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# Existence and properties of pseudo-inverses for Bessel and related processes

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**Abstract** It is shown that the tail probability of a Bessel process is the distribution function of a random time which is related to first and last passage times of Bessel processes.

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**Key words** Bessel processes, Pseudo-inverses, Generalized Black-Scholes formulae.

## 1 Introduction. Definition of a pseudo-inverse

**1.0** The aim of this paper is to give some mathematical meaning to the assertion :

"the stochastic process  $(R_t, t \geq 0)$ , with values in  $\mathbb{R}_+$ , has a tendency to increase" (1.0)

and, more precisely, to "measure this tendency" when  $(R_t, t \geq 0)$  is a Bessel process, by showing the existence of a "pseudo-inverse" of  $R$  and by studying its properties.

First, here are several possible interpretations of (1.0) :

(1.a)  $(R_t, t \geq 0)$  is a.s. increasing ; then, it admits an increasing inverse process.

(1.b)  $(R_t, t \geq 0)$  is a submartingale ; under some adequate conditions, it admits a Doob-Meyer decomposition :

$$R_t = M_t + A_t \quad (t \geq 0)$$

with  $(A_t, t \geq 0)$  an increasing process and  $(M_t, t \geq 0)$  a martingale. Since a martingale is a "well balanced" process,  $(R_t, t \geq 0)$  has a tendency to increase, which is measured by  $(A_t, t \geq 0)$ .

(1.c)  $(E(R_t), t \geq 0)$  is an increasing function.

(1.d)  $(R_t, t \geq 0)$  is "stochastically increasing", i.e. : for every  $y \geq 0$ ,  $P(R_t \geq y)$  is an increasing function of  $t$  (in the case  $R_0 = x > 0$ , we need to modify the previous assertion by taking  $y \geq x$ , to get a meaningful assertion).

We note that, trivially :  $(1.a) \implies (1.b) \implies (1.c) \iff (1.d)$  (when  $x = 0$ ).

It is the assertion (1.d) which we have in mind throughout this study. It leads us to the definition of a pseudo-inverse of  $(R_t, t \geq 0)$ , which we now present.

**1.1** Let  $(\Omega, (R_t, t \geq 0), P_x(x \in \mathbb{R}_+))$  denote a process taking values on  $\mathbb{R}_+$ , which is a.s. continuous and such that  $P_x(R_0 = x) = 1$ .

**Definition 1.1**  $(R_t, t \geq 0)$  admits an increasing pseudo-inverse (resp. decreasing pseudo-inverse) if :

$$i) \text{ for every } y > x, \lim_{t \rightarrow \infty} P_x(R_t \geq y) = 1 \quad (1.1)$$

$$ii) \text{ for every } y > x, \text{ the application from } \mathbb{R}_+ \text{ into } [0, 1] : t \rightarrow P_x(R_t \geq y) \text{ is increasing} \quad (1.2)$$

(resp. if :

$$i') \text{ for every } y < x, \lim_{t \rightarrow \infty} P_x(R_t \leq y) = 1 \quad (1.1')$$

$$ii') \text{ for every } y < x, \text{ the application from } \mathbb{R}_+ \text{ into } [0, 1] : t \rightarrow P_x(R_t \leq y) \text{ is increasing} \quad (1.2')$$

**Definition 1.2** Assume that (1.1) and (1.2) (resp. (1.1') and (1.2')) are satisfied. Then, there exists a family of positive r.v.'s  $(Y_{x,y}, y > x)$  (resp.  $(Y_{x,y}, x > y)$ ) such that :

$$P_x(R_t \geq y) = P(Y_{x,y} \leq t) \quad (t \geq 0) \quad (1.3)$$

$$\text{resp. } P_x(R_t \leq y) = P(Y_{x,y} \leq t) \quad (t \geq 0) \quad (1.3')$$

We call the family of positive r.v.'s  $(Y_{x,y}, y > x)$  (resp.  $(Y_{x,y}, y < x)$ ) the increasing (resp. decreasing) pseudo-inverse of the process  $(R_t, t \geq 0)$ .

We note that, a priori, the family  $(Y_{x,y}, y > x)$  (resp.  $(Y_{x,y}, y < x)$ ) is only a family of r.v.'s, and does not constitute a process.

Note that formulae (1.3) and (1.3') may be read, either considering  $x$  as fixed and  $y$  varying or by considering the two parameters  $x$  and  $y$  as varying.

**Remark 1.3**

i) If  $(R_t, t \geq 0)$  admits an increasing (resp. decreasing) pseudo-inverse, then  $R_t \xrightarrow[t \rightarrow \infty]{} +\infty$  in probability (resp.  $R_t \xrightarrow[t \rightarrow \infty]{} 0$  in probability).

ii) If  $(R_t, t \geq 0)$  admits an increasing pseudo-inverse, then, for every  $\alpha > 0$  (resp. for every  $\alpha < 0$ ), the process  $(R_t^\alpha, t \geq 0)$  admits an increasing (resp. decreasing) pseudo-inverse. If  $(R_t, t \geq 0)$  admits a decreasing pseudo-inverse, for every  $\alpha > 0$  (resp. for every  $\alpha < 0$ ), the process  $(R_t^\alpha, t \geq 0)$  admits a decreasing (resp. increasing) pseudo-inverse.

iii) Justifying the term "pseudo-inverse"

Condition (1.2) indicates that the process  $(R_t, t \geq 0)$  "has a tendency to increase". In case this process is indeed increasing, we introduce  $(\tau_l, l \geq 0)$  its right-continuous inverse :

$$\tau_l := \inf\{t \geq 0 ; R_t > l\}$$

Then, from (1.3), and for  $y > x$  ;

$$P_x(R_t \geq y) = P_x(\tau_y \leq t) = P(Y_{x,y} \leq t). \quad (1.4)$$

Thus  $Y_{x,y} \stackrel{(\text{law})}{=} \tau_y$  under  $P_x$ .

Hence, in this case, we may choose for the family  $(Y_{x,y}, y > x)$  the process  $(\tau_y, y > x)$ , and this justifies our terminology of pseudo-inverse.

*iv) A simple example*

Let  $(B_t, t \geq 0)$  be a Brownian motion started at 0, and  $a > 0$ . Define :  $R_t := \exp(B_t + at)$ . Then, the submartingale  $(R_t, t \geq 0)$  (which satisfies  $R_0 = 1$ ) admits an increasing pseudo-inverse. Indeed, for every  $y > 1$  :

$$\begin{aligned} P(R_t \geq y) &= P(B_t + at \geq \log y) = P(\sqrt{t} B_1 \geq \log y - at) \quad (\text{by scaling}) \\ &= P\left(B_1 \geq \frac{\log y}{\sqrt{t}} - a\sqrt{t}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\log y}{\sqrt{t}} - a\sqrt{t}}^{\infty} e^{-\frac{x^2}{2}} dx \end{aligned}$$

Obviously,  $P(R_0 \geq y) = 0$  and  $\lim_{t \rightarrow \infty} P(R_t \geq y) = 1$ . Moreover :

$$\frac{\partial}{\partial t} P(R_t \geq y) = \frac{1}{2} \left( \frac{\log y}{\sqrt{2\pi t^3}} + \frac{a}{\sqrt{2\pi t}} \right) e^{-\frac{1}{2t}(\log y - at)^2} \geq 0$$

Thus, with  $(Y_z, z > 0)$  defined as or, rather, satisfying :

$$P(Y_{\log y} \leq t) = P(R_t \geq y)$$

we have :

$$\begin{aligned} P(Y_{\log y} \in dt) &= \frac{1}{2} \left( \frac{\log y}{\sqrt{2\pi t^3}} + \frac{a}{\sqrt{2\pi t}} \right) \exp \left\{ -\frac{1}{2t} (\log y - at)^2 \right\} dt \\ &= \frac{1}{2} \{ P(T_{\log y}^{(a)} \in dt) + P(G_{\log y}^{(a)} \in dt) \} \end{aligned}$$

with :

$$G_{\log y}^{(a)} := \sup\{t \geq 0 ; B_t + at = \log y\}, \quad T_{\log y}^{(a)} = \inf\{t \geq 0 ; B_t + at = \log y\}$$

since as is well-known (see e.g. [MRY])

$$\begin{aligned} P(G_{\log y}^{(a)} \in dt) &= \frac{a}{\sqrt{2\pi t}} \exp \left\{ -\frac{1}{2t} (\log y - at)^2 \right\} dt \\ P(T_{\log y}^{(a)} \in dt) &= \frac{\log y}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{1}{2t} (\log y - at)^2 \right\} dt \quad (\log y, a > 0) \end{aligned}$$

## 1.2 Pseudo-inverses and generalised Black-Scholes formulae

Our main reason to introduce this notion of a pseudo-inverse is, independently from the fact that we think it is of interest by itself, because it plays an important role in the set-up of some generalised Black-Scholes formulae. More precisely, let us denote by  $(M_t, t \geq 0; \mathcal{F}_t, t \geq 0; P_x, x \geq 0)$  a Markov process taking values in  $\mathbb{R}_+$ , such that  $(M_t, t \geq 0)$  is a  $(\mathcal{F}_t, t \geq 0, P_x)$  positive local martingale, hence a supermartingale, which satisfies  $P_x(M_0 = x) = 1$  and

$$\lim_{t \rightarrow \infty} M_t = 0 \quad P_x \quad \text{a.s.} \quad (1.5)$$

Let  $\Pi_M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$  denote the put quantity associated with  $(M_t, t \geq 0)$ , that is :

$$\Pi_M(K, t) := E_x((K - M_t)^+) \quad (K, t \geq 0) \quad (1.6)$$

Then, it follows from our assumptions that, for every  $x > 0$

i)  $\left(\frac{1}{x} \Pi_M(y, \infty), 0 \leq y \leq x\right) = \left(\frac{y}{x}, 0 \leq y \leq x\right)$  is the distribution function of a uniform r.v.  $U_x$  on  $[0, x]$

ii)  $\left(\frac{1}{x} \Pi_M(x, t), t \geq 0\right)$  is the distribution function of a positive r.v., since  $\frac{1}{x} \Pi_M(x, 0) = 0$ ,  $\frac{1}{x} \Pi_M(x, \infty) = 1$  and the application  $t \rightarrow \frac{1}{x} \Pi_M(x, t) = \frac{1}{x} E_x((x - M_t)^+)$  is an increasing function of  $t$ ,  $(x - M_t)^+$  being a submartingale (since  $(x - y)^+$  is a convex function of  $y$ ). More precisely, it is proven in [MRY] that :

$$\frac{1}{x} \Pi_M(x, t) = P_x(G_x \leq t) \quad (1.6)$$

with :

$$G_x := \sup\{t \geq 0; M_t = x\} \quad (1.7)$$

(= 0 if this set is empty)

A natural question is then :  $x$  being fixed, is the function of  $K$  and  $t$   $\left(\frac{1}{x} \Pi_M(K, t), K \leq x; t < \infty\right)$ , the distribution function of a probability on  $[0, x] \times [0, \infty]$ , which we denote, if it exists, by  $\gamma_{M^{(x)}}$  ?

If so, one has

$$\frac{1}{x} \Pi_M(K, t) = \gamma_{M^{(x)}}([0, K] \times [0, t]) \quad (K \leq x, t \geq 0) \quad (1.8)$$

On the other hand, note that, by Fubini, one has :

$$\frac{1}{x} \Pi_M(K, t) = \frac{1}{x} \int_0^K P_x(M_t \leq y) dy \quad (1.9)$$

Hence :

$$\frac{\partial}{\partial K} \left( \frac{1}{x} \Pi_M(K, t) \right) = \frac{1}{x} P_x(M_t \leq K) \quad (K \leq x) \quad (1.10)$$

Thus, the existence of the probability  $\gamma_{M^{(x)}}$  is equivalent to :

$$t \rightarrow P_x(M_t \leq K) \text{ is an increasing function of } t \quad (\text{for } K < x)$$

(for  $K < x$ ). Clearly, this condition is equivalent to the existence of a decreasing pseudo-inverse  $(Y_{x,y}, y < x)$  for the process  $(M_t, t \geq 0; P_x, x \geq 0)$ .

At this point, we have obtained :

**Proposition 1.4** *For every  $x > 0$ , there exists a probability  $\gamma_{M^{(x)}}$  on  $[0, x] \times [0, \infty[$  such that :*

$$\frac{1}{x} \Pi_M(K, t) = \gamma_{M^{(x)}}([0, K] \times [0, t]) \quad (K < x, t \geq 0) \quad (1.11)$$

*if and only if  $(M_t, t \geq 0; P_x, x \geq 0)$  admits a decreasing pseudo-inverse  $(Y_{x,y}, y < x)$ , i.e. if, for every  $0 < y < x$  and  $t \geq 0$ , one has :*

$$P_x(M_t \leq y) = P(Y_{x,y} \leq t) \quad (1.12)$$

Of course, if (1.11) is satisfied, then  $\gamma_{M^{(x)}}$  is the law of a couple  $(U_x, \Lambda_x)$  where  $U_x$  is uniform on  $[0, x]$  and  $\Lambda_x \stackrel{(\text{law})}{=} G_x$ , but  $U_x$  and  $\Lambda_x$  are dependent.

Moreover, if  $\Pi_{M^{(x)}}$  is a regular function of  $K$  and  $t$ , then, denoting by  $f_{\gamma_{M^{(x)}}}$  the density of  $\gamma_{M^{(x)}}$  :

$$f_{\gamma_{M^{(x)}}}(K, t) = \frac{\partial^2}{\partial K \partial t} \frac{1}{x} \Pi_M(K, t) = \frac{1}{x} \frac{\partial}{\partial t} P_x(M_t \leq y) \quad (1.13)$$

(from (1.10))

$$= \frac{1}{x} \frac{\partial}{\partial t} P(Y_{x,y} \leq t) = \frac{1}{x} f_{Y_{x,y}}(t) \quad (K \leq x) \quad (1.14)$$

where  $f_{Y_{x,y}}$  denotes the density of  $Y_{x,y}$ .

### 1.3 Our aim in this paper

In the set-up of the classical Black-Scholes formulae, i.e. when  $M_t = \mathcal{E}_t := \exp\left(B_t - \frac{t}{2}\right)$ , where under  $P_x$ ,  $(B_t, t \geq 0)$  is a Brownian motion starting from  $(\log x)$ , we have proven in [MRY] the existence of a decreasing pseudo-inverse of  $(M_t, t \geq 0)$  and thus established the existence of a probability  $\gamma_{\mathcal{E}^{(x)}}$  characterized by :

$$\frac{1}{x} E_x((K - \mathcal{E}_t)^+) = \gamma_{\mathcal{E}^{(x)}}([0, K] \times [0, t]) \quad (K \leq x, t \geq 0) \quad (1.15)$$

In fact, in [MRY], the study was done for  $x = 1$ , but reducing the study to the case  $x = 1$  is easy and we have also described this probability  $\gamma_{\mathcal{E}(x)}$  in detail.

Now, let  $(\Omega, (R_t, \mathcal{F}_t), t \geq 0; P_x^{(\nu)}, x \geq 0)$  denote a Bessel process of index  $\nu$  ( $\nu \geq -1$ ). We shall prove in Section 2 (see Theorem 2.1) the following results :

- when  $\nu \geq -\frac{1}{2}$ , the existence of an increasing pseudo-inverse  $(Y_{x,y}^{(\nu)}, y > x)$  for  $(R_t, t \geq 0, P_x^{(\nu)}, x \geq 0)$
- when  $\nu = -1$ , the existence of a decreasing pseudo-inverse  $(Y_{x,y}^{(-1)}, y < x)$  for  $(R_t, t \geq 0, P_x^{(-1)}, x \geq 0)$
- when  $\nu \in ]-1, -\frac{1}{2}[$  :
  - i*) if  $x = 0$ , the existence of an increasing pseudo-inverse
  - ii*) if  $x > 0$ , the non-existence of a pseudo-inverse.

In Section 3, we describe the laws of the r.v.'s  $(Y_{x,y}^{(\nu)}, y > x)$ . In particular, we obtain explicitly the Laplace transform of  $(Y_{x,y}^{(\nu)}, y > x)$  :

$$E(e^{-\lambda Y_{x,y}^{(\nu)}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \quad \left(\nu \geq -\frac{1}{2}\right) \quad (1.16)$$

where  $I_\nu$  and  $K_\nu$  are the classical modified Bessel functions (see [Leb], p. 108). We also answer (partially) the question : are the r.v.'s  $Y_{x,y}^{(\nu)}$  infinitely divisible ?

For  $\nu > 0$ , we apply (end of Section 2) the existence of a pseudo-inverse for the process  $(R_t, t \geq 0)$  to the generalised Black-Scholes formula relative to the local martingale  $(R_t^{-2\nu}, t \geq 0)$ , when  $(R_t, t \geq 0)$  denotes a  $\delta = 2(\nu + 1)$  dimensional Bessel process.

In Section 4, we present two other families of Markovian positive submartingales which admit an increasing pseudo-inverse ; they are :

- i*) the Bessel processes with drift, studied by S. Watanabe [Wata] with infinitesimal generator :

$$L^{(\nu,a)} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{2\nu + 1}{2x} + a \frac{I_{\nu+1}(ax)}{I_\nu(ax)} \right) \frac{d}{dx}$$

for  $\nu \geq -\frac{1}{2}$ , and  $a \geq 0$  ;

- ii*) the generalized squares of Ornstein-Uhlenbeck processes (see, e.g., [PY2]), also called Cox-Ingersoll-Ross ( : CIR) processes, with infinitesimal generator :

$$L^{(\nu,\beta)} = 2x \frac{d^2}{dx^2} + (2\beta x + 2(\nu + 1)) \frac{d}{dx},$$

again, with  $\nu \geq -\frac{1}{2}$ , and  $\beta \geq 0$ .

Of course, letting  $a \rightarrow 0$ , resp.  $\beta \rightarrow 0$ , in *i*), resp. *ii*), we recover the result for the Bessel processes.

In Section 5, we exhibit a two parameter family  $(Y^{(\nu,\alpha)}, \nu \geq 0, 0 \leq \alpha \leq 1)$  of variables (indexed by  $(x, y)$ ,  $x < y$ ) which extends the family  $Y^{(\nu)}$ , corresponding to  $\alpha = 1$ .

The following paragraph aims at relating the results of Section 3, which presents the properties of the r.v.'s  $Y_{x,y}^{(\nu)}$ , and those of Section 5, where the r.v.'s  $Y_{x,y}^{(\nu,\alpha)}$  are defined and studied.

Consider  $(R_t^{(\nu)}, t \geq 0)$  and  $(R_t^{(\nu')}, t \geq 0)$  two independent Bessel processes starting from 0, and with respective indices  $\nu$  and  $\nu'$ , greater than  $-1$ , or dimensions :  $d = 2(\nu + 1)$  and  $d' = 2(\nu' + 1)$ . From the additivity property of squares of Bessel processes, the process :

$$R_t^{(\nu+\nu'+1)} := \left( (R_t^{(\nu)})^2 + (R_t^{(\nu')})^2 \right)^{\frac{1}{2}}, \quad t \geq 0,$$

is a Bessel process with index  $\nu + \nu' + 1$  (or dimension :  $d + d'$ ), starting from 0. Let, for  $y > 0$  :

$$\begin{aligned} G_y^{(\alpha)} &:= \sup\{t \geq 0 : R_t^{(\alpha)} = y\} \\ T_y^{(\alpha)} &:= \inf\{t \geq 0 : R_t^{(\alpha)} = y\} \end{aligned} \quad (\alpha = \nu, \nu + \nu' + 1)$$

It is then clear that :

$$T_y^{(\nu+\nu'+1)} \leq T_y^{(\nu)} \leq G_y^{(\nu)}, \quad \text{and} \quad G_y^{(\nu+\nu'+1)} \leq G_y^{(\nu)}.$$

These inequalities invite to look for some r.v.'s  $Z_y^{(i)}$  ( $i = 1, 2, 3$ ), such that :

$$G_y^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+\nu'+1)} + Z_y^{(1)} \tag{1.17}$$

$$G_y^{(\nu)} \stackrel{(\text{law})}{=} T_y^{(\nu)} + Z_y^{(2)} \tag{1.18}$$

$$T_y^{(\nu)} \stackrel{(\text{law})}{=} T_y^{(\nu+\nu'+1)} + Z_y^{(3)} \tag{1.19}$$

where the r.v.'s featured on the right-hand side of (1.17), (1.18) and (1.19) are independent. The existence of these r.v.'s  $Z_y^{(i)}$ , ( $i = 1, 2, 3$ ) is obtained in Sections 3 and 5 as a sub product of the existence of pseudo-inverses for Bessel processes. Precisely, the identities (1.17), (1.18) and (1.19) are shown respectively as (5.12), in Proposition 5.1, resp. as (5.8), with  $y = z$  and  $z = 0$ , resp. as (5.11) in Proposition 5.1. In Theorem 3.1 (point *iv*) and *v*) and Theorem 5.2, other equalities in law of the type of (1.17), (1.18) and (1.19), are established, when the starting points of the Bessel processes differ from 0.

Section 6 concludes with a small list of the ingredients we have used throughout the paper to obtain these pseudo-inverses.

## 2 Existence of pseudo-inverses for Bessel processes

### 2.1 Statement of our main result

Let  $\nu \geq -1$  and  $(R_t, t \geq 0; P_x^{(\nu)}, x \geq 0)$  a Bessel process with index  $\nu$ , i.e. with dimension  $\delta = 2\nu + 2$ , with  $\delta \geq 0$ .

#### Theorem 2.1

i) If  $\nu \geq -\frac{1}{2}$ ,  $(R_t, t \geq 0)$  admits an increasing pseudo-inverse ; that is for every  $y > x > 0$ , there exists a positive r.v.  $Y_{x,y}^{(\nu)}$  such that, for every  $t \geq 0$  :

$$P_x^{(\nu)}(R_t \geq y) = P(Y_{x,y}^{(\nu)} \leq t). \quad (2.1)$$

ii) Let  $\nu \in ]-1, -\frac{1}{2}[$  ; if  $x > 0$ ,  $(R_t, t \geq 0)$  does not admit a pseudo-inverse. If  $x = 0$ ,  $(R_t, t \geq 0)$  admits an increasing pseudo-inverse (see point 2.3.2 of the proof of Theorem 2.1).

iii) If  $\nu = -1$ ,  $(R_t, t \geq 0)$  admits a decreasing pseudo-inverse, that is there exists, for every  $x > y > 0$ , a positive r.v.  $Y_{x,y}^{(-1)}$  such that, for every  $t \geq 0$  :

$$P_x^{(-1)}(R_t \leq y) = P(Y_{x,y}^{(-1)} \leq t) \quad (2.2)$$

Thus, in Theorem 2.1, the value  $\nu = -\frac{1}{2}$  appears as a critical value. This is, in fact, rather natural. Indeed, for  $\nu \geq -\frac{1}{2}$ ,  $(R_t, t \geq 0)$  is a submartingale - therefore it is, in a way, increasing in  $t$  - and it may be written as :

$$R_t = x + B_t + \frac{2\nu + 1}{2} \int_0^t \frac{ds}{R_s} \quad \text{if } \nu > -\frac{1}{2}$$

where  $(B_t, t \geq 0)$  denotes a Brownian motion starting from 0, and for  $\nu = -\frac{1}{2}$ ,  $R_t = |x + B_t| = |x| + \beta_t + L_t^{-x}$ , where  $(L_t^{-x}, t \geq 0)$  denotes the local time of  $B$  at level  $-x$  and  $(\beta_t, t \geq 0)$  is a Brownian motion.

For  $\nu \in ]-1, -\frac{1}{2}[$ , such a representation is no longer true. In fact :

$$R_t = x + \beta_t + \frac{2\nu + 1}{2} k_t$$

with

$$k_t := \text{p.v.} \int_0^t \frac{ds}{R_s} := \int_0^\infty a^{2\nu} (L_t^a - L_t^0) da$$

where  $(L_t^a, t \geq 0, a \geq 0)$  denotes the jointly continuous family of diffusion local times associated with  $(R_t, t \geq 0)$  (see [RY], Exerc. 1.26, Chap. XI, p. 451) ; in this case,  $(R_t, t \geq 0)$  is no longer a semimartingale, but it is a Dirichlet process. (See Bertoin [Ber] for a deep study of excursion theory for the process  $((R_t, k_t), t \geq 0)$ ).

In order to prove Theorem 2.1, we gather a few results about Bessel processes.

## 2.2 A summary of some results about Bessel processes

**2.2.a** The density of the Bessel semi-group (see [RY], Chap. XI)

We denote by  $p^{(\nu)}(t, x, y)$  the density of  $R_t$  under  $P_x^{(\nu)}$  ( $\nu > -1$ ) ; one has :

$$p^{(\nu)}(t, x, y) = \frac{y}{t} \left(\frac{y}{x}\right)^\nu \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \quad (\nu > -1) \quad (2.3)$$

if  $x > 0$  whereas for  $x = 0$

$$p^{(\nu)}(t, 0, y) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu + 1) y^{2\nu+1} \exp\left(-\frac{y^2}{2t}\right) \quad (\nu > -1) \quad (2.4)$$

where  $I_\nu$  denotes the modified Bessel function with index  $\nu$  (see [Leb], p. 108).

**2.2.b** Density of last passage times (See [PY1])

Let

$$G_y := \sup\{t \geq 0 ; R_t = y\} \quad (2.5)$$

(= 0 if this set is empty)

Then, for  $x \leq y$  :

$$P_x^{(\nu)}(G_y \in dt) = \frac{\nu}{y} p^{(\nu)}(t, x, y) dt \quad (\nu > 0) \quad (2.6)$$

It follows easily from (2.6) and (2.3) that, under  $P_0^{(\nu)}$ , the law of  $G_y$  is that of  $\frac{y^2}{2\gamma_\nu}$ , where  $\gamma_\nu$  is a gamma r.v. with parameter  $\nu$  ( $\nu > 0$ ) :

$$G_y \stackrel{(\text{law})}{=} \frac{y^2}{2\gamma_\nu} \quad (G_y \text{ being considered under } P_0^{(\nu)}) \quad (2.7)$$

and :

$$E_x^{(\nu)}(e^{-\lambda G_y}) = 2\nu \left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda}) K_\nu(y\sqrt{2\lambda}) \quad (0 < x < y ; \nu > 0)$$

$$E_0^{(\nu)}(e^{-\lambda G_y}) = \frac{1}{2^{\nu-1}\Gamma(\nu)} (y\sqrt{2\lambda})^\nu K_\nu(y\sqrt{2\lambda}) \quad (y > 0 ; \nu > 0)$$

**2.2.c** Laplace transform of first hitting times (See [K] and [PY1] )

Let

$$T_y := \inf\{t \geq 0 ; R_t = y\} \quad (2.8)$$

(=  $+\infty$  if this set is empty)

Then :

$$E_x^{(\nu)}(e^{-\lambda T_y}) = \left(\frac{y}{x}\right)^\nu \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \quad (x < y, \lambda \geq 0) \quad (2.9)$$

In particular, for  $x = 0$

$$E_0^{(\nu)}(e^{-\lambda T_y}) = \frac{1}{2^\nu \Gamma(\nu + 1)} \frac{(y\sqrt{2\lambda})^\nu}{I_\nu(y\sqrt{2\lambda})} \quad (2.10)$$

#### 2.2.d Resolvent kernel (see [B.S])

Let

$$u_\lambda^{(\nu)}(x, y) = \int_0^\infty e^{-\lambda t} p^{(\nu)}(t, x, y) dt \quad (\lambda \geq 0)$$

Then, for every positive Borel function  $f$  :

$$\int_x^\infty u_\lambda^{(\nu)}(x, y) f(y) dy = \frac{2}{x^\nu} I_\nu(x\sqrt{2\lambda}) \int_x^\infty y^{\nu+1} K_\nu(y\sqrt{2\lambda}) f(y) dy \quad (2.11)$$

and

$$u_\lambda^{(\nu)}(x, y) = 2y \left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda}) K_\nu(y\sqrt{2\lambda}) \quad (x < y)$$

where  $K_\nu$  denotes the Bessel-Mc Donald function with index  $\nu$  (see [Leb], p. 108).

#### 2.2.e Bessel realisations of the Hartman-Watson laws (See [Y] and [PY1])

It was established by Hartman-Watson [H.W] that, for any  $\nu \geq 0$  and  $r > 0$ , the ratio

$$\lambda \longrightarrow \frac{I_{\sqrt{\lambda+\nu^2}}}{I_\nu}(r)$$

is the Laplace transform of a probability, say  $\theta_r^{(\nu)}$  on  $\mathbb{R}_+$ , which may then be called Hartman-Watson distribution. In [Y] and [PY1] these laws were shown to be those of the Bessel clocks  $\int_0^t \frac{ds}{R_s^2}$  for Bessel bridges  $(R_s, s \leq t)$  conditioned at both ends. Precisely :

For  $\nu > -1$  and  $\mu \neq 0$ , one has :

$$E_x^{(\nu)} \left( \exp \left( -\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) = \frac{I_{\sqrt{\mu^2+\nu^2}}}{I_\nu} \left( \frac{xy}{t} \right) \quad (x, y \geq 0) \quad (2.12)$$

This formula may be obtained from the particular case  $\nu = 0$  :

$$E_x^{(0)} \left( \exp \left( -\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) = \frac{I_{|\mu|}}{I_0} \left( \frac{xy}{t} \right)$$

together with the absolute continuity relationship :

$$P_x^{(\nu)}|_{\mathcal{R}_t \cap (t < T_0)} = \left( \frac{R_t}{x} \right)^\nu \exp \left( -\frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \cdot P_x^{(0)}|_{\mathcal{R}_t} \quad (2.13)$$

where  $(\mathcal{R}_t, t \geq 0)$  denotes the natural filtration of  $(R_t, t \geq 0)$ . If  $\nu > 0$ ,  $\mathcal{R}_t \cap (t < T_0)$  may be replaced by  $\mathcal{R}_t$  in formula (2.13) since then  $T_0 = \infty$  a.s., whereas if  $\nu \in ]-1, 0[$ , then, for  $\mu^2 > 0$  :

$$E_x^{(\nu)} \left( \exp \left( -\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) = E_x^{(\nu)} \left( \exp \left( -\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) 1_{T_0 > t} \middle| R_t = y \right)$$

since, if  $T_0 < t$ ,  $\int_0^t \frac{ds}{R_s^2} = +\infty$ .

We make several remarks :

i) For  $\nu < 0$ , letting  $\mu \rightarrow 0$  in (2.12), we get :

$$P_x^{(\nu)}(T_0 > t | R_t = y) = \frac{I_{-\nu}}{I_\nu} \left( \frac{xy}{t} \right) \quad (2.14)$$

ii) For  $\nu > -\frac{1}{2}$ , formula (2.12) becomes, taking  $\mu^2 = 2\nu + 1$  :

$$E_x^{(\nu)} \left( \exp \left( -\left( \nu + \frac{1}{2} \right) \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) = \frac{I_{\nu+1}}{I_\nu} \left( \frac{xy}{t} \right) \quad \left( \nu > -\frac{1}{2} \right) \quad (2.15)$$

From (2.15), we deduce that, for  $\nu \geq -\frac{1}{2}$  and  $z \geq 0$  :

$$I_{\nu+1}(z) \leq I_\nu(z) \quad (2.16)$$

(2.16) is an immediate consequence of (2.15) for  $\nu > -\frac{1}{2}$  and for  $\nu = -\frac{1}{2}$  :

$$I_{\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sinh(z) \leq I_{-\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cosh(z) \quad ([\text{Leb}], \text{ p. } 112).$$

**2.3.f** The Hirsch-Song formula (see [H.S])

**Proposition 2.2** *Let  $\nu > -\frac{1}{2}$ .*

$$i) \quad \frac{\partial}{\partial x} p^{(\nu)}(t, x, y) = -\frac{\partial}{\partial y} \left( \frac{x}{y} p^{(\nu+1)}(t, x, y) \right) \quad (2.17)$$

ii) *Let  $H$  denote the operator defined on the space of  $C^1$  functions, with derivative equal to 0 at 0, and which are bounded, as well as their derivative :*

$$Hf(x) = \frac{1}{x} f'(x) \quad (2.18)$$

Let  $(P_t^{(\nu)}, t \geq 0)$  denotes the Bessel semi-group with index  $\nu$ . Then :

$$HP_t^{(\nu)} = P_t^{(\nu+1)} H \quad (2.19)$$

iii) Let  $(Q_t^{(\nu)}, t \geq 0)$  denotes the semi-group of the Bessel squared process, with index  $\nu$ . Then :

$$DQ_t^{(\nu)} = Q_t^{(\nu+1)}D \quad (2.20)$$

where  $D$  denotes the differentiation operator :  $Df(x) = f'(x)$ , with domain the space of  $C^1$  functions, bounded as well as their derivative.

It is easily verified that (2.19) and (2.20) are equivalent (since  $Q_t^{(\nu)}f(z) = P_t^{(\nu)}(\tilde{f})(\sqrt{z})$ , with  $\tilde{f}(z) = f(z^2)$ ).

On the other hand, denoting by  $\bar{L}^{(\nu)}$  the infinitesimal generator of the semi-group  $Q_t^{(\nu)}$  : (see point 2.2.g below)

$$\bar{L}^{(\nu)}f(x) = 2x f''(x) + 2(\nu + 1)f'(x) \quad (2.21)$$

an easy computation allows to obtain  $DL^{(\nu)} = L^{(\nu+1)}D$ , hence (2.20).

We show (2.17), following [HS]

For  $f$  regular with compact support, one has ;

$$\begin{aligned} P_t^{(\nu)}f(x) &= \int_0^\infty p^{(\nu)}(t, x, y)f(y)dy, \text{ hence :} \\ \frac{\partial}{\partial x}P_t^{(\nu)}f(x) &= \int_0^\infty \frac{\partial}{\partial x}p^{(\nu)}(t, x, y)f(y)dy \end{aligned} \quad (2.22)$$

On the other hand, with obvious notations :

$$\frac{\partial}{\partial x}P_t^{(\nu)}f(x) = \frac{\partial}{\partial x}E^{(\nu)}[f(R_t^x)] = E^{(\nu)}\left[f'(R_t^x)\frac{\partial R_t^x}{\partial x}\right]$$

However, since :

$$R_t^x = x + B_t + \frac{2\nu + 1}{2} \int_0^t \frac{ds}{R_s^x} \quad \left(\nu > -\frac{1}{2}\right)$$

we obtain, by differentiation with respect to  $x$  :

$$\frac{\partial R_t^x}{\partial x} = 1 - \frac{2\nu + 1}{2} \int_0^t \frac{ds}{(R_s^x)^2} \frac{\partial R_s^x}{\partial x} \quad (2.23)$$

The linear equation (2.23) may be integrated, and we obtain :

$$\frac{\partial R_t^x}{\partial x} = \exp\left(-\frac{2\nu + 1}{2} \int_0^t \frac{ds}{(R_s^x)^2}\right) \quad (2.24)$$

Thus :

$$\begin{aligned}
\int_0^\infty \frac{\partial}{\partial x} p^{(\nu)}(t, x, y) f(y) dy &= E_x^{(\nu)} \left[ f'(R_t) \exp \left( -\frac{2\nu + 1}{2} \int_0^t \frac{ds}{R_s^2} \right) \right] \\
&= \int_0^\infty E_x^{(\nu)} \left( \exp \left( -\frac{2\nu + 1}{2} \int_0^t \frac{ds}{R_s^2} \right) \Big|_{R_t = y} \right) f'(y) p^{(\nu)}(t, x, y) dy \\
&= \int_0^\infty p^{(\nu)}(t, x, y) f'(y) \frac{I_{\nu+1}}{I_\nu} \left( \frac{xy}{t} \right) dy \quad (\text{from (2.15)}) \\
&= - \int_0^\infty f(y) \frac{\partial}{\partial y} \left( \frac{I_{\nu+1}}{I_\nu} \left( \frac{xy}{t} \right) p^{(\nu)}(t, x, y) \right) dy \\
&= - \int_0^\infty f(y) \frac{\partial}{\partial y} \left( \frac{x}{y} p^{(\nu+1)}(t, x, y) \right) dy \quad (\text{from (2.3)})
\end{aligned} \tag{2.25}$$

The comparison of (2.25) and (2.22) then implies point *i*) of Proposition 2.2.

### 2.2.g The Fokker-Planck formula

The infinitesimal generator  $L^{(\nu)}$  of the Bessel semi-group  $(P_t^{(\nu)}, t \geq 0)$  with index  $\nu$  is given by :

$$L^{(\nu)} f(x) = \frac{1}{2} f''(x) + \frac{2\nu + 1}{2x} f'(x) \tag{2.26}$$

Its domain is the space of functions  $f$  such that  $L^{(\nu)} f$  is bounded continuous and satisfies :  $\lim_{x \rightarrow 0} x^{2\nu+1} f'(x) = 0$ . One has

$$\begin{aligned}
\frac{\partial}{\partial t} p^{(\nu)}(t, x, y) &= L^{(\nu)}(p^{(\nu)}(t, \cdot, y))(x) \\
&= L^{(\nu)*}(p^{(\nu)}(t, x, \cdot))(y) \quad (\text{Fokker - Planck})
\end{aligned} \tag{2.27}$$

where the operator  $L^{(\nu)*}$ , the adjoint of  $L^{(\nu)}$ , is defined by :

$$L^{(\nu)*} f(x) = \frac{1}{2} f''(x) - \frac{\partial}{\partial x} \left( \frac{2\nu + 1}{2x} f(x) \right) \tag{2.28}$$

The infinitesimal generator  $\bar{L}^{(\nu)}$  of the semi-group  $(Q_t^{(\nu)}, t \geq 0)$  of the squared Bessel process with index  $\nu$ , is given by :

$$\bar{L}^{(\nu)} f(x) = 2x f''(x) + (2\nu + 2) f'(x) \tag{2.29}$$

and its adjoint  $\bar{L}^{(\nu)*}$  is defined by :

$$\bar{L}^{(\nu)*} f(x) = \frac{\partial^2}{\partial x^2} (2x f(x)) - (2\nu + 2) f'(x) \tag{2.30}$$

**2.2.h** Sojourn time below level  $y$  (see [PY1], also [MY], Th. 11.6, p. 180)

Let

$$A_y^- = \int_0^\infty 1_{(R_t \leq y)} dt \left( = \int_0^{G_y} 1_{(R_t \leq y)} dt \right)$$

Then :

$$E_y^{(\nu)}(e^{-\lambda A_y^-}) = \frac{2\nu}{y\sqrt{2\lambda}} \frac{I_\nu(y\sqrt{2\lambda})}{I_{\nu-1}(y\sqrt{2\lambda})} \quad (\nu > 0 ; \lambda > 0) \quad (2.31)$$

## 2.3 Proof of Theorem 2.1

### 2.3.1 A useful Lemma

**Lemma 2.3** *Let  $I_\nu$  denote the modified Bessel function with index  $\nu$  (see [Leb], p. 108)*

*i) If  $\nu \geq -\frac{1}{2}$ , then, for every  $z \geq 0$ ,*

$$I_\nu(z) \geq I_{\nu+1}(z) \quad (2.32)$$

*ii) If  $\nu = -1$ , then, for every  $z \geq 0$ ,*

$$I_1(z) = I_{-1}(z) \leq I_0(z) \quad (2.33)$$

*iii) If  $\nu \in ]-1, -\frac{1}{2}[$  for  $z$  small enough  $I_\nu(z) > I_{\nu+1}(z)$  whereas for  $z$  large enough,  $I_\nu(z) < I_{\nu+1}(z)$*

### **Proof of Lemma 2.3**

Point *i)* has been proven : it is relation (2.16).

We now prove point *ii)*

It follows from the previous point since  $I_{-1} = I_1$ .

We prove point *iii)*

In the neighborhood of 0, one has :

$$\frac{I_\nu(z)}{I_{\nu+1}(z)} \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \frac{\Gamma(\nu+2)}{\left(\frac{z}{2}\right)^{\nu+1}} = \frac{2}{z} (\nu+1) \xrightarrow{z \rightarrow 0} +\infty \quad (\nu > -1) \quad (2.34)$$

In the neighborhood of  $+\infty$ , one has (see [Leb], p. 122 and 123) :

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{4\mu^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right) \right)$$

Thus, with  $\mu = -\nu$  and  $\nu \in \left] \frac{1}{2}, 1 \right[$  :

$$\begin{aligned} I_{-\nu}(z) &= \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{4\nu^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right) \right) \\ I_{-\nu+1}(z) &= \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{4(1-\nu)^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right) \right) \end{aligned}$$

Now, for  $\nu \in \left] \frac{1}{2}, 1 \right[$ , one has  $4\nu^2 - 1 \geq 4(1-\nu)^2 - 1$  ; indeed, this is equivalent to  $\nu^2 \geq 1 + \nu^2 - 2\nu$ , i.e.  $\nu \geq \frac{1}{2}$ .

Hence, for  $\nu \in \left] \frac{1}{2}, 1 \right[$ , the ratio  $\frac{I_{-\nu}(z)}{I_{-\nu+1}(z)} \xrightarrow{z \rightarrow \infty} 1_-$  as  $z \rightarrow \infty$ . As a conclusion, from this point and (2.34), for  $\nu \in \left] -1, -\frac{1}{2} \right[$  :

- for  $z$  large enough,  $\frac{I_\nu(z)}{I_{\nu+1}(z)} < 1$  ;
- for  $z$  small enough  $\frac{I_\nu(z)}{I_{\nu+1}(z)} > 1$ .

### 2.3.2 The case $x = 0$

When  $x = 0$  and  $\nu > -1$ , we shall show the existence of an increasing process  $(J_t, t \geq 0)$  such that  $J_t \xrightarrow[t \rightarrow \infty]{} +\infty$  a.s. and

$$P_0^{(\nu)}(R_t \geq y) = P(J_t \geq y) \tag{2.35}$$

This formula obviously implies the existence of a process which is a pseudo-inverse of  $(R_t, t \geq 0)$ . Let us prove (2.35), which hinges in fact simply on the scaling property of  $(R_t, t \geq 0)$ . Indeed, one has :

$$\begin{aligned} P_0^{(\nu)}(R_t \geq y) &= P_0^{(\nu)} \left( \sqrt{t} \geq \frac{y}{R_1} \right) \quad (\text{by scaling}) \\ &= P_0^{(\nu)} \left( t \geq \frac{y^2}{R_1^2} \right) \\ &= P \left( t \geq \frac{y^2}{2\gamma_{\nu+1}} \right) \end{aligned}$$

(since  $R_1^2 \stackrel{(\text{law})}{=} 2\gamma_{\nu+1}$  under  $P_0^{(\nu)}$ , where  $\gamma_{\nu+1}$  is a gamma variable with parameter  $\nu + 1$ ).

$$\begin{aligned} &= P_0^{(\nu+1)}(G_y \leq t) \quad (\text{from (2.7)}) \\ &= P_0^{(\nu+1)} \left( \inf_{u \geq t} R_u \geq y \right) = P_0^{(\nu+1)}(J_t \geq y) \end{aligned}$$

with  $J_t := \inf_{u \geq t} R_u$ . Clearly  $(J_t, t \geq 0)$  is an increasing process and  $J_t \xrightarrow[t \rightarrow \infty]{} +\infty$   $P_0^{(\nu+1)}$  a.s. since  $\nu + 1 > 0$ .

### 2.3.3 We now prove point $i$ ) of Theorem 2.1

Since, from the comparison theorem, (applied to Bessel processes) for  $\nu \geq -\frac{1}{2}$ , one has :

$$\begin{aligned} P_x^{(\nu)}(R_t \geq y) &\geq P_0^{(-\frac{1}{2})}(R_t \geq y) \\ &= P(|B_t| \geq y) = P_0\left(|B_1| > \frac{y}{\sqrt{t}}\right) \xrightarrow[t \rightarrow \infty]{} 1 \end{aligned}$$

where  $(B_t, t \geq 0)$  is a Brownian motion started from 0.

It suffices, in order to prove point  $i$ ) of Theorem 2.1 to show that :  $t \rightarrow P_x^{(\nu)}(R_t \geq y)$  is an increasing function of  $t$ . Now, one has :

$$\begin{aligned} \frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) &= \int_y^\infty \frac{\partial}{\partial t} p^{(\nu)}(t, x, z) dz \\ &= \int_y^\infty \left[ \frac{1}{2} \frac{\partial^2}{\partial z^2} p^{(\nu)}(t, x, z) - \frac{\partial}{\partial z} \left( \frac{2\nu + 1}{2z} p^{(\nu)}(t, x, z) \right) \right] dz \\ &\quad \text{(from Fokker-Planck)} \\ &= -\frac{1}{2} \frac{\partial}{\partial y} p^{(\nu)}(t, x, y) + \frac{2\nu + 1}{2y} p^{(\nu)}(t, x, y) \end{aligned}$$

Now, from (2.3) :

$$\frac{\partial}{\partial y} p^{(\nu)}(t, x, y) = p^{(\nu)}(t, x, y) \left[ \frac{\nu + 1}{y} - \frac{y}{t} + \frac{x}{t} \frac{I'_\nu\left(\frac{xy}{t}\right)}{I_\nu\left(\frac{xy}{t}\right)} \right]$$

Hence :

$$\begin{aligned} \frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) &= p^{(\nu)}(t, x, y) \left[ \frac{\nu}{2y} + \frac{y}{2t} - \frac{x}{2t} \frac{I'_\nu\left(\frac{xy}{t}\right)}{I_\nu\left(\frac{xy}{t}\right)} \right] \\ &= p^{(\nu)}(t, x, y) \left[ \frac{\nu}{2y} + \frac{y}{2t} - \frac{x}{2t} \left( \frac{I_{\nu+1}}{I_\nu}\left(\frac{xy}{t}\right) + \frac{\nu t}{xy} \right) \right] \\ &\quad \left( \text{since } \frac{I'_\nu(z)}{I_\nu(z)} = \frac{I_{\nu+1}}{I_\nu}(z) + \frac{\nu}{z} \quad ([\text{Leb}], \text{ p. 110}) \right) \\ &= \frac{p^{(\nu)}(t, x, y)}{2t I_\nu\left(\frac{xy}{t}\right)} \left[ y I_\nu\left(\frac{xy}{t}\right) - x I_{\nu+1}\left(\frac{xy}{t}\right) \right] \end{aligned} \tag{2.36}$$

It now follows from Lemma 2.3 that  $\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y)$  is positive, since  $y > x$  and  $I_\nu(z) \geq I_{\nu+1}(z)$ .

**2.3.4** We now prove point *ii*) of Theorem 2.1

This follows immediately from (2.36) and from point *iii*) of Lemma 2.3 : for  $x \neq 0$ ,  $\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y)$  does not have a fixed sign, hence  $t \rightarrow P_x^{(\nu)}(R_t \geq y)$  cannot be a monotone function.

**2.3.5** We now prove point *iii*) of Theorem 2.1

We need to show that, for every  $x > 0$  and  $0 < y < x$ ,  $P_x^{(-1)}(R_t \leq y)$  is an increasing function of  $t$ . We may of course replace  $R_t$  by  $R_t^2$ . Now, the law of  $R_t^2$ , when  $R_0^2 = x$ , is given by :

$$P(R_t^2 \in dy) = \exp\left(-\frac{x}{2t}\right) \cdot \delta_0(dy) + q(t, x, y)dy$$

with

$$q(t, x, y) := \frac{1}{2t} \sqrt{\frac{x}{y}} \exp\left(-\frac{x+y}{2t}\right) I_1\left(\frac{xy}{t}\right) \quad (2.37)$$

Then, we need to show that, for every  $y < x$ , one has :

$$\frac{\partial}{\partial t} \left(\exp -\frac{x}{2t}\right) + \frac{\partial}{\partial t} \int_0^y q(t, x, z)dz \geq 0$$

But, since  $\frac{\partial}{\partial t} \left(\exp -\frac{x}{2t}\right) \geq 0$ , it suffices to show that :

$$\frac{\partial}{\partial t} \int_0^y q(t, x, z)dz \geq 0.$$

Now, one has :

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^y q(t, x, z)dz &= \int_0^y \frac{\partial}{\partial t} q(t, x, z)dz = \int_0^y \frac{\partial^2}{\partial z^2} (2zq(t, x, z))dz \\ &\quad \text{(from Fokker-Planck)} \\ &= \frac{\partial}{\partial y} (2y q(t, x, y)) \\ &= 2q(t, x, y) + 2y q(t, x, y) \left[ -\frac{1}{2y} - \frac{1}{2t} + \frac{1}{2t} \sqrt{\frac{x}{y}} \frac{I_1'}{I_1} \left( \frac{\sqrt{xy}}{t} \right) \right] \\ &\quad \text{(from (2.37))} \\ &= q(t, x, y) \left[ 1 - \frac{y}{t} + \frac{\sqrt{xy}}{t} \left( \frac{I_0}{I_1} \left( \frac{\sqrt{xy}}{t} \right) - \frac{t}{\sqrt{xy}} \right) \right] \\ &\quad \left( \text{since } \frac{I_1'}{I_1}(z) = \frac{I_0}{I_1}(z) - \frac{1}{z} \right) \\ &= \frac{\sqrt{y}}{t} q(t, x, y) \left[ \sqrt{x} \frac{I_0}{I_1} \left( \frac{\sqrt{xy}}{t} \right) - \sqrt{y} \right] \geq 0 \end{aligned}$$

from point *ii*) of Lemma 3.2 and since  $y < x$ .

This ends the proof of Theorem 2.1.

**Remark 2.4**

*i*) The property :  $t \rightarrow P_x^{(\nu)}(R_t \geq y)$  is an increasing function of  $t$  ( $y > x$ ) is equivalent to : for every Borel increasing function  $\varphi$  with support in  $[x, \infty[$ ,  $t \rightarrow E_x^{(\nu)}[\varphi(R_t)]$  is an increasing function of  $t$ . Indeed, we have, for such a  $\varphi$  :

$$\begin{aligned} E_x^{(\nu)}[\varphi(R_t)] &= E_x^{(\nu)}\left[\int_x^{R_t} d\varphi(y)\right] = E_x^{(\nu)}\int_x^\infty d\varphi(y)1_{(R_t \geq y)} \\ &= \int_x^\infty d\varphi(y)P_x^{(\nu)}(R_t \geq y) \end{aligned}$$

*ii*) Around formula (2.36)

Formula (2.36) :

$$\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) = \frac{p^{(\nu)}(t, x, y)}{2t I_\nu\left(\frac{xy}{t}\right)} \left[ y I_\nu\left(\frac{xy}{t}\right) - x I_{\nu+1}\left(\frac{xy}{t}\right) \right]$$

may still be written, with the help of (2.3) :

$$\frac{\partial}{\partial t} P(Y_{x,y}^{(\nu)} \leq t) = \frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) = \frac{y}{2t} p^{(\nu)}(t, x, y) - \frac{x^2}{2ty} p^{(\nu+1)}(t, x, y) \quad (2.38)$$

We denote by  $\theta^{(\nu)}(t, x, y)$  the RHS of (2.38). We now check directly that this positive function of  $t$  integrates to 1 i.e. : it is a density of probability :

$$\int_0^\infty \theta^{(\nu)}(t, x, y) dt = 1 \quad (2.39)$$

For this purpose, we use the classical Lipschitz-Hankel formula (see, e.g. [Wat], p. 384), for  $\nu > 0$  :

$$\nu \int_0^\infty \frac{du}{u} \exp(-au) I_\nu(u) = \frac{1}{(a + \sqrt{a^2 - 1})^\nu} \quad (a > 1) \quad (2.40)$$

Differentiating both sides with respect to  $a$ , we obtain :

$$\int_0^\infty \exp(-au) \cdot I_\nu(u) du = \frac{1}{(a + \sqrt{a^2 - 1})^\nu} \cdot \frac{1}{\sqrt{a^2 - 1}} \quad (a > 1, \nu > 0) \quad (2.41)$$

In order to check (2.39), we first prove that :

$$\Lambda_{x,y}^{(\nu)} := \int_0^\infty \frac{dt}{t} p^{(\nu)}(t, x, y) = \frac{2y}{y^2 - x^2} \quad (y > x) \quad (2.42)$$

Note that (2.42) (which we assume for a moment) shows that  $\Lambda_{x,y}^{(\nu)}$  does not depend on  $\nu$  ! We deduce from (2.42) that :

$$\begin{aligned} & \int_0^\infty \left[ \frac{y}{2t} p^{(\nu)}(t, x, y) - \frac{x^2}{2ty} p^{(\nu+1)}(t, x, y) \right] dt \\ &= \frac{y}{2} \Lambda_{x,y}^{(\nu)} - \frac{x^2}{2y} \Lambda_{x,y}^{(\nu+1)} = \left( \frac{y}{2} - \frac{x^2}{2y} \right) \frac{2y}{y^2 - x^2} = 1 \end{aligned}$$

which is the desired result (2.39).

We now prove (2.42)

From (2.3), we get :

$$\begin{aligned} \Lambda_{x,y}^{(\nu)} &= \left( \frac{y}{x} \right)^\nu y \int_0^\infty \frac{dt}{t^2} \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \\ &= \left( \frac{y}{x} \right)^\nu y \int_0^\infty du \exp\left(-\frac{x^2 + y^2}{2} u\right) I_\nu(xy u) \\ &\quad \text{(after the change of variable } \frac{1}{t} = u) \\ &= \left( \frac{y}{x} \right)^\nu \frac{y}{xy} \int_0^\infty dv \exp\left(-\frac{x^2 + y^2}{2xy} v\right) I_\nu(v) \\ &\quad \text{(after the change of variable } xy u = v) \\ &= \left( \frac{y}{x} \right)^\nu \frac{1}{x} \frac{1}{(a + \sqrt{a^2 - 1})^\nu} \frac{1}{\sqrt{a^2 - 1}} \end{aligned}$$

with the help of (2.41), where  $a := \frac{x^2 + y^2}{2xy}$  ( $> 1$  since  $x < y$ ). On the other hand, there are the elementary identities :

$$\begin{aligned} \sqrt{a^2 - 1} &= \frac{y^2 - x^2}{2xy}, \quad a + \sqrt{a^2 - 1} = \frac{y}{x}, \quad \text{which implies :} \\ \Lambda_{x,y}^{(\nu)} &= \frac{1}{x} \frac{1}{\frac{y^2 - x^2}{2xy}} = \frac{2y}{y^2 - x^2} \end{aligned}$$

which is the desired result (2.42). We shall come back to this computation, in Section 3, in order to establish certain properties of the r.v.'s  $Y_{x,y}^{(\nu)}$ .

## 2.4 Interpretation in terms of the local martingales $(R_t^{-2\nu}, t \geq 0)$

We assume here  $\nu > 0$ . We know that  $(R_t, t \geq 0)$  solves the S.D.E. :

$$R_t = r + B_t + \frac{2\nu + 1}{2} \int_0^t \frac{ds}{R_s} \tag{2.43}$$

where  $(B_t, t \geq 0)$  is a Brownian motion starting from 0. Let  $M_t^{(\nu)} := R_t^{-2\nu}$  and denote  $M_0^{(\nu)} = x (= r^{-2\nu})$ . Itô formula yields :

$$M_t^{(\nu)} = x - 2\nu \int_0^t (M_s^{(\nu)})^{\frac{2\nu+1}{2\nu}} dB_s. \quad (2.44)$$

We denote by  $\mathbb{Q}_x^{(\nu)}$  the law of the solution of (2.44). Thus  $(M_t^{(\nu)}, t \geq 0; \mathbb{Q}_x^{(\nu)}, x \geq 0)$  is the family of laws of a Markov process and  $(M_t^{(\nu)}, t \geq 0)$  is a positive local martingale, hence, it is a supermartingale, which converges a.s. to 0 as  $t \rightarrow \infty$ . Gathering the results of Theorem 2.1, point *ii*) of Remark 1.3 and Proposition 1.4, we have obtained :

**Theorem 2.4**

*i)* For every  $\nu > 0$  and  $x > 0$ , there exists a probability measure  $\gamma_{M^{(\nu,x)}}$  on  $[0, x] \times [0, \infty[$  such that :

$$\frac{1}{x} \mathbb{Q}_x^{(\nu)}((K - M_t^{(\nu)})^+) = \gamma_{M^{(\nu,x)}}([0, K] \times [0, t]) \quad (K \leq x, t \geq 0) \quad (2.45)$$

*ii)* There exists a family of positive r.v.'s  $(\bar{Y}_{x,y}^{(\nu)}, y < x)$  such that :

$$\mathbb{Q}_x^{(\nu)}(M_t \leq y) = P(\bar{Y}_{x,y}^{(\nu)} \leq t) \quad (y < x, t \geq 0) \quad (2.46)$$

**Remark 2.5**

*i)* Since :

$$\mathbb{Q}_x^{(\nu)}(M_t \leq y) = P_{x^{-\frac{1}{2\nu}}}^{(\nu)}\left(\frac{1}{R_t^{2\nu}} \leq y\right) = P_{x^{-\frac{1}{2\nu}}}^{(\nu)}(R_t \geq y^{-\frac{1}{2\nu}}) \quad (2.47)$$

we have :

$$\bar{Y}_{x,y}^{(\nu)} \stackrel{(\text{law})}{=} Y_{x^{-\frac{1}{2\nu}}, y^{-\frac{1}{2\nu}}}^{(\nu)} \quad (x > y) \quad (2.48)$$

In Section 3, we shall study in details the laws of the r.v.'s  $(Y_{x,y}^{(\nu)}, y > x)$  which, thanks to (2.48), allows to obtain easily the corresponding properties of the r.v.'s  $(\bar{Y}_{x,y}^{(\nu)}, y < x)$ .

*ii)* When  $\nu \in ]-1, 0[$ ,  $(R_t^{-2\nu}, t \geq 0)$  is no longer a local martingale, but it is a submartingale (see [D.M, RVY]) such that :

$$R_t^{-2\nu} = N_t^{(\nu)} + L_t^{(\nu)} \quad (2.49)$$

where  $(N_t^{(\nu)}, t \geq 0)$  is a martingale and  $(L_t^{(\nu)}, t \geq 0)$  an increasing process such that  $dL_t^{(\nu)} = 1_{(R_t=0)} dL_t^{(\nu)}$ .

### 3 Some properties of the r.v.'s $(Y_{x,y}^{(\nu)}, y > x)$ ( $\nu \geq -\frac{1}{2}$ )

We recall that these r.v.'s  $Y_{x,y}^{(\nu)}$  (or, more exactly, their laws) are defined (see Theorem 2.1) via :

$$P_x^{(\nu)}(R_t \geq y) = P(Y_{x,y}^{(\nu)} \leq t) \quad (3.1)$$

**Theorem 3.1** Here are a few properties of the laws of  $Y_{x,y}^{(\nu)}$  ( $\nu \geq -\frac{1}{2}, y > x$ )

i) Laplace transform of  $Y_{x,y}^{(\nu)}$

$$\begin{aligned} \bullet \quad E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \quad (x > 0, \lambda \geq 0) \\ &= (y\sqrt{2\lambda}) \left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda}) K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned} \quad (3.2)$$

For  $x = 0$ , we get :

$$\bullet \quad E(e^{-\lambda Y_{0,y}^{(\nu)}}) = \frac{1}{2^\nu \Gamma(\nu+1)} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \quad (3.3)$$

ii) Scaling property

• The law of  $\frac{Y_{x,y}^{(\nu)}}{xy}$  depends only on the ratio  $\frac{y}{x}$  ( $> 1$ ). In particular :

$$\bullet \quad Y_{x,y}^{(\nu)} = x^2 Y_{1, \frac{y}{x}} \quad (y > x) \quad (3.4)$$

and

$$\bullet \quad E(e^{-\frac{\lambda Y_{x,y}^{(\nu)}}{xy}}) = \sqrt{2\lambda} b^{\nu+\frac{1}{2}} I_\nu \left( \sqrt{\frac{2\lambda}{b}} \right) K_{\nu+1}(\sqrt{2\lambda b}) \quad (3.5)$$

where  $b = \frac{y}{x} > 1$ .

iii) Further results about the laws of  $Y_{x,y}^{(\nu)}$

$$\bullet \quad Y_{0,y}^{(\nu)} \stackrel{(\text{law})}{=} \frac{y^2}{2\gamma_{\nu+1}} \stackrel{(\text{law})}{=} G_y^{(\nu+1)} \quad (3.6)$$

where  $G_y^{(\nu+1)}$  denotes the r.v.  $G_y$  under  $P_0^{(\nu+1)}$ .

The following relations hold :

$$\bullet \quad P(Y_{x,y}^{(\nu)} \leq t) = P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{y} p^{(\nu+1)}(t, z, y) dz \quad (3.7)$$

- $\frac{\partial}{\partial x} P(Y_{x,y}^{(\nu)} \leq t) = \frac{x}{y} p^{(\nu+1)}(t, x, y) = \frac{x}{\nu+1} f_{G_{x,y}^{(\nu+1)}}(t)$  (3.8)

where  $f_{G_{x,y}^{(\nu+1)}}$  denotes the density of the r.v.  $G_y$  under  $P_x^{(\nu+1)}$  ( $x < y$ )

- The density  $f_{Y_{x,y}^{(\nu)}}$  of  $Y_{x,y}^{(\nu)}$  is given by :

$$f_{Y_{x,y}^{(\nu)}}(t) = \frac{1}{2t} \left[ y p^{(\nu)}(t, x, y) - \frac{x^2}{y} p^{(\nu+1)}(t, x, y) \right] \quad (3.9)$$

- $E(Y_{x,y}^{(\nu)}) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)} \quad (\nu > 0)$  (3.10)

iv) An equation satisfied by  $Y_{x,y}^{(\nu)}$

$$Y_{x,z}^{(\nu)} \stackrel{(\text{law})}{=} T_{x,y}^{(\nu)} + Y_{y,z}^{(\nu)} \quad (x < y < z) \quad (3.11)$$

In particular :

$$Y_{0,z}^{(\nu)} \stackrel{(\text{law})}{=} T_{0,y}^{(\nu)} + Y_{y,z}^{(\nu)} \quad (0 < y < z) \quad (3.12)$$

The r.v.'s which occur on the RHS of (3.11) and (3.12) are independent and  $T_{x,y}^{(\nu)}$  is the first hitting time of level  $y$  by the process  $(R_t, t \geq 0)$  starting from  $x$ .

- v)  $T_{0,x}^{(\nu)} + Y_{x,y}^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+1)} \quad (0 < x < y)$  (3.13)

The r.v.'s which occur on the LHS of (3.13) are independent (see (2.5), (2.6) and (2.7) for the definition of  $G_y^{(\nu+1)}$ ).

### Proof of Theorem 3.1

i) We compute the Laplace transform of  $Y_{x,y}^{(\nu)}$

From (3.1), we deduce :

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P_x^{(\nu)}(R_t \geq y) dt &= \int_0^\infty dt \int_y^\infty e^{-\lambda t} p^{(\nu)}(t, x, z) dz \\ &= \int_y^\infty dz \int_0^\infty e^{-\lambda t} p^{(\nu)}(t, x, z) dt = \int_y^\infty u_\lambda^{(\nu)}(x, z) dz \quad (\text{see (2.2.d)}) \\ &= E \left( \int_0^\infty e^{-\lambda t} 1_{(t \geq Y_{x,y}^{(\nu)})} dt \right) \\ &= \frac{1}{\lambda} E(e^{-\lambda Y_{x,y}^{(\nu)}}) \end{aligned}$$

Hence :

$$\begin{aligned} E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= \lambda \int_y^\infty u_\lambda^{(\nu)}(x, z) dz \\ &= \frac{2\lambda}{x^\nu} I_\nu(x\sqrt{2\lambda})(2\lambda)^{-\frac{\nu}{2}-1} \int_{y\sqrt{2\lambda}}^\infty K_\nu(h) h^{\nu+1} dh \quad (\text{from (2.11)}) \end{aligned}$$

Now, since :

$$-z^{\nu+1}K_\nu(z) = \frac{\partial}{\partial z}(z^{\nu+1}K_{\nu+1}(z)) \quad ([\text{Leb}], \text{ p. 110})$$

we obtain :

$$\begin{aligned} E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= (2\lambda)^{-\frac{1}{2}} \frac{I_\nu(x\sqrt{2\lambda})}{x^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \\ &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned}$$

Formula (3.3) may be obtained by letting  $x$  tend to 0 in (3.2) and (3.10) follows from (3.2) by differentiation.

ii) Proof of the scaling property

• It is an immediate consequence of (3.2). The fact that the law of  $\frac{Y_{x,y}^{(\nu)}}{xy}$  depends only on the ratio  $\left(\frac{y}{x}\right)$  ( $\frac{y}{x} > 1, x > 0$ ) may also be obtained with the help of (2.38). Indeed, we deduce from (2.36) and (2.3) that :

$$P\left(\frac{Y_{x,y}^{(\nu)}}{xy} \leq t\right) = P\left(\frac{xy}{Y_{x,y}^{(\nu)}} \geq \frac{1}{t}\right) = \int_0^{txy} \frac{du}{2u} \left[ y p^{(\nu)}(u, x, y) - \frac{x^2}{y} p^{(\nu+1)}(u, x, y) \right]$$

or, equivalently :

$$P\left(\frac{xy}{Y_{x,y}^{(\nu)}} \geq t\right) = \int_0^{\frac{xy}{t}} \frac{du}{2u} \left[ y p^{(\nu)}(u, x, y) - \frac{x^2}{y} p^{(\nu+1)}(u, x, y) \right]$$

Given (2.3), this formula yields the density  $f_{\frac{xy}{Y_{x,y}^{(\nu)}}}$  of the r.v.  $\frac{xy}{Y_{x,y}^{(\nu)}} :$

$$f_{\frac{xy}{Y_{x,y}^{(\nu)}}}(u) = \frac{1}{2} \left(\frac{y}{x}\right)^{\nu+1} e^{-a(x,y)u} \left( I_\nu(u) - \frac{x}{y} I_{\nu+1}(u) \right)$$

where  $a(x, y) := \frac{x^2 + y^2}{2xy} = \frac{1}{2} \left( \frac{x}{y} + \frac{y}{x} \right)$ .

Thus, the law of  $\frac{xy}{Y_{x,y}^{(\nu)}}$ , hence that of  $\frac{Y_{x,y}^{(\nu)}}{xy}$  only depends on the ratio  $\left(\frac{y}{x}\right)$ .

• We may also prove this scaling property as a direct consequence of the scaling property of the Bessel process. Indeed, we have :

$$\begin{aligned}
P(Y_{x,y}^{(\nu)} \leq txy) &= P_x^{(\nu)}(R_{txy} \geq y) \\
&= P_x^{(\nu)}\left(\frac{1}{\sqrt{xy}} R_{txy} \geq \frac{y}{\sqrt{xy}}\right) \\
&= P_{\frac{x}{\sqrt{xy}}}^{(\nu)}\left(R_t \geq \sqrt{\frac{y}{x}}\right) \quad (\text{by scaling of the Bessel process}) \\
&= P_{\sqrt{\frac{x}{y}}}^{(\nu)}\left(R_t \geq \sqrt{\frac{y}{x}}\right).
\end{aligned}$$

*iii) Proof of (3.6)*

This formula (3.6) has been obtained during the proof of Theorem 2.1 (point 2.3.2, in the case  $x = 0$ ). We note that one can deduce (3.3) from (3.6). Indeed, from (3.6), since

$P(Y_{0,y}^{(\nu)} \leq t) = P\left(\frac{y^2}{2\gamma_{\nu+1}} \leq t\right)$ , one has :

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} P(Y_{0,y}^{(\nu)} \leq t) dt &= \int_0^\infty e^{-\lambda t} P\left(\frac{y^2}{2\gamma_{\nu+1}} \leq t\right) dt, \quad \text{i.e. :} \\
E(e^{-\lambda Y_{0,y}^{(\nu)}}) &= E\left(e^{-\frac{\lambda}{2} \frac{y^2}{\gamma_{\nu+1}}}\right) \\
&= \frac{1}{\Gamma(\nu+1)} \int_0^\infty z^\nu \exp\left\{-\frac{1}{2}\left(2z + \frac{\lambda y^2}{z}\right)\right\} dz \\
&= \frac{2^{\frac{1-\nu}{2}}}{\Gamma(\nu+1)} \lambda^{\frac{\nu+1}{2}} y^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \quad ([\text{Leb}], \text{p.119}) \\
&= \frac{1}{2^\nu \Gamma(\nu+1)} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda})
\end{aligned}$$

*iv) We now prove (3.8) and (3.7)*

Since, from the Hirsch-Song formula (2.17), one has :

$$\frac{\partial}{\partial x} p^{(\nu)}(t, x, y) = -\frac{\partial}{\partial y} \left(\frac{x}{y} p^{(\nu+1)}(t, x, y)\right)$$

then :

$$\begin{aligned}
\frac{\partial}{\partial x} P(Y_{x,y}^{(\nu)} \leq t) &= \frac{\partial}{\partial x} P_x^{(\nu)}(R_t \geq y) \\
&= \int_y^\infty \frac{\partial}{\partial x} p^{(\nu)}(t, x, z) dz \\
&= -\int_y^\infty \frac{\partial}{\partial z} \left(\frac{x}{z} p^{(\nu+1)}(t, x, z)\right) dz \\
&= \frac{x}{y} p^{(\nu+1)}(t, x, y)
\end{aligned}$$

and the relation :

$$\frac{x}{y} p^{(\nu+1)}(t, x, y) = \frac{x}{\nu+1} f_{G_{x,y}^{(\nu+1)}}(t)$$

follows from (2.6). Formula (3.7) is obtained by integration of (3.8) (with respect to  $x$ ) :

$$\begin{aligned} P(Y_{x,y}^{(\nu)} \leq t) &= P(Y_{0,y}^{(\nu)} \leq t) + \int_0^x \frac{\partial}{\partial z} P(Y_{z,y}^{(\nu)} \leq t) dz \\ &= P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{y} p^{(\nu+1)}(t, z, y) dz \end{aligned}$$

from (3.6) and (3.8). We note that (3.7) may also be written, from (3.8) :

$$P(Y_{x,y}^{(\nu)} \leq t) = P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{\nu+1} f_{G_{x,y}^{(\nu+1)}}(t) dz \quad (3.14)$$

We also remark that the computation of  $E(Y_{x,y}^{(\nu)})$ , given by (3.10) when  $\nu > 0$ , may be obtained from (3.14). Indeed we deduce from (3.14) that :

$$P(Y_{x,y}^{(\nu)} \geq t) = P\left(\gamma_{\nu+1} \leq \frac{y^2}{2t}\right) - \int_0^x \frac{z}{\nu+1} f_{G_{z,y}^{(\nu+1)}}(t) dz$$

Thus, integrating this relation in  $t$  from 0 to  $\infty$ , for  $\nu > 0$ , we obtain :

$$\begin{aligned} P(Y_{x,y}^{(\nu)} \geq t) &= \int_0^\infty dt \int_0^{\frac{y^2}{2t}} \frac{1}{\Gamma(\nu+1)} e^{-z} z^\nu dz - \frac{x^2}{2(\nu+1)} \\ &= \frac{1}{\Gamma(\nu+1)} \int_0^\infty e^{-z} z^\nu \frac{y^2}{2z} dz - \frac{x^2}{2(\nu+1)} = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)} \end{aligned}$$

*v)* Formula (3.9) is an immediate consequence of (3.13).

*vi)* We now prove that  $Y_{x,y}^{(\nu)}$  satisfies equation (3.11)

Indeed, this follows from a simple application of the Markov property. Let  $x < y < z$ . Since the process  $(R_t, t \geq 0)$  starting from  $x$  needs to pass through  $y$  to reach  $z$ , we obtain :

$$\begin{aligned} P_x^{(\nu)}(R_t \geq z) &= P_x^{(\nu)}(T_y < \infty ; R_t \geq z) \\ &= P_x^{(\nu)}(1_{T_y \leq t} \widehat{P}_y^{(\nu)}(\widehat{R}_{t-T_y} \geq z)) \end{aligned}$$

where in the expression  $\widehat{P}_y^{(\nu)}(\widehat{R}_{t-T_y} \geq z)$  the term  $T_y$  is frozen. Hence, conditioning with respect to  $T_y = u$ , we obtain :

$$\begin{aligned} P_x^{(\nu)}(R_t \geq z) &= \int_0^t P_x^{(\nu)}(T_y \in du) P_y^{(\nu)}(R_{t-u} \geq z) du, \quad \text{i.e. :} \\ P(Y_{x,z}^{(\nu)} \leq t) &= \int_0^t P_x^{(\nu)}(T_y \in du) P(Y_{y,z}^{(\nu)} \leq t-u) du \end{aligned}$$

hence :

$$Y_{x,z}^{(\nu)} \stackrel{(\text{law})}{=} T_{x,y}^{(\nu)} + Y_{y,z}^{(\nu)}$$

It is also possible to obtain (3.11) by using the Laplace transforms of  $Y_{x,y}^{(\nu)}$  and  $T_{x,y}^{(\nu)}$ . Indeed, from (3.2) :

$$E(e^{-\lambda Y_{x,z}^{(\nu)}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (z\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(z\sqrt{2\lambda})$$

whereas, from (3.2) and (2.9) :

$$\begin{aligned} E(e^{-\lambda(T_{x,y}^{(\nu)} + Y_{y,z}^{(\nu)})}) &= \left(\frac{y}{x}\right)^\nu \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \cdot \frac{I_\nu(y\sqrt{2\lambda})}{(y\sqrt{2\lambda})^\nu} (z\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(z\sqrt{2\lambda}) \\ &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (z\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(z\sqrt{2\lambda}) = E(e^{-\lambda Y_{x,z}^{(\nu)}}) \end{aligned}$$

*vii)* The proof of (3.13) hinges on the same arguments as previously and, from (2.7), on :

$$E(e^{-\lambda G_y^{(\nu+1)}}) = \frac{1}{2^\nu \Gamma(\nu+1)} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda})$$

Indeed, from (2.6), :

$$\begin{aligned} E(e^{-\lambda G_y^{(\nu+1)}}) &= E(e^{-\lambda \frac{y^2}{2\gamma_{\nu+1}}}) \\ &= \frac{1}{\Gamma(\nu+1)} \int_0^\infty e^{-\frac{\lambda y^2}{2z}} e^{-z} z^\nu dz = \frac{1}{2^\nu \Gamma(\nu+1)} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned}$$

(see [Leb], p. 119, (5.10.25)).

This ends the proof of Theorem 3.1.

### Remark 3.2

*i)* We note that the r.v.'s  $(Y_{x,y}^{(\nu)}, y > x)$  are not the only ones which satisfy equation (3.11). Indeed, let, for  $x < y$  :

$$A_{x,y}^{(\nu),-} := \int_0^\infty 1_{(R_x^{(\nu)}(s) \leq y)} ds \left( = \int_0^{G_y} 1_{(R_x^{(\nu)}(s) \leq y)} ds \right)$$

where here  $(R_x^{(\nu)}(s), s \geq 0)$  denotes the Bessel process with index  $\nu$  starting from  $x$ . Then, an application of the Markov property yields, for  $x < y < z$  :

$$A_{x,z}^{(\nu),-} \stackrel{(\text{law})}{=} T_{x,y}^{(\nu)} + A_{y,z}^{(\nu),-} \tag{3.15}$$

We note that, although the r.v.'s  $(A_{x,y}^{(\nu),-}, y > x)$  satisfy the same equation as the r.v.'s  $(Y_{x,y}^{(\nu)}, y > x)$ , they do not have the same law. Indeed, for  $x = 0$ , from Ciesielski-Taylor ([CT], 1962)

$$A_{0,y}^{(\nu),-} \stackrel{(\text{law})}{=} T_{0,y}^{(\nu-1)} \quad (\nu > 0) \quad (3.16)$$

whereas  $Y_{0,y}^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+1)}$  (from (3.6)) and the laws of  $T_{0,y}^{(\nu-1)}$  and  $G_y^{(\nu+1)}$  differ ; indeed  $E(e^{-\lambda T_{0,y}^{(\nu-1)}}) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \frac{(y\sqrt{2\lambda})^{\nu-1}}{I_{\nu-1}(y\sqrt{2\lambda})}$ , from (2.10), whereas, from (3.3) and (3.6),

$$E(e^{-\lambda Y_{0,y}^{(\nu)}}) = E_0^{(\nu+1)}(e^{-\lambda G_y}) = \frac{1}{2^\nu \Gamma(\nu+1)} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda})$$

ii) For  $\nu > 0$ , the equality :

$$E(Y_{x,y}^{(\nu)}) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)}$$

may be obtained in a different manner than the previously developed ones, by making use this time of the r.v.'s  $A_{x,y}^{(\nu),-}$ . Indeed, for  $x < y$  :

$$\begin{aligned} E(Y_{x,y}^{(\nu)}) &= \int_0^\infty P(Y_{x,y}^{(\nu)} \geq t) dt = E_x^{(\nu)} \left( \int_0^\infty 1_{(R_t \leq y)} dt \right) \\ &= E(A_{x,y}^{(\nu),-}) \end{aligned} \quad (3.17)$$

Thus, we need to compute  $E(A_{x,y}^{(\nu),-})$ . Itô's formula, for  $\nu > 0$ , implies, from (2.43) :

$$(R_t \wedge y)^2 = x^2 + 2 \int_0^t R_s 1_{(R_s < y)} \left( dB_s + \frac{2\nu+1}{2R_s} ds \right) - y L_t^y + \int_0^t 1_{(R_s < y)} ds \quad (3.18)$$

where  $(L_t^y, t \geq 0)$  denotes the local time of  $(R_t, t \geq 0)$  at level  $y$ . Hence, taking expectation in (3.18) and letting  $t$  tend to  $\infty$ , we obtain :

$$y^2 = x^2 + (2\nu+2)E(A_{x,y}^{(\nu),-}) - y E_x^{(\nu)}(L_\infty^y) \quad (3.19)$$

However :

$$\begin{aligned} E_x^{(\nu)}(L_\infty^y) &= E_y^{(\nu)}(L_\infty^y) = \int_0^\infty p^{(\nu)}(t, y, y) dt \\ &= \int_0^\infty \frac{y}{t} \exp\left(-\frac{y^2}{t}\right) I_\nu\left(\frac{y^2}{t}\right) dt \quad (\text{from (2.3)}) \\ &= y \int_0^\infty I_\nu(z) e^{-z} \frac{dz}{z} \quad \left(\text{after the change of variable } \frac{y^2}{t} = z\right) \\ &= \frac{y}{\nu} \end{aligned} \quad (3.20)$$

from the Lipschitz-Hankel formula (see (2.40). Finally, from (3.20), (3.19) and (3.17) :

$$E(Y_{x,y}^\nu) = E(A_{x,y}^{(\nu,-)}) = \frac{1}{2\nu+2} \left( y^2 \left( 1 + \frac{1}{\nu} \right) - x^2 \right) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)}$$

iii) It may be of interest to express the law of the r.v.  $Y_{x,y}^{(\nu)}$  in terms of the only process  $(R_t, t \geq 0; P_x^{(\nu)})$ . Here is our result :

**Proposition 3.3** *Let  $\nu > 0$  and  $h$  a generic positive Borel function. Then :*

$$E(h(Y_{x,y}^{(\nu)})) = \frac{y^2}{2\nu} E_x^{(\nu)} \left[ \frac{h(G_y)}{G_y} \left( 1 - \frac{x}{y} \cdot \exp \left( - \left( \nu + \frac{1}{2} \right) \int_0^{G_y} \frac{ds}{R_s^2} \right) \right) \right] \quad (3.21)$$

$$= \frac{y(y-x)}{2\nu} E_x^{(\nu)} \left( \frac{h(G_y)}{G_y} \right) + \frac{xy}{2\nu} E_x^{(\nu)} \left( \frac{h(G_y)}{G_y} \left( 1 - e^{-\left(\nu+\frac{1}{2}\right)\int_0^{G_y} \frac{ds}{R_s^2}} \right) \right) \quad (3.22)$$

### Proof of Proposition 3.3

The density of the r.v.  $Y_{x,y}^{(\nu)}$ ,  $f_{Y_{x,y}^{(\nu)}}$ , equals, from (2.36) :

$$\begin{aligned} f_{Y_{x,y}^{(\nu)}}(t) &= \frac{p^{(\nu)}(t, x, y)}{2t} \left( y - \frac{I_{\nu+1}}{I_\nu} \left( \frac{xy}{t} \right) \right) \\ &= \frac{y p^{(\nu)}(t, x, y)}{2t} \left( 1 - \frac{x}{y} E_x^{(\nu)} \left( \exp \left( - \left( \nu + \frac{1}{2} \right) \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) \right) \quad (3.23) \\ &\quad (\text{from (2.15)}) \end{aligned}$$

On the other hand, from (2.6), the density  $f_{G_{x,y}^{(\nu)}}$  of  $G_y$  under  $P_x^{(\nu)}$  equals :

$$f_{G_{x,y}^{(\nu)}}(t) = \frac{\nu p^{(\nu)}(t, x, y)}{y} \quad (3.24)$$

Thus, since for every positive and predictable process  $H$ , one has (see [FPY]) :

$$E_x^{(\nu)}(H_{G_y}) = \int_0^\infty P_x^{(\nu)}(G_y \in dt) E_x^{(\nu)}(H_t | R_t = y) \quad (3.25)$$

we derive from (3.22) that :

$$\begin{aligned} E[h(Y_{x,y}^{(\nu)})] &= \int_0^\infty \frac{h(t)}{2t} p^{(\nu)}(t, x, y) y \left( 1 - \frac{x}{y} E_x^{(\nu)} \left( \exp \left( - \left( \nu + \frac{1}{2} \right) \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) \right) \\ &= \frac{y^2}{2\nu} \int_0^\infty \frac{h(t)}{t} P_x^{(\nu)}(G_y \in dt) \left( 1 - \frac{x}{y} E_x^{(\nu)} \left( \exp \left( - \left( \nu + \frac{1}{2} \right) \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t = y \right) \right) \\ &= \frac{y^2}{2\nu} E_x^{(\nu)} \left( \frac{h(G_y)}{G_y} \left( 1 - \frac{x}{y} \left( \exp - \left( \nu + \frac{1}{2} \right) \int_0^{G_y} \frac{ds}{R_s^2} \right) \right) \right) \end{aligned}$$

from (3.24). This is formula (3.21), which, after some mild rearrangement, yields formula (3.22).

We note that both terms in (3.22) :

$$\frac{y(y-x)}{2\nu} E_x^{(\nu)} \left( \frac{h(G_y)}{G_y} \right) \quad \text{and} \quad \frac{xy}{2\nu} E_x^{(\nu)} \left[ \frac{h(G_y)}{G_y} \left( 1 - e^{-(\nu+\frac{1}{2}) \int_0^{G_y} \frac{ds}{R_s^2}} \right) \right]$$

are positive measures (viewed via the integration of  $h$ ). The first one has total mass :

$$\begin{aligned} \frac{y(y-x)}{2\nu} E_x^{(\nu)} \left( \frac{1}{G_y} \right) &= \frac{y(y-x)}{2\nu} \int_0^\infty \frac{1}{ty} \nu p^{(\nu)}(t, x, y) dt \quad (\text{from (3.23)}) \\ &= \frac{y(y-x)}{2\nu} \frac{\nu}{y} \frac{2y}{y^2 - x^2} = \frac{y}{y+x} \quad (\text{from (2.42)}) \end{aligned}$$

whereas the second has total mass  $\frac{x}{y+x}$ . In particular, for  $x = 0$ , we recover :

$$E[h(Y_{0,y}^{(\nu)})] = \frac{y^2}{2\nu} E_0^{(\nu)} \left( \frac{h(G_y)}{G_y} \right)$$

and the second term in (3.22) vanishes.

iv) The r.v.'s  $\bar{Y}_{x,y}^{(-1)}$  ( $x > y$ )

In the same way that point *i*) of Theorem 2.1 allows to define the r.v.'s  $\left( Y_{x,y}^{(\nu)}, y > x; \nu \geq -\frac{1}{2} \right)$ , point *iii*) of this Theorem allows to define the positive r.v.'s  $(\bar{Y}_{x,y}^{(-1)}, x > y)$  characterised by :

$$Q_x^{(-1)}(M_t \leq y) = P(\bar{Y}_{x,y}^{(-1)} \leq t)$$

where  $(M_t, t \geq 0)$  under  $Q_x^{(-1)}$  is a Bessel square process with index  $-1$  started from  $x$ . With arguments similar to those used to prove Theorem 3.1, we obtain :

$$E(e^{-\lambda \bar{Y}_{x,y}^{(-1)}}) = K_1(\sqrt{2\lambda x}) \left[ \sqrt{2\lambda x} + \frac{y}{\sqrt{x}} \sqrt{2\lambda} I_2(\sqrt{2\lambda y}) \right] \quad (3.26)$$

v) Use of additivity property of squares of Bessel processes

We have shown, in Theorem 2.1, that for  $\nu \geq -\frac{1}{2}$  and for  $y > 1$  :  $t \rightarrow P_1^{(\nu)}(R_t \geq y)$  is an increasing function of  $t$ . It is the distribution function of the r.v.  $Y_{1,y}^{(\nu)}$  (the general case, with  $x < y$ , may be deduced from this one, by scaling). On the other hand, it is well known (see, e.g., [RY], Chap. XI) that  $Q_x^{(d)} * Q_{x'}^{(d')} = Q_{x+x'}^{(d+d')}$ , where  $Q_x^{(d)}$  (resp.  $Q_{x'}^{(d')}$ ) denotes the law of a squared Bessel process, with dimension  $d = 2\nu + 2$ , starting from  $x$  (resp. with dimension  $d' = 2\nu' + 2$  starting from  $x'$ ). Hence

$$P_1^{(\nu)}(R_t \geq \sqrt{y}) = P_1^{(\nu)}(R_t^2 \geq y) = P(R_0^2(t) + X(t) \geq y)$$

where  $(R_0^2(t), t \geq 0)$  is a squared Bessel process with dimension  $d = 2\nu + 2$  starting from 0 and  $(X_t, t \geq 0)$  is a squared Bessel process with dimension 0 started from 1. Hence :

$$P_1^{(\nu)}(R_t \geq \sqrt{y}) = P\left(\gamma_{\frac{d}{2}} \geq \frac{1}{2t}(y - X_t)_+\right) \quad (3.27)$$

since  $R_0^2(t) \stackrel{(\text{law})}{=} t R_0^2(1)$  by scaling and  $R_0^2(1) \stackrel{(\text{law})}{=} 2\gamma_{\frac{d}{2}}$ . Consequently :

$$P_1^{(\nu)}(R_t \geq \sqrt{y}) = \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty P\left(\frac{1}{2t}(y - X_t)_+ \leq z\right) z^{\frac{d}{2}-1} e^{-z} dz \quad (3.28)$$

Consequently, from (3.27), Theorem 2.1 hinges only upon the properties of the process  $(X_t, t \geq 0)$ , which is a squared Bessel process started from 1.

For  $d = 2$  (i.e.  $\nu = 0$ ), (3.28) becomes :

$$P_1^{(0)}(R_t \geq \sqrt{y}) = E\left(\exp -\frac{1}{2t}(y - X_t)_+\right) \quad (3.29)$$

since  $\gamma_{\frac{d}{2}} = \gamma_1$  is then a standard exponential variable. We note that, Theorem 2.1, applied for  $\nu = 0$  ( $> -\frac{1}{2}$ !) implies that the RHS of (3.29) is a distribution function (i.e. : that of the r.v.  $Y_{1,y}^{(0)}$ ).

For  $d = 1$  (i.e.  $\nu = -\frac{1}{2}$ ), (3.28) becomes :

$$P_1^{(-\frac{1}{2})}(R_t \geq y) = \sqrt{\frac{2}{\pi}} E\left(\int_{\sqrt{\frac{(y-X_t)_+}{t}}}^\infty e^{-\frac{u^2}{2}} du\right) \quad (3.30)$$

which is also a distribution function with respect to  $t$ , that of the r.v.  $Y_{1,y}^{(-\frac{1}{2})}$ . This remark invites to ask the following questions :

- Which are the probabilities  $\pi$  on  $\mathbb{R}_+$  such that :  $E\left(\bar{\pi}\left(\frac{1}{2t}(y - X_t)_+\right)\right)$  is a distribution function in  $t$ , with  $\bar{\pi}(x) = \pi(]x, \infty[)$  ? Theorem 2.1 implies that it is the case when  $\pi$  is a mixture of gamma laws, with parameter  $\frac{d}{2} \geq \frac{1}{2}$  but that is not true if  $\frac{d}{2} < \frac{1}{2}$ .
- More generally, which are the properties of the process  $\left(Z_t = \frac{1}{2t}(y - X_t)_+, t \geq 0\right)$ , with  $y > 1$  which may explain the above increase property ?

vi) Around infinite divisibility properties for  $Y_{x,y}^{(\nu)}$

### Proposition 3.4

i) Let  $x > 0$  and  $V_x$  a positive,  $\frac{1}{2}$  stable r.v. such that  $E(e^{-\lambda V_x}) = \exp(-x\sqrt{2\lambda})$  ( $\lambda \geq 0$ )

If  $\nu > \frac{1}{2}$ , then  $V_x + Y_{x,y}^{(\nu)}$  (with  $V_x$  and  $Y_{x,y}^{(\nu)}$  independent) is infinitely divisible.

ii) If  $\nu > \frac{1}{2}$  and  $\frac{y}{x}$  is large enough, then  $Y_{x,y}^{(\nu)}$  is infinitely divisible.

**Proof of Proposition 3.4**

We prove i) : By scaling we may suppose  $x = 1$  and  $y > 1$ . From (3.2) we have :

$$\begin{aligned} E(e^{-\lambda Y_{1,y}^{(\nu)}}) &= y^{\nu+1} \sqrt{2\lambda} I_\nu(\sqrt{2\lambda}) K_{\nu+1}(y\sqrt{2\lambda}) \\ &:= \exp(-h(\lambda)) \end{aligned} \quad (3.31)$$

Hence :

$$\begin{aligned} h'(\lambda) &= -\frac{1}{2\lambda} - \frac{1}{\sqrt{2\lambda}} \frac{I'_\nu(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} - \frac{y}{\sqrt{2\lambda}} \frac{K'_{\nu+1}(y\sqrt{2\lambda})}{K_{\nu+1}(y\sqrt{2\lambda})} \\ &= \frac{y}{\sqrt{2\lambda}} \frac{K_\nu}{K_{\nu+1}}(y\sqrt{2\lambda}) - \frac{1}{\sqrt{2\lambda}} \frac{I_{\nu+1}}{I_\nu}(\sqrt{2\lambda}) \end{aligned} \quad (3.32)$$

(since  $\frac{\partial}{\partial x}(x^\nu K_\nu(x)) = -x^\nu K_{\nu-1}(x)$ ,  $\frac{\partial}{\partial x}(x^{-\nu} I_\nu(x)) = x^{-\nu} I_{\nu+1}(x)$  ; see [Leb], p. 110). On the other hand (see Ismail [I]) :

$$\frac{1}{\sqrt{2\lambda}} \frac{K_\nu(\sqrt{2\lambda})}{K_{\nu+1}(\sqrt{2\lambda})} = \frac{4}{\pi^2} \int_0^\infty \frac{1}{2\lambda + t^2} \frac{dt}{t(J_{\nu+1}^2(t) + Y_{\nu+1}^2(t))} \quad (3.33)$$

$$\text{and : } \frac{I_{\nu+1}(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} = \sqrt{2\lambda} \sum_{n=1}^\infty \frac{2}{2\lambda + j_{\nu,n}^2} \quad (3.34)$$

where  $J_{\nu+1}, Y_{\nu+1}$  are the Bessel functions with index  $\nu$  (see [Leb], p. 98) and  $(j_{\nu,n}, n \geq 1)$  the increasing sequence of the positive zeroes of  $J_\nu$ . Hence :

$$\frac{1}{2} h'(\lambda) = \frac{1}{\pi} \int_0^\infty \frac{y^2}{2\lambda y^2 + t^2} \frac{2}{\pi t} \frac{dt}{J_{\nu+1}^2(t) + Y_{\nu+1}^2(t)} - \sum_{n \geq 1} \frac{1}{2\lambda + j_{\nu,n}^2} \quad (3.35)$$

Let now  $Z_y := V_1 + Y_{1,y}^{(\nu)}$  ( $y > 1$ ). We have, from (3.35) :

$$E(e^{-\lambda Z_y}) = \exp(-h(\lambda) - \sqrt{2\lambda}) := \exp(-g(\lambda))$$

$$\text{with : } \frac{1}{2} g'(\lambda) = \frac{1}{2} h'(\lambda) + \frac{1}{2\sqrt{2\lambda}}$$

$$\begin{aligned} &= \left[ \frac{1}{\pi} \int_0^\infty \frac{y^2}{2\lambda y^2 + t^2} \frac{2}{\pi t} \frac{dt}{J_{\nu+1}^2(t) + Y_{\nu+1}^2(t)} \right] + \left[ \frac{1}{\pi} \int_0^\infty \frac{dt}{2\lambda + t^2} - \sum_{n \geq 1} \frac{1}{2\lambda + j_{\nu,n}^2} \right] \\ &:= (1) + (2) \end{aligned} \quad (3.36)$$

(we used  $\frac{1}{\sqrt{2\lambda}} = \frac{2}{\pi} \int_0^\infty \frac{dt}{2\lambda + t^2}$ ).

One needs, to prove *i*), to show that  $g'$  is completely monotone. For (1), it is clear since it is the Stieltjes transform of a positive measure. For (2), one has :

$$\begin{aligned} & \frac{1}{\pi} \int_0^\infty \frac{dt}{2\lambda + t^2} - \sum_{n \geq 1} \frac{1}{2\lambda + j_{\nu,n}^2} = \frac{1}{\pi} \left( \sum_{n=1}^\infty \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{dt}{2\lambda + t^2} - \pi \sum_{n=1}^\infty \frac{1}{2\lambda + j_{\nu,n}^2} \right) \\ &= \frac{1}{\pi} \sum_{n=1}^\infty \left\{ \int_{j_{\nu,n-1}}^{j_{\nu,n}} \left( \frac{1}{2\lambda + t^2} - \frac{1}{2\lambda + j_{\nu,n}^2} \right) dt + (j_{\nu,n} - j_{\nu,n-1} - \pi) \frac{1}{2\lambda + j_{\nu,n}^2} \right\} \\ &= \frac{1}{\pi} \sum_{n=1}^\infty \left\{ \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{j_{\nu,n}^2 - t^2}{(2\lambda + t^2)(2\lambda + j_{\nu,n}^2)} dt + (j_{\nu,n} - j_{\nu,n-1} - \pi) \frac{1}{2\lambda + j_{\nu,n}^2} \right\} \end{aligned}$$

which is clearly completely monotone since (see [I], p. 357),  $j_{\nu,n} - j_{\nu,n-1} > \pi$  as  $\nu > \frac{1}{2}$ .

*ii) We now prove ii)*

One needs to show that  $h'(\lambda)$ , as given by (3.24) is the Laplace transform, for  $y$  large enough, of a positive measure. Now, one has ;

$$\frac{1}{2} h'(\lambda) = \frac{1}{\pi} \int_0^\infty \frac{1}{2\lambda + v^2} \frac{2}{\pi v} \frac{dv}{J_{\nu+1}^2(yv) + Y_{\nu+1}^2(yv)} - \sum_{n \geq 1} \frac{1}{2\lambda + j_{\nu,n}^2} \quad (3.37)$$

(after the change of variable  $t = yv$  in (3.35))

$$\begin{aligned} &= \frac{1}{\pi} \left( \sum_{n=1}^\infty \int_{j_{\nu,n-1}}^{j_{\nu,n}} \left( \frac{1}{2\lambda + v^2} - \frac{1}{2\lambda + j_{\nu,n}^2} \right) \frac{2}{\pi v} \frac{dv}{J_{\nu+1}^2(yv) + Y_{\nu+1}^2(yv)} \right. \\ & \quad \left. + \pi \sum_{n=1}^\infty \int_{j_{\nu,n-1}}^{j_{\nu,n}} \left( \frac{2}{\pi v} \frac{1}{J_{\nu+1}^2(yv) + Y_{\nu+1}^2(yv)} - 1 \right) dv \cdot \frac{1}{2\lambda + j_{\nu,n}^2} \right) \quad (3.38) \end{aligned}$$

But, from Watson ([Wat], p. 449), one has :

$$J_{\nu+1}^2(z) + Y_{\nu+1}^2(z) \underset{y \rightarrow \infty}{\sim} \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{2}{\pi v} \frac{\pi v y}{2} dv$$

Hence :

$$\begin{aligned} & \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{2}{\pi v} \frac{dv}{J_{\nu+1}^2(yv) + Y_{\nu+1}^2(yv)} \underset{y \rightarrow \infty}{\sim} \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{2}{\pi v} \frac{\pi v y}{2} dv \\ & \underset{z \rightarrow \infty}{\sim} y(j_{\nu,n} - j_{\nu,n-1}) \quad \text{uniformly in } n \end{aligned}$$

Thus, for  $y$  large enough, and since  $\nu > \frac{1}{2}$  implies  $j_{\nu,n} - j_{\nu,n-1} > \pi$ , one has :

$$\begin{aligned} \frac{1}{2} h'(\lambda) &= \frac{1}{\pi} \sum_{n=1}^\infty \int_{j_{\nu,n-1}}^{j_{\nu,n}} \frac{j_{\nu,n}^2 - v^2}{(2\lambda + v^2)(2\lambda + j_{\nu,n}^2)} \frac{2}{\pi v} \frac{dv}{J_{\nu+1}^2(yv) + Y_{\nu+1}^2(yv)} \\ & \quad + \sum_{n=1}^\infty \alpha_n(y) \frac{1}{2\lambda + j_{\nu,n}^2} \end{aligned}$$

where the  $\alpha_n(y)$  are all positive. It is then clear that  $h'(\lambda)$  is completely monotone.

*vii) On the negative moments of  $Y_{x,y}^{(\nu)}$  ( $x < y$ )*

For every  $m > 0$ , we have :

$$E \left( \frac{1}{(Y_{x,y}^{(\nu)})^m} \right) < \infty \quad (3.39)$$

Indeed, we have, for  $m > 0$  :

$$E \left( \frac{1}{(Y_{x,y}^{(\nu)})^m} \right) = \frac{1}{\Gamma(m)} \int_0^\infty E(e^{-\lambda Y_{x,y}^{(\nu)}}) \lambda^{m-1} d\lambda$$

and, from (3.2),  $E(e^{-\lambda Y_{x,y}^{(\nu)}}) \underset{\lambda \rightarrow \infty}{\sim} \frac{1}{2} \left( \frac{y}{x} \right)^\nu \frac{1}{\sqrt{xy}} e^{-(y-x)\sqrt{2\lambda}}$  (see [Leb], p. 123).

## 4 Two extensions of Bessel processes with increasing pseudo-inverses

### 4.1 Bessel processes with index $\nu \geq -\frac{1}{2}$ and drift $a > 0$

S. Watanabe (see, in particular [Wata], p. 117 and 118) introduced the Bessel processes  $((R_t, t \geq 0), P_x^{(\nu,a)}, x \geq 0)$  with index  $\nu$  and drift  $a$ , with extended infinitesimal generator :

$$\frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{2\nu + 1}{2x} + a \frac{I_{\nu+1}(ax)}{I_\nu(ax)} \right) \frac{d}{dx} \quad (4.1)$$

(This is the expression (2.1) in [Wata], where we replaced  $\alpha$  by  $(2\nu + 1)$ , and  $\sqrt{2c}$  by  $a$ ). We recall that, for integer dimensions  $d = 2(\nu + 1)$ , these processes may be obtained by taking  $|\vec{B}_t + \vec{m} \cdot t|$ , where  $(\vec{B}_t)$  is a  $d$ -dimensional Brownian motion, starting from 0, and  $a = |\vec{m}|$ , for some  $\vec{m} \in \mathbb{R}^d$ .

Here is a first generalization of our Theorem 2.1 (recovered by letting  $a \rightarrow 0$ ).

**Theorem 4.1 :** *For  $\nu \geq -\frac{1}{2}$ , and  $a > 0$ , the process  $((R_t, t \geq 0), (P_x^{(\nu,a)}, x \geq 0))$  admits an increasing pseudo-inverse.*

**Sketch of the proof of Theorem 4.1** (It is very similar to that of Theorem 2.1).

We need to show that, for  $y > x$ ,  $\frac{\partial}{\partial t} P_x^{(\nu,a)}(R_t \geq y) \geq 0$ . From Fokker-Planck formula, we get :

$$\frac{\partial}{\partial t} p(t, x, y) = -\frac{1}{2} \frac{\partial}{\partial y} p(t, x, y) + \left( \frac{2\nu + 1}{2y} + a \frac{I_{\nu+1}(ay)}{I_\nu(ay)} \right) p(t, x, y) \quad (4.2)$$

where  $p(t, x, y)$ , the density with respect to Lebesgue measure of  $R_t$  under  $P_x^{(\nu, a)}$ , equals, from ([Wata], p. 117-118) :

$$p(t, x, y) = y \frac{I_\nu(ay)}{I_\nu(ax)} \frac{e^{-\frac{a^2 t}{2}}}{t} \exp - \left( \frac{x^2 + y^2}{2t} \right) I_\nu \left( \frac{xy}{t} \right) \quad (4.3)$$

Hence :

$$\frac{\partial}{\partial y} p(t, x, y) = p(t, x, y) \left[ \frac{1}{y} + a \frac{I'_\nu(ay)}{I_\nu(ay)} - \frac{y}{t} + \frac{x}{t} \frac{I'_\nu\left(\frac{xy}{t}\right)}{I_\nu\left(\frac{xy}{t}\right)} \right]$$

Hence, we deduce from :

$$\frac{I'_\nu(z)}{I_\nu(z)} = \frac{I_{\nu+1}(z)}{I_\nu(z)} + \frac{\nu}{z}, \quad \text{and from (4.2),}$$

that :

$$\begin{aligned} \frac{\frac{\partial}{\partial t} P_x^{(\nu, a)}(R_t \geq y)}{p(t, x, y)} &= -\frac{1}{2y} - \frac{a}{2} \left( \frac{I_{\nu+1}(ay)}{I_\nu(ay)} + \frac{\nu}{ay} \right) + \frac{y}{2t} - \frac{x}{2t} \left( \frac{I_{\nu+1}\left(\frac{xy}{t}\right)}{I_\nu\left(\frac{xy}{t}\right)} + \frac{\nu t}{xy} \right) \\ &\quad + \frac{2\nu + 1}{2y} + a \frac{I_{\nu+1}(ay)}{I_\nu(ay)} \\ &= \frac{a}{2} \frac{I_{\nu+1}(ay)}{I_\nu(ay)} + \frac{1}{2t} \left[ y - x \frac{I_{\nu+1}\left(\frac{xy}{t}\right)}{I_\nu\left(\frac{xy}{t}\right)} \right] \geq 0 \end{aligned} \quad (4.4)$$

since  $y \geq x$  and  $I_{\nu+1}(z) \leq I_\nu(z)$ , from Lemma 2.3.

## 4.2 Squares of generalized Ornstein-Uhlenbeck processes, also called CIR processes in mathematical finance

Let  $((R_t, t \geq 0), (Q_x^{(\nu, \beta)}, x \geq 0))$  denote the square of a generalized Ornstein-Uhlenbeck process, with infinitesimal generator :

$$2x \frac{d^2}{dx^2} + (2\beta x + 2(\nu + 1)) \frac{d}{dx} \quad (4.5)$$

For  $d = 2(\nu + 1)$  an integer, this process may be constructed as the square of the Euclidian norm of the  $d$ -dimensional Ornstein-Uhlenbeck, with parameter  $\beta > 0$ , that is the solution of :

$$X_t = x_0 + B_t + \beta \int_0^t X_s ds$$

where  $(B_t)$  denotes a  $d$ -dimensional Brownian motion. See, e.g., Pitman-Yor [PY2] for results about this family of diffusions.

**Theorem 4.2** For  $\nu \geq -\frac{1}{2}$ , and  $\beta \geq 0$ , the process  $((R_t, t \geq 0); (Q_x^{(\nu, \beta)}, x \geq 0))$  admits an increasing pseudo-inverse.

**Sketch of the proof of Theorem 4.2 :**

We need to show that  $\frac{\partial}{\partial t} Q_x^{(\nu, \beta)}(R_t \geq y) \geq 0$  for every  $y \geq x$ . From the Fokker-Planck formula, we get :

$$\frac{\partial}{\partial t} Q_x^{(\nu, \beta)}(R_t \geq y) = -2y \frac{\partial}{\partial y} p(t, x, y) + (2\beta y + 2\nu)p(t, x, y) \quad (4.6)$$

where  $p(t, x, y)$ , the density with respect to the Lebesgue measure of  $R_t$  under  $Q_x^{(\nu, \beta)}$ , is given by (see [PY 2]) :

$$p(t, x, y) = \frac{\beta}{2 \sinh \beta t} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \left\{ \exp -\beta \left[ (1 - \nu)t + \frac{xe^{\beta t} + ye^{-\beta t}}{2 \sinh \beta t} \right] \right\} \cdot I_\nu \left( \frac{\beta \sqrt{xy}}{\sinh \beta t} \right) \quad (4.7)$$

Hence :

$$\frac{\frac{\partial}{\partial y} p(t, x, y)}{p(t, x, y)} = \frac{\nu}{2y} - \beta \frac{e^{-\beta t}}{2 \sinh \beta t} + \frac{1}{2} \sqrt{\frac{x}{y}} \frac{\beta}{\sinh \beta t} \frac{I'_\nu}{I_\nu} \left( \frac{\beta \sqrt{xy}}{\sinh \beta t} \right) \quad (4.8)$$

Thus, from (4.8) and (4.6) :

$$\frac{\frac{\partial}{\partial t} Q_x^{(\nu, \beta)}(R_t \geq y)}{p(t, x, y)} = 2\beta y + \nu + \beta y \frac{e^{-\beta t}}{\sinh \beta t} - \frac{\beta}{\sinh \beta t} \sqrt{xy} \frac{I'_\nu}{I_\nu} \left( \frac{\beta \sqrt{xy}}{\sinh \beta t} \right)$$

and, using again the relation :

$$\frac{I'_\nu}{I_\nu}(z) = \frac{I_{\nu+1}}{I_\nu}(z) + \frac{\nu}{z},$$

we have :

$$\begin{aligned} \Delta_t &:= \frac{\frac{\partial}{\partial t} Q_x^{(\nu, \beta)}(R_t \geq y)}{p(t, x, y)} \\ &= 2\beta y + \nu + \beta y \frac{e^{-\beta t}}{\sinh \beta t} - \frac{\beta}{\sinh \beta t} \sqrt{xy} \left( \frac{I_{\nu+1}}{I_\nu} \left( \frac{\beta \sqrt{xy}}{\sinh \beta t} \right) + \frac{\nu \sinh \beta t}{\beta \sqrt{xy}} \right) \\ &= 2\beta y + \frac{\beta y e^{-\beta t}}{\sinh \beta t} - \frac{\beta}{\sinh \beta t} \sqrt{xy} \frac{I_{\nu+1}}{I_\nu} \left( \frac{\beta \sqrt{xy}}{\sinh \beta t} \right) \end{aligned}$$

Denoting

$$z := \frac{\beta \sqrt{xy}}{\sinh \beta t}, \text{ and using } y > x \implies z \leq \frac{\beta y}{\sinh \beta t}, \text{ we have :}$$

$$\begin{aligned} \Delta_t &\geq 2z \sinh \beta t + z \sqrt{\frac{y}{x}} e^{-\beta t} - z \frac{I_{\nu+1}}{I_\nu}(z) \quad (\text{since } y \geq x) \\ &\geq z \left( e^{\beta t} - e^{-\beta t} + e^{-\beta t} - \frac{I_{\nu+1}}{I_\nu}(z) \right) \\ &= z \left( e^{\beta t} - \frac{I_{\nu+1}}{I_\nu}(z) \right) \geq 0 \end{aligned}$$

as  $\beta \geq 0$  implies  $e^{\beta t} \geq 1 \geq \frac{I_{\nu+1}}{I_\nu}(z)$ , from Lemma 2.3

**Remarks :** 1) When  $d$  is an integer, the previous computation may be obtained in a simpler manner, using the fact that  $R_t$  is the square of the norm of a  $d$ -dimensional Gaussian variable with mean  $x_0 \exp(\beta t)$ , and (common) variance  $\frac{e^{2\beta t} - 1}{2\beta}$ .

2) We also deduce, from the comparison theorem for SDE's, that, for  $\nu' \geq \nu$  and  $\beta' \geq \beta$  :

$$Q_x^{(\nu', \beta')}(R_t \geq y) \geq Q_x^{(\nu, \beta)}(R_t \geq y) \quad (y \geq x)$$

hence, with obvious notation, for  $x \leq y$ , and  $t \geq 0$  :

$$P(Y_{x,y}^{(\nu', \beta')} \leq t) \geq P(Y_{x,y}^{(\nu, \beta)} \leq t)$$

which states that the r.v.'s.  $Y_{x,y}^{(\nu, \beta)}$  are stochastically decreasing in the parameters  $\nu$  and  $\beta$ . These variables  $Y_{x,y}^{(\nu, \beta)}$  are different from those discussed in Section 5.

### 4.3 A third example : the process :

$$X_t^{(\nu)} = \int_0^t ds \exp 2[(B_t - B_s) + \nu(t - s)], \quad t \geq 0$$

Here,  $(B_t, t \geq 0)$  denotes a 1-dimensional Brownian motion, starting from 0. The process  $(X_t^{(\nu)}, t \geq 0)$  is easily shown to be Markov, since, thanks to Itô's formula, we obtain :

$$dX_t^{(\nu)} = (2(\nu + 1)X_t^{(\nu)} + 1)dt + 2X_t^{(\nu)}dB_t.$$

**Theorem 4.3** For  $\nu + 1 \geq 0$ , the submartingale  $(X_t^{(\nu)}, t \geq 0)$  admits an increasing pseudo-inverse.

**Proof of Theorem 4.3 :** Using time reversal (from  $t$ ), we get : for any fixed  $t \geq 0$ ,

$$X_t^{(\nu)} \stackrel{(\text{law})}{=} A_t^{(\nu)} := \int_0^t du \exp (2(B_u + \nu u)).$$

Now, the process  $(A_t^{(\nu)}, t \geq 0)$  is increasing, and, if we denote by  $(\tau_y^{(\nu)}, y \geq 0)$  its inverse, we obtain :

$$P(X_t^{(\nu)} \geq y) = P(A_t^{(\nu)} \geq y) = P(\tau_y^{(\nu)} \leq t) \quad (y \geq 0)$$

We note that, here, we may define the pseudo-inverse process  $(Y_y^{(\nu)}, y > 0)$  of the process  $(X_t^{(\nu)}, t \geq 0)$  as a "time" process, precisely the process  $(\tau_y^{(\nu)}, y \geq 0)$ .

## 5 The more general family $(Y_{x,y}^{(\nu,\alpha)} ; x < y, \nu \geq 0, 0 \leq \alpha \leq 1)$

In Section 2, we introduced the r.v.'s  $(Y_{x,y}^{(\nu)}, x < y ; \nu \geq -\frac{1}{2})$  and studied their properties in Section 3. We shall now introduce further positive r.v.'s  $(Y_{x,y}^{(\nu,\alpha)} ; x < y, \nu \geq 0, 0 \leq \alpha \leq 1)$  which extends the family  $(Y_{x,y}^{(\nu)})$ , corresponding to  $\alpha = 1$ , and we shall describe some of their properties. Let us insist again that variables have nothing to do with those introduced in Remark 2 following Theorem 4.2.

**5.1** For ease of the reader, we recall some notation and formulae which we have already used and which shall be useful to us below.

$T_{x,y}^{(\nu)}$  : a r.v. whose law, under  $P_x^{(\nu)}$ , is that of  $\inf\{t \geq 0 ; R_t = y\}$

$T_y^{(\nu)} = T_{0,y}^{(\nu)}$

$G_{x,y}^{(\nu)}$  : a r.v. whose law, under  $P_x^{(\nu)}$ , is that of  $\sup\{t \geq 0 ; R_t = y\}$

$G_y^{(\nu)} = G_{0,y}^{(\nu)}$

Then :

$$E(e^{-\lambda T_{x,y}^{(\nu)}}) = \left(\frac{y}{x}\right)^\nu \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \quad (x < y) \quad (5.1)$$

$$E(e^{-\lambda T_y^{(\nu)}}) = \frac{1}{2^\nu \Gamma(\nu + 1)} \frac{(y\sqrt{2\lambda})^\nu}{I_\nu(y\sqrt{2\lambda})} \quad (y > 0) \quad (5.2)$$

$$E(e^{-\lambda G_{x,y}^{(\nu)}}) = 2\nu \left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda}) K_\nu(y\sqrt{2\lambda}) \quad (x < y) \quad (5.3)$$

$$E(e^{-\lambda G_y^{(\nu)}}) = \frac{1}{2^{\nu-1} \Gamma(\nu)} (y\sqrt{2\lambda})^\nu K_\nu(y\sqrt{2\lambda}) \quad (0 < y, \nu > 0) \quad (5.4)$$

$$E(e^{-\lambda Y_{x,y}^{(\nu)}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \quad (5.5)$$

From these formulae, we deduced, in Theorem 3.1 :

$$T_x^{(\nu)} + Y_{x,y}^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+1)} \quad (5.6)$$

$$Y_{x,z}^{(\nu)} \stackrel{(\text{law})}{=} T_{x,y}^{(\nu)} + Y_{x,z}^{(\nu)} \quad (x < y < z) \quad (5.7)$$

On the other hand, it is obvious that :

$$T_{x,y}^{(\nu)} + G_{y,z}^{(\nu)} \stackrel{(\text{law})}{=} G_{x,z}^{(\nu)} \quad (x < y < z) \quad (5.8)$$

**5.2 Definition of the r.v.'s** ( $G_y^{(\nu+\theta,\nu)}, y > 0, \nu, \theta \geq 0$ ) **and** ( $T_y^{(\nu+\theta,\nu)}, y > 0, \nu, \theta \geq 0$ )

From Ismail-Kelker [IK], or Pitman-Yor ([PY1], p. 336, formula (9.b.1)), there exists, for every  $y > 0, \nu, \theta \geq 0$ , a positive r.v.  $G_y^{(\nu+\theta,\nu)}$  such that :

$$E(e^{-\lambda G_y^{(\nu+\theta,\nu)}}) = \frac{\Gamma(\nu + \theta)}{\Gamma(\nu)} \frac{2^\theta}{(y\sqrt{2\lambda})^\theta} \frac{K_\nu(y\sqrt{2\lambda})}{K_{\nu+\theta}(y\sqrt{2\lambda})} \quad (5.9)$$

Likewise from [IK], or [PY1], p. 336, formula (9.a.1), there exists, for every  $y > 0, \nu, \theta \geq 0$ , a positive r.v.  $T_y^{(\nu+\theta,\nu)}$  such that :

$$E(e^{-\lambda T_y^{(\nu+\theta,\nu)}}) = \frac{\Gamma(\nu + \theta + 1)}{\Gamma(\nu + 1)} \frac{2^\theta}{(y\sqrt{2\lambda})^\theta} \frac{I_{\nu+\theta}(y\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \quad (5.10)$$

From the relations (5.2) and (5.10) on one hand, and relations (5.4) and (5.9) on the other hand, we immediately deduce the following Proposition which completes the results of Pitman-Yor [PY1] :

**Proposition 5.1**

i) For every  $y > 0, \nu \geq -\frac{1}{2}$  and  $\theta \geq 0$  :

$$T_y^{(\nu+\theta)} + T_y^{(\nu+\theta,\nu)} \stackrel{(\text{law})}{=} T_y^{(\nu)} \quad (5.11)$$

ii) For every  $y > 0, \nu > 0$  and  $\theta \geq 0$  :

$$G_y^{(\nu+\theta)} + G_y^{(\nu+\theta,\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu)} \quad (5.12)$$

Of course, it is desirable to give a "more probabilistic" proof of (5.11) and (5.12). Here is such a proof for relation (5.11).

**Another proof of (5.11)**

To simplify, we take  $y = 1$ . Let  $(R_t^{(\nu)}, t \geq 0)$  and  $(R_t^{(\theta-1)}, t \geq 0)$  two Bessel processes starting from 0, independent, with respective indices  $\nu$  and  $\theta - 1$  (i.e. with dimension resp.  $2\nu + 2$  and  $2\theta$ ). Let  $R_t^{(\nu+\theta)} := (\sqrt{(R_t^{(\nu)})^2 + (R_t^{(\theta-1)})^2}, t \geq 0) \cdot (R_t^{(\nu+\theta)}, t \geq 0)$  is a Bessel process with index  $\nu + \theta$ , i.e. with dimension  $2\nu + 2\theta + 2$ , started at 0. Let  $T_1^{(\nu+\theta)} := \inf\{t \geq 0; R_t^{(\nu+\theta)} = 1\}$ . It is clear that  $T_1^{(\nu+\theta)} \leq T_1^{(\nu)}$  (with  $T_1^{(\nu)} := \inf\{t \geq 0; R_t^{(\nu)} = 1\}$ ) and that  $R_{T_1^{(\nu+\theta)}}^{(\nu)} \leq 1$ . Thus :

$$T_1^{(\nu)} = T_1^{(\nu+\theta)} + \inf\{u \geq 0; R_{T_1^{(\nu+\theta)}+u}^{(\nu)} = 1\} \quad (5.13)$$

On the other hand, it follows from the intertwining properties of the Bessel semi-groups (see [CPY] or [DRVY]) that :

$$E_0^{(\nu+\theta)} [f(R_t^{(\nu)}) | \mathcal{R}_t^{(\nu+\alpha)}] = E[f(r\sqrt{\beta_{\nu+1,\theta}})] \quad (5.14)$$

with, on the right-hand side of (5.14)  $r = R_t^{(\nu+\alpha)}$  and  $\beta_{\nu+1,\theta}$  a beta variable with parameter  $\nu + 1$  and  $\theta$ . We then deduce from (5.14) (which is valid for the  $\mathcal{R}_t^{(\nu+\theta)}$  stopping time  $T = T_1^{(\nu+\alpha)}$ ) that :

$$R_{T_1^{(\nu+\theta)}}^{(\nu)} \stackrel{(\text{law})}{=} \sqrt{\beta_{\nu+1,\theta}} \quad (5.15)$$

and that  $T_1^{(\nu+\theta)}$  and  $(R_{T_1^{(\nu+\theta)}+u}^{(\nu)}, u \geq 0)$  are independent. It then follows from (5.13) that :

$$E[e^{-\lambda T_1^{(\nu)}}] = E(e^{-\lambda T_1^{(\nu+\theta)}}) \cdot \int_0^1 \frac{x^\nu(1-x)^{\theta-1}}{B(\nu+1,\theta)} E(e^{-\lambda T_{\sqrt{x},1}^{(\nu)}}) dx \quad (5.16)$$

Taking (5.2) into account, we shall have proven (5.11) once we establish :

$$\int_0^1 \frac{x^\nu(1-x)^{\theta-1}}{B(\nu+1,\theta)} \left(\frac{1}{\sqrt{x}}\right)^\nu \frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda})} dx = \frac{2^\theta \Gamma(\nu+\theta+1)}{\Gamma(\nu+1)} \frac{1}{(\sqrt{2\lambda})^\theta} \frac{I_{\nu+\theta}(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} \quad (5.17)$$

Now (5.17) is easily established, by using the series expansion :

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \quad (\text{see [Leb], p. 108})$$

and then integrating term by term on the left hand side of (5.17) and using the formula :

$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0).$$

Concerning (5.12) a proof close to the preceding one seems harder since  $G_1^{(\nu+\theta)}$  is not a stopping time. Nonetheless, it may be possible to use the enlarged filtration  $(\mathcal{G}_t^{(\nu+\theta)}, t \geq 0)$ , i.e. the smallest filtration which contains the natural filtration  $(\mathcal{R}_t^{(\nu+\theta)}, t \geq 0)$  of  $(R_t^{(\nu+\theta)}, t \geq 0)$  and which makes  $G_1^{(\nu+\theta)}$  a  $(\mathcal{G}_t^{(\nu+\theta)}, t \geq 0)$  stopping time. Explicit computations in this new filtration are made possible from the knowledge of the Azéma supermartingale  $(P(G_1^{(\nu+\theta)} \geq t | \mathcal{R}_t^{(\nu+\theta)}), t \geq 0) = ((R_t^{(\nu+\theta)})^{-2(\nu+\theta)} \wedge 1, t \geq 0)$ . The enlargement formulae allowing to express the process  $(R_t^{(\nu+\theta)}, t \geq 0)$  in this new filtration lead to complicated formulae which do not seem to provide us with the desired explicit result.

**5.3** Here is now the main result of this Section 5, i.e. the existence and the properties of the r.v.'s  $(Y_{x,y}^{(\nu,\alpha)}; x < y; \nu \geq 0, \alpha \in [0,1])$ .

**Theorem 5.2**

*i) For every  $\nu \geq 0, x < y$  and  $\alpha \in [0,1]$ , there exists a positive r.v.  $Y_{x,y}^{(\nu,\alpha)}$  such that :*

$$E(e^{-\lambda Y_{x,y}^{(\nu,\alpha)}}) = \frac{2^{1-\alpha} \Gamma(\nu+1)}{\Gamma(\nu+\alpha)} \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda}) \quad (5.18)$$

Moreover :

ii) The r.v.'s  $Y_{x,y}^{(\nu,\alpha)}$  interpolate between  $Y_{x,y}^{(\nu)}$  and  $G_{x,y}^{(\nu)}$ , i.e. :

$$Y_{x,y}^{(\nu,1)} \stackrel{\text{(law)}}{=} Y_{x,y}^{(\nu)}, \quad Y_{x,y}^{(\nu,0)} \stackrel{\text{(law)}}{=} G_{x,y}^{(\nu)} \quad (5.19)$$

$$\text{iii)} \quad Y_{x,y}^{(\nu,\alpha)} \stackrel{\text{(law)}}{=} Y_{x,y}^{(\nu)} + G_y^{(\nu+1,1-\alpha)} \quad (5.20)$$

and, for  $(1-\nu)^+ \leq \alpha \leq 1$

$$Y_{x,y}^{(\nu,\alpha)} \stackrel{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu,1-\alpha)} \quad (5.21)$$

$$\text{iv)} \quad T_x^{(\nu)} + Y_{x,y}^{(\nu,\alpha)} \stackrel{\text{(law)}}{=} G_y^{(\nu+\alpha)} \stackrel{\text{(law)}}{=} T_x^{(\nu+\alpha-1)} + Y_{x,y}^{(\nu+\alpha-1)} \quad (5.22)$$

$$\text{v)} \quad G_{x,y}^{(\nu)} = G_y^{(\nu+\alpha,\alpha)} + Y_{x,y}^{(\nu,\alpha)} \quad (5.23)$$

In particular, for  $x = 0$  we recover (5.12) :

$$G_y^{(\nu)} \stackrel{\text{(law)}}{=} G_y^{(\nu+\alpha,\alpha)} + G_y^{(\nu+\alpha)} \quad (5.24)$$

$$\text{vi)} \quad Y_{x,y}^{(\nu)} + G_y^{(\nu+1,1-\alpha)} \stackrel{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu,1-\alpha)} \stackrel{\text{(law)}}{=} Y_{x,y}^{(\nu,\alpha)} \quad (5.25)$$

Observe that, from (5.20) and (5.24), the r.v.'s  $Y_{x,y}^{(\nu,\alpha)}$  are stochastically decreasing in  $\alpha$ .

### Proof of Theorem 5.2

i) We show (5.18). We have :

$$\begin{aligned} & \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)} \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda}) \\ &= \left\{ \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \right\} \\ & \quad \cdot \left\{ \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)} (y\sqrt{2\lambda})^{\alpha-1} \frac{K_{\nu+\alpha}(y\sqrt{2\lambda})}{K_{\nu+1}(y\sqrt{2\lambda})} \right\} \\ &= E(e^{-\lambda Y_{x,y}^{(\nu)}}) \cdot E(e^{-\lambda G_y^{(\nu+1,1-\alpha)}}) \quad (\text{from (5.5) and (5.9)}) \end{aligned}$$

This proves (5.18) and (5.20).

ii) We now show (5.21). We have :

$$\begin{aligned} E(e^{-\lambda Y_{x,y}^{(\nu,\alpha)}}) &= \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)} \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu} (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+1}(y\sqrt{2\lambda}) \\ &= \left\{ \frac{I_{\nu+\alpha-1}(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^{\nu+\alpha-1}} (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda}) \right\} \\ & \quad \cdot \left\{ \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)} \frac{1}{(x\sqrt{2\lambda})^{1-\alpha}} \frac{I_\nu(x\sqrt{2\lambda})}{I_{\nu+\alpha-1}(x\sqrt{2\lambda})} \right\} \\ &= E(e^{-\lambda Y_{x,y}^{(\nu+\alpha-1)}}) \cdot E(e^{-\lambda T_x^{(\nu,1-\alpha)}}) \quad (\text{from (5.5) and (5.10)}) \end{aligned}$$

This shows (5.21).

*iii)* The relation  $T_x^{(\nu)} + Y_{x,y}^{(\nu,\alpha)} \stackrel{(\text{law})}{=} G_y^{(\nu+\alpha)}$  follows immediately from (5.2), (5.18) and (5.4). The relation  $G_y^{(\nu+\alpha)} \stackrel{(\text{law})}{=} T_x^{(\nu+\alpha-1)} + Y_{x,y}^{(\nu+\alpha-1)}$  follows from (5.6) (or from the preceding relation where we replace  $\nu$  by  $\nu+\alpha-1$  and we observe, as is obvious, that  $Y_{x,y}^{(\nu+\alpha-1)} \stackrel{(\text{law})}{=} Y_{x,y}^{(\nu+\alpha-1, 1)}$ ).

*iv)* We now show (5.23). We deduce, from (5.8), (5.24) and (5.22) :

$$\begin{aligned} T_x^{(\nu)} + G_{x,y}^{(\nu)} &\stackrel{(\text{law})}{=} G_y^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+\alpha, \alpha)} + G_y^{(\nu+\alpha)} \\ &\stackrel{(\text{law})}{=} G_y^{(\nu+\alpha, \alpha)} + T_x^{(\nu)} + Y_{x,y}^{(\nu+\alpha)} \end{aligned}$$

Hence :

$$G_{x,y}^{(\nu)} \stackrel{(\text{law})}{=} G_y^{(\nu+\alpha, \alpha)} + Y_{x,y}^{(\nu,\alpha)}$$

We might also have proven this last relation by using (5.3), (5.9) and (5.18).

*v)* The relation (5.19) obviously holds.

*vi)* Finally, relation (5.25) follows from (5.21) and (5.20). Indeed, from (5.21) and (5.20) :

$$\begin{aligned} Y_{x,y}^{(\nu,\alpha)} &\stackrel{(\text{law})}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu, 1-\alpha)} \\ &\stackrel{(\text{law})}{=} T_{x,y}^{(\nu)} + G_y^{(\nu+1, 1-\alpha)} \quad (x < y, (1-\nu)^+ \leq \alpha \leq 1) \end{aligned}$$

## 6 Concluding remarks : our tools to show existence of pseudo-inverses

**6.1** At the beginning of this paper (see point *iv*) of Remark 1.3) we showed that the geometric Brownian motion  $(\exp(B_t + \nu t), t \geq 0)$ , with  $\nu > 0$ , admits an increasing pseudo-inverse. On the other hand, Lamperti's representation of geometric Brownian motion is (see [RY], Exerc. 1.28, Chap. XI, p. 452) :

$$\begin{aligned} \exp(B_t + \nu t) &= R^{(\nu)} \left( \int_0^t (\exp 2(B_s + \nu s)) ds \right) \\ &= R^{(\nu)}(C^{(\nu)}(t)) \end{aligned} \tag{6.1}$$

with  $C^{(\nu)}(t) := \int_0^t (\exp 2(B_s + \nu s)) ds$  and  $(R^{(\nu)}(u), u \geq 0)$  a Bessel process with index  $\nu$ . This formula (6.1) may induce the reader in thinking that Theorem 2.1 – the existence of an increasing pseudo-inverse for  $(R^{(\nu)}(u), u \geq 0)$  – may be deduced from (6.1) and point *iv*) of Remark 1.3. This is not at all clear since the clock  $(C^{(\nu)}(t), t \geq 0)$  is random.

**6.2** For the proof of Theorem 2.1, we have used, essentially, among the properties of Bessel processes :

- the Fokker-Planck formula, which allowed us to compute  $\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y)$
- the Pitman-Yor formula (2.15), which allowed to show that  $\frac{I_{\nu+1}}{I_\nu}(z) \leq 1$ , a key point to prove that  $P_x^{(\nu)}(R_t \geq y)$  is increasing in  $t$
- the formula due to Hirsch-Song, which allowed to give several expressions of the law of the r.v.  $Y_{x,y}^{(\nu)}$  (for  $\nu \geq -\frac{1}{2}$ )
- the formula for the resolvent kernel, thanks to which we were able to compute explicitly the Laplace transform of the r.v.'s  $Y_{x,y}^{(\nu)}$ .

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