Chebyshev Knots

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Abstract

A Chebyshev knot is a knot which admits a parametrization of the form $x(t) = T_a(t);\ y(t) = T_b(t);\ z(t) = T_c(t+\varphi)$, where a, b, c are pairwise coprime, $T_n(t)$ is the Chebyshev polynomial of degree n, and $\varphi \in \mathbf{R}$. Chebyshev knots are non compact analogues of the classical Lissajous knots. We show that there are infinitely many Chebyshev knots with $\varphi = 0$. We also show that every knot is a Chebyshev knot.

keywords: Polynomial curves, Chebyshev polynomials, Chebyshev curves, Lissajous knots, long knots, braids
Mathematics Subject Classification 2000: 14H50, 57M25, 14P99

1 Introduction

A Lissajous knot is a knot which admits a one to one parametrization of the form

$$x = \cos(at);\ y = \cos(bt + \varphi);\ z = \cos(ct + \psi)$$

where $0 \leq t \leq 2\pi$ and where a, b, c are pairwise coprime integers. These knots, first defined in [BHJS], have been studied by many authors: V. F. R. Jones, J. Przytycki, C. Lamm, J. Hoste and L. Zirbel. Most known properties of Lissajous knots are deduced from their symmetries, which are easy to see (see [JP, La, HZ]).

On the other hand Vassiliev considered polynomial knots, i.e. non singular polynomial embeddings $\mathbf{R} \to \mathbf{R}^3$ (see [Va, Sh, RS, KP1, KP2]).

In this paper we study a polynomial analogue of Lissajous knots. The classical Chebyshev polynomials $T_n(t)$ are defined by the trigonometric identity $\cos(n\,\theta) = T_n(\cos\,\theta)$. It is natural to use these polynomials instead of cosine functions to define our Chebyshev knots.

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Definition 1 *A knot in* $\mathbf{R}^3 \subset \mathbf{S}^3$ *is a Chebyshev knot if it admits a parametrization of the form*

$$x = T_a(t);\ y = T_b(t);\ z = T_c(t + \varphi)$$

where $t \in \mathbf{R}$, a, b, c *are pairwise coprime integers, and* φ *is a real constant.*

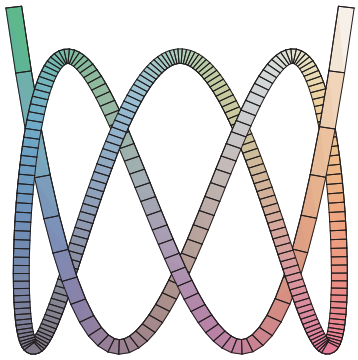


Figure 1: The mirror image of the 7_7 knot is Chebyshev.

We begin with the study of plane Chebyshev curves which are projections of Chebyshev knots on the (x, y)-plane. We conclude this paragraph with a theorem of Hoste and Zirbel [HZ] describing these curves in terms of particular braid projections.

Then, we study some families of Chebyshev knots with $\varphi = 0$, called harmonic knots. We prove that for a, b coprime integers, $c = ab - a - b$, the harmonic knot $\mathrm{H}(a, b, c)$ has an alternating projection on the (x, y)-plane. We deduce that there are infinitely many harmonic knot types. This is similar to a theorem of C. Lamm concerning Lissajous knots (see [La]). We also prove that the torus knots $T(2, 2n + 1)$ are harmonic knots. On the other hand, we observe that the symmetries of harmonic knots are quite different from those of Lissajous knots. There are infinitely many amphicheiral harmonic knots and infinitely many strongly invertible harmonic knots. The trefoil and the figure-eight knot are harmonic knots but are not Lissajous. Some knots are both Lissajous and harmonic knots, e.g. 5_2 and 7_5.

We conclude the paper with our principal result: *every knot is a Chebyshev knot.* This is done by showing first that every knot has a plane projection which is a Chebyshev curve, the use of some classical results of braid theory and a density argument based on Kronecker's theorem.

At the end we give Chebyshev diagrams of the first 2-bridge harmonic knots.

2 Geometry of plane Chebyshev curves

Chebyshev curves were defined in [Fi] to replace the older denomination of "doubly parametrized Lissajous curves". Their double points are easier to study than those of Lissajous

curves. It will be convenient to consider also the case of implicit Chebyshev curves.

Proposition 1 *Let a, b be integers, a being odd. The affine Chebyshev curve $\mathcal{C}$ defined by*

$$\mathcal{C}: \quad T_b(x) - T_a(y) = 0$$

has $\frac{1}{2}(a-1)(b-1)$ singular points which are crossing points. These points form two rectangular grids contained in the open square $Q =]-1,1[^2$, $R = \{(x,y) \in Q,\ T_b(x) = T_a(y) = 1\}$, and $R' = \{(x,y) \in Q,\ T_b(x) = T_a(y) = -1\}$.

Proof. The singular points of $\mathcal{C}$ are obtained for $T_b'(x) = 0,\ T_a'(y) = 0,\ T_b(x) = T_a(y)$. ¿From $T_a(\cos\theta) = \cos a\theta$, we deduce that T_a has degree a and $T_a'(\cos\theta) = a\dfrac{\sin a\theta}{\sin\theta}$. T_a' has $a-1$ simple roots in $]0,1[$: $y_k = \cos\left(k\dfrac{\pi}{a}\right)$, $k = 1, \ldots, a-1$. At these points, we have $T_a(y_k) = (-1)^k$. T_b' has $b-1$ roots in $]0,1[$: $x_1, \ldots, x_{b-1}$. For each x_i there are exactly $\frac{1}{2}(a-1)$ values y_j satisfying $T_a'(y_j) = 0, T_a(y_j) = T_b(x_i)$. Hence the number of singular points is $\frac{1}{2}(a-1)(b-1)$, and they form two rectangular grids. Since the roots of $T_b'(x) = 0$ are simple, we see that these points are crossing points. □

Remark. It follows from our proof that $|R| = \frac{1}{2}\left[\frac{b-1}{2}\right](a-1)$, $|R'| = \frac{1}{2}\left[\frac{b}{2}\right](a-1)$.

Proposition 2 *Let a and b are coprime integers, a being odd. Let the Chebyshev curve $\mathcal{C}$ be defined by the equation $T_b(x) - T_a(y) = 0$. Then $\mathcal{C}$ admits the parametrization $x = T_a(t),\ y = T_b(t)$. The pairs (t,s) giving a crossing point are*

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi,\ s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,$$

where k, h are positive integers such that $\dfrac{k}{a} + \dfrac{h}{b} < 1$.

Proof. Since $T_a \circ T_b = T_b \circ T_a = T_{ab}$, the rational curve $\mathcal{C}'$ parametrized by $x = T_a(t),\ y = T_b(t)$ is contained in $\mathcal{C}$. These two curves intersect the line $\{x = x_0\}$ in one point if $|x_0| > 1$, in a points if $|x_0| < 1$ and in $\frac{1}{2}(a+1)$ points if $x_0 = \pm 1$. Consequently, they are equal.

The $\frac{1}{2}(a-1)(b-1)$ pairs

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi,\ s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,$$

give rise to double points of $\mathcal{C}' = \mathcal{C}$. Because the number of singular points of $\mathcal{C}$ is $\frac{1}{2}(a-1)(b-1)$, we see that there is no other singular point. □

Remarks. Note that for these values we get $T_b(x) = T_a(y) = (-1)^{ah+bk}$.
We observe that the crossing points are obtained for the $(a-1)(b-1)$ elements of

$$E = \{t_u = \cos\frac{u}{ab}\pi,\ 0 \le u \le ab,\ a \not| u,\ b \not| u\}. \tag{1}$$

For these values, we get $T_b(x(t_u)) = T_a(y(t_u)) = (-1)^u$. Note that t_u and $t_{u'}$ correspond to the same point when $u \equiv -u' \pmod{2b}$ and $u \equiv u' \pmod{2a}$.

The following proposition will be useful to consider Chebyshev curves as trajectories in a rectangular billiard (see [JP]).

Proposition 3 *Let $\mathcal{C}$ be the Chebyshev curve: $T_b(x) - T_a(y) = 0$. There exists an homeomorphism from the square $I = [-1, 1]^2$ to the rectangle $[0, b] \times [0, a]$, such that the image of $\mathcal{C} \bigcap I$ is the union of line segments with slopes ± 1. These segments form broken lines (the billiard trajectories) through the points with coordinates $x = b,\ y = a - 2k, 0 \leq 2k \leq a$.*

Proof. Consider the mapping $F(x, y) = (X, Y)$ with $\pi X = b(\pi - \arccos x),\ \pi Y = a(\pi - \arccos y)$. By trigonometry, it is not hard to check that F has the announced properties. □

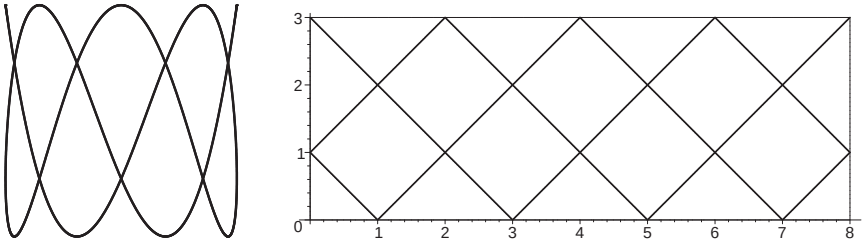


Figure 2: $T_8(x) = T_3(y)$ and its billiard picture

We shall now present a description of Chebyshev curves using braids. Let B_n be the group of braids on n strings. For practical purposes we shall draw these braids horizontally, the strings being numbered from the bottom to the top. The standard braid generators are denoted $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. The braid σ_i exchanges the strings $i + 1$ and i, the string $i + 1$ passing over the string i. In this paragraph we shall be interested in plane projections of braids, called plane braids. We shall also consider the composition of such plane braids. Let $a = n$. Let s_i denote the plane braid which is the plane projection of σ_i. This plane braid has one crossing point.

Following Hoste and Zirbel [HZ], let us define the plane braids s_{even} and s_{odd} as

$$s_{\text{even}} = s_2 s_4 \cdots s_E, \quad s_{\text{odd}} = s_1 s_3 \cdots s_O,$$

where E and O are the largest even and odd integers less than $n = a$.

Proposition 4 *Let a, b be integers, a being odd. Let $\mathcal{C}$ be the Chebyshev curve $T_b(x) - T_a(y) = 0$. Let $\varepsilon > 0$ small enough and consider the rectangle $R_\varepsilon = \{|x| < 1 - \epsilon,\ |y| \leq 1\}$. Then there is a homeomorphism between the pairs $(R_\varepsilon, \mathcal{C})$ and (R_ε, ρ) where $\rho = (s_{\text{odd}}\, s_{\text{even}})^{\frac{b-1}{2}}$ if b is odd and $\rho = (s_{\text{even}}\, s_{\text{odd}})^{\frac{b-2}{2}}\, s_{\text{even}}$ if b is even.*

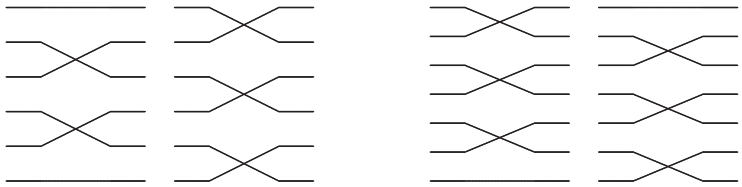


Figure 3: s_{even} and s_{odd} for $n = 6, 7$.

Proof. Following the proof of prop. (1), the $\frac{1}{2}(a-1)(b-1)$ singular points of $\mathcal{C}$ are in R_ε when ε is small enough. For each $k = 1, \ldots, b-1$, there are $\frac{1}{2}(a-1)$ singular points

$$(x_k, y_l) = \left(\cos k\frac{\pi}{b}, \cos l\frac{\pi}{a}\right), \ k + l \equiv 0 \pmod 2.$$

It means that over a neighborhood over x_k, the curve $\mathcal{C}$ is then isotopic to s_{even} if k is odd, and isotopic to s_{odd} if k is even. This proves the result. □

We can define the plat closure of a plane horizontal braid with $2m$ strings labelled $0, 1, \ldots, 2m-1$, to be the plane curve obtained by connecting the right ends 0 to 1, ..., $2m-2$ to $2m-1$, and the left ends in the same order.

Corollary 1 *Let a be an odd integer, and b an even integer. Let ρ' be the plane braid with $a+1$ strings obtained by adding a free string numbered $a+1$ over $\rho = (s_{\text{even}}\, s_{\text{odd}})^{\frac{b-2}{2}}\, s_{\text{even}}$. Then the Chebyshev curve $T_b(x) - T_a(y) = 0$ is isotopic (in $\mathbf{S}^2$) to the plat closure of the plane braid ρ'.*

Proof. Let us illustrate this by looking at the curve $T_{10}(x) - T_5(y) = 0$, which is reducible. We see on fig. (4) that it is the plat closure of $(s_{\text{even}}\ s_{\text{odd}})^4 s_{\text{even}}$. □

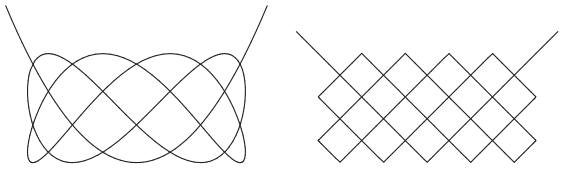


Figure 4: The Chebyshev curve $T_{10}(x) - T_5(y) = 0$ and its billiard picture

3 Harmonic knots

In this paragraph we shall study Chebyshev knots with $\varphi = 0$. Comstock (1897) found the number of crossing points of the harmonic curve parametrized by $x = T_a(t), y = T_b(t), z =$

$T_c(t)$. In particular, he proved that this curve is non singular if and only if a, b, c are pairwise coprime integers [Com]. Such curves will be named harmonic knots $\mathrm{H}(a, b, c)$.

We see that $\mathrm{H}(a, b, 1)$ is the unknot because the height function is monotonic. We can also obtain the unknot in a less trivial way.

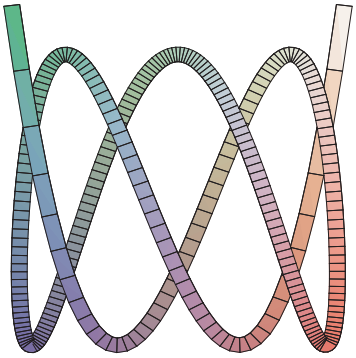


Figure 5: The knot $\mathrm{H}(3, 8, 11)$ is trivial.

Proposition 5 *Let a, b be coprime integers and $c = a + b$. The harmonic knot $\mathrm{H}(a, b, c)$ is trivial.*

Proof. Let $t \in [-1, 1],\ t = \cos\theta$. We have $x = T_a(t) = \cos a\,\theta, y = \cos b\,\theta, z = \cos(a + b)\,\theta$. By trigonometry, we see that the bounded part of our knot is on the surface

$$S = \{(x, y, z) \in \mathbf{R}^3,\ |x| \leq 1, |y| \leq 1, z = xy \pm \sqrt{(1 - x^2)(1 - y^2)}\}.$$

Since S is the union of two sheets that are homeomorphic to the square $[0, 1]^2$ glued along their boundaries, we see that it is homeomorphic to a sphere. Consequently the genus of $\mathrm{H}(a, b, c)$ is zero, hence it is the unknot. □

Note that the surface S has the symmetries of a regular tetrahedron. It is contained in the cubic surface $\{x^2 + y^2 + z^2 = 1 + 2xyz\}$ which has the same symmetries.

Let $\mathcal{C}$ be a plane projection of a parametrized knot. Consider a crossing point of $\mathcal{C}$ obtained for the parameter pair (t, s), and suppose that the tangents at this point are not parallel to the coordinates axis. It is not hard to check that the nature of this crossing point depends only of the sign of the expression $D = \Big(z(t) - z(s)\Big)x'(t)y'(t)$ (it is not the usual definition of the sign of oriented crossings, see fig. 6).

Lemma 1 *Let $\mathrm{H}(a, b, c)$ be a harmonic knot. The nature of the crossing point of parameter $t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi$, is given by*

$$\mathrm{sign}(D) = \mathrm{sign}\Big(-(-1)^{h+k}\sin(\frac{ah}{b}\pi)\ \sin(\frac{bk}{a}\pi)\sin(\frac{ch}{b}\pi)\ \sin(\frac{ck}{a}\pi)\Big).$$

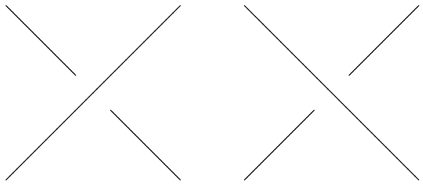


Figure 6: The two cases of crossings

Proof. The crossing points of the plane projection $\{x = T_a(t),\ y = T_b(t)\}$ are obtained for the parameters $t = \cos\tau,\ s = \cos\sigma$, where $\tau = \left(\frac{k}{a} + \frac{h}{b}\right)\pi,\ \sigma = \left(\frac{k}{a} - \frac{h}{b}\right)\pi$.

Using trigonometry and the values of t, s, we get $x'(t) = a\frac{\sin a\tau}{\sin\tau},\ y'(t) = b\frac{\sin b\tau}{\sin\tau}$, so

$$x'(t)y'(t) = (-1)^{h+k}\frac{ab}{\sin^2\tau}\ \sin(\frac{ah}{b}\pi)\ \sin(\frac{bk}{a}\pi). \tag{2}$$

We have also

$$z(t) - z(s) = T_c(t) - T_c(s) = -2\ \sin\big(\frac{ch}{b}\pi\big)\ \sin\big(\frac{ck}{a}\pi\big). \tag{3}$$

□

Alternate harmonic knots

The following theorem is the analogue of a theorem of Lamm [La] concerning Lissajous knots.

Theorem 1 (Alternate harmonic knots) *Let a, b be relatively prime integers, $c = ab - a - b$. The harmonic knot $\mathrm{H}(a, b, c)$ is alternate.*

Proof. Using eq. (3), we get

$$\begin{aligned} z(t) - z(s) &= -2\sin\big(\frac{ch}{b}\pi\big)\sin\big(\frac{ck}{a}\pi\big) \\ &= -2\sin\big((a-1)h\pi - \frac{ah}{b}\pi\big)\sin\big((b-1)k\pi - \frac{bk}{a}\pi\big) \\ &= -2(-1)^{h(a-1)+k(b-1)}\sin\big(\frac{ah}{b}\pi\big)\sin\big(\frac{bk}{a}\pi\big). \end{aligned}$$

Using eq. (2), we deduce that $\mathrm{sign}\,\big(D\big) = -(-1)^{ah+bk}$. The crossing points are obtained for the $(a-1)(b-1)$ elements of $E = \{t_u = \cos\frac{u}{ab}\pi,\ 0 \le u \le ab,\ a \not| u,\ b \not| u\}$.

Note that $t_u < t_{u-1}$ and that at the crossing point corresponding to $t_u \in E$ one has $\text{sign}\big(D\big) = -(-1)^u$.

The polynomial $x'(t)y'(t)$ has $(a+b-2)$ simple roots for $t_u = \cos\frac{u}{ab}\pi$, where a or b divides $u = 1, \ldots, ab-1$. For these parameters, the billiard curve corresponding to the (x,y)-plane projection bounces on a wall.

Three cases may occur because at least one of three consecutive t_u belongs to E.

1. $t_{h+1} \in E$ and $t_h \in E$. Then $\text{sign}\big(x'(t_h)y'(t_h)\big) = \text{sign}\big(x'(t_{h+1})y'(t_{h+1})\big)$ and since the sign of D changes, we conclude that the sign of $z(t) - z(s)$ changes between the 2 consecutive parameters t_{h+1} and t_h.

2. $t_{h+1} \in E$, $t_h \notin E$, $t_{h-1} \notin E$. We have $x'(t)y'(t) = 0$ at t_h and t_{h-1}. For $t_{h-1} < t < t_{h-2}$, we have $\text{sign}\big(x'(t)y'(t)\big) = \text{sign}\big(x'(t_{h+1})y'(t_{h+1})\big)$, so $\text{sign}\big(x'(t_{h-2})y'(t_{h-2})\big) = \text{sign}\big(x'(t_{h+1})y'(t_{h+1})\big)$. Hence we see that the sign of $z(t) - z(s)$ changes between the 2 consecutive parameters t_{h+1} and t_{h-2}.

3. $t_{h+1} \in E$, $t_h \notin E$, $t_{h-1} \in E$. We have $x'(t)y'(t) = 0$ at t_h, so $\text{sign}\big(x'(t_{h-1})y'(t_{h-1})\big) = -\text{sign}\big(x'(t_{h+1})y'(t_{h+1})\big)$. Hence we see that the sign of $z(t) - z(s)$ changes between the 2 consecutive parameters t_{h+1} and t_{h-1}.

In conclusion, the knot is alternate. □

Symmetries and harmonic knots

A knot K in $\mathbf{S}^3$ is strongly $(-)$amphicheiral if there is an involution of $(\mathbf{S}^3, K)$ which reverses the orientation of both $\mathbf{S}^3$ and K. A knot K in $\mathbf{S}^3$ is strongly invertible if there is an involution of $(\mathbf{S}^3, K)$ which preserves the orientation of $\mathbf{S}^3$ and reverses the orientation of K (see [Kaw], pp. 127-128).

Proposition 6 *The harmonic knot* $\mathrm{H}(a,b,c)$ *is either strongly* $(-)$*amphicheiral if abc is odd, or strongly invertible if abc is even.*

Proof. It is immediate from the parity of Chebyshev polynomials.

Corollary 2 *There are infinitely many amphicheiral harmonic knots. There are infinitely many strongly invertible harmonic knots.*

Proof. Since the harmonic knot $\mathrm{H}(a,b,c)$, $c = ab - a - b$ is alternate, its crossing number is $\frac{1}{2}(a-1)(b-1)$. From this we conclude that there is an infinity of such knots with a, b, c odd, or with abc even. □

If σ is any permutation of $\{a,b,c\}$ then the harmonic knot $\mathrm{H}(\sigma(a),\sigma(b),\sigma(c))$ is either $H(a,b,c)$ if σ is an even permutation or its mirror image if σ is an odd permutation.

Proposition 7 *Let a, b be coprime integers.* $\mathrm{H}(a,b,c)$, $\mathrm{H}(a,b,2ab-c)$ *and* $\mathrm{H}(a,b,2ab+c)$ *are the same knot.*

Proof. The expression of sign $\big(D\big)$ (eqs. (2,3)) for a given pair of parameters (t,s) corresponding to crossing points in the (x,y)-plane projection is invariant under the transformation $c \mapsto c+2ab$ and $c \mapsto 2ab-c$. □

We can therefore suppose that $a<b$ and $0<c<ab$ to consider all cases.

Proposition 8 *Let a, b, c be relatively prime integers. There exists c' such that* $\mathrm{H}(a,b,c')$ *is the mirror image of* $\mathrm{H}(a,b,c)$.

Proof. Because a and b are relatively prime, one can write $c=\alpha a+\beta b$ where α and β are integers. Let us consider $c'=-\alpha a+\beta b$. We have $c' \equiv c \pmod{2a}$ and $c' \equiv -c \pmod{2b}$. For any crossing point of the Chebyshev curve $\mathcal{C} : T_b(x)=T_a(y)$ corresponding to diagrams of both $\mathrm{H}(a,b,c)$ and $\mathrm{H}(a,b,c')$, we see that sign $\big(D\big)$ changes to opposite when c is replaced by c'. □

Corollary 3 $\mathrm{H}(a,b,ab+a-b)$ *is the mirror image of the alternate knot* $\mathrm{H}(a,b,ab-a-b)$.

Corollary 4 *Let a, b be relatively prime integers. There are at most $\varphi(a)\varphi(b)$ different harmonic knots* $\mathrm{H}(a,b,c)$.

Proof. The number of c in $[1,ab]$ that are relatively prime to a and b is $\varphi(a)\varphi(b)$ where φ is the Euler function. □

Remarks. Because of prop. (7) and prop. (8), for each c there is exists $c'<ab$ giving the mirror image. We have at most $\frac{1}{2}\varphi(a)\varphi(b)$ different knots when we identify the knots and their mirror images.
When $a+1<b$, $H(a,b,1)$ and its mirror image, $H(a,b,a+b)$ and $H(a,b,b-a)$ are trivial.

The simplest alternate harmonic knots

It is remarkable that for $a=3$ the curves are drawn in "Conway normal form" for 2-bridge knots [Mu]. Then their Conway notation is $\mathrm{H}(3,n,2n-3) = C(1,1,\ldots,1) = C(2,1,1,\ldots,1,2)$ when n is not a multiple of 3. Turner [Tu] did name these knots Fibonacci knots, because their determinant are Fibonacci numbers. For $n=4$ we obtain the trefoil, for $n=5$ the figure eight, for $n=7$ the 6_3 knot, for $n=8$ the $\overline{7}_7$ knot...

The Fibonacci knots with an even crossing number are 2-bridge amphicheiral knots. By a theorem of Hartley and Kawauchi [HK] they cannot be strongly positive amphicheiral, and then they are not Lissajous [BHJS].

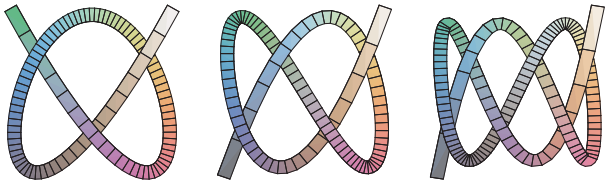


Figure 7: The trefoil, the figure eight knot, and the 6_3 knot.

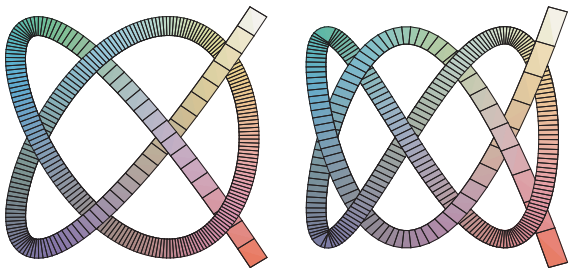


Figure 8: The 6_2 and 9_{20} knots .

For $a = 4$, we also obtain 2-bridge knots. Following the classical method ([Mu] p. 183-187), we see that their Conway notation is $\mathrm{H}(4, n, 3n-4) = C(-1, -2, \ldots, -1, -2) = C(-3, -1, -2, \ldots, -1, -2)$, ($n$ odd). For $n = 5$ we obtain the 6_2 knot, for $n = 7$ a symmetric picture of the 9_{20} knot (compare with Rolfsen's table [Ro]).

For $n \geq 5$ we may obtain p-bridge knots, with $p \geq 3$. For example the harmonic knot $\mathrm{H}(5, 6, 19)$ is the mirror image of 10_{116} knot in Rolfsen's table (amazingly, with exactly the same picture). Its bridge number is known to be 3.

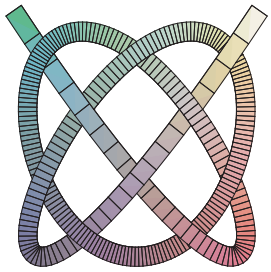


Figure 9: The mirror image of the 10_{116} knot.

Note that the torus knot $T(2, 2n+1)$ cannot be obtained as an alternate harmonic knot, except for the trefoil knot, as it is proved in [KP1]. Nevertheless it can be obtained as an harmonic knot.

The torus knots $\mathbf{T(2, 2n + 1)}$

Theorem 2 *The torus knot* $T(2, 2n + 1)$ *is harmonic. It is the harmonic knot* $\mathrm{H}(3, 3n + 1, 3n + 2)$ *if* n *is even, or the harmonic knot* $\mathrm{H}(3, 3n + 2, 3n + 1)$ *if* n *is odd.*

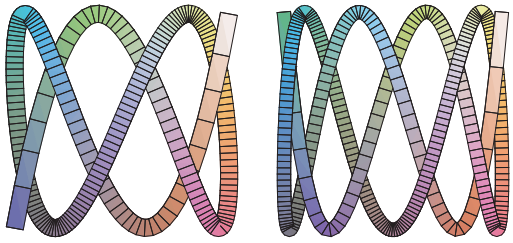


Figure 10: The torus knots 5_1: $\mathrm{H}(3, 7, 8)$ and $\overline{7_1}$: $\mathrm{H}(3, 10, 11)$

Proof. We shall determine the Conway normal form of the harmonic knot $\mathrm{H}(3, b, c)$, $b = 3n + 1, c = b + 1$. The crossing points of the plane projection of $\mathrm{H}(3, b, c)$ are obtained for pairs of values (t, s) where $t = \cos\tau$, $s = \cos\sigma$, and $\tau = \dfrac{m}{3b}\pi$, $\sigma = \dfrac{m'}{3b}\pi$.

For $k = 0, \ldots, n - 1$, let us consider

A_k be obtained for $m = 3k + 1$, $m' = 2b - m$.

B_k be obtained for $m = 2b + 3k + 2$, $m' = m - 2b$.

C_k be obtained for $m = 2b + 3k + 3$, $m' = 4b - m$.

Then we have

$$x(A_k) = \cos(\frac{3k+1}{b}\pi),\ y(A_k) = \tfrac{1}{2}(-1)^k.$$

$$x(B_k) = \cos(\frac{3k+2}{b}\pi),\ y(B_k) = \tfrac{1}{2}(-1)^{k+1}.$$

$$x(C_k) = \cos(\frac{3k+3}{b}\pi),\ y(C_k) = \tfrac{1}{2}(-1)^k.$$

Hence our $3n$ points satisfy

$$x(A_{k-1}) > x(B_{k-1}) > x(C_{k-1}) > x(A_k) > x(B_k) > x(C_k),\ k = 1, \ldots, n - 1.$$

Let us determine the nature of the crossing points. Using the identity $T'_a(\cos\tau) = a\dfrac{\sin a\tau}{\sin\tau}$, we get

$$\operatorname{sign}\big(x'(t)y'(t)\big) = \operatorname{sign}\big(\sin 3\tau \sin b\tau\big).$$

We get

$$\begin{aligned}
\text{for } A_k\text{:}\quad \operatorname{sign}\big(x'(t)y'(t)\big) &= \operatorname{sign}\big(\sin(\frac{3k+1}{b}\pi)\sin(\frac{3k+1}{3}\pi)\big) = (-1)^k.\\
\text{for } B_k\text{:}\quad \operatorname{sign}\big(x'(t)y'(t)\big) &= \operatorname{sign}\big(\sin\big(\frac{2b+3k+2}{b}\pi\big)\sin\big(\frac{2b+3k+2}{3}\pi\big)\big)\\
&= (-1)^{k+1}\operatorname{sign}\big(\sin\big(\frac{3k+2}{b}\pi\big)\sin\frac{\pi}{3}\big) = (-1)^{k+1}.\\
\text{for } C_k\text{:}\quad \operatorname{sign}\big(x'(t)y'(t)\big) &= \operatorname{sign}\big(\sin(\frac{2b+3k+3}{b}\pi)\sin(\frac{2b+3k+3}{3}\pi)\big)\\
&= (-1)^{k+1}\operatorname{sign}\big(\sin(\frac{3k+3}{b}\pi)\sin\frac{2\pi}{3}\big) = (-1)^{k+1}.
\end{aligned}$$

Now, let us compute the sign of

$$T_c(t) - T_c(s) = -2\sin(c\frac{\tau+\sigma}{2})\sin(c\frac{\tau-\sigma}{2}) = -2\sin\Big(\frac{c}{6b}(m-m')\Big)\sin\Big(\frac{c}{6b}(m+m')\Big)$$

We have, with $c = b+1 = 3n+2$,

$$\begin{aligned}
\text{for } A_k\text{:}\quad z(t) - z(s) &= -2\sin c\frac{\pi}{3}\sin\big(c\frac{m-b}{3b}\pi\big)\\
&= -2(-1)^n\sin\frac{2\pi}{3}\sin\big(c\frac{n-k}{b}\pi\big)\\
&= 2(-1)^n\sin\frac{2\pi}{3}(-1)^{n-k}\sin\big(\frac{n-k}{b}\pi\big)
\end{aligned}$$

so $\operatorname{sign}\big(z(t)-z(s)\big) = (-1)^k$.

$$\begin{aligned}
\text{for } B_k\text{:}\quad z(t) - z(s) &= -2\sin\big(c\frac{b+3k+2}{3b}\pi\big)\sin(c\frac{\pi}{3})\\
&= -2\sin\big((n+k+1+\frac{n+k+1}{b}\pi\big)\pi(-1)^n\sin\frac{\pi}{3}\\
&= 2(-1)^k\sin\big(\frac{n+k+1}{b})\pi\big)\sin\frac{\pi}{3}.
\end{aligned}$$

so $\operatorname{sign}\big(z(t)-z(s)\big) = (-1)^k$.

$$\begin{aligned}
\text{for } B_k\text{:}\quad z(t) - z(s) &= -2\sin\big(c\frac{2\pi}{3}\big)\sin(c\frac{3k+3}{3b}\pi\big)\\
&= -2\sin\big(\frac{4\pi}{3}\big)\sin\big((b+1)\frac{k+1}{b}\pi\big)\\
&= 2\sin\big(\frac{4\pi}{3}\big)(-1)^{k+1}\sin\big(\frac{k+1}{b}\pi\big)
\end{aligned}$$

so $\operatorname{sign}\big(z(t)-z(s)\big) = (-1)^{k+1}$.

Collecting these results we finally get

$$\operatorname{sign}\big(D(A_k)\big) = 1, \quad \operatorname{sign}\big(D(B_k)\big) = -1, \quad \operatorname{sign}\big(D(C_k)\big) = 1.$$

Crossing points of $\mathcal{C}$ are alternatively on the two rows: $y = \frac{1}{2}$ and $y = -\frac{1}{2}$. We must multiply the signs of crossings by the sign of y in order to get the Conway normal form. We get therefore $(-1)^k, (-1)^k, (-1)^k$ for A_k, B_k, C_k. We are now able to conclude the proof. If n

is odd the Conway normal form of our 2-bridge knot is $C(1,1,1,-1,-1,-1,\ldots,1,1,1)$. Its Schubert fraction is equal to $\dfrac{2n+1}{2n} \sim -(2n+1)$ and our knot is the mirror image of the torus knot $T(2,2n+1)$. If n is even, the Conway normal form is $C(-1,-1,-1,1,1,1,\ldots,1,1,1)$, its Schubert fraction is $\dfrac{2n+1}{-2n} \sim (2n+1)$, and our knot is the torus knot $T(2,2n+1)$. □

Remark: Note that $\mathrm{H}(3,3n+1,3(3n+1)-1)$ is the mirror image of $\mathrm{H}(3,3n+1,3n+2)$. (see prop. (8)).

Remark: In [KP2], we obtained the torus knot $T(2,2n+1)$ as an alternate polynomial knot where $x(t)=T_3(t)$, $y(t)=P(t)$, $z(t)=Q(t)$ are polynomials and $\deg P = 3n+1$, $\deg Q = 3n+2$, that is to say the same degrees.

Because of their definitions, the symmetries of the harmonic knots are easy to find. They are either strongly negative amphicheiral if a,b,c are odd, or strongly invertible. So that not every knot is an harmonic knot. We can also remark that harmonic Knots are billiard knots in a convex (compact) billiard (in fact a truncated cube)[JP].

On the other hand, it is not difficult to see that if we change the nature of one crossing point in the diagram of the 10_{116} knot, we can obtain the famous 8_{17} knot.

In the next paragraph, we shall see that it is possible to choose the nature of the crossing points with a (shifted) Chebyshev polynomial as height function.

4 Every knot is a Chebyshev knot

Let us denote B_n the group of n-braids and S_n the symmetric group. The group of pure braids P_n is the kernel of the morphism $\pi : B_n \to S_n$. If α is a braid, we shall denote $\rho(\alpha)$ its plane projection. In the next theorem, which is analogous to a theorem of Lamm for Lissajous curves (see [BDHZ]), we show that

Theorem 3 *Every knot has a projection which is a Chebyshev plane curve.*

This is a consequence of the following proposition.

Proposition 9 *Let K be a knot, $br(K)$ its bridge number. Let $m \geq \mathrm{br}(K)$ be an integer. Then K has a projection which is a Chebyshev curve $x=T_a(t)$, $y=T_b(t)$, where $a=2m-1$, and $b \equiv 2 \pmod{2a}$.*

Proof. Let K be a knot. Let D be a regular diagram of K such that the abscissa has only two extremal values reached at m maxima and m minima. It means that K is the plat closure of a horizontal braid t with $2m$ strings, such that the upper string is unbraided. Furthermore, reordering if necessary the ordinates of the $2m$ extrema, we can suppose that $\pi(t) = (2,3)\cdots(2m-2,2m-1) = \pi(\sigma_2\,\sigma_4\cdots\sigma_{2m-2})$. Let us denote $s_{\mathrm{even}} = \rho(\sigma_2\cdots\sigma_{2m-2})$, and $s_{\mathrm{odd}} = \rho(\sigma_1\cdots\sigma_{2m-3})$.

As $\pi(t) = \pi(\sigma_2 \cdots \sigma_{2m-2})$ we see that there exists $l \in \ker(\pi) = P_{2m-1}$, the group of pure braids, such that $t = l \cdot \sigma_2 \cdots \sigma_{2m-2}$. As the braids $A_{i\,j} = x^{-1}\sigma_i^2 x$, where $x = \sigma_{i+1} \cdots \sigma_j$, generate P_m, the braid l is a composition of such elementary braids. It is not difficult to see that there is a braid $\alpha_{i\,j}$ equivalent to $A_{i\,j}$ with plane projection $\rho(\alpha_{i\,j}) = (s_{\text{even}} s_{\text{odd}})^a$. Consequently, the braid t is equivalent to a braid projecting on $(s_{\text{even}}\, s_{\text{odd}})^{ka} s_{\text{even}}$. Using the braid description of Chebyshev curves (corollary 1), we conclude that our knot K is equivalent to a knot projecting upon the Chebyshev curve $x = T_a(t),\ y = T_b(t),\ a = \text{odd},\ b \equiv 2 \pmod{2a}$. □

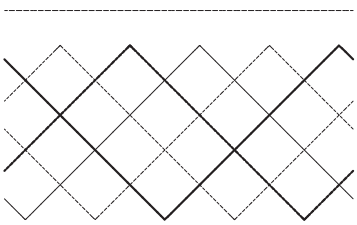


Figure 11: The "plane braid" $(s_{\text{even}}\, s_{\text{odd}})^5$ is the projection of the braid $\alpha_{2,5} \in B_5$.

We shall prove our principal result with a density argument based on Kronecker's theorem ([HW] th. 443 p. 382.) Let us recall this theorem.

Theorem 4 (Kronecker) *If $\theta_1, \theta_2, \ldots, \theta_k$, 1 are linearly independent over $\mathbf{Q}$, then the set of points $((n\,\theta_1), (n\,\theta_2), \ldots, (n\,\theta_k))$ is dense in the unit cube.* Here (x) denotes the fractional part of x.

We shall need the following lemma.

Lemma 2 *Let $c_1, \ldots, c_k$ be real numbers such that $-1 < c_1 < c_2 < \cdots < c_k < 1$. There exists a positive number $e < 1 - c_k$ such that the numbers $\arccos(c_1 + e), \arccos(c_2 + e), \ldots, \arccos(c_k + e)$, 1 are linearly independent over $\mathbf{Q}$.*

Proof. First, we shall prove, by induction on k, that the functions $\arccos(c_i + x)$, 1 are linearly independent over $\mathbf{R}$. Let $\lambda_0 + \sum_{i=1}^k \lambda_i \arccos(c_i + x) = 0$ be a linear relation between these functions. We get by derivating

$$\sum_{i=1}^{k} \lambda_i \frac{1}{\sqrt{1 - (c_i + x)^2}} = 0.$$

Then, when $x \to 1 - c_k$, we get $\lambda_k = 0$, and the result follows by induction.

Suppose now that for each $e < 1 - c_k$ there exists a relation

$$\lambda_0 + \sum_{i=1}^{k} \lambda_i \arccos(c_i + e) = 0, \qquad \lambda_i \in \mathbf{Q}, \qquad \sum_{i=1}^{k} |\lambda_i| = 1.$$

By cardinality, there are infinitely many e in $]0, 1-c_k[$ with the same collection of λ_i. This means that the analytic function $\lambda(x) = \lambda_0 + \sum_{i=1}^{k} \lambda_i \arccos(c_i + x)$ has an infinity of zeroes in the interval $]0, 1-c_k[$, which is absurd. □

Theorem 5 *Every knot is a Chebyshev knot.*

Proof. Let K be a knot projecting on the Chebyshev curve $\{x = T_a(t),\ y = T_b(t)\}$. The crossing points of the projection are obtained for the distinct pairs of values

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi,\ s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,\ \frac{k}{a} + \frac{h}{b} < 1.$$

Let us denote these values by $(t_i, s_i),\ i = 1 \dots n = \frac{1}{2}(a-1)(b-1)$. By our lemma, let $e < 1 - \cos\dfrac{\pi}{ab}$ be a positive number such that the $2n+1$ numbers $1,\ \tau_i = \arccos(t_i + e),\ \sigma_i = \arccos(s_i + e),\ i = 1, \dots n,$ are linearly independent over $\mathbf{Q}$. Let us define the function $Z(t) = T_c(t + e)$ (depending on the integer c). We have

$$Z(t_i) - Z(s_i) = \cos c\,\tau_i - \cos c\,\sigma_i.$$

Since the numbers $1, \tau_i, \sigma_i$ are linearly independent over $\mathbf{Q}$, the numbers $c\,\tau_i \pmod{2\pi}$ and $c\,\sigma_i \pmod{2\pi}$ are dense in $[0,\ 2\pi]^n$ by Kronecker's theorem. So that we can choose arbitrarily the signs of $Z(t_i) - Z(s_i)$, that is, the over/under nature of the crossing points. □

Example 1: the knot 6_1

Let us consider the curve parametrized by

$$x = T_3(t),\ y = T_8(t),\ z = T_{10}(t + \frac{1}{100}).$$

Computing sign $\big(D\big)$ for the $\frac{1}{2}(3-1)(8-1) = 7$ crossing points, we find the Conway normal form: $C(-1, -1, -1, -1, 1, 1, 1)$. Its Schubert fraction (see [Mu]) is then

$$-1 + \cfrac{1}{-1 + \cfrac{1}{-1 + \cfrac{1}{-1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}} = \frac{9}{-5} \sim \frac{9}{4}.$$

This knot is the knot 6_1.

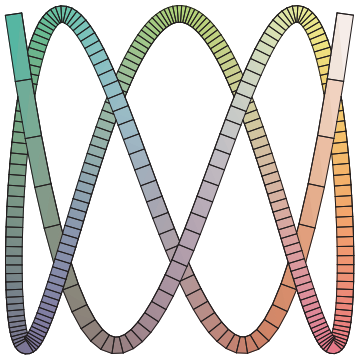


Figure 12: The knot 6_1 is a Chebyshev knot

Example 2: the knot 8_{17}

The famous 8_{17} knot is non invertible and strongly $(-)$amphicheiral (see [Kaw] p. 128). It is a 3-bridge knot. The Chebyshev curve $T_6(x) = T_5(y)$ is one of its diagrams. It can

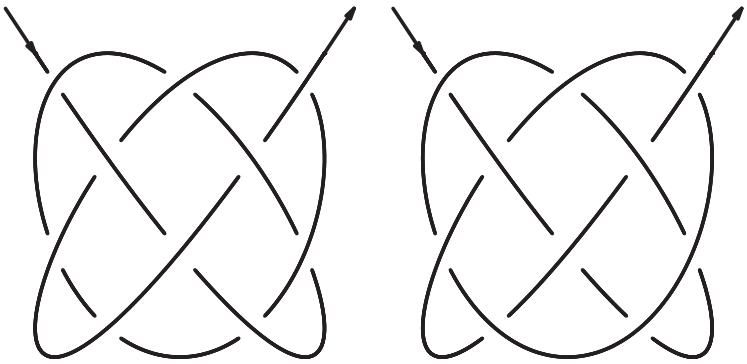
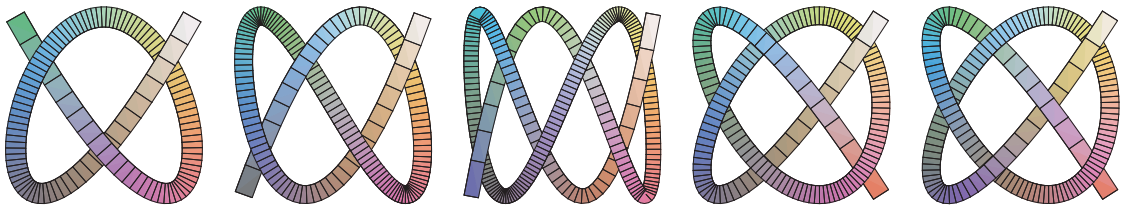


Figure 13: The knots 8_{17} and its reverse as Chebyshev knots

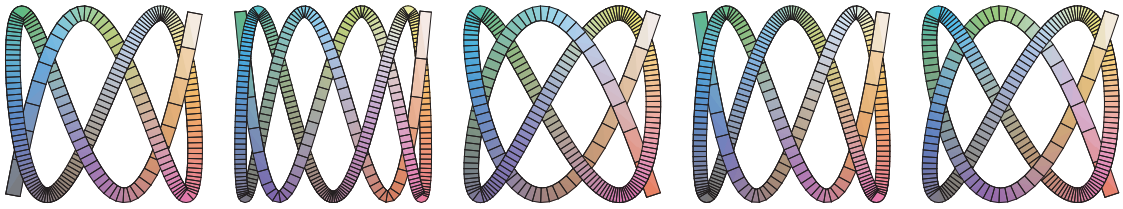
be parametrized by $x = T_5(t), y = T_6(t), z = T_{33}(t + \frac{1}{67})$. The knot parametrized by $x = T_5(t), y = T_6(t), z = T_{33}(t - \frac{1}{67})$ is the reversed image of 8_{17} by a half-turn about the y axis.

Conclusion

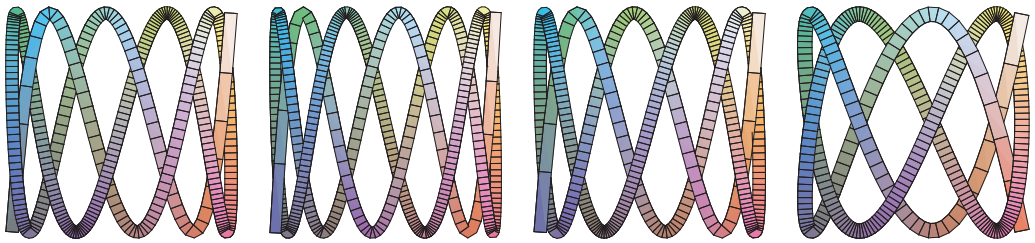
Since every knot is a Chebyshev knot, we shall give here some diagrams of 2-bridge harmonic knots, with their Conway-Rolfsen numbering. A bar over a knot name indicates mirror image.



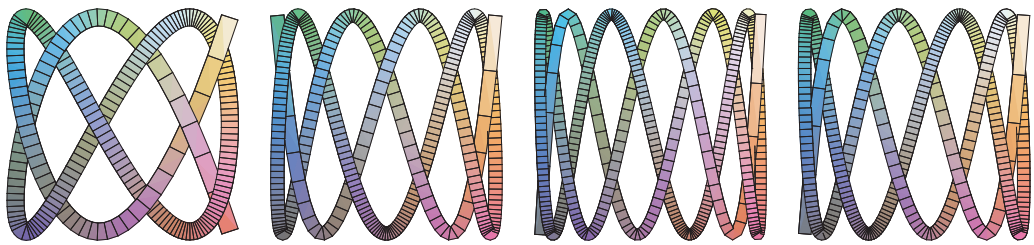
$\overline{3}_1$ H(3, 4, 5) | 4_1 H(3, 5, 7) | 5_1 H(3, 7, 8) | $\overline{5}_2$ H(4, 5, 7) | 6_2 H(4, 5, 11)



6_3 H(3, 7, 11) | $\overline{7}_1$ H(3, 10, 11) | 7_5 H(4, 7, 9) | $\overline{7}_7$ H(3, 8, 13) | 8_7 H(4, 7, 13)

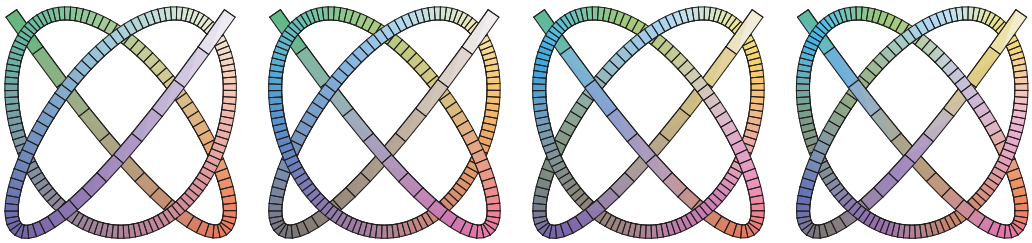


8_3 H(3, 11, 13) | 9_1 H(3, 13, 14) | 9_{17} H(3, 11, 16) | $\overline{9}_{18}$ H(4, 9, 11)



9_{20} H(4, 7, 17) | $\overline{9}_{31}$ H(3, 10, 17) | 10_{37} H(3, 13, 17) | 10_{45} H(3, 11, 19)

We shall now give the complete list of harmonic knots H(5, 6, c). We get $4 = \frac{1}{2}\varphi(5)\varphi(6)$ different types up to mirror symmetry.



0 H(5, 6, 1) | 5_2 H(5, 6, 7) | 10_{159} H(5, 6, 13) | $\overline{10}_{116}$ H(5, 6, 19)

The 10_{159} knot is the harmonic knot H(5, 6, 13). Its bridge number is equal to 3. Our Chebyshev parametrization provides an easy proof that it is strongly invertible (compare [Kaw], Appendix F, p. 254).

In conclusion, we have found a great number of harmonic knots. The classification of these knots would be an interesting problem.

On the other hand, we showed how to construct a Chebyshev model for any knot. We hope that these models which have simple equations and simple geometric properties will be useful to the study of knots.

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