Delta V_f = \rho_f := \int_{\mathbb{R}^N} f\, dv \text{ in } (0,\infty)\times\Omega, \qquad V_f = 0 \text{ on } (0,\infty)\times\partial\Omega. \tag{6.2}$$

We do not give the explicit expression for $Q(f,f)$ that we may find in [26, 37] for example. The precise assumptions we make on the cross section are those introduced in [37]. We only recall that the collision operator splits into a gain term and a loss term, $Q(f,f) = Q^+(f,f) - Q^-(f,f)$, that it has the following collision invariants

$$\int_{\mathbb{R}^3} Q(f,f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0, \tag{6.3}$$

and that the so-called *entropy production term* $e(f) \geq 0$ satisfies

$$\int_{\mathbb{R}^3} e(f)\, dv = -\int_{\mathbb{R}^3} Q(f,f)\, \log f\, dv. \tag{6.4}$$

Moreover, it has been established in [37] the following estimate

$$\forall\, R>0\ \ \exists C_R<\infty \qquad \int_{B_R} \frac{Q^\pm(f,f)}{1+f}\, dv \leq C_R \int_{\mathbb{R}^N} \left[(1+|v|^2)\, f + e(f)\right] dv, \tag{6.5}$$

and in [65] (we also refer to [52] for a related result) the more accurate estimate

$$\forall\, R>0\ \ \exists C_R<\infty \qquad \int_{B_R} \frac{|Q(f,f)|}{\sqrt{1+f}}\, dv \leq C_R \int_{\mathbb{R}^N} \left[(1+|v|^2)\, f + e(f)\right] dv. \tag{6.6}$$

We assume furthermore that f satisfies the boundary condition (1.1) and the initial condition (1.8), where f_{in} is assumed to verify (1.9) and

$$\int_\Omega |\nabla_x V_{f_{in}}|^2\, dx < \infty \quad \text{ with } \quad -\Delta_x V_{f_{in}} = \int_{\mathbb{R}^3} f_{in}(x,v)\, dv \text{ on } \Omega, \quad V_{f_{in}} = 0 \text{ on } \partial\Omega. \tag{6.7}$$

Lemma 6.1 *For any nonnegative initial datum f_{in} such that (1.9)-(6.7) holds and any time $T \in (0,\infty)$ there exists a constant $C_T \in (0,\infty)$ (only depending on T and on f_{in} through the quantities*

C_0 and $\|\nabla_x V_{f_{in}}\|_{L^2}$) such that any solution f to the initial boundary value problem (6.1)-(6.2)-(1.1)-(1.8) satisfies (at least formally)

$$\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f\left(1+|v|^2+|\log f|\right)dvdx+\int_\Omega |\nabla_x V_f|^2\,dx\Big\} + \int_0^T\iint_{\mathcal{O}}\big(e(f)+\nu\,\frac{|\nabla_v f|^2}{f}\big)\,dvdxdt \le C_T, \tag{6.8}$$

as well as

$$\int_0^T\int_{\partial\Omega}\Big\{\mathcal{E}\left(\frac{\gamma_+ f}{M}\right)+\sqrt{\widetilde{\gamma_+ f}}\,\Big\}\,d\sigma_x dt \le C_T, \tag{6.9}$$

where $\mathcal{E}$ is defined in (1.13). It is worth mentioning that the second estimate in (6.9) is an "a posteriori" estimate which we deduce from the interior estimate (6.8) and a Green formula.

Proof of (6.8) in Lemma 6.1. We claim that for f sufficiently regular and decreasing at the infinity all the integrations (by parts) that we shall perform are allowed.
First, we simply integrate the equation (6.1) over all variables, and we get the conservation of mass

$$\iint_{\mathcal{O}} f(t,.)\,dvdx=\iint_{\mathcal{O}} f_{in}\,dvdx \qquad \forall t\ge 0.$$

Next, setting $h_M(s)=s\,\log(s/M)$ and $E=\nabla_x V_f$, we compute

$$\begin{aligned}\frac{\partial}{\partial t}h_M(f)+v\cdot\nabla_x h_M(f)+\mathrm{div}_v\big((E+\lambda\,v)h_M(f)\big)-\nu\,\Delta_v h_M(f)=\\ =h'_M(f)\,Q(f,f)-\nu\,h''_M(f)\,|\nabla_v f|^2-f\,(E+\lambda\,v)\cdot\nabla_v(\log M)\\ +\lambda\,(h_M(f)-f\,h'_M(f))+2\,\nu\,\nabla_v f\cdot\nabla_v(\log M)+\nu\,f\,\Delta_v(\log M),\end{aligned}$$

where $h'_M(s)=1+\log(s/M)$. We integrate this equation over the x,v variables using the collision invariants (6.3) and the entropy production identity (6.4), to obtain

$$\begin{aligned}\frac{d}{dt}\iint_{\mathcal{O}} h_M(f)\,dvdx+\iint_{\mathcal{O}}\big(e(f)+\nu\,\frac{|\nabla_v f|^2}{f}\big)\,dvdx+\iint_\Sigma h_M(\gamma f)\,v\cdot n(x)\,dvd\sigma_x=\\ =\int_\Omega E\cdot\frac{j}{\Theta}\,dx+\iint_{\mathcal{O}}\Big\{\lambda\,\big(\frac{|v|^2}{\Theta}-1\big)+\frac{\nu}{\Theta}\Big\}\,f\,dvdx,\end{aligned} \tag{6.10}$$

where

$$j(t,x)=\int_{\mathbb{R}^3} v\,f(t,x,v)\,dv.$$

We first remark that integrating equation (6.1) in the velocity variable we have

$$\frac{\partial}{\partial t}\rho+div_x\,j=0 \quad \text{ on } \ (0,\infty)\times\Omega,$$

and therefore

$$-\int_\Omega E\cdot\frac{j}{\Theta}\,dx=\int_\Omega\nabla V_f\cdot\frac{j}{\Theta}\,dx=\int_\Omega\frac{V_f}{\Theta}\,\frac{\partial\rho}{\partial t}\,dx=\frac{d}{dt}\int_\Omega\frac{|\nabla_x V_f|^2}{2\,\Theta}\,dx. \tag{6.11}$$

Next, combining (6.10), (6.11) and the boundary estimate (2.5) we obtain

$$\begin{aligned}\frac{d}{dt}\Big\{\iint_{\mathcal{O}} h_M(f)\,dvdx+\int_\Omega\frac{|\nabla_x V_f|^2}{2\,\Theta}\,dx\Big\}+\iint_{\mathcal{O}}\big[e(f)+\nu\,\frac{|\nabla_v f|^2}{f}\big]\,dvdx\\ +\bar\alpha\int_{\partial\Omega}\mathcal{E}(\gamma_+ f)\,d\sigma_x\le C_{\lambda,\nu}\iint_{\mathcal{O}}(1+|v|^2)\,f\,dvdx.\end{aligned}$$

Here and below, we set $\bar\alpha = \alpha$ in the case of the constant accommodation coefficient (1.2) and $\bar\alpha$ is defined just after equation (1.6) in the case of mass flux dependent accommodation coefficient. Using the elementary estimate (2.6) and (2.7) we conclude that (6.8) holds, as well as the first estimate in (6.9).

In order to prove the second estimate in (6.9), we fix $\chi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \chi \le 1$, $\chi = 1$ on B_1 and $\mathrm{supp}\,\chi \subset B_2$ and we apply the Green formula (4.9) written with $\phi = n(x)\cdot v\,\chi(v)$ and $\beta(s) = \sqrt{1+s}$. We get

$$
\begin{aligned}
(6.12)\qquad \int_0^T\!\!\iint_\Sigma \sqrt{1+\gamma f}\,(n(x)\cdot v)^2\,\chi\,dvd\sigma_x dt &= \Big[\iint_{\mathcal{O}} \sqrt{1+\gamma f}\,\phi\,dvdx\Big]_T^0 \\
&+ \int_0^T\!\!\iint_{\mathcal{O}} \Big(\sqrt{1+f}\,\Big(v\cdot\nabla_x + (\nabla_x V_f + \lambda\, v)\cdot\nabla_v + \nu\,\Delta_v + N\,\lambda\Big)\phi\,dvdxdt \\
&+ \int_0^T\!\!\iint_{\mathcal{O}} \Big(\frac{Q(f,f)}{2\,(1+f)^{1/2}} + \frac{\nu}{4}\,\frac{|\nabla_v f|^2}{(1+f)^{3/2}}\Big)\phi\,dvdxdt.
\end{aligned}
$$

Thanks to (6.8) and (6.6) and because $\nabla_x\phi \in L^\infty$, $D_v^2\phi \in L^\infty$, we see that the right hand side term in (6.12) is bounded by a constant denoted by C_T' and which only depends on C_T defined in (6.8). On the other hand, from the boundary condition (1.1)-(1.2) or (1.1)-(1.6), we have $\gamma_- f \ge \bar\alpha\, M(v)\,\widetilde{\gamma_+ f}$ on $(0,T)\times\Sigma_-$. Therefore there is a constant $C_\chi > 0$ such that

$$
\begin{aligned}
C_\chi \int_0^T\!\!\int_{\partial\Omega} \sqrt{\widetilde{\gamma_+ f}}\,d\sigma_x dt &\le \int_0^T\!\!\iint_{\Sigma_-} \sqrt{\widetilde{\gamma_+ f}}\,\bar\alpha^{1/2}\,M^{1/2}(v)\,\chi\,(n(x)\cdot v)^2\,dvd\sigma_x dt \\
&\le \int_0^T\!\!\iint_{\Sigma_-} \sqrt{\gamma_- f}\,\chi\,(n(x)\cdot v)^2\,dvd\sigma_x dt \le C_T',
\end{aligned}
$$

which ends the proof of (6.9). □

We can now specify the sense of the solution we deal with. With DiPerna and Lions [35], [37, 38], [52] we say that $0 \le f \in C([0,\infty); L^1(\mathcal{O}))$ is a renormalized solution of (6.1)-(6.2)-(1.1)-(1.8) if first f satisfies the *a priori* physical bound (6.8) and is a solution of

$$
\begin{aligned}
(6.13)\qquad &\frac{\partial}{\partial t}\beta(f) + v\cdot\nabla_x\beta(f) + (\nabla_x V_f + \lambda\, v)\cdot\nabla_v\beta(f) - \nu\,\Delta_v\beta(f) = \\
&\qquad = \beta'(f)\,(Q(f,f) + \lambda\,N\,f) - \nu\,\beta''(f)\,|\nabla_v f|^2 \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathcal{O}),
\end{aligned}
$$

for all time $T > 0$, and all $\beta \in \mathcal{B}_5$, the class of all functions $\beta \in C^2(\mathbb{R})$ such that $|\beta''(s)| \le C/(1+s)$, $|\beta'(s)| \le C/\sqrt{1+s}$, $\forall s \ge 0$. Thanks to (6.8) (and (6.6)) we see that each term in equation (6.13) makes sense. Next, the trace functions $f(0,.)$ and γf defined by Theorem 4.5 through the Green formula (4.9) must satisfy (1.8) and (1.1), say almost everywhere. Finally, we will always assume that γf satisfies the additional bound (6.9).

Our main result is the following stability or compactness result. Once again, in order to do not repeat several times the proof, we establish our result for the full VPFPB system and the full VPB system, the same holds for the same equation with less terms.

Theorem 6.2 *Let (f_n) be a sequence of renormalized solutions to equation (6.1)-(6.2) such that the associated trace functions γf_n satisfy (1.1), with the linear reflexion operator (1.2) when $\nu = 0$ and a possibly mass flux depending accommodation coefficient (1.6) when $\nu > 0$ (FP type models). Let furthermore assume that both the sequence of solutions (f_n) and the trace sequence (γf_n) satisfy (uniformly in n) the natural physical a priori bounds*

$$
\begin{aligned}
(6.14)\qquad &\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f_n\,\big(1+|v|^2+|\log f_n|\big)\,dvdx + \int_\Omega |\nabla_x V_{f_n}|^2\,dx\Big\} \\
&\qquad + \int_0^T\!\!\iint_{\mathcal{O}} \big(e(f_n) + \nu\,\frac{|\nabla_v f_n|^2}{f_n}\big)\,dvdxdt + \int_0^T\!\!\int_{\partial\Omega} \mathcal{E}\left(\frac{\gamma_+ f_n}{M}\right)\,d\sigma_x dt \le C_T.
\end{aligned}
$$

If $f_n(0,.)$ converges to f_{in} weakly in $L^1(\mathcal{O})$ then, up to the extraction of a subsequence, f^n converges weakly in $L^p(0,T;L^1(\mathcal{O}))$ for all $T>0$ and $p\in[1,\infty)$ (the convergence being strong in the FP type models case) to a renormalized solution f to (6.1)-(6.2) with initial value f_{in} and which satisfies the physical estimates (6.8). Furthermore, for any $\varepsilon>0$ and $T>0$, there exists a measurable set $A\subset(0,T)\times\partial\Omega$ such that meas$\,((0,T)\times\partial\Omega\setminus A)<\varepsilon$ and

$$\gamma_+ f_n \rightharpoonup \gamma_+ f \quad \textit{weakly in} \quad L^1(A\times\mathbb{R}^N, d\lambda_1),$$

the convergence being strong in the case of the Fokker-Planck type equations. As a consequence we can pass to the limit in the boundary reflection condition (1.1) (and (1.6) in the case of the FP type equations), so that the trace condition is fulfilled and the trace estimate (6.9) holds.

Proof of the Theorem 6.2. From (6.14) we deduce, extracting a subsequence if necessary, that f_n converges weakly in $L^p(0,T;L^1(\mathcal{O}))$ $(\forall p\in[1,\infty))$ to a function f and that the local mass density $\rho_n=\rho_{f_n}$ satisfies (see [52])

$$\sup_{[0,T]}\int_\Omega \rho_n(1+|\log\rho_n|)\,dx\le C_T.$$

In the case $\nu=0$, using the velocity averaging lemma of [42, 39] and the standard properties of the Poisson equation, we also show (see for instance [52] and [62])

$$\rho^n \underset{n\to\infty}{\longrightarrow} \rho_f \text{ in } L^p(0,T;L^1(\Omega)) \quad\text{and}\quad \nabla_x V_{f_n} \underset{n\to\infty}{\longrightarrow} \nabla_x V_f \text{ in } L^p(0,T;W^{1,1}\cap L^a(\Omega))$$

for all $T\in(0,\infty)$, $p\in[1,\infty)$ and $a\in[1,2)$. It is also shown in [52] that

$$\frac{Q^\pm(f_n,f_n)}{1+\delta\, f^n} \underset{n\to\infty}{\rightharpoonup} \bar{Q}^\pm_\delta \text{ weakly in } L^1((0,T)\times\mathcal{O}_R) \quad\text{and}\quad \bar{Q}^\pm_\delta \nearrow Q^\pm(f,f) \text{ a.e.}$$

In the case $\nu>0$, since the term on the right hand side of equation (6.13) is bounded in L^1, thanks to the uniform estimate (6.14), and since $\Lambda_{E_{f_n}}$ is an hypoelliptic operator (see [35], [15], [49]), we obtain that, say, $\log(1+f^n)$ and next f^n converge a.e. (see [14] and [35]). We conclude that $f^n\to f$ strongly in $L^p(0,T;L^1(\mathcal{O}))$, $\forall p\in[1,\infty)$. It is also shown in [35] that

$$\frac{Q(f_n,f_n)}{1+f_n}\to\frac{Q(f,f)}{1+f} \quad \text{strongly in} \quad L^1((0,T)\times\mathcal{O}_R).$$

Therefore, using Theorem 5.1 or Theorem 5.2, we obtain that f satisfies the renormalized equation (6.13) (first for renormalizing function $\beta\in\mathcal{B}_4$ and next $\beta\in\mathcal{B}_5$) and that

$$\gamma f_n \overset{r}{\rightharpoonup} \gamma f \quad \text{in the renormalized sense on } (0,T)\times\Sigma,$$

as well as

$$\gamma f_n\to\gamma f \quad \text{a.e. on } (0,T)\times\Sigma,$$

when $\nu>0$. It is worth mentioning that f also satisfies the physical estimate (6.8), see [35, 38, 52] Next, from (1.1) we have

$$\widetilde{\gamma_+ f_n}\le \bar{\alpha}^{-1}\,M^{-1}(v)\,\gamma_- f_n \quad \text{on} \quad (0,T)\times\Sigma_-,$$

so that

$$\widetilde{\gamma_+ f_n}\overset{r}{\rightharpoonup}\psi \quad \text{in} \quad (0,T)\times\partial\Omega, \quad \text{with} \quad \psi\le\bar{\alpha}^{-1}\,M^{-1}(v)\,\gamma_- f.$$

Furthermore, repeating the proof of Lemma 6.1 we get that $\psi\in L^{1/2}((0,T)\times\partial\Omega)$. Now, we can apply Theorem 3.9 (with $m(v)=M(v)$, $y=(t,x)$, $d\varpi_y(v)=\mathbf{1}_{\Sigma^x_+}\,|n(x)\cdot v|\,dv$, $\phi_n=\gamma_+ f_n$ and $d\nu(y)=d\sigma_x dt$), which says that for every $\varepsilon>0$ there is $A=A_\varepsilon\subset(0,T)\times\partial\Omega$ such that meas$\,((0,T)\times\partial\Omega\backslash A)<\varepsilon$ and

$$\gamma_+ f_n\rightharpoonup\gamma_+ f \quad \text{weakly in} \quad L^1(A\times\mathbb{R}^N).$$

In the case $\nu > 0$, since we already know the a.e. convergence, this convergence is in fact strong in $L^1(A \times \mathbb{R}^N)$. There is no difficulty in passing to the limit in the boundary condition so that f satisfies (1.1) and f satisfies the same physical estimate (6.9) thanks to the convexity argument of Theorem 2.5. □

Remark 6.3 *For the Boltzmann equation and the FPB equation, as well as for the VP system and the VPFP system when the Poisson equation (6.2) is provided with Neumann condition, we can prove the additional a priori estimate (2.3) on the trace function. As a consequence, we may also establish the a priori physical bound (6.8) for a time and position dependent wall temperature* $\Theta = \Theta(t,x)$ *which satisfies*

$$0 < \Theta_0 \leq \Theta(t,x) \leq \Theta_1 < \infty.$$

Therefore, the stability result and the corresponding existence result can be generalized to these kind of boundary conditions. We refer to [6] and [56] for more details.

Remark 6.4 *Let consider the general reflection operator*

$$\mathcal{R}\,\phi = \int_{v'\cdot n(x)>0} k(v,v')\,\phi(v')\,v'\cdot n(x)\,dv' \tag{6.15}$$

where the measurable function k *satisfies the usual non-negativity, normalization and reciprocity conditions*

$$k \geq 0, \qquad \int_{v\cdot n(x)<0} k(v,v')\,dv = 1, \qquad \mathcal{R}\,M = M, \tag{6.16}$$

where M *is the normalized Maxwellian (1.3). For that reflection operator (6.15), we can prove that a solution* f *to equations (6.1)-(6.2)-(1.1) formally satisfies the a priori physical estimate (6.8)-(6.9) with* $\mathcal{E}$ *replaced by*

$$\mathcal{E}_k(\phi/M) := \int_{v\cdot n(x)>0} \big[\, h\big(\frac{\phi}{M}\big) - h\big(\frac{\mathcal{R}\,\phi}{M}\big)\,\big]\, M\, v\cdot n(x)\,dv.$$

By Jensen inequality one can prove that $\mathcal{E}_k$ *is non negative, see [40], [30], [43]. However, we do not know if our analysis can be adapted to this general kernel. Nevertheless, considering a sequence* (f_n) *of solutions which satisfies the uniform interior estimate in (6.14), we can pass to the limit in (1.1) with the help of Theorem 5.1 or Theorem 5.2 and of Proposition 3.5.4, and we get that the limit function* f *is a solution which trace* γf *satisfies the relaxed boundary condition (1.10). That extends and generalizes previous results known for the Boltzmann equation, see for instance [6], [29], [56].*

A Appendix: More about the renormalized convergence

We come back to the notions of renormalized convergence and mainly discuss its relationship with the biting-L^1 weak convergence.

Remark A.1 *1. Hypothesis* $\psi \in L^0(Y)$ *in Theorem 3.7 (and (3.4)) is fundamental, since for example, the sequence* (ψ_n) *defined by* $\psi_n = \psi \equiv +\infty\ \forall n$ *does converge in the renormalized sense to* ψ*, but* (ψ_n) *does not converge (and none of its subsequence!) in the biting* L^1*-weak sense.*

2. The (asymptotically) boundedness of (ψ_n) *in* L^0 *does not guarantee that* (ψ_n) *satisfies, up to the extraction of a subsequence, (3.3) or (3.4). An instructive example is the following: we define* $u(y) = 1/y$ *on* $Y = [0,1]$ *that we extend by* 1*-periodicity to* $\mathbb{R}$*, and we set* $\psi_n(y) = u(n\,y)$ *for* $y \in Y$*. Therefore,* (ψ_n) *is obviously bounded in* $L^a(Y)$ *for all* $a \in [0,1)$ *and converges to* $\psi \equiv +\infty$ *in the renormalized sense.*

Proposition A.2 *1. There exists (ϕ_n) which r-converges but does not b-converges.*
2. There exists (ϕ_n) which b-converges but does not r-converges.
3. Given a sequence (ϕ_n), the property

(A.1) *for any sub-sequence $(\phi_{n'})$ there exists a sub-sequence $(\phi_{n''})$ of $(\phi_{n'})$ such that $\phi_{n''} \overset{ww}{\rightharpoonup} \phi$*

does not imply $\phi_n \overset{ww}{\rightharpoonup} \phi$, where $\overset{ww}{\rightharpoonup}$ denotes either the b-convergence or the r-convergence As a consequence, the b-convergence and the r-convergence are not associated to any Hausdorff (separated) topological structure.

Proof of Proposition A.2. Points 1 & 3. Let (ϕ_n) be the sequence defined by $\phi_n = \phi_{p,k} = p\,\mathbf{1}_{[k/p,(k+1)/p]}$ where $p \in \mathbb{N}^*$, $0 \le k \le p-1$ and $n = 1 + 2 + ... + p + k$. Then (ϕ_n) is bounded in L^1 and clearly r-converges to 0, but does not b-converge. Moreover, for any subsequence $(\phi_{n'})$ we can find a second subsequence $(\phi_{n''})$ such that $\phi_{n''}$ b-converges to 0.

Points 2 & 3. Let consider $\mu_y = \mu$ and $\nu_y = \nu$ two Young measures on $Y = [0,1]$ such that

$$\int_{\mathbb{R}} \lambda\,\mu(\lambda) = \int_{\mathbb{R}} \lambda\,\nu(\lambda) =: \phi \in L^1(Y),$$
$$\int_{\mathbb{R}} T_M(\lambda)\,\mu(\lambda) \neq \int_{\mathbb{R}} T_M(\lambda)\,\nu(\lambda) \quad \forall\, M > 0,$$

and define (u_n) (resp. (v_n)) a sequence of L^1 associated to μ (resp. ν), such that for any $f \in C(\mathbb{R})$

$$f(u_n) \rightharpoonup \bar{f} := \int_{\mathbb{R}} f(\lambda)\,\mu_y(d\lambda) \qquad \big(\text{resp.}\quad f(v_n) \rightharpoonup \tilde{f} := \int_{\mathbb{R}} f(\lambda)\,\nu_y(d\lambda)\big),$$

see [63, Theorem 5], [69]. Then define (ϕ_n) by setting $\phi_{2\,n} = u_n$, $\phi_{2\,n+1} = v_n$. In such a way, we have constructed a sequence (ϕ_n) which does not r-converge (for instance does not (T_M)-renormalized converge) but converges to ϕ in the weak L^1 sense, and thus b-converges to ϕ. Moreover, for any sub-sequence $(\phi_{n'})$, there exists a sub-sequence $(\phi_{n''})$ which either converges to $\bar{T}_M$ (if $\{n'\}$ contain an infinity of even integer numbers) or to $\tilde{T}_M$ (if $\{n'\}$ contain an infinity of odd integer numbers). Because $\bar{T}_M \nearrow \phi$ and $\tilde{T}_M \nearrow \phi$ when $M \nearrow \infty$, in both case $\phi_{n''}$ r-converges to ϕ, and (A.1) holds. □

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