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# Stochastic Heat Equation with Multiplicative Fractional-Colored Noise

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## Abstract

We consider the stochastic heat equation with multiplicative noise  $u_t = \frac{1}{2}\Delta u + u \diamond \dot{W}$  in  $\mathbb{R}_+ \times \mathbb{R}^d$ , where  $\diamond$  denotes the Wick product, and the solution is interpreted in the mild sense. The noise  $\dot{W}$  is fractional in time (with Hurst index  $H \geq 1/2$ ), and colored in space (with spatial covariance kernel  $f$ ). We prove that if  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , then the sufficient condition for the existence of the solution is  $d \leq 2 + \alpha$  (if  $H > 1/2$ ), respectively  $d < 2 + \alpha$  (if  $H = 1/2$ ), whereas if  $f$  is the heat kernel or the Poisson kernel, then the equation has a solution for any  $d$ . We give a representation of the  $k$ -th order moment of the solution, in terms of an exponential moment of the “convoluted weighted” intersection local time of  $k$  independent  $d$ -dimensional Brownian motions.

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*Key words and phrases:* stochastic heat equation, Gaussian noise, multiple stochastic integrals, chaos expansion, Skorohod integral, fractional Brownian motion, local time

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# 1 Introduction

The study of stochastic partial differential equations (s.p.d.e.'s) driven by a Gaussian noise which is white in time and has a non-trivial correlation structure in space (called "color"), constitutes now a classical line of research. These equations represent an alternative to the standard s.p.d.e.'s driven by a space-time white noise. A first step in this direction has been made in [7], where the authors identify the necessary and sufficient condition for the existence of the solution of the stochastic wave equation (in spatial dimension  $d = 2$ ), in the space of real-valued stochastic processes. The fundamental reference in this area is Dalang's seminal article [6], in which the author gives the necessary and sufficient conditions under which various s.p.d.e.'s with a white-colored noise (e.g. the wave equation, the damped heat equation, the heat equation) have a process solution, in arbitrary spatial dimension. The methods used in this article exploit the temporal martingale structure of the noise, and cannot be applied when the noise is "colored" in time. Other related references are: [31], [19], [28], [8] and [9].

Recently, there has been a growing interest in studying s.p.d.e.'s driven by a Gaussian noise which has the covariance structure of the fractional Brownian motion (fBm) in time, combined with a white (or colored) spatial covariance structure. (Recall that an fBm is a centered Gaussian process  $(B_t)_{t \geq 0}$  with covariance  $E(B_t B_s) = R_H(t, s) := (t^{2H} + s^{2H} - |t - s|^{2H})/2$ , with  $H \in (0, 1)$ . The Brownian motion is an fBm of index  $H = 1/2$ . We refer the reader to the expository article [23], for a comprehensive account on the fBm.) This interest comes from the large number of applications of the fBm in practice. To list only a few examples of the appearance of fractional noises in practical situations, we mention [16] for biophysics, [3] for financial time series, [10] for electrical engineering, and [5] for physics.

In the present article, we consider the stochastic heat equation with a multiplicative Gaussian noise, which is fractional (or white) in time with Hurst index  $H > 1/2$  (respectively  $H = 1/2$ ), and has a non-trivial spatial covariance structure given by a kernel  $f$ . As in [6], we assume that  $f$  is the Fourier transform of a tempered measure  $\mu$ . (Note that the particular case of a spatially white noise arises when  $f = \delta_0$ .) More precisely, we consider the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \Delta u + u \diamond \dot{W}, \quad t > 0, x \in \mathbb{R}^d \\ u_{0,x} &= u_0(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

where  $u_0 \in C_b(\mathbb{R}^d)$  is non-random,  $\diamond$  denotes the Wick product, and  $\dot{W}$  is a formal writing for the noise  $W = \{W(h); h \in \mathcal{HP}\}$  (to be introduced rigorously in Section 2).

Before discussing the multiplicative case, we recall briefly the known results related to the existence of the solution of the stochastic heat equation with additive noise:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \dot{W}, \quad t > 0, x \in \mathbb{R}^d \tag{2}$$

$$u_{0,x} = 0, \quad x \in \mathbb{R}^d,$$

When  $H = 1/2$  and  $f = \delta_0$ , equation (2) admits a solution in the space of real-valued processes, if and only if  $d = 1$ . This phenomenon is called in [15] the “curse of dimensionality” and can be explained intuitively by saying that, while the Laplacian smooths, the white noise roughens. If the spatial dimension  $d$  is larger than 2, then the roughness effect of the white noise overcomes the smoothness influence of the Laplacian.

What happens when the space-time white noise is replaced by a noise which is fractional in time, but continues to be white in space? This situation has been studied in several papers such as [11], [20], [25], [29] and recently in [2]. In this case, a necessary and sufficient condition for the existence of the solution of (2) is  $d < 4H$ , which allows us to consider the cases  $d = 1, 2$  or  $3$ , for suitable values of  $H$ . This can be interpreted by saying that for  $H > 1/2$ , the noise roughens a little bit less, and the smoothness influence of the Laplacian overcomes the roughness of the noise. If the noise is colored in space, the conditions for the existence of the solution of (2) depend on the noise regularity in space. For example, if  $f$  the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha$ , the necessary and sufficient condition for the existence of the solution of (2) is  $d < 4H + \alpha$ , whereas if  $f$  is the heat kernel or the Poisson kernel, the solution exists for any  $d \geq 1$  and  $H > 1/2$  (see [2], as well as Appendix B for a correction of the result of [2]).

Another explanation of this curse of dimensionality is given in [15], and it is related to the local time of the stochastic processes associated with the differential operator of the s.p.d.e. In the particular case of the stochastic heat equation driven by a space-time white noise, the solution exists only in dimension  $d = 1$  because this is the only case when the  $d$ -dimensional Brownian motion has a local time.

We now return to the discussion of equation (1). This equation has been studied recently in [14], when the noise is fractional in time, and white in space. In this article, it is proved that a *sufficient* condition for the existence of the solution (in the space of square-integrable processes) is  $d \leq 2$ : if  $d = 1$ , then equation (1) has a solution in any time interval  $[0, T]$ , but if  $d = 2$ , this equation has a solution only up to a critical point  $T_0$  (i.e. it has a solution in any interval  $[0, T]$ , with  $T < T_0$ ). It is not known if this condition is necessary as well. There still is a connection with the local time, in the sense that the second-order moment of the solution is equal to the exponential moment of the “weighted” intersection local time  $L_t$  of two independent  $d$ -dimensional Brownian motion  $B^1$  and  $B^2$ , written formally as:

$$L_t := H(2H - 1) \int_0^t \int_0^t |r - s|^{2H-2} \delta_0(B_r^1 - B_s^2) dr ds.$$

In the present article, we consider equation (1) driven by the Gaussian noise introduced in [2]. This noise is fractional in time with Hurst index  $H \geq 1/2$ , and colored in space, with covariance kernel  $f$  chosen among the following: the Riesz kernel, the Bessel kernel, the heat kernel, or the Poisson kernel (see Examples

2.1-2.4). The case of the fractional kernel  $f(x) = \prod_{i=1}^d H_i(2H_i - 1)|x_i|^{2H_i - 2}$  with  $1/2 < H_i < 1$  has been examined in [13], using methods that rely on the product form of  $f$ . These methods cannot be used in the present article, since in our case (except the heat kernel),  $f$  is not of product type. For the fractional kernel, it was proved in [13] that the sufficient condition for the existence of the solution is  $d < 2/(2H - 1) + \sum_{i=1}^d H_i$ .

As in the case of equation (2) with additive noise, we find that the existence of the solution depends on the roughness of the noise. If  $H > 1/2$ , and  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$  (which are “rough” kernels), then a sufficient condition for the existence of the solution is  $d \leq 2 + \alpha$ : if  $d < 2 + \alpha$  the solution exists in any time interval  $[0, T]$ , whereas if  $d = 2 + \alpha$ , the solution exists only up to a critical point  $T_0$ . If  $f$  is the heat or the Poisson kernel (which are “smooth” kernels), the solution exists in any time interval, for any  $d \geq 1$  and  $H \geq 1/2$ . If  $f$  is one of the “rough” kernels mentioned above, we prove that if the solution exists, then  $d < 4H + \alpha$ . This shows that for  $H = 1/2$ , the necessary and sufficient condition for the existence of solution is  $d < 2 + \alpha$ . It remains an open problem to identify the necessary and sufficient condition for the existence of the solution, in the case of  $H > 1/2$ .

The existence of the solution is connected to the “convoluted weighted” intersection local time  $L_t$ , written formally as:

$$L_t = H(2H - 1) \int_0^t \int_0^t \int_{\mathbb{R}^d} |r - s|^{2H-2} \delta_0(B_r^1 - B_s^2 - y) f(y) dy dr ds.$$

More precisely, the second-order moment of the solution can be expressed as:

$$E[u_{t,x}^2] = E[u_0(x + B_t^1)u_0(x + B_t^2) \exp(L_t)].$$

As in [14], this expression can be extended to the moments of order  $k \geq 2$ , using  $k$  independent  $d$ -dimensional Brownian motions.

This article is organized as follows. Section 2 contains some preliminaries related to the Wick product and the Skorohod integral. In Sections 3, we discuss the existence of the solution. In Section 4, we examine the relationship with the “convoluted weighted” intersection local time. Appendix A contains a technical lemma. Appendix B contains a correction to Theorem 3.13 of [2] (which does not concern the case of the Riesz kernel).

## 2 Preliminaries

We begin by describing the kernel which gives the spatial covariance of the noise. As in [6], let  $f$  be the Fourier transform of a tempered distribution  $\mu$  on  $\mathbb{R}^d$ , i.e.

$$f(x) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \mu(d\xi), \quad \forall x \in \mathbb{R}^d,$$

where  $\xi \cdot x$  denotes the scalar product in  $\mathbb{R}^d$ . Let  $\mathcal{P}(\mathbb{R}^d)$  be the completion of  $\{1_A; A \in \mathcal{B}_b(\mathbb{R}^d)\}$ , where  $\mathcal{B}_b(\mathbb{R}^d)$  denotes the class of bounded Borel sets in  $\mathbb{R}^d$ ,

with respect to the inner product

$$\langle 1_A, 1_B \rangle_{\mathcal{P}(\mathbb{R}^d)} = \int_A \int_B f(x-y) dy dx.$$

We consider some examples of kernel functions  $f$ . In what follows,  $|x|$  denotes the Euclidian norm of  $x \in \mathbb{R}^d$ .

**Example 2.1** If  $\mu(d\xi) = |\xi|^{-\alpha} d\xi$  for some  $0 < \alpha < d$ , then  $f$  is the Riesz kernel of order  $\alpha$ :

$$f(x) = \gamma_{\alpha,d} |x|^{-d+\alpha},$$

where  $\gamma_{\alpha,d} = \Gamma((d-\alpha)/2) 2^{-\alpha} \pi^{-d/2} / \Gamma(\alpha/2)$ .

**Example 2.2** If  $\mu(d\xi) = (1 + |\xi|^2)^{-\alpha/2} d\xi$  for some  $\alpha > 0$ , then  $f$  is the Bessel kernel of order  $\alpha$ :

$$f(x) = \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/(4w)} dw,$$

where  $\gamma'_\alpha = (4\pi)^{\alpha/2} \Gamma(\alpha/2)$ . In this case,  $\mathcal{P}(\mathbb{R}^d)$  coincides with  $\mathcal{H}^{-\alpha/2}(\mathbb{R}^d)$ , the fractional Sobolev space of order  $-\alpha/2$ ; see e.g. p.191, [12].

**Example 2.3** If  $\mu(d\xi) = e^{-\pi^2 \alpha |\xi|^2} d\xi$  for some  $\alpha > 0$ , then  $f$  is the heat kernel of order  $\alpha$ :

$$f(x) = (2\pi\alpha)^{-d/2} e^{-|x|^2/(2\alpha)}.$$

**Example 2.4** If  $\mu(d\xi) = e^{-4\pi^2 \alpha |\xi|} d\xi$  for some  $\alpha > 0$ , then  $f$  is the Poisson kernel of order  $\alpha$ :

$$f(x) = C_d \alpha (|x|^2 + \alpha^2)^{-(d+1)/2},$$

where  $C_d = \pi^{-(d+1)/2} \Gamma((d+1)/2)$ .

As in [2] (see also []), if  $H > 1/2$ , we let  $\mathcal{HP}$  be the Hilbert space defined as the completion of  $\{1_{[0,t] \times A}; t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$  with respect to the inner product

$$\langle 1_{[0,t] \times A}, 1_{[0,s] \times B} \rangle_{\mathcal{HP}} = \alpha_H \int_0^t \int_0^s \int_A \int_B |u-v|^{2H-2} f(x-y) dy dx dv du,$$

where  $\alpha_H = H(2H-1)$ . The space  $\mathcal{HP}$  is isomorphic to  $\mathcal{H} \otimes \mathcal{P}(\mathbb{R}^d)$ , where  $\mathcal{H}$  is the completion of  $\{1_{[0,t]}; t \geq 0\}$  with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} dv du.$$

If  $H = 1/2$ , we let  $\mathcal{HP}$  be the completion of  $\{1_{[0,t] \times A}; t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$  with respect to the inner product

$$\langle 1_{[0,t] \times A}, 1_{[0,s] \times B} \rangle_{\mathcal{HP}} = (t \wedge s) \int_A \int_B f(x-y) dy dx.$$

In this case, the space  $\mathcal{HP}$  is isomorphic to  $L^2(\mathbb{R}_+) \otimes \mathcal{P}(\mathbb{R}^d)$ .

We note that in both cases, the space  $\mathcal{HP}$  may contain distributions.

Let  $W = \{W(h); h \in \mathcal{HP}\}$  be a zero-mean Gaussian process, defined on a probability space  $(\Omega, \mathcal{F}, P)$ , with covariance

$$E(W(h)W(g)) = \langle h, g \rangle_{\mathcal{HP}}.$$

The process  $W$  introduce formally the noise perturbing the stochastic heat equation. This noise is considered to be “colored” in space, with the color given by the kernel  $f$ . If  $H > 1/2$ , the noise is fractional in time, whereas if  $H = 1/2$  the noise is white in time.

We now introduce the basic elements of Malliavin calculus with respect to  $W$ , which are needed in the sequel. For a comprehensive account on this subject, we refer the reader to [22] and [24].

We begin with a brief description of the multiple Wiener (or Wiener-Itô) integral with respect to  $W$ . Let  $\mathcal{F}^W$  be the  $\sigma$ -field generated by  $\{W(h); h \in \mathcal{HP}\}$ ,  $H_n(x)$  be the  $n$ -th order Hermite polynomial, and  $\mathcal{HP}_n$  be the closed linear span of  $\{H_n(W(h)); h \in \mathcal{HP}\}$  in  $L^2(\Omega, \mathcal{F}^W, P)$ . The space  $\mathcal{HP}_n$  is called the  **$n$ -th Wiener chaos** of  $W$ .

It is known that  $L^2(\Omega, \mathcal{F}^W, P) = \bigoplus_{n=0}^{\infty} \mathcal{HP}_n$ , and hence every  $F \in L^2(\Omega, \mathcal{F}^W, P)$  admits the following **Wiener chaos expansion**:

$$F = \sum_{n=0}^{\infty} J_n(F), \quad (3)$$

where  $J_n : L^2(\Omega, \mathcal{F}^W, P) \rightarrow \mathcal{HP}_n$  is the orthogonal projection. By convention,  $\mathcal{HP}_0 = \mathbb{R}$  and  $J_0(F) = E(F)$ .

For each  $n \geq 1$ , and for each  $h \in \mathcal{HP}$  with  $\|h\|_{\mathcal{HP}} = 1$ , we define

$$I_n(h^{\otimes n}) = n! H_n(W(h)).$$

By polarization, we extend  $I_n$  to elements of the form  $h_1 \otimes \dots \otimes h_n$  (see p. 230 of [13]; e.g.  $h_1 \otimes h_2 = [(h_1 + h_2)^{\otimes 2} - (h_1 - h_2)^{\otimes 2}]/4$ ). By linearity and continuity, we extend the definition of  $I_n$  to the space  $\mathcal{HP}^{\otimes n}$ . (Note that if  $\{e_i; i \geq 1\}$  is a CONS in  $\mathcal{HP}$ , then  $\{e_{i_1} \otimes \dots \otimes e_{i_n}; i_j \geq 1\}$  is a CONS in  $\mathcal{HP}^{\otimes n}$ .) For any  $h \in \mathcal{HP}^{\otimes n}$ , we say that

$$I_n(h) := \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^n} h(t_1, x_1, \dots, t_n, x_n) dW_{t_1, x_1} \dots dW_{t_n, x_n}$$

is the **multiple Wiener integral** of  $h$  with respect to  $W$ . We have

$$E(I_n(h)I_n(g)) = n! \langle \tilde{h}, \tilde{g} \rangle_{\mathcal{HP}^{\otimes n}}, \quad \forall h, g \in \mathcal{HP}^{\otimes n},$$

where  $\tilde{h}(t_1, x_1, \dots, t_n, x_n) = (n!)^{-1} \sum_{\sigma \in S_n} h(t_{\sigma(1)}, x_{\sigma(1)}, \dots, t_{\sigma(n)}, x_{\sigma(n)})$  is the symmetrization of  $h$  with respect to the  $n$  variables  $(t_1, x_1), \dots, (t_n, x_n)$ , and  $S_n$  is the set of all permutations of  $\{1, \dots, n\}$ . By convention, we set  $I_0(x) = x$ .

The map  $I_n : \mathcal{HP}^{\otimes n} \rightarrow \mathcal{HP}_n$  is surjective. Moreover, for any  $F_n \in \mathcal{HP}_n$ , there exists a unique  $f_n \in \mathcal{HP}^{\otimes n}$  symmetric, such that  $I_n(f_n) = F_n$ . Using (3), we conclude that any  $F \in L^2(\Omega, \mathcal{F}^W, P)$  can be written as:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad (4)$$

where  $f_0 = E(F)$  and  $f_n \in \mathcal{HP}^{\otimes n}$  is symmetric and uniquely determined by  $F$ . We have:

$$E|F|^2 = \sum_{n=1}^{\infty} E|I_n(f_n)|^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{\mathcal{HP}^{\otimes n}}^2.$$

We now introduce the Skorohod integral with respect to  $W$ . Let  $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$  be an  $\mathcal{F}^W$ -measurable square-integrable process. By (4), for any  $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , we have

$$u_{t,x} = E(u_{t,x}) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)), \quad (5)$$

where  $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$  is symmetric and uniquely determined by  $u_{t,x}$ . For each  $n \geq 1$ , let  $\tilde{f}_n$  be the symmetrization of  $f_n$  with respect to all  $n+1$  variables. Let  $\tilde{f}_0 = E(u)$ . We say that  $u$  is **Skorohod integrable** (with respect to  $W$ ) if  $\tilde{f}_n \in \mathcal{HP}^{\otimes(n+1)}$  for every  $n \geq 0$ , and  $\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$  converges in  $L^2(\Omega)$ . In this case, we have:

$$\delta(u) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \quad \text{and} \quad E|\delta(u)|^2 = \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{\mathcal{HP}^{\otimes(n+1)}}^2.$$

### 3 Existence of the solution

In this section, we give conditions for the existence of the solution of equation (1).

Let  $p_t(x)$  be the heat kernel on  $\mathbb{R}^d$ , i.e.

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, x \in \mathbb{R}^d.$$

For any bounded Borel function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ , let  $p_t\varphi(x) = \int_{\mathbb{R}^d} p_t(x-y)\varphi(y)dy$ .

For each  $t > 0$ , let  $\mathcal{F}_t$  be the  $\sigma$ -field generated by  $\{W(1_{[0,s] \times A}); s \in [0, t], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ .

The solution of equation (1) is interpreted in the mild sense using the Skorohod integral with respect to  $W$ . More precisely, we have the following definition.



**Definition 3.1** An  $(\mathcal{F}_t)_t$ -adapted square-integrable process  $u = \{u_{t,x}; (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$  is a **solution to (1)** if for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , the process  $\{p_{t-s}(x-y)u_{s,y}1_{[0,t]}(s); (s, y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$  is Skorohod integrable, and

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u_{s,y}1_{[0,t]}(s)\delta W_{s,y}.$$

As in [14] (see also [4], [17], [21], [26], [27], [30]), one can find a closed formula for the kernels  $f_n(\cdot, t, x)$  appearing in the Wiener chaos expansion (5) of a solution  $u = \{u_{t,x}; (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ . More precisely, we have the following result.

**Proposition 3.2** In order that equation (1) possesses a solution it is necessary and sufficient that for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_{\mathcal{H}\mathcal{P}^{\otimes n}}^2 < \infty, \quad (6)$$

where

$$f_n(t_1, x_1, \dots, t_n, x_n, t, x) = \frac{1}{n!} \prod_{j=1}^n p_{t_{\rho(j+1)} - t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}) p_{t_{\rho(1)}} u_0(x_{\rho(1)}),$$

$\rho$  denotes the permutation of  $\{1, 2, \dots, n\}$  such that  $t_{\rho(1)} < t_{\rho(2)} < \dots < t_{\rho(n)}$ ,  $t_{\rho(n+1)} = t$  and  $x_{\rho(n+1)} = x$ . In this case, the solution  $u$  is unique, admits the Wiener chaos decomposition (5), and

$$E|u_{t,x}|^2 = \sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_{\mathcal{H}\mathcal{P}^{\otimes n}}^2. \quad (7)$$

We begin to examine condition (6). Note that

$$\alpha_n(t, x) := (n!)^2 \|f_n(\cdot, t, x)\|_{\mathcal{H}\mathcal{P}^{\otimes n}}^2 \leq \|u_0\|_{\infty}^2 \alpha_n(t), \quad (8)$$

with equality if  $u_0 = 1$ , and hence

$$E|u_{t,x}|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t, x) \leq \|u_0\|_{\infty}^2 \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t),$$

where

$$\alpha_n(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} \psi^{*(n)}(\mathbf{s}, \mathbf{t}) ds dt & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi^{*(n)}(\mathbf{s}, \mathbf{s}) ds & \text{if } H = 1/2 \end{cases} \quad (9)$$

and

$$\begin{aligned} \psi^{*(n)}(\mathbf{s}, \mathbf{t}) &:= \int_{\mathbb{R}^{2nd}} \prod_{j=1}^n f(x_j - y_j) \prod_{j=1}^n p_{t_{\rho(j+1)} - t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}) \\ &\quad \prod_{j=1}^n p_{s_{\sigma(j+1)} - s_{\sigma(j)}}(y_{\sigma(j+1)} - y_{\sigma(j)}) dx dy. \end{aligned} \quad (10)$$

In the above integrals, we denoted  $\mathbf{s} = (s_1, \dots, s_n)$ ,  $\mathbf{t} = (t_1, \dots, t_n)$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$ , and we chose the permutations  $\rho$  and  $\sigma$  of  $\{1, \dots, n\}$  such that

$$0 < t_{\rho(1)} < t_{\rho(2)} < \dots < t_{\rho(n)} \quad \text{and} \quad 0 < s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(n)}, \quad (11)$$

with  $t_{\rho(n+1)} = s_{\sigma(n+1)} = t$  and  $x_{\rho(n+1)} = y_{\sigma(n+1)} = x$ .

Note that

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) = \langle g_{\mathbf{s}}^{(n)}, g_{\mathbf{t}}^{(n)} \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}}, \quad \forall \mathbf{t}, \mathbf{s} \in [0, t]^n,$$

where

$$\begin{aligned} g_{\mathbf{t}}^{(n)}(x_1, \dots, x_n) &= \prod_{j=1}^n p_{t_{\rho(j+1)} - t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}) \\ g_{\mathbf{s}}^{(n)}(y_1, \dots, y_n) &= \prod_{j=1}^n p_{s_{\sigma(j+1)} - s_{\sigma(j)}}(y_{\sigma(j+1)} - y_{\sigma(j)}), \end{aligned}$$

and the permutations  $\rho$  and  $\sigma$  are chosen such that (11) holds.

As in [2], for any  $y, z \in \mathbb{R}^d$  and  $u, v > 0$ , we denote

$$J_f(u, v, y, z) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_u(x - y) p_v(x' - z) f(x - x') dx dx'.$$

**Lemma 3.3** (i) *If  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , then*

$$J_f(u, v, y, z) \leq D_{\alpha, d} (u + v)^{-(d-\alpha)/2}, \quad \forall y, z \in \mathbb{R}^d,$$

where  $D_{\alpha, d}$  is a positive constant depending on  $\alpha$  and  $d$ .

(ii) *If  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ , then*

$$J_f(u, v, y, z) \leq C_{\alpha, d}, \quad \forall y, z \in \mathbb{R}^d,$$

where  $C_{\alpha, d}$  is a positive constant depending on  $\alpha$  and  $d$ .

**Proof:** Note that  $J_f(u, v, y, z) = E[f(y - z + \sqrt{u}Y - \sqrt{v}Z)]$ , where  $Y$  and  $Z$  are independent  $d$ -dimensional standard normal random vectors. We use the following inequality: (see (3.24) of [2])

$$E[e^{-|y-z+\sqrt{2u}Y-\sqrt{2v}Z|^2/(4w)}] \leq \left(1 + \frac{u+v}{w}\right)^{-d/2}. \quad (12)$$

(i) In the case of the Riesz kernel, this inequality has been shown in the proof of Theorem 3.13 of [2]. Suppose now that  $f$  is the Bessel kernel of order  $\alpha < d$ . Using (12),

$$J_f(u, v, y, z) = \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} E[e^{-|y-z+\sqrt{u}Y-\sqrt{v}Z|^2/(4w)}] dw$$

$$\begin{aligned}
&\leq \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} \left(1 + \frac{u+v}{2w}\right)^{-d/2} dw \\
&= \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \left(w + \frac{u+v}{2}\right)^{-d/2} dw = \gamma'_\alpha I_{\alpha,d} \left(\frac{u+v}{2}\right)
\end{aligned}$$

where  $I_{\alpha,d}(x) := \int_0^\infty w^{\alpha/2-1} e^{-w} (w+x)^{-d/2} dw$ . The result follows, since

$$\begin{aligned}
I_{\alpha,d}(x) &\leq x^{-d/2} \int_0^x w^{\alpha/2-1} e^{-w} dw + \int_x^\infty w^{\alpha/2-1} e^{-w} (w+x)^{-d/2} dw \\
&= x^{-d/2} x^{\alpha/2} \int_0^1 y^{\alpha/2-1} e^{-xy} dy + x^{-(d-\alpha)/2} \int_1^\infty y^{\alpha/2-1} e^{-xy} (y+1)^{-d/2} dy \\
&\leq x^{-(d-\alpha)/2} \int_0^1 y^{\alpha/2-1} dy + x^{-(d-\alpha)/2} \int_1^\infty y^{-(d-\alpha)/2-1} dy = K_{\alpha,d} x^{-(d-\alpha)/2},
\end{aligned}$$

where  $K_{\alpha,d} = 2/\alpha + 2/(d-\alpha) = 2d/[\alpha(d-\alpha)]$ , and we used the fact that  $\alpha < d$ .

(ii) If  $f$  is the heat kernel, using (12), we obtain:

$$\begin{aligned}
J_f(u, v, y, z) &= (2\pi\alpha)^{-d/2} E[e^{-|y-z+\sqrt{u}Y-\sqrt{v}Z|^2/(2\alpha)}] \\
&\leq (2\pi\alpha)^{-d/2} \left(1 + \frac{u+v}{\alpha}\right)^{-d/2} \\
&= (2\pi)^{-d/2} (\alpha+u+v)^{-d/2} \leq (2\pi\alpha)^{-d/2}.
\end{aligned}$$

If  $f$  is the Poisson kernel, we have:

$$\begin{aligned}
J_f(u, v, y, z) &= C_d \alpha E[(|y-z+\sqrt{u}Y-\sqrt{v}Z|^2 + \alpha^2)^{-(d+1)/2}] \\
&\leq C_d \alpha (\alpha^2)^{-(d+1)/2} = C_d \alpha^{-d}.
\end{aligned}$$

□

**Lemma 3.4** (i) If  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , then for any  $\mathbf{s}, \mathbf{t} \in [0, t]^n$ ,

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) \leq \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\right)^n [\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4},$$

where  $\beta(\mathbf{s}) := \prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)})$ ,  $\beta(\mathbf{t}) := \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)})$ , and the permutations  $\rho$  and  $\sigma$  are chosen such that (11) holds.

(ii) If  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ , then for any  $\mathbf{s}, \mathbf{t} \in [0, t]^n$ ,

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) \leq C_{\alpha,d}^n,$$

where  $C_{\alpha,d}$  is a constant depending on  $\alpha$  and  $d$ .

**Proof:** By the Cauchy-Schwartz inequality,

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) \leq \psi^{*(n)}(\mathbf{s}, \mathbf{s})^{1/2} \psi^{*(n)}(\mathbf{t}, \mathbf{t})^{1/2}.$$

To find an upper bound for  $\psi^{*(n)}(\mathbf{s}, \mathbf{s})$ , we use Lemma 3.3 to estimate the following integrals:

$$\begin{aligned} I_j &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{u_j}(x_{\sigma(j+1)} - x_{\sigma(j)}) p_{u_j}(y_{\sigma(j+1)} - y_{\sigma(j)}) f(x_{\sigma(j)} - y_{\sigma(j)}) dx_{\sigma(j)} dy_{\sigma(j)} \\ &= J_f(u_j, u_j, x_{\sigma(j+1)}, y_{\sigma(j+1)}), \quad \text{with } u_j = s_{\sigma(j+1)} - s_{\sigma(j)}, \quad j = 1, \dots, n. \end{aligned}$$

(i) In this case,  $I_j \leq D_{\alpha, d} [2(s_{\sigma(j+1)} - s_{\sigma(j)})]^{-(d-\alpha)/2}$  and

$$\psi^{*(n)}(\mathbf{s}, \mathbf{s}) \leq D_{\alpha, d}^n 2^{-n(d-\alpha)/2} \left[ \prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)}) \right]^{-(d-\alpha)/2}.$$

(ii) In this case,  $I_j \leq C_{\alpha, d}$  and  $\psi^{*(n)}(\mathbf{s}, \mathbf{s}) \leq C_{\alpha, d}^n$ .  $\square$

If  $H > 1/2$ , it was proved in [18] that there exists  $\beta_H > 0$  such that

$$\alpha_H \int_0^\infty \int_0^\infty \varphi(s) \varphi(t) |t - s|^{2H-2} ds dt \leq \beta_H^2 \left( \int_0^\infty |\varphi(t)|^{1/H} dt \right)^{2H},$$

for any  $\varphi \in L^{1/H}(\mathbb{R}_+)$ . Hence,

$$\alpha_H^n \int_{\mathbb{R}_+^{2n}} \varphi(\mathbf{s}) \varphi(\mathbf{t}) \prod_{j=1}^n |t_j - s_j|^{2H-2} ds dt \leq \beta_H^{2n} \left( \int_{\mathbb{R}_+^n} |\varphi(\mathbf{t})|^{1/H} dt \right)^{2H}, \quad (13)$$

for any  $\varphi \in L^{1/H}(\mathbb{R}_+^n)$ . If  $H = 1/2$ , we let  $\beta_H = 1$ .

Using Lemma 3.4 and (13), we obtain the following estimate for  $\alpha_n(t)$ .

**Proposition 3.5** *Suppose that  $H \geq 1/2$  and let  $\alpha_n(t)$  be given by (9).*

(i) *If  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , and*

$$H > \frac{d - \alpha}{4}, \quad (14)$$

*then there exists a constant  $C_{H, d, \alpha} > 0$  depending on  $H, d$  and  $\alpha$ , such that*

$$\alpha_n(t) \leq C_{H, d, \alpha} \left\{ D_{\alpha, d} 2^{-(d-\alpha)/2} \beta_H^2 \Gamma \left( 1 - \frac{d-\alpha}{4H} \right)^{2H} t^{2H-(d-\alpha)/2} \right\}^n (n!)^{(d-\alpha)/2},$$

*for any  $t > 0$  and  $n \geq 1$ .*

(ii) *If  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ , then*

$$\alpha_n(t) \leq \{C_{\alpha, d} t^{2H}\}^n, \quad \text{for any } t > 0 \text{ and } n \geq 1.$$

**Remark 3.6** Condition (14) is necessary and sufficient for the existence of the solution of the stochastic heat equation with additive noise  $u_t = \frac{1}{2} \Delta u + \dot{W}$ ; see Appendix B for a corrected version of Theorem 3.13 of [2].

**Proof:** We only give the proof in the case  $H > 1/2$ , the case  $H = 1/2$  being similar. We use the definition (9) of  $\alpha_n(t)$ .

(i) By Lemma 3.4.(i) and (13), we obtain:

$$\begin{aligned}\alpha_n(t) &\leq \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n \alpha_H^n \int_{([0,t]^2)^n} \prod_{j=1}^n |t_j - s_j|^{2H-2} [\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4} ds dt \\ &\leq \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n \beta_H^{2n} \left(\int_{[0,t]^n} \beta(\mathbf{s})^{-(d-\alpha)/(4H)} ds\right)^{2H} \\ &= \left(D_{\alpha,d}2^{-(d-\alpha)/2}\beta_H^2\right)^n (n!)^{2H} I_n(t,h)^{2H}, \quad \text{with } h := -\frac{d-\alpha}{4H},\end{aligned}$$

where

$$I_n(t,h) := \int_{T_n} [(t-s_n)(s_n-s_{n-1})\dots(s_2-s_1)]^h ds,$$

and  $T_n = \{\mathbf{s} = (s_1, \dots, s_n); 0 < s_1 < s_2 < \dots < s_n < t\}$ .

Using Lemma A.1 (Appendix A), and letting  $C_{H,d,\alpha} := \Gamma(1+h)^{2H}$ , we obtain:

$$\begin{aligned}\alpha_n(t) &\leq C_{H,d,\alpha} \left(D_{\alpha,d}2^{-(d-\alpha)/2}\beta_H^2\right)^n (n!)^{2H} \left\{\frac{\Gamma(1+h)^n}{\Gamma(n(1+h)+1)} t^{n(1+h)}\right\}^{2H} \\ &= C_{H,d,\alpha} \left\{B_f 2^{-(d-\alpha)/2}\beta_H^2 \Gamma(1+h)^{2H} t^{2Hn(1+h)}\right\}^n \left(\frac{n!}{\Gamma(n(1+h)+1)}\right)^{2H}.\end{aligned}$$

The result follows since  $\Gamma(na+1) \sim (n!)^a$  for  $a \in (0,1)$ .

(ii) By Lemma 3.4.(ii),

$$\alpha_n(t) \leq C_{\alpha,d}^n \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} ds dt = C_{\alpha,d}^n t^{2Hn}.$$

□

Using Proposition 3.5, we examine the existence of the solution of equation (1). The next result is an extension of Proposition 4.3 of [14] to the case of a colored noise  $W$ .

**Proposition 3.7** (i) *Let  $f$  be the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ . Suppose that either*

$$H > 1/2 \quad \text{and} \quad d \leq 2 + \alpha, \quad (15)$$

or

$$H = 1/2 \quad \text{and} \quad d < 2 + \alpha. \quad (16)$$

Then (1) has a unique solution in  $[0, T] \times \mathbb{R}^d$ , provided that  $T < T_0$  where

$$T_0 = \begin{cases} \left\{ D_{\alpha,d} 2^{-1} \beta_H^2 \Gamma\left(1 - \frac{1}{2H}\right)^{2H} \right\}^{-1/(2H-1)} & \text{if } d = 2 + \alpha \\ \infty & \text{if } d < 2 + \alpha \end{cases} \quad (17)$$

(ii) Let  $H \geq 1/2$ , and  $f$  be the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ . Then (1) has a unique solution in  $\mathbb{R}_+ \times \mathbb{R}^d$ .

**Remark 3.8** Either one of conditions (15) or (16) is stronger than (14).

**Remark 3.9** Proposition 3.7 shows that in the case  $H = 1/2$ , there is no “critical” dimension  $d$ , i.e. there is a strict inequality in (16).

**Proof:** We apply Proposition 3.2, using Proposition 3.5.

(i) Let

$$K_{\alpha,d}(t) := D_{\alpha,d} 2^{-(d-\alpha)/2} \beta_H^2 \Gamma\left(1 - \frac{d-\alpha}{4H}\right)^{2H} t^{2H-(d-\alpha)/2}.$$

We have:

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_{\mathcal{H}\mathcal{P}^{\otimes n}}^2 \leq \|u_0\|_{\infty}^2 \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t) \leq \|u_0\|_{\infty}^2 C_{H,f,d} \sum_{n=0}^{\infty} \frac{[K_{\alpha,d}(t)]^n}{(n!)^{1-(d-\alpha)/2}}.$$

If  $d - \alpha = 2$ , then the last sum is finite if  $K(t) < 1$ , which is equivalent to saying that  $t < T_0$ . If  $d - \alpha < 2$ , then the last sum is finite for any  $t > 0$ , by Stirling’s formula and D’Alembert criterion.

(ii) Let  $K_{\alpha,d}(t) = C_{\alpha,d} t^{2H}$ . We have:

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_{\mathcal{H}\mathcal{P}^{\otimes n}}^2 \leq \|u_0\|_{\infty}^2 \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t) \leq \|u_0\|_{\infty}^2 \sum_{n=0}^{\infty} \frac{[K_{\alpha,d}(t)]^n}{n!} < \infty.$$

□

The next result shows that (14) is a necessary condition for the existence of the solution.

**Proposition 3.10** Suppose that  $H \geq 1/2$  and  $f$  is either the Riesz kernel or order  $\alpha$ , or the Bessel kernel of order  $\alpha$ . If equation (1) with  $u_0 = 1$  has a solution in  $\mathbb{R}_+ \times \mathbb{R}^d$ , then (14) holds.

**Proof:** Note that  $E|u_{t,x}|^2 = \sum_{n=0}^{\infty} \alpha_n(t)/n! < \infty$  implies that  $\alpha_1(t) < \infty$ , which in turn implies (14) (see Appendix B). □

**Remark 3.11** Proposition 3.7 and Proposition 3.10 show that, if  $H = 1/2$  and  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , then the condition  $d < \alpha + 2$  is necessary and sufficient for the existence of the solution of (1). It remains an open problem to see if this condition is necessary, when  $H > 1/2$ .

**Remark 3.12** If  $H = 1/2$  and  $f$  is an arbitrary kernel, it was proved in [6] (using different methods) that the sufficient condition for the existence of the solution in  $\mathbb{R}_+ \times \mathbb{R}^d$  of (1) with vanishing initial conditions (i.e.  $u_0 = 0$ ), is:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty. \quad (18)$$

(see Remark 14 of [6]). When  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha$ , (18) holds if and only if  $d < \alpha + 2$ . Combining this with Remark 3.11, we conclude that, in the case of the two kernels, condition (18) is also necessary for the existence of the solution. For an arbitrary kernel  $f$ , it is not known if (18) remains a necessary condition for the existence of the solution.

## 4 Relationship with the Local Time

In this section, we identify a random variable  $L_t$ , defined formally as a “convoluted intersection local time” of two independent  $d$ -dimensional standard Brownian motions, such that

$$\alpha_n(t) = E(L_t^n), \quad \forall n \geq 1. \quad (19)$$

An immediate consequence of (7), (8) and (19) is that the second moment of  $u_{t,x}$  is bounded by the exponential moment of  $L_t$ :

$$E|u_{t,x}|^2 \leq \|u_0\|_\infty^2 \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t) = \|u_0\|_\infty^2 \sum_{n=0}^{\infty} \frac{1}{n!} E(L_t^n) = \|u_0\|_\infty^2 E(e^{L_t}),$$

with equality if  $u_0 = 1$ .

To show (19), we approximate  $\alpha_n(t)$  by  $\{\alpha_{n,\varepsilon}(t)\}_{\varepsilon>0}$ , when  $\varepsilon \rightarrow 0$ , where the constants  $\alpha_{n,\varepsilon}(t)$  are chosen such that:

$$\alpha_{n,\varepsilon}(t) = E(L_{t,\varepsilon}^n), \quad \forall n \geq 1,$$

for a certain random variable  $L_{t,\varepsilon}$ .

To identify the approximation constants  $\alpha_{n,\varepsilon}(t)$ , we recall the definition (9), which says that  $\alpha_n(t)$  is the weighted integral of the function  $\psi^{*(n)}(\mathbf{s}, \mathbf{t})$ . The next lemma gives the exact calculation for the integrand  $\psi^{*(n)}(\mathbf{s}, \mathbf{t})$ .

**Lemma 4.1** *We have:*

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp \left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk}^* \xi_j \cdot \xi_k \right\} \mu(d\xi_1) \dots \mu(d\xi_n),$$

where  $\sigma_{jk}^* := (t - s_j) \wedge (t - s_k) + (t - t_j) \wedge (t - t_k)$ .

**Remark 4.2** Lemma 4.1 gives a generalization -and a minor correction- to Lemma 4.2 of [14]. The correction refers to the fact that the result of [14] is stated incorrectly with the constant  $\sigma_{jk} = s_j \wedge s_k + t_j \wedge t_k$ , instead of  $\sigma_{jk}^*$ . However, a trivial change of variables  $s'_j := t - s_j, t'_j := t - t_j$  in the definition (9) of  $\alpha_n(t)$  shows that this minor error does not affect the calculation of  $\alpha_n(t)$ . We have indeed:

$$\alpha_n(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} \psi^{(n)}(\mathbf{s}, \mathbf{t}) ds dt & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi^{(n)}(\mathbf{s}, \mathbf{s}) ds & \text{if } H = 1/2 \end{cases} \quad (20)$$

with

$$\begin{aligned}\psi^{(n)}(\mathbf{s}, \mathbf{t}) &= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp \left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk} \xi_j \cdot \xi_k \right\} \mu(d\xi_1) \dots \mu(d\xi_n) \\ &= \psi^{*(n)}(t\mathbf{1} - \mathbf{s}, t\mathbf{1} - \mathbf{t}), \quad \text{where } \mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n.\end{aligned}$$

**Proof of Lemma 4.1:** Note that

$$\langle \varphi, \psi \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}} = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \mathcal{F}\varphi(\xi_1, \dots, \xi_n) \overline{\mathcal{F}\psi(\xi_1, \dots, \xi_n)} \mu(d\xi_1) \dots \mu(d\xi_n),$$

where  $\mathcal{F}$  denotes the Fourier transform. Hence,

$$\begin{aligned}\psi^{*(n)}(\mathbf{s}, \mathbf{t}) &= \langle g_{\mathbf{s}}^{(n)}, g_{\mathbf{t}}^{(n)} \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}} \\ &= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \mathcal{F}g_{\mathbf{s}}^{(n)}(\xi_1, \dots, \xi_n) \overline{\mathcal{F}g_{\mathbf{t}}^{(n)}(\xi_1, \dots, \xi_n)} \mu(d\xi_1) \dots \mu(d\xi_n).\end{aligned}$$

It was shown in the proof of Lemma 4.2 of [14] that:

$$\begin{aligned}\mathcal{F}g_{\mathbf{s}}(\xi_1, \dots, \xi_n) &= E \left[ \prod_{j=1}^n e^{-i\xi_j \cdot [x - (B_t^1 - B_{s_j}^1)]} \right] \\ \mathcal{F}g_{\mathbf{t}}(\xi_1, \dots, \xi_n) &= E \left[ \prod_{j=1}^n e^{-i\xi_j \cdot [x - (B_t^2 - B_{t_j}^2)]} \right],\end{aligned}$$

where  $B^1 = (B_t^1)_{t \geq 0}$  and  $B^2 = (B_t^2)_{t \geq 0}$  are independent  $d$ -dimensional standard Brownian motions. Hence,

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} E \left[ \prod_{j=1}^n e^{-i\xi_j \cdot [(B_{s_j}^1 - B_t^1) - (B_{t_j}^2 - B_t^2)]} \right] \mu(d\xi_1) \dots \mu(d\xi_n)$$

We begin to evaluate the integrand of the above integral. We denote  $\xi_j = (\xi_{j,1}, \dots, \xi_{j,d})$ ,  $B_t^1 = (B_{t,1}^1, \dots, B_{t,d}^1)$  and  $B_t^2 = (B_{t,1}^2, \dots, B_{t,d}^2)$ . We observe that for any  $j = 1, \dots, n$  fixed, the random variables

$$(B_{s_j,l}^1 - B_{t,l}^1) - (B_{t_j,l}^2 - B_{t,l}^2), \quad l = 1, \dots, d, \quad \text{are i.i.d.,}$$

with the same distribution as  $(b_{s_j}^1 - b_t^1) - (b_{t_j}^2 - b_t^2)$ , where  $b^1 = (b_t^1)_{t \geq 0}$  and  $b^2 = (b_t^2)_{t \geq 0}$  are independent 1-dimensional standard Brownian motions. Hence,

$$E \left[ \prod_{j=1}^n e^{-i\xi_j \cdot [(B_{s_j}^1 - B_t^1) - (B_{t_j}^2 - B_t^2)]} \right] = \prod_{l=1}^d E \left[ \prod_{j=1}^n e^{-i\xi_{j,l} [(B_{s_j,l}^1 - B_{t,l}^1) - (B_{t_j,l}^2 - B_{t,l}^2)]} \right]$$



$$\begin{aligned}
&= \prod_{l=1}^d E \left[ \prod_{j=1}^n e^{-i\xi_{j,l}[(b_{s_j}^1 - b_t^1) - (b_{t_j}^2 - b_t^2)]} \right] = \prod_{l=1}^d \exp \left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk}^* \xi_{j,l} \xi_{k,l} \right\} \\
&= \exp \left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk}^* \xi_j \cdot \xi_k \right\},
\end{aligned}$$

where for the second last equality we used the fact that the vector

$$((b_{s_1}^1 - b_t^1) - (b_{t_1}^2 - b_t^2), \dots, (b_{s_n}^1 - b_t^1) - (b_{t_n}^2 - b_t^2))$$

has a normal distribution with mean zero and covariance matrix  $(\sigma_{jk}^*)_{1 \leq j,k \leq n}$ . This concludes the proof of the lemma.  $\square$

In what follows, we use the alternative definition (20) of  $\alpha_n(t)$ , given in Remark 4.2. The idea is to find a suitable approximation for the integrand  $\psi^{(n)}(\mathbf{s}, \mathbf{t})$ , by replacing the Dirac function  $\delta_0(x)$  with the heat kernel  $p_\varepsilon(x)$ . This approximation turns out to be:

$$\psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{t}) := E \left[ \int_{(\mathbb{R}^d)^n} p_\varepsilon(B_{s_1}^1 - B_{t_1}^2 - y_1) \dots p_\varepsilon(B_{s_n}^1 - B_{t_n}^2 - y_n) f(y_1) \dots f(y_n) d\mathbf{y} \right],$$

where  $B^1 = (B_t^1)_{t \geq 0}$  and  $B^2 = (B_t^2)_{t \geq 0}$  are independent  $d$ -dimensional standard Brownian motions, and we denote  $\mathbf{y} = (y_1, \dots, y_n)$ .

More precisely, we have the following result.

**Lemma 4.3** *Suppose that  $\mu(d\xi) = g(\xi)d\xi$ , i.e.  $f = \mathcal{F}g$ . Then*

$$\psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp \left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk} \xi_j \cdot \xi_k - \frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2 \right\} \mu(d\xi_1) \dots \mu(d\xi_n).$$

**Proof:** We first calculate the inverse Fourier transform of  $p_\varepsilon * f$ :

$$\mathcal{F}^{-1}(p_\varepsilon * f)(\xi) = \mathcal{F}^{-1}p_\varepsilon(\xi) \mathcal{F}^{-1}f(\xi) = (2\pi)^{-d} e^{-\varepsilon|\xi|^2/2} g(\xi).$$

This shows that  $(p_\varepsilon * f)(x) = (2\pi)^{-d} \mathcal{F}[e^{-\varepsilon|\xi|^2/2} g(\xi)](x)$ , i.e.

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|x-y|^2/(2\varepsilon)} f(y) dy = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-\varepsilon|\xi|^2/2} g(\xi) d\xi. \quad (21)$$

Using (21) with  $x = B_{s_j}^1 - B_{t_j}^2$ , we obtain:

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|B_{s_j}^1 - B_{t_j}^2 - y_j|^2/(2\varepsilon)} f(y_j) dy_j = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi_j \cdot (B_{s_j}^1 - B_{t_j}^2)} e^{-\varepsilon|\xi_j|^2/2} g(\xi_j) d\xi_j.$$

Therefore,

$$\begin{aligned}
\psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{t}) &= E \left[ \prod_{j=1}^n \int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|B_{s_j}^1 - B_{t_j}^2 - y_j|^2 / (2\varepsilon)} f(y_j) dy_j \right] \\
&= (2\pi)^{-nd} E \left[ \prod_{j=1}^n \int_{\mathbb{R}^d} e^{-i\xi_j \cdot (B_{s_j}^1 - B_{t_j}^2)} e^{-\varepsilon|\xi_j|^2 / 2} g(\xi_j) d\xi_j \right] \\
&= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} E \left[ \prod_{j=1}^n e^{-i\xi_j \cdot (B_{s_j}^1 - B_{t_j}^2)} \right] e^{-\varepsilon \sum_{j=1}^n |\xi_j|^2 / 2} \mu(d\xi_1) \dots \mu(d\xi_n) \\
&= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} e^{-\sum_{j,k=1}^n \sigma_{jk} \xi_j \cdot \xi_k / 2} e^{-\varepsilon \sum_{j=1}^n |\xi_j|^2 / 2} \mu(d\xi_1) \dots \mu(d\xi_n).
\end{aligned}$$

□

We are now ready to define the approximation constants  $\alpha_{n,\varepsilon}(t)$ :

$$\alpha_{n,\varepsilon}(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{t}) ds dt & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{s}) ds & \text{if } H = 1/2 \end{cases}.$$

Note that  $\alpha_{n,\varepsilon}(t) = E(L_{t,\varepsilon}^n)$ , where

$$L_{t,\varepsilon} := \begin{cases} \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} |r - s|^{2H-2} p_\varepsilon(B_r^1 - B_s^2 - y) f(y) dy dr ds & \text{if } H > 1/2 \\ \int_0^t \int_{\mathbb{R}^d} p_\varepsilon(B_s^1 - B_s^2 - y) f(y) dy ds & \text{if } H = 1/2 \end{cases}$$

The random variable  $L_{t,\varepsilon}$  is an approximation of the ‘‘convoluted intersection local time’’  $L_t$ , written formally as:

$$L_t = \begin{cases} \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} |r - s|^{2H-2} \delta_0(B_r^1 - B_s^2 - y) f(y) dy dr ds & \text{if } H > 1/2 \\ \int_0^t \int_{\mathbb{R}^d} \delta_0(B_s^1 - B_s^2 - y) f(y) dy ds & \text{if } H = 1/2 \end{cases}$$

More generally (and for the sake of a result encountered later in the sequel), if  $\eta : [0, t]^2 \rightarrow \mathbb{R}_+$  is an arbitrary function such that  $\eta(r, s) = \eta(t - r, t - s)$  for all  $r, s \in [0, t]$ , we define

$$L_{t,\varepsilon}(\eta) := \int_0^t \int_0^t \int_{\mathbb{R}^d} \eta(r, s) p_\varepsilon(B_r^1 - B_s^2 - y) f(y) dy dr ds.$$

Then  $\alpha_{n,\varepsilon}(t, \eta) = E(L_{t,\varepsilon}(\eta)^n)$ , where  $\alpha_{n,\varepsilon}(t, \eta) := \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) \psi_\varepsilon^{(n)}(\mathbf{s}, \mathbf{t}) ds dt$ .

Let

$$\alpha_n(t, \eta) := \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) \psi^{(n)}(\mathbf{s}, \mathbf{t}) ds dt, \quad (22)$$

and note that  $\alpha_{n,\varepsilon}(t, \eta) \leq \alpha_n(t, \eta)$  for all  $\varepsilon > 0$ .

**Lemma 4.4** *Let  $t > 0$  be arbitrary. a) If  $\alpha_2(t, \eta) < \infty$ , then*

$$\lim_{\varepsilon, \delta \downarrow 0} E(L_{t, \varepsilon}(\eta)L_{t, \delta}(\eta)) = \alpha_2(t, \eta), \quad (23)$$

*and there exists a random variable  $L_t(\eta) := \lim_{\varepsilon \downarrow 0} L_{t, \varepsilon}(\eta)$  in  $L^2(\Omega)$ . We denote*

$$L_t(\eta) = \int_0^t \int_0^t \int_{\mathbb{R}^d} \eta(r, s) \delta_0(B_r^1 - B_s^2 - y) f(y) dy dr ds.$$

*b) If  $\alpha_n(t, \eta) < \infty$  for all  $n \geq 1$ , then*

$$\lim_{\varepsilon \downarrow 0} E|L_{t, \varepsilon}(\eta) - L_t(\eta)|^p = 0, \quad \text{for all } p \geq 2.$$

*In particular,  $E(L_t(\eta)^n) = \lim_{\varepsilon \downarrow 0} E(L_{t, \varepsilon}(\eta)^n) = \alpha_n(t, \eta)$  for all  $n \geq 1$ .*

**Proof:** As in [14], the proof follows by classical methods. We include it for the sake of completeness.

a) Note that  $E(L_{t, \varepsilon}(\eta)L_{t, \delta}(\eta)) = \int_{[0, t]^4} \eta(s_1, t_1) \eta(s_2, t_2) \psi_{\varepsilon, \delta}^{(2)}(\mathbf{s}, \mathbf{t}) ds dt$ , where

$$\begin{aligned} \psi_{\varepsilon, \delta}^{(2)}(\mathbf{s}, \mathbf{t}) &:= E \left[ \int_{(\mathbb{R}^d)^2} p_\varepsilon(B_{s_1}^1 - B_{t_1}^2 - y_1) p_\varepsilon(B_{s_2}^1 - B_{t_2}^2 - y_2) f(y_1) f(y_2) dy_1 dy_2 \right] \\ &= (2\pi)^{-2d} \int_{(\mathbb{R}^d)^2} \exp \left\{ -\frac{1}{2} \sum_{j, k=1}^2 \sigma_{jk} \xi_j \cdot \xi_k - \frac{\varepsilon}{2} |\xi_1|^2 - \frac{\delta}{2} |\xi_2|^2 \right\} \mu(d\xi_1) \mu(d\xi_2). \end{aligned}$$

Then  $\lim_{\varepsilon, \delta \downarrow 0} \psi_{\varepsilon, \delta}^{(2)}(\mathbf{s}, \mathbf{t}) = \psi^{(2)}(\mathbf{s}, \mathbf{t})$ . Relation (23) follows by the Dominated Convergence Theorem, since  $\psi_{\varepsilon, \delta}^{(2)}(\mathbf{s}, \mathbf{t}) \leq \psi^{(2)}(\mathbf{s}, \mathbf{t})$  for all  $\varepsilon, \delta > 0$ , and

$$\int_{[0, t]^4} \eta(s_1, t_1) \eta(s_2, t_2) \psi^{(2)}(\mathbf{s}, \mathbf{t}) = \alpha_2(t) < \infty.$$

Hence,  $\lim_{\varepsilon, \delta \downarrow 0} E|L_{t, \varepsilon}(\eta) - L_{t, \delta}(\eta)|^2 = 0$ , i.e.  $(L_{t, \varepsilon}(\eta))_{\varepsilon > 0}$  is Cauchy in  $L^2(\Omega)$ .

b) Note that  $\lim_{\varepsilon \downarrow 0} L_{t, \varepsilon}(\eta) = L_t(\eta)$  a.s. (along a subsequence), and  $(L_{t, \varepsilon}(\eta))_{\varepsilon > 0}$  is uniformly integrable, since

$$\sup_{\varepsilon > 0} E(L_{t, \varepsilon}(\eta)^n) = \sup_{\varepsilon > 0} \alpha_{n, \varepsilon}(t, \eta) \leq \alpha_n(t, \eta) < \infty, \quad \text{for } n \geq 2.$$

Therefore,  $\lim_{\varepsilon \downarrow 0} E|L_{t, \varepsilon}(\eta) - L_t(\eta)|^p = 0$  for any  $p \geq 2$ .  $\square$

The next two results are the analogues of Propositions 3.1 and 3.2 of [14] in the case of a colored noise.

**Proposition 4.5** *Suppose that  $\eta : [0, t]^2 \rightarrow \mathbb{R}_+$  satisfies the following condition: for all  $t > 0$*

$$\|\eta\|_{1, t} := \max \left( \sup_{s \in [0, t]} \int_0^t \eta(r, s) dr, \sup_{r \in [0, t]} \int_0^t \eta(r, s) ds \right) < \infty. \quad (24)$$

If either one of the following two conditions holds:

(i)  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha < d$ , and  $d < 2 + \alpha$ ;

(ii)  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ , then  $\lim_{\varepsilon \downarrow 0} L_{t,\varepsilon}(\eta) = L_t(\eta)$  exists in  $L^p(\Omega)$ , for all  $p \geq 2$ , and

$$E[\exp(\lambda L_{t,\varepsilon}(\eta))] < \infty, \quad \text{for all } \lambda > 0, \varepsilon > 0.$$

**Proof:** We use the definition (22) of  $\alpha_n(t, \eta)$ , and Lemma 3.4 for the estimation of  $\psi^{*(n)}(\mathbf{s}, \mathbf{t})$ .

(i) Using the Cauchy-Schwartz inequality, condition (24), and Lemma A.1 (Appendix A), we get:

$$\begin{aligned} \alpha_n(t, \eta) &\leq \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\right)^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) [\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4} ds dt \\ &\leq \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\right)^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) [\beta(\mathbf{s})]^{-(d-\alpha)/2} ds dt \\ &\leq \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\right)^n \|\eta\|_{1,t}^n \int_{[0,t]^n} [\beta(\mathbf{s})]^{-(d-\alpha)/2} ds \\ &= \left(D_{\alpha,d} 2^{-(d-\alpha)/2} \|\eta\|_{1,t}\right)^n n! I_n(t, h) \quad \text{with } h = -\frac{d-\alpha}{2} \\ &\leq C_{H,d,\alpha} C(t)^n (n!)^{(d-\alpha)/2}, \end{aligned}$$

where  $C_{H,d,\alpha} = \Gamma(1+h)$  and  $C(t) = D_{\alpha,d} 2^{-(d-\alpha)/2} \|\eta\|_{1,t} \Gamma(1+h) t^{1+h}$ . The first statement follows by Lemma 4.4. The second statement follows since,

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \alpha_{n,\varepsilon}(t, \eta) \leq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \alpha_n(t, \eta) \leq C_{H,d,\alpha} \sum_{n=0}^{\infty} \frac{[\lambda C(t)]^n}{(n!)^{1-(d-\alpha)/2}},$$

and the last sum is finite for all  $\lambda > 0$ , since  $d - \alpha < 2$ .

(ii) In this case,

$$\alpha_n(t, \eta) \leq C_{\alpha,d}^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) ds dt \leq C_{\alpha,d}^n \|\eta\|_{1,t}^n t^n,$$

and the conclusion follows.  $\square$

**Proposition 4.6** Suppose that  $\eta : [0, t]^2 \rightarrow \mathbb{R}_+$  satisfies the following condition: there exist  $\gamma > 0$  and  $1/2 < H < 1$ , such that

$$\eta(r, s) \leq \gamma |r - s|^{2H-2}, \quad \forall r, s \in [0, t]. \quad (25)$$

(i) If  $f$  is the Riesz kernel of order  $\alpha$ , or  $f$  is the Bessel kernel of order  $\alpha < d$ , and  $d \leq 2 + \alpha$ , then  $\lim_{\varepsilon \downarrow 0} L_{t,\varepsilon}(\eta) = L_t(\eta)$  exists in  $L^p(\Omega)$ , for all  $p \geq 2$ , and

$$\sup_{\varepsilon > 0} E[\exp(\lambda L_{t,\varepsilon}(\eta))] < \infty, \quad \text{for all } 0 < \lambda < \lambda_0(t),$$

where

$$\lambda_0(t) = \begin{cases} D_{\alpha,d}^{-1} 2\alpha_H \gamma^{-1} \beta_H^{-2} \Gamma\left(1 - \frac{1}{2H}\right)^{-2H} t^{1-2H} & \text{if } d = 2 + \alpha \\ \infty & \text{if } d < 2 + \alpha \end{cases}$$

(ii) If  $f$  is the heat kernel of order  $\alpha$  or the Poisson kernel of order  $\alpha$ , then the same conclusion as in part (i) holds, with  $\lambda_0(t) = \infty$ .

**Proof:** The proof is similarly to Proposition 3.5. We use the definition (22) of  $\alpha_n(t, \eta)$ , Lemma 3.4 and condition (25).

(i) We have:

$$\alpha_n(t, \eta) \leq C_{H,d,f} C(t)^n (n!)^{(d-\alpha)/2} < \infty,$$

where

$$C(t) := D_{\alpha,d} 2^{-(d-\alpha)/2} \frac{\gamma}{\alpha_H} \beta_H^2 \Gamma\left(1 - \frac{d-\alpha}{4H}\right)^{2H} t^{2H-(d-\alpha)/2}.$$

The first statement follows by Lemma 4.4. The other statement follows, since

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \alpha_{n,\varepsilon}(t, \eta) \leq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \alpha_n(t, \eta) \leq C_{H,d,f} \sum_{n=0}^{\infty} \frac{[C(t)\lambda]^n}{(n!)^{1-(d-\alpha)/2}}.$$

If  $d - \alpha = 2$ , then the last sum is finite for all  $0 < \lambda < \lambda_0(t) := 1/C(t)$ . If  $d - \alpha < 2$ , then the last sum is finite for all  $\lambda > 0$ .

(ii) The result follows, since:

$$\begin{aligned} \alpha_n(t, \eta) &\leq C_{\alpha,d}^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j, t_j) ds dt \leq C_{\alpha,d}^n \gamma^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} ds dt \\ &= \left( C_{\alpha,d} \frac{\gamma}{\alpha_H} t^{2H} \right)^n. \end{aligned}$$

□

Using the approximation technique of [14], for any  $\varepsilon, \delta > 0$ , we set:

$$\dot{W}_{t,x}^{\varepsilon,\delta} = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\varepsilon(x-y) dW_{s,y},$$

where  $\varphi_\delta(t) = \delta^{-1} 1_{[0,\delta]}(t)$ . We consider the following equation:

$$\begin{aligned} \frac{\partial u^{\varepsilon,\delta}}{\partial t} &= \frac{1}{2} \Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta} \diamond \dot{W}^{\varepsilon,\delta}, \quad t > 0, x \in \mathbb{R}^d \\ u_{0,x}^{\varepsilon,\delta} &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (26)$$

Using the Feynman-Kac formula, one can prove that the process  $u^{\varepsilon,\delta} = \{u_{t,x}^{\varepsilon,\delta}; (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$  defined by:

$$u_{t,x}^{\varepsilon,\delta} := E^B \left[ u_0(x + B_t) \exp \left( \int_0^t \int_{\mathbb{R}^d} A_{r,y}^{\varepsilon,\delta,B} dW_{r,y} - \frac{1}{2} \|A^{\varepsilon,\delta,B}\|_{\mathcal{H}^{\mathcal{P}}}^2 \right) \right] \quad (27)$$

is a solution of (26), where  $A_{r,y}^{\varepsilon,\delta,B} = \int_0^t \varphi_\delta(t-s-r)p_\varepsilon(x+B_s-y)ds$ .

Let  $B^i = (B_t^i)_{t \in [0,T]}$ ,  $i \geq 1$  be independent  $d$ -dimensional standard Brownian motions, independent of  $W$ . Suppose that either (15) or (16) hold. For any pair  $(i, j)$  with  $i \neq j$ , let

$$L_t^{B^i, B^j} = \begin{cases} \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} |r-s|^{2H-2} \delta_0(B_r^i - B_s^j - y) f(y) dy dr ds & \text{if } H > 1/2 \\ \int_0^t \int_{\mathbb{R}^d} \delta_0(B_s^i - B_s^j - y) f(y) dy ds & \text{if } H = 1/2 \end{cases}$$

be the random variable defined in Lemma 4.4, with  $\eta(r, s) = \alpha_H |r-s|^{2H-2}$  if  $H > 1/2$ , respectively  $\eta(r, s) = 1_{\{r=s\}}$  if  $H = 1/2$ . (By Proposition 3.5,  $\alpha_n(t) < \infty$  for all  $n \geq 1$ , and hence the random variable  $L_t^{B^i, B^j}$  is well-defined.)

The following result is the analogue of Theorem 5.3 of [14] in the case of the colored noise.

**Theorem 4.7** (i) *Suppose that  $f$  is the Riesz kernel of order  $\alpha$  or the Bessel kernel of order  $\alpha < d$ , and either (15) or (16) holds. Then, for any integer  $k \geq 2$ , we have:*

$$\sup_{\varepsilon, \delta > 0} E[(u_{t,x}^{\varepsilon,\delta})^k] < \infty, \quad \text{for all } 0 < t < t_0(k), \quad (28)$$

where

$$t_0(k) = \begin{cases} \left[ k(k-1) D_{\alpha,d} 2^{-2H} \beta_H^2 \Gamma\left(1 - \frac{1}{2H}\right)^{2H} \right]^{-1/(2H-1)} & \text{if } d = 2 + \alpha \\ \infty & \text{if } d < 2 + \alpha \end{cases}$$

For any  $0 < t < t_0(2)$  and  $x \in \mathbb{R}^d$ , the limit  $u_{t,x} := \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} u_{t,x}^{\varepsilon,\delta}$  exists in  $L^2(\Omega)$ , the process  $u = \{u_{t,x}; (t, x) \in [0, t_0(2)) \times \mathbb{R}^d\}$  is a solution of (1), and

$$E[u_{t,x}^2] = E \left[ u_0(x + B_t^1) u_0(x + B_t^2) \exp \left( L_t^{B^1, B^2} \right) \right] := \gamma_2(t, x).$$

If  $t < t_0(M)$  for some  $M \geq 3$ , then  $\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E|u_{t,x}^{\varepsilon,\delta} - u_{t,x}|^p = 0$  for all  $2 \leq p < M$ , and for any integer  $2 \leq k \leq M-1$ ,

$$E[u_{t,x}^k] = E \left[ \prod_{j=1}^k u_0(x + B_t^j) \exp \left( \sum_{1 \leq i < j \leq k} L_t^{B^i, B^j} \right) \right] := \gamma_k(t, x). \quad (29)$$

(ii) *Suppose that  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ . Then the conclusion same as in part (i) holds, with  $t_0(k) = \infty$  for all  $k \geq 2$ .*

**Proof:** Let  $k \geq 2$  be a fixed integer. Using (27), one can show that:

$$E[(u_{t,x}^{\varepsilon,\delta})^k] = E \left[ \prod_{k=1}^k u_0(x + B_t^j) \exp \left( \sum_{i < j} \langle A^{\varepsilon,\delta, B^i}, A^{\varepsilon,\delta, B^j} \rangle_{\mathcal{H}\mathcal{P}} \right) \right]. \quad (30)$$

Using the fact that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\varepsilon(x+B_{s_1}^i-y_1)p_\varepsilon(x+B_{s_2}^j-y_2)f(y_1-y_2)dy_2dy_1 = \int_{\mathbb{R}^d} p_{2\varepsilon}(B_{s_1}^i-B_{s_2}^j-y)f(y)dy,$$

it follows that

$$\begin{aligned} \langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}} &= \int_0^t \int_0^t \int_{\mathbb{R}^d} \eta_\delta(s_1-s_2)p_{2\varepsilon}(B_{s_1}^i-B_{s_2}^j-y)f(y)dyds_1ds_2 \\ &= L_{t,2\varepsilon}^{B^i,B^j}(\eta_\delta), \end{aligned} \quad (31)$$

where

$$\eta_\delta(s_1, s_2) = \begin{cases} \alpha_H \int_0^t \int_0^t \varphi_\delta(t-s_1-r_1)\varphi_\delta(t-s_2-r_2)|r_1-r_2|^{2H-2}dr_1dr_2 & \text{if } H > 1/2 \\ \int_0^t \varphi_\delta(t-s_1-r)\varphi_\delta(t-s_2-r)dr & \text{if } H = 1/2 \end{cases}.$$

Note that, for any continuous function  $g : [0, t]^2 \rightarrow \mathbb{R}$ ,

$$\lim_{\delta \downarrow 0} \int_0^t \int_0^t \eta_\delta(s_1, s_2)g(s_1, s_2)ds_1ds_2 = \begin{cases} \alpha_H \int_0^t \int_0^t |s_1-s_2|^{2H-2}g(s_1, s_2)ds_1ds_2 & \text{if } H > 1/2 \\ \int_0^t g(s, s)ds & \text{if } H = 1/2 \end{cases}$$

Taking  $g(s_1, s_2) = \int_{\mathbb{R}^d} p_{2\varepsilon}(B_{s_1}^i-B_{s_2}^j-y)f(y)dy$ , we get:  $\lim_{\delta \downarrow 0} \langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}} = L_{t,2\varepsilon}^{B^i,B^j}$ . Using Lemma 4.4, we obtain:

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}} = \lim_{\varepsilon \downarrow 0} L_{t,2\varepsilon}^{B^i,B^j} = L_t^{B^i,B^j} \quad \text{a.s.} \quad (32)$$

If  $H = 1/2$ , then  $\eta_\delta$  satisfies condition (24), whereas if  $H > 1/2$ , then  $\eta_\delta$  satisfies condition (25) with  $\gamma = \alpha_H 2^{2-2H}$  (see p.28 of [14]). By applying Proposition 4.5 (if  $H = 1/2$ ), or Proposition 4.6 (if  $H > 1/2$ ), and using (31), it follows that:

$$\sup_{\delta, \varepsilon > 0} E \left[ \exp \left( \lambda \langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}} \right) \right] < \infty, \quad \text{for all } 0 < \lambda < \lambda_0(t). \quad (33)$$

where  $\lambda_0(t) = \infty$  if  $H = 1/2$ , and  $\lambda_0(t)$  is defined in Proposition 4.6, if  $H > 1/2$ . Since

$$\sup_{\varepsilon, \delta > 0} E[(u_{t,x}^{\varepsilon,\delta})^k] \leq \|u_0\|_\infty^k \sup_{\varepsilon, \delta > 0} E \left[ \exp \left( \frac{k(k-1)}{2} \langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}} \right) \right],$$

and

$$\frac{k(k-1)}{2} < \lambda_0(t) \quad \text{if and only if} \quad t < t_0(k),$$

relation (28) follows. From (33), we also infer that, if  $1 < \lambda_0(t)$  (i.e.  $t < t_0(2)$ ), then

$$\{\exp(\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \rangle_{\mathcal{HP}})\}_{\varepsilon, \delta > 0} \quad \text{is uniformly integrable,} \quad (34)$$

Using (30), (32) and (34), we obtain:

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E[(u_{t,x}^{\varepsilon,\delta})^k] = \gamma_k(t, x), \quad \text{if } t < t_0(2). \quad (35)$$

Using a similar argument, one can prove that  $\lim_{\varepsilon, \varepsilon' \downarrow 0} \lim_{\delta, \delta' \downarrow 0} E[u_{t,x}^{\varepsilon,\delta} u_{t,x}^{\varepsilon',\delta'}] = \gamma_2(t, x)$ , if  $t < t_0(2)$ . To see this, note that

$$E[u_{t,x}^{\varepsilon,\delta} u_{t,x}^{\varepsilon',\delta'}] = E[u_0(x + B_t^1) u_0(x + B_t^2) \exp(\langle A^{\varepsilon,\delta, B^1}, A^{\varepsilon',\delta', B^2} \rangle_{\mathcal{HP}})],$$

and  $\lim_{\varepsilon, \varepsilon' \downarrow 0} \lim_{\delta, \delta' \downarrow 0} \langle A^{\varepsilon,\delta, B^1}, A^{\varepsilon',\delta', B^2} \rangle_{\mathcal{HP}} = \lim_{\varepsilon, \varepsilon' \downarrow 0} L_{t, \varepsilon + \varepsilon'}^{B^1, B^2} = L_t^{B^1, B^2}$ .

It follows that, for any  $t < t_0(2)$  and  $x \in \mathbb{R}^d$ ,  $\{u_{t,x}^{\varepsilon,\delta}\}_{\varepsilon, \delta > 0}$  is Cauchy in  $L^2(\Omega)$ , and there exists  $u_{t,x} := \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} u_{t,x}^{\varepsilon,\delta}$  in  $L^2(\Omega)$ . The fact that  $u = \{u_{t,x}; (t, x) \in [0, t_0(2)) \times \mathbb{R}^d\}$  is a solution of (1) follows as in [14], by checking that  $u_{t,x} = p_t u_0(x) + \delta(p_{t-} \cdot (x - \cdot) u1_{[0,t]})$ , i.e.

$$E(F u_{t,x}) = E(F) p_t u_0(x) + E(\langle p_{t-} \cdot (x - \cdot) u1_{[0,t]}, DF \rangle_{\mathcal{HP}}), \quad \forall F \in \mathbb{D}^{1,2}.$$

If  $t < t_0(M)$  for some  $M \geq 3$ , then  $\sup_{\varepsilon, \delta > 0} E[(u_{t,x}^{\varepsilon,\delta})^M] < \infty$ , and hence,  $\{(u_{t,x}^{\varepsilon,\delta})^p\}_{\varepsilon, \delta > 0}$  is uniformly integrable, for any  $2 \leq p < M$ . Since  $u_{t,x}^{\varepsilon,\delta} - u_{t,x} \rightarrow 0$  a.s. (along a subsequence), we conclude that  $E|u_{t,x}^{\varepsilon,\delta} - u_{t,x}|^p \rightarrow 0$ . In particular,  $\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E[(u_{t,x}^{\varepsilon,\delta})^k] = E[u_{t,x}^k]$  for any  $2 \leq k < M$ , and (29) follows by (35).  $\square$

## A The calculation of $I_n(t, h)$

**Lemma A.1** *Let*

$$I_n(t, h) := \int_{T_n} [(t - s_n)(s_n - s_{n-1}) \dots (s_2 - s_1)]^h ds,$$

where  $T_n = \{\mathbf{s} = (s_1, \dots, s_n); 0 < s_1 < s_2 < \dots < s_n < t\}$ . Then  $I_n(t, h) < \infty$  if and only if  $1 + h > 0$ . In this case,

$$I_n(t, h) = \frac{\Gamma(1+h)^n}{\Gamma(n(1+h)+1)} t^{n(1+h)}.$$

**Proof:** First note that  $\int_0^{s_2} (s_2 - s_1)^h ds_1 = s_2^{h+1}/(h+1)$ , and then

$$\begin{aligned} \int_0^{s_3} (s_3 - s_2)^h s_2^{h+1} ds_2 &= s_3^{2h+2} \beta(h+2, h+1) \\ &= s_3^{2(h+1)} \beta((h+1)+1, h+1) \end{aligned}$$

where  $\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$  is the beta function, and we used the change of variables  $s_2/s_3 = z$ . In this way, the integral  $I_n(t, h)$  becomes

$$I_n(t, h) = \frac{1}{h+1} \beta((h+1)+1, h+1) \dots \beta((n-2)(h+1)+1, h+1)$$



$$\begin{aligned}
& \int_0^t s_n^{(n-1)(h+1)} (t-s_n)^h ds_n \\
&= t^{n(h+1)} \beta((h+1)+1, h+1) \dots \beta((n-1)(h+1)+1, h+1).
\end{aligned}$$

Using the fact that  $\beta(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  for  $a, b > 0$  and  $\Gamma(z+1) = z\Gamma(z)$  for any  $z > 0$ , we obtain the desired conclusion.  $\square$

## B Correction to Theorem 3.13 of [2]

Theorem 3.13 of [2] gives the necessary and sufficient condition for the existence of the solution of the equation:  $u_t = \frac{1}{2}\Delta u + \dot{W}$  in  $(0, T) \times \mathbb{R}^d$ , with  $u(0, \cdot) = 0$ . This condition is equivalent to saying that  $\|g_{tx}\|_{\mathcal{HP}} < \infty$ , where  $g_{tx}(s, y) = [2\pi(t-s)]^{-d/2} \exp\{-|x-y|^2/[2(t-s)]\} = p_{t-s}(x-y)$ . The condition is incorrectly stated in the case of the Bessel kernel, the heat kernel, and the Poisson kernel.

We give below a correction of Theorem 3.13 of [2].

**Theorem B.1** (i) *If  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha$ , then  $\|g_{tx}\|_{\mathcal{HP}} < \infty$  if and only if  $H > (d-\alpha)/4$ .*

(ii) *If  $f$  is the heat kernel of order  $\alpha$ , or the Poisson kernel of order  $\alpha$ , then  $\|g_{tx}\|_{\mathcal{HP}} < \infty$  for any  $H > 1/2$  and  $d \geq 1$ .*

**Proof:** Note that  $\|g_{tx}\|_{\mathcal{HP}}^2 = \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} I(r, s) dr ds = \alpha_1(t)$ , where

$$I(r, s) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_1 - y_1) p_{t-s}(x - x_1) p_{t-r}(x - y_1) dx_1 dy_1 = \psi^{*(1)}(r, s).$$

(i) In the case of the Riesz kernel, the result has been correctly proved in [2]. Suppose that  $f$  is the Bessel kernel of order  $\alpha$ . Then

$$I(r, s) = \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} (w+2t-r-s)^{-d/2} dw,$$

and

$$\begin{aligned}
& \|g_{tx}\|_{\mathcal{HP}}^2 \\
&= \alpha_H \gamma'_\alpha \int_0^t \int_0^t |r-s|^{2H-2} \int_0^\infty w^{\alpha/2-1} e^{-w} (w+r+s)^{-d/2} dw dr ds \\
&\geq \alpha_H \gamma'_\alpha 2^{-d/2} \int_0^t \int_0^t |r-s|^{2H-2} (r+s)^{-d/2} \left( \int_0^{r+s} w^{\alpha/2-1} e^{-w} dw \right) dr ds \\
&= \alpha_H \gamma'_\alpha 2^{-d/2+1} \int_0^t \int_0^r (r-s)^{2H-2} (r+s)^{-d/2} \left( \int_0^{r+s} w^{\alpha/2-1} e^{-w} dw \right) ds dr \\
&= \alpha_H \gamma'_\alpha 2^{-d/2+1} \int_0^t r^{2H-1-d/2} \int_0^1 (1-x)^{2H-2} (1+x)^{-d/2} \left( \int_0^{r(1+x)} w^{\alpha/2-1} e^{-w} dw \right) dx dr
\end{aligned}$$

$$\begin{aligned}
&\geq 2\alpha_H \gamma'_\alpha \int_0^t r^{2H-1-d/2} \int_0^1 (1-x)^{2H-2} \left( \int_0^r w^{\alpha/2-1} e^{-w} dw \right) dx dr \\
&= 2H \gamma'_\alpha \int_0^t r^{2H-1-d/2} \gamma(\alpha/2, r) dr,
\end{aligned}$$

where

$$\gamma(a, x) = \int_0^x w^{a-1} e^{-w} dw, \quad a > 0, x > 0$$

is the *imcomplete Gamma function*. It is known that: (see [1], Section 6.5, pages 260-263)

$$\lim_{x \rightarrow 0} \frac{\gamma(a, x)}{x^a} = 1. \quad (36)$$

From (36) it follows that the function  $\gamma(\alpha/2, r)$  behaves as  $r^{\alpha/2}$ , for  $r$  close to zero. The fact that the integral  $\int_0^t r^{2H-1-d/2} \gamma(\alpha/2, r) dr$  is finite forces the condition  $d < 4H + \alpha$ .

Suppose now that  $H > (d - \alpha)/4$ . We prove that  $\|g_{tx}\|_{\mathcal{HP}} < \infty$ . We have:

$$\begin{aligned}
&\|g_{tx}\|_{\mathcal{HP}}^2 \\
&= 2\alpha_H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \int_0^t \int_s^t (r-s)^{2H-2} (w+r+s)^{-d/2} dr ds dw \\
&= 2\alpha_H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \int_0^{2t} \int_0^t u^{2H-2} (w+v)^{-d/2} \mathbf{1}_{\{u \leq v\}} \mathbf{1}_{\{u \leq 2t-v\}} du dv dw \\
&= 2H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \left[ \int_0^t v^{2H-1} (w+v)^{-d/2} dv + \int_0^t v^{2H-1} (w+2t-v)^{-d/2} dv \right] dw \\
&:= 2H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} [I_1(w) + I_2(w)] dw,
\end{aligned}$$

and

$$\begin{aligned}
I_1(w) &\leq \int_0^t (w+v)^{2H-1-d/2} dv = \int_w^{w+t} v^{2H-1-d/2} dv \\
I_2(w) &\leq \int_0^t (w+2t-v)^{2H-1-d/2} dv = \int_{w+t}^{w+2t} v^{2H-1-d/2} dv.
\end{aligned}$$

If  $H > d/4$ , then  $I_1(w) + I_2(w) \leq [(w+t)^{2H-d/2} + (w+2t)^{2H-d/2}]/(2H-d/2)$  and

$$\|g_{tx}\|_{\mathcal{HP}}^2 \leq \frac{2H \gamma'_\alpha}{2H - d/2} \int_0^\infty w^{\alpha/2-1} e^{-w} [(w+t)^{2H-d/2} + (w+2t)^{2H-d/2}] dw < \infty.$$

If  $(d-\alpha)/4 < H < d/4$ , then  $I_1(w) + I_2(w) \leq [w^{2H-d/2} + (w+t)^{2H-d/2}]/(d/2 - 2H)$  and

$$\|g_{tx}\|_{\mathcal{HP}}^2 \leq \frac{2H \gamma'_\alpha}{d/2 - 2H} \int_0^\infty w^{\alpha/2-1} e^{-w} [w^{2H-d/2} + (w+t)^{2H-d/2}] dw < \infty.$$

If  $H = d/4$ , then  $I_1(w) + I_2(w) \leq \ln(w + 2t) - \ln w$  and

$$\|g_{tx}\|_{\mathcal{HP}}^2 \leq C \int_0^\infty w^{\alpha/2-1} e^{-w} [\ln(w + 2t) - \ln w] dw < \infty$$

(ii) If  $f$  is the heat kernel of order  $\alpha$ , then  $I(r, s) = (2\pi)^{-d/2} (\alpha + 2t - r - s)^{-d/2}$  and

$$\begin{aligned} \|g_{tx}\|_{\mathcal{HP}}^2 &= (2\pi)^{-d/2} \alpha_H \int_0^t \int_0^t |r - s|^{2H-2} (\alpha + r + s)^{-d/2} dr ds \\ &\leq (2\pi\alpha)^{-d/2} \alpha_H \int_0^t \int_0^t |r - s|^{2H-2} dr ds = (2\pi\alpha)^{-d/2} t^{2H} < \infty. \end{aligned}$$

If  $f$  is the Poisson kernel of order  $\alpha$ , then:

$$I(r, s) = C_d \int_0^\infty w^{d/2-1} e^{-w/2} [2(2t - r - s) + \alpha^2]^{-(d+1)/2} dw$$

and

$$\begin{aligned} \|g_{tx}\|_{\mathcal{HP}}^2 &= C_d \alpha_H \int_0^\infty w^{d/2-1} e^{-w/2} \int_0^t \int_0^t |r - s|^{2H-2} [2(r + s) + \alpha^2]^{-(d+1)/2} dr ds dw \\ &\leq C_d \alpha_H \alpha^{-(d+1)} \int_0^\infty w^{d/2-1} e^{-w/2} \int_0^t \int_0^t |r - s|^{2H-2} dr ds dw < \infty. \end{aligned}$$

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