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Partitions versus sets : a case of duality

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Abstract

In a recent paper, Amini et al. introduced a general framework to prove duality theorems between tree decompositions and their dual combinatorial object. They unify all known ad-hoc proofs in one duality theorem based on submodular partition functions. This general theorem remains however a bit technical and relies on this particular submodularity property. Instead of partition functions, we propose here a simple combinatorial property of set of partitions which also gives these duality results. Our approach is both simpler, and a little bit more general.

1 Introduction

In the past 30 years, several decompositions of graphs and discrete structures such as tree-decompositions and branch-decompositions of graphs [5,6]

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and matroids [3,4] have been introduced. Most of these decompositions admit some dual combinatorial object (brambles, tangles...), in the sense that a decomposition exists if and only if the dual object does not.

In [1], the authors present a general framework for proving these duality relations. Precisely, a *partitioning tree* on a finite set E is a tree T which leaves are identified to the elements of E in a one-to-one way. Every internal node v of T corresponds to the partition of E which parts are the set of leaves of the subtrees obtained by deleting v . Such a partition is a *node-partition*. A partitioning tree T is *compatible* with a set of partitions \mathcal{P} of E if every node-partition of T belong to \mathcal{P} . For some specific sets of partitions \mathcal{P} , one can get classical tree decompositions. Let for instance $G = (V, E)$ be a graph. The *border* of a partition μ of E is the set of vertices incident with edges in at least two parts of μ . For every integer k , let \mathcal{P}_k be the set of partitions of E whose border contain at most $k + 1$ vertices. Now, there exists a partitioning tree compatible with \mathcal{P}_k if and only if the tree-width of G is at most k .

The dual objects of partitioning trees are *brambles*. A *non-principal \mathcal{P} -bramble* is a nonempty set of pairwise intersecting subsets of E containing no singleton and which contains a part of every partition of \mathcal{P} . A non-principal \mathcal{P} -bramble and a partitioning tree compatible with \mathcal{P} cannot both exist at the same time, but there may be none of them.

In [1], the authors propose a sufficient condition for a set of partitions \mathcal{P} to be such that there exists a partitioning tree compatible with \mathcal{P} if and only if no non-principal \mathcal{P} -bramble does (duality property). The condition they introduced is expressed by the mean of weight functions on partitions. Precisely, they prove that if a partition function is (*weakly*) *submodular*, the set of partitions with weight bounded by a fixed constant enjoys the duality property. For example, the weight function corresponding to tree-width (the size of the border of a partition) is submodular. Therefore, if the tree-width of G is more than k , there is no partitioning tree compatible with \mathcal{P}_k , hence a bramble exists. This provides a alternative proof of [1], also presented in [2]. This kind of argument provides duals for some other tree-decompositions.

While [1]'s framework unifies several ad-hoc proof techniques of duality between decompositions and their dual objects, its core theorem mimics a proof of [6]. The argument is quite technical and does not give a real insight of the reason why the duality property holds. Moreover, at least one partition function, the function \max_f which corresponds to branchwidth, is not weakly submodular. Since this function is a limit of weakly submodular functions, Amini et al. also manage to apply their theorem to branchwidth.

The goal of this paper is twofold. First we give a simpler proof of the duality theorem, then we slightly extend (and simplify) the definition of *weak*

submodularity so that the function \max_f becomes weakly submodular.

To do so, we consider *partial partitioning trees*, in which the leaves of a tree T are labelled by the parts of some partition of E , called the *displayed partition* of T . When the displayed partition consists of singletons, we have our previous definition of partitioning trees. The set of displayed partitions of partial partitioning trees compatible with \mathcal{P} (i.e. such that every node-partition belongs to \mathcal{P}) is denoted by \mathcal{P}^\uparrow . Observe that in T , internal nodes of degree two can be simplified, so we can assume that all internal nodes have degree at least three.

We do not make any distinction between principal and non principal \mathcal{P} -brambles. Instead we define a *set of small sets* to be a subset of 2^E closed under taking subset, and whose elements are *small*. We say that a set of partitions \mathcal{P}^\uparrow is *dualising* if for any set of small sets \mathcal{S} , there exists a *big bramble* (i.e. a bramble containing no part in \mathcal{S}) if and only if \mathcal{P}^\uparrow contains no *small partition* (i.e. a partition whose parts all belong to \mathcal{S}). Thus the classical duality results are derived when \mathcal{S} consists of the empty set and the singletons. Note that since a \mathcal{P} -bramble Br meets all partitions in \mathcal{P} , if \mathcal{P} contains a small partition, Br cannot contain only big parts. Hence, a class of partitions cannot both admit a big bramble and a small partition.

In Section 2, we fix some notations and give some basic definitions. In Section 3, we give an equivalent and yet easier notion than the dualising property: the refining property. In Section 4, we give a sufficient condition on \mathcal{P} so that \mathcal{P}^\uparrow is refining (and thus dualising). Finally, in Section 5, we extend the definition of *weak submodularity* to match our sufficient condition for duality, and we prove that the partition function \max_f is weakly submodular and thus, that branchwidth fully belongs to the unifying framework.

2 Brambles

Let E be a finite set. We denote by 2^E the set of subsets of E . A partition of E is a set of pairwise nonempty subsets of E which cover E . The set \mathcal{P} always denote a set of partitions of E . Greek letters α, β, \dots denote sets of nonempty subsets of E , while capital letters A, B, \dots denote nonempty subsets of E (apart the sets of indices I and J). We write X^c for the complement $E \setminus X$ of X . We denote a finite union $\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_p$ by $(\alpha_1 | \alpha_2 | \dots | \alpha_p)$ and also shorten $(\{A\} | \alpha | \{B\})$ into $(A | \alpha | B)$. The *size* of a subset α of 2^E is just the number of sets in α .

Starting with some subset β of 2^E , one can perform two operations:

- (Deletion) Suppress an element in some set of β . Precisely, if $\beta = (B|\gamma)$ and $b \in B$, the result of the deletion operation is $(B \setminus \{b\}|\gamma)$.
- (Partition) Partition some set of β . Precisely, if $\beta = (B|\gamma)$ and δ is a partition of B , the result of the partition operation is $(\delta|\gamma)$.

We say that α is *finer* than β if it can be obtained from β by a sequence of deletions and partitions, and that α is *strongly finer* than β if we only use deletions. When we write that $(\alpha_1|\dots|\alpha_p)$ is finer (resp. strongly finer) than $(\beta_1|\dots|\beta_q)$, with $p \leq q$, we will usually mean that each α_i is finer (resp. strongly finer) than β_i . Observe that in some cases, the deletion operation can result in emptyset, in which case we simply delete the set, since we do not allow the emptyset in our families of sets.

For any F , $\alpha \setminus F$ denotes the set $\{A_i \setminus F; i \in I\}$. The *overlap* of α is the set $ov(\alpha)$ of the elements which belong to at least two parts of α . Observe that if α is finer than β , then $ov(\alpha)$ is included in $ov(\beta)$.

A \mathcal{P} -*bramble* or just *bramble* when no confusion can occur is a set Br of subsets of E such that

- Br contains a part of every $\mu \in \mathcal{P}$ (Br meets every $\mu \in \mathcal{P}$);
- the elements of Br are pairwise intersecting.

If Br is a \mathcal{P} -bramble, we say that \mathcal{P} admit the bramble Br .

3 Dualising and refining sets of partitions

A set \mathcal{S} of *small* sets is just a subset of 2^E which is closed under taking subsets. A set which does not belong to \mathcal{S} is *big*. By extension, a *big bramble* is a bramble consisting of big sets, while a *small partition* is a partition consisting of small parts.

A set of partitions \mathcal{P} is *dualising* if for any set of small sets \mathcal{S} , either there exists a big \mathcal{P} -bramble, or \mathcal{P} contains a small partition.

A set of partitions \mathcal{P} is *refining* (resp. *strongly refining*) if for any $(\alpha|A)$, $(B|\beta) \in \mathcal{P}$ with A disjoint from B , there exists a partition of \mathcal{P} finer (resp. strongly finer) than the covering $(\alpha|\beta)$.

Theorem 1 *If \mathcal{P} is refining, then \mathcal{P} is dualising.*

PROOF. Suppose that \mathcal{P} is refining and contains no small partition for some set of small sets. There exists a set closed under taking superset which contains

a big part from every partition of \mathcal{P} (just consider for this the set of all big sets). We claim that such a set Br , chosen inclusion-wise minimal, is a big bramble.

If not, there exists two disjoint sets A and B in Br . Choose them inclusion-wise minimal. Since $Br \setminus \{A\}$ is upward closed, Br being minimal, there exists $(\alpha|A) \in \mathcal{P}$ which contain no part of $Br \setminus \{A\}$. Similarly, there exists $(B|\beta) \in \mathcal{P}$ which contain no part of $Br \setminus \{B\}$. Hence Br does not meet $(\alpha|\beta)$, but since \mathcal{P} is refining, it contains λ finer than $(\alpha|\beta)$ which is not met by Br , a contradiction. \square

Conversely,

Theorem 2 *If \mathcal{P} is dualising, then \mathcal{P} is refining.*

PROOF. Assume for contradiction that \mathcal{P} is not refining. Let $(\alpha|A)$ and $(B|\beta)$ in \mathcal{P} where A is disjoint from B and such that \mathcal{P} contains no partition finer than $(\alpha|\beta)$. Our set of small sets consists of all sets included in some part of $(\alpha|\beta)$. We now claim that \mathcal{P} is not dualising. Indeed

- Since \mathcal{P} has no partition finer than $(\alpha|\beta)$, there is no small partition.
- Since a bramble cannot both contain A and B , it must contain a small set to meet both $(\alpha|A)$ and $(B|\beta)$. Hence there is no big bramble. \square

We would like to emphasise that only Theorem 1 will be used in the following of the paper.

4 Pushing sets of partitions

We denote by \mathcal{P}^\dagger the least superset of \mathcal{P} such that whenever $(\alpha|A) \in \mathcal{P}^\dagger$ and $(A^c|\beta) \in \mathcal{P}^\dagger$, we have $(\alpha|\beta) \in \mathcal{P}^\dagger$. Observe that \mathcal{P}^\dagger is the set of all displayed partitions of partial partitioning trees compatible with \mathcal{P} .

Lemma 3 *For any $(\alpha|A) \in \mathcal{P}^\dagger \setminus \mathcal{P}$, there exists $(\gamma|C) \in \mathcal{P}$ and $(C^c|\mu|A) \in \mathcal{P}^\dagger$ such that $(\alpha|A) = (\gamma|\mu|A)$, where $(\gamma|C)$ has at least three parts.*

PROOF. Let T be some partial partitioning tree which displayed partition is $(\alpha|A)$. Since $(\alpha|A)$ does not belong to \mathcal{P} , T has at least two internal nodes.

The partition $(\gamma|C)$ can be any node-partition of an internal node of T which is adjacent to only one internal node and not adjacent to the leaf A . \square

We say that such a partition $(\gamma|C)$ *decomposes* $(\alpha|A)$. To extend this notion to \mathcal{P}^\dagger , we also say that $(\alpha|A)$ *decomposes* $(\alpha|A)$, when $(\alpha|A) \in \mathcal{P}$.

We now introduce a property on \mathcal{P} which implies that \mathcal{P}^\dagger is refining and thus, by Theorem 1, that \mathcal{P}^\dagger is dualising.

A set of partitions \mathcal{P} is *pushing* if for every pair of partitions $(\alpha|A)$ and $(B|\beta)$ in \mathcal{P} with $A^c \cap B^c \neq \emptyset$, there exists a nonempty $F \subseteq A^c \cap B^c$ such that $(\alpha \setminus F|A \cup F) \in \mathcal{P}$ or $(B \cup F|\beta \setminus F) \in \mathcal{P}$.

Theorem 4 *If \mathcal{P} is pushing, then \mathcal{P}^\dagger is strongly refining.*

PROOF. Suppose for a contradiction that \mathcal{P} is pushing, that $(\alpha|A)$, $(B|\beta)$ both belong to \mathcal{P}^\dagger with A disjoint from B , and yet \mathcal{P}^\dagger contain no partition strongly finer than $(\alpha|\beta)$. Choose $(\alpha|\beta)$ with minimal size, and then with minimum overlap among counter-examples with minimal size. Let $O = A^c \cap B^c$ be the overlap of $(\alpha|\beta)$. Observe that since $(\alpha|\beta)$ is not a partition of E , O is nonempty.

We claim that there exist no $(\gamma|C)$, $(D|\delta) \in \mathcal{P}^\dagger$ with C disjoint from D , such that $(\gamma|\delta)$ is strongly finer than $(\alpha|\beta)$ and has an overlap which is a strict subset of O . If not, our choice of $(\alpha|A)$, $(B|\beta)$ implies that \mathcal{P}^\dagger contains a partition λ which is strongly finer than $(\gamma|\delta)$ and thus λ is strongly finer than $(\alpha|\beta)$, a contradiction.

Let $(\gamma|C)$ and $(D|\delta)$ be respectively decomposing $(\alpha|A)$ and $(B|\beta)$. Since $A \subseteq C$ and $B \subseteq D$, we have $C^c \cap D^c \subseteq O$. If $C^c \cap D^c$ is nonempty, since \mathcal{P} is pushing there exists a nonempty subset F of O such that, say, $(\gamma \setminus F, C \cup F) \in \mathcal{P}$. If C^c and D^c are disjoint, C^c and D^c cannot both contain O . There thus exists a non empty $F \subseteq O$ which is disjoint from, say, C^c , and therefore $(\gamma|C) = (\gamma \setminus F, C \cup F)$ since $F \subseteq C$. In both cases, $(\gamma \setminus F, C \cup F) \in \mathcal{P}$.

- If $(\gamma|C) = (\alpha|A)$, then $(\gamma \setminus F|\beta)$ is strongly finer than $(\alpha|\beta)$ and its overlap is $O \setminus F$, which is strictly included in O , a contradiction.
- If $(\gamma|C) \neq (\alpha|A)$, we consider $(C^c|\mu|A) \in \mathcal{P}^\dagger$ such that $(\gamma|\mu|A) = (\alpha|A)$. Since $(C^c|\mu|A)$ has fewer parts than $(\alpha|A)$, there exists $(C'|\mu'|\beta') \in \mathcal{P}^\dagger$ strongly finer than $(C^c|\mu|\beta)$. We assume that $C' \subseteq C$ is nonempty, since $(\mu'|\beta') \in \mathcal{P}^\dagger$ would be strongly finer than $(\alpha|\beta)$. If $O \not\subseteq C'^c$, then $(\gamma|\mu'|\beta')$ is strongly finer than $(\alpha|\beta)$, with an overlap strictly included in O , a contradiction. If $O \subseteq C'^c$, then C' and $C \cup F$ are disjoint. But then $(\gamma \setminus F|\mu|\beta)$

is strongly finer than $(\alpha|\beta)$, and its overlap (which is a subset of $O \setminus F$) is a strict subset of O , a contradiction. \square

The following result suggests that for an algorithmic point of view, there is no need to consider \mathcal{P}^\dagger -brambles. Just asking to meet every partition of \mathcal{P} is sufficient to characterize a bramble.

Lemma 5 *A set B_r is a \mathcal{P} -bramble if and only if it is a \mathcal{P}^\dagger -bramble.*

PROOF. Clearly, every \mathcal{P}^\dagger -bramble is also a \mathcal{P} -bramble. Conversely, we prove the result by induction on the number of nodes of the partial partitioning tree, based on the following observation. If B_r contains a part of both $(\alpha|A) \in \mathcal{P}^\dagger$ and $(A^c|\beta) \in \mathcal{P}^\dagger$, then B_r contains a part of $(\alpha|\beta)$ since B_r cannot both contain A and A^c . \square

5 Submodular partition functions

A *partition function* is a function from the set of partitions of E into $\mathbb{R} \cup \{+\infty\}$. In [1], the authors define *submodular* partition functions Ψ such that for every partitions $(\alpha|A)$ and $(B|\beta)$, we have:

$$\Psi(\alpha|A) + \Psi(B|\beta) \geq \Psi(\alpha \setminus B^c|A \cup B^c) + \Psi(\beta \setminus A^c|B \cup A^c).$$

It is routine to observe that if Ψ is partition submodular, then for every k , the set \mathcal{P}_k of partitions with Ψ value at most k is pushing, just consider for this $F = A^c \cap B^c$ in the definition of the pushing property. Hence \mathcal{P}_k^\dagger is dualising as soon as Ψ is submodular. From this follows the duality theorems for tree-width of matroids and graphs, as explicated in [1].

However, in order to also obtain duality for branchwidth, the authors introduced *weakly submodular* partition functions as partition functions such that for every partitions $(\alpha|A)$ and $(B|\beta)$, at least one of the following holds:

- there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$.

Since $(\beta|B)$ and $(\beta \setminus A^c|B \cup A^c)$ are equal when $A^c \cap B^c = \emptyset$, this definition is only really interesting when $A^c \cap B^c \neq \emptyset$.

We introduce now a more convenient property, still called *weak submodularity*, where partition functions satisfy that for every $(\alpha|A)$ and $(B|\beta)$ with $A^c \cap B^c \neq$

\emptyset , there exists a nonempty $F \subseteq A^c \cap B^c$ such that at least one of the following holds:

- $\Psi((\alpha|A)) \geq \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$.

This definition indeed generalises the previous one.

- Suppose that there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$. Set $F' := F \cap (A^c \cap B^c)$. Since $F = F' \cup A$, $(\alpha \setminus F|A \cup F) = (\alpha \setminus F'|A \cup F')$. Thus $\Psi((\alpha|A)) > \Psi((\alpha \setminus F'|A \cup F'))$ and F' is certainly nonempty.
- Suppose that $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$. Set $F := A^c \cap B^c$. Since $(\beta \setminus A^c|B \cup A^c) = (\beta \setminus F|B \cup F)$, $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$ and F is nonempty.

Obviously, given a weakly submodular partition function Ψ , the class of partitions $\mathcal{P}_k = \{\alpha ; \Psi(\alpha) \leq k\}$, for some $k \in \mathbb{R}$, is pushing. Conversely if \mathcal{P} is pushing, then defining Ψ as $\Psi(\alpha) = 0$ if $\alpha \in \mathcal{P}$ and $\Psi(\alpha) = 1$ otherwise, we obtain a weakly submodular partition function.

A *connectivity function* is a function $f : 2^E \mapsto \mathbb{R} \cup \{+\infty\}$ which is *symmetric* (i.e. for any $A \subseteq E$, $f(A) = f(A^c)$) and *submodular* (i.e. for any $A, B \subseteq E$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$). For any connectivity function f , we define the partition function \max_f by $\max_f(\alpha) = \max\{f(A) ; A \in \alpha\}$ (α a partition of E). The weak submodularity of the \max_f function gives the duality theorems concerning branchwidth and rank-width.

Lemma 6 *The function \max_f is a weakly submodular partition function.*

PROOF. Let $(\alpha|A)$ and $(B|\beta)$ be two partitions of E such that $A^c \cap B^c$ is nonempty. Let F with $A \setminus B \subseteq F \subseteq (B \setminus A)^c$ be such that $f(F)$ is minimum. We claim that $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F|A \cup F))$.

Indeed, we have $f(F \cap A) \geq f(F)$ by definition of F , and by submodularity, since $f(F) + f(A) \geq f(A \cap F) + f(A \cup F)$, we have $f(A) \geq f(A \cup F)$. For every X in α , we have by submodularity of f :

$$f(X) + f(F^c) \geq f(X \cap F^c) + f(X \cup F^c) \quad (1)$$

Since $f(F)$ is minimum, $f(F) \leq f(F \setminus X)$, and thus f being symmetric:

$$f(X \cup F^c) \geq f(F^c) \quad (2)$$

Adding (1) and (2), we obtain $f(X) \geq f(X \cap F^c)$. Thus $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F, A \cup F))$, as claimed.

Similarly, $\max_f((B|\beta)) \geq \max_f((B \cup F^c|\beta \setminus F^c))$. Now at least one of $F_A := F \cap (A^c \cap B^c)$ and $F_B := F^c \cap (A^c \cap B^c)$, say F_A , is nonempty. Since $(\alpha \setminus F|A \cup F) = (\alpha \setminus F_A|A \cup F_A)$, there exists a nonempty $F_A \subseteq A^c \cap B^c$ with $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F_A, A \cup F_A))$ which proves that \max_f is weakly submodular. \square

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