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# ON A CONJECTURE ON EXPONENTIAL DIOPHANTINE EQUATIONS

MIHAI CIPU AND MAURICE MIGNOTTE

ABSTRACT. We deal with a conjecture of Terai (1994) asserting that if  $a, b, c$  are fixed coprime integers with  $\min(a, b, c) > 1$  such that  $a^2 + b^2 = c^r$  for a certain odd integer  $r > 1$ , then the equation  $a^x + b^y = c^z$  has only one solution in positive integers with  $\min(x, y, z) > 1$ . Co-operation man-machine is needed for the proof.

## 1. THE PROBLEM

Let  $a, b, c$  be fixed coprime integers with  $\min(a, b, c) > 1$ . In 1933, Mahler [15] developed a  $p$ -adic equivalent of the Thue-Siegel method to prove that the equation

$$(1) \quad a^x + b^y = c^z$$

has finitely many solutions  $(x, y, z)$  in positive integers. His method is ineffective in the sense that it gives no indication on the number of possible solutions for a fixed triple  $(a, b, c)$ . Such an information has been obtained only in particular instances. Thus, Sierpiński [20] showed that  $(2, 2, 2)$  is the unique solution in positive integers to the equation  $3^x + 4^y = 5^z$ . In the same journal, Jeśmanowicz [10] conjectured the unicity of the solution to Eq. (1) in case  $(a, b, c)$  is a Pythagorean triple. This conjecture is still open, despite the efforts of many authors.

In analogy to Jeśmanowicz's conjecture, Terai [23] stated that Eq. (1) always has at most one solution in positive integers. Simple examples disproving this statement have been found by Cao [4], who attempted to remedy the situation by adding the hypothesis  $\max(a, b, c) > 7$ . It turns out that this condition is not sufficient to entail the thought-for unicity. A family of counterexamples have been pointed out by Le [14], who also stated the following variant of Terai's conjecture.

**Conjecture 1.1.** *For given coprime integers  $a, b, c > 1$ , the Diophantine equations (1) has at most one solution in integers  $x, y, z > 1$ .*

Much work has been devoted to the case when Eq. (1) has a solution of the form  $(2, 2, r)$ , with  $r$  greater than 1 and odd. This implies in particular that

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$c$  is odd and exactly one of  $a, b$  is even. For the sake of definiteness, suppose that  $a$  is even and therefore  $b$  is odd. Most of the recent results concerns the case  $a \equiv 2 \pmod{4}$ ,  $b \equiv 3 \pmod{4}$ . The conjecture is established in this case under one of the following additional hypotheses:

- ( $\alpha$ ) (Terai [25])  $b \equiv 3 \pmod{8}$ ,  $b \geq 30a$  and  $\left(\frac{a}{d}\right) = -1$ , where  $d > 1$  is a divisor of  $b$  and  $\left(\frac{a}{d}\right)$  denotes the Jacobi symbol,
- ( $\beta$ ) (Cao [4])  $c$  is a prime power,
- ( $\gamma$ ) (Cao-Dong [5])  $b \geq 25.1a$ ,
- ( $\delta$ ) (Le [14])  $c > 3 \cdot 10^{27}$  and  $r > 7200$ .

Further partial confirmations of the conjecture are referred to in the papers cited above.

Contrary to what it is claimed in [14], the last result quoted above does not imply that the conjecture holds with the exception of finitely many pairs  $(c, r)$ . One of the aims of this paper is to prove that indeed there are at most finitely many values for which the conjecture can be refuted. On the way we shall prove other results for the positive solutions to the Diophantine simultaneous equations

$$(2) \quad a^2 + b^2 = c^r, \quad a^2 + b^y = c^z,$$

where

$$(3) \quad r, z > 1 \text{ are odd, } a \equiv 2 \pmod{4}, b \equiv 3 \pmod{4}, \text{ and } \gcd(a, b) = 1.$$

As a consequence of our deliberations, improvements on the results ( $\alpha$ )–( $\delta$ ) are obtained. Our proofs approach these cases from a different perspective and are much shorter than the published ones, although they involve a harder computational component. We give here a rough description of our procedure. In the hypotheses of our work,  $c$  is a sum of two coprime squares. We generate all such decompositions for  $c$  up to  $4 \cdot 10^{10}$  with the help of Cornacchia's algorithm (see [18] and [2] for very simple proofs of its correctness). We notice that, when compared to the obvious method (for  $c$  fixed and  $1 \leq u < \sqrt{c}$  test whether  $c - u^2$  is a square), for our range of values Cornacchia's algorithm is more than ten times faster.

This description is vague; details are given in the third section, after we recall classical facts, some of them going back at least to Lagrange. Additional information on the putative solutions of the Diophantine system (2) are given in Section 3. Section 4 contains the proofs of our main results, among which are the following.

**Theorem 1.2.** *If the Diophantine equation  $X^x + Y^y = Z^z$  has a solution with  $X = a \equiv 2 \pmod{4}$ ,  $Y = b \equiv 3 \pmod{4}$ ,  $Z = c$ ,  $x = 2$ ,  $y = 2$  and  $z = r$  odd, where  $\gcd(a, b) = 1$ , then this is the only solution in positive integers, with the possible exception of finitely many values  $(c, r)$ .*

**Theorem 1.3.** *If  $a$  or  $b$  is a prime power then the system (2) has no solutions subject to restrictions from (3).*

The last part of the paper is devoted to improvements of bounds on the parameters associated to a putative solution to system (2). They are meant to shrink the search domains for the components of a solution to a manageable size according to the present-day technology.

Although Terai's conjecture remains open, we have pushed the analysis further than ever before; and there is significant hope that our results can be improved by either complementing them with brand new ideas or dedicated computations.

## 2. ARITHMETIC RESTRICTIONS

We use a result of Lagrange (1741), *Leçons sur le calcul des fonctions*, which makes recurrent appearance in the study of Diophantine equations, as well as in the theory of finite fields, Chebyshev polynomials and many other areas of mathematics. For the sake of completeness, we sketch its proof.

**Lemma 2.1.** *Let  $X$  and  $Y$  two commuting indeterminates and let  $n \geq 1$  be a positive integer. Then*

$$X^n + Y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} c_{n,j} (-XY)^j (X + Y)^{n-2j},$$

where the  $c_{n,j}$  are nonnegative integers which are defined recursively by

$$c_{n,j} = 0 \quad \text{if} \quad j < 0 \quad \text{or} \quad j > \lfloor n/2 \rfloor,$$

$$c_{1,0} = 1, \quad c_{2,0} = 1, \quad c_{2,1} = 2, \quad \text{and} \quad c_{n+1,j} = c_{n,j} + c_{n-1,j-1} \quad \text{for} \quad n \geq 2.$$

More precisely,

$$c_{n,j} = \frac{(n-j-1)! n}{(n-2j)! j!}.$$

*Proof.* The result is obvious for  $n \leq 2$ , including the initial values  $c_{1,0} = 1$ ,  $c_{2,0} = 1$ ,  $c_{2,1} = 2$ . The general case can be obtained by induction from the formula

$$X^{n+1} + Y^{n+1} = (X^n + Y^n)(X + Y) - XY(X^{n-1} + Y^{n-1}),$$

which implies the recursive relation

$$c_{n+1,j} = c_{n,j} + c_{n-1,j-1} \quad \text{for} \quad n \geq 2.$$

By completely working out the details, one can get the closed form for the coefficients  $c_{n,j}$ .  $\square$

In the present situation, using Corollary 3.3 below, we get expressions for  $a$ ,  $b$  and  $b^{y/2}$ .

**Corollary 2.2.** *The values of  $a$  and  $b$  satisfy*

$$a = \pm u \sum_{j=0}^{(r-1)/2} c_{r,j} (-c)^j (4v^2)^{(r-1)/2-j}, \quad b = \pm v \sum_{j=0}^{(r-1)/2} c_{r,j} c^j (-4v^2)^{(r-1)/2-j}$$

and

$$a = \pm u_1 \sum_{j=0}^{(z-1)/2} c_{z,j} (-c)^j (4v_1^2)^{(r-1)/2-j}, \quad b^{y/2} = \pm v_1 \sum_{j=0}^{(z-1)/2} c_{z,j} c^j (-4v_1^2)^{(z-1)/2-j}.$$

From the last formula it follows that

$$b^{y/2} \equiv \pm v_1 z c^{(z-1)/2} \pmod{v_1^3},$$

in particular

$$p \mid v_1 \implies v_p(v_1) \geq y/2 \geq 3 \quad \text{if } \gcd(p, z) = 1.$$

### 3. BOUNDS FOR $a$ , $b$ , $c$

From our standard hypotheses on  $a$ ,  $b$ ,  $c$  stated in Eq. (2) and (3) it follows that  $c \equiv 5 \pmod{8}$ . As it is well-known (see, e.g., Lemma 3.2 below), the first equation from (2) implies that there exist positive integers  $u$  and  $v$  such that

$$(4) \quad c = u^2 + v^2.$$

By [4], we also may suppose that  $c$  has at least two prime divisors. Then it is easily seen that one has  $c \geq 85$ .

Other useful facts are given by the next result, proved in several places, for instance in [14].

**Lemma 3.1.** *With the above notation and hypotheses, let  $(x, y, z)$  be a solution to (1) with  $(x, y, z) \neq (2, 2, r)$ . Then  $x = 2$ ,  $y \equiv 2 \pmod{4}$ ,  $y \geq 6$  and  $z$  is odd.*

We shall repeatedly use the well-known structure of integers satisfying the first equation from (2).

**Lemma 3.2.** *If  $X$ ,  $Y$  and  $Z$  are coprime positive integers such that*

$$X^2 + Y^2 = Z^n,$$

*where  $n$  is an odd integer and  $X$  is even, then there exist coprime positive integers  $u$  and  $v$ , with  $u$  even and  $v$  odd, and  $\lambda_1, \lambda_2 \in \{-1, 1\}$  such that*

$$X + Y\sqrt{-1} = \lambda_1 \varepsilon^n, \quad \varepsilon = u + v\lambda_2\sqrt{-1}, \quad Z = u^2 + v^2.$$

*Moreover, if  $\varepsilon = |\varepsilon| e^{\theta\sqrt{-1}/2}$  then*

$$X = Z^{n/2} |\cos(n\theta/2)|, \quad Y = Z^{n/2} |\sin(n\theta/2)|.$$

The former part is proved as in Mordell's book [17, pp.122–123]; the later assertion is an obvious consequence of the preceding formulas.

In the present case, using the relations  $a^2 + b^2 = c^r$  and  $a^2 + b^y = c^z$ , we get exponential expressions for  $a$ ,  $b$  and  $b^{y/2}$ .

**Corollary 3.3.** *There are positive integers  $u, v, u_1, v_1$ , with  $u, u_1$  even and  $v, v_1$  odd, such that  $c = u^2 + v^2 = u_1^2 + v_1^2$  and*

$$a = \frac{1}{2}|\varepsilon^r + \bar{\varepsilon}^r| = c^{r/2}|\cos(r\xi)|, \quad b = \frac{1}{2}|\varepsilon^r - \bar{\varepsilon}^r| = c^{r/2}|\sin(r\xi)|$$

and

$$a = \frac{1}{2}|\varepsilon_1^z + \bar{\varepsilon}_1^z| = c^{z/2}|\cos(z\xi_1)|, \quad b^{y/2} = \frac{1}{2}|\varepsilon_1^z - \bar{\varepsilon}_1^z| = c^{z/2}|\sin(z\xi_1)|,$$

where  $\varepsilon = u + v\sqrt{-1}$ ,  $\varepsilon_1 = u_1 + v_1\sqrt{-1}$ ,  $\tan \xi = v/u$ , and  $\tan \xi_1 = v_1/u_1$ .

Let us come back to the notation of the above lemma and put  $\varepsilon = e^{i\theta/2}$  and

$$\alpha := \varepsilon/\bar{\varepsilon} = \frac{u^2 - v^2 + 2uv\sqrt{-1}}{u^2 + v^2} e^{\theta\sqrt{-1}}.$$

Since  $\alpha$  is a root of the irreducible integer polynomial

$$(u^2 + v^2)T^2 - 2(u^2 - v^2)T + u^2 + v^2,$$

whose Mahler's measure is equal to  $u^2 + v^2$ , the absolute logarithmic height of  $\alpha$  is

$$h(\alpha) = \frac{1}{2} \log(u^2 + v^2) = \frac{1}{2} \log Z.$$

We also have

$$\min\{X, Y\} \geq \frac{Z^{n/2}}{\pi} \min_{k' \in \mathbb{Z}} |n\theta - k'\pi|.$$

Let  $k$  be an integer such that  $\min_{k' \in \mathbb{Z}} |n\theta - k'\pi| = |n\theta - k\pi|$  and put

$$\Lambda = n \log \alpha - k \log(-1).$$

Then

$$\min\{X, Y\} \geq \frac{Z^{n/2}}{\pi} |\Lambda| \quad \text{and} \quad \min\{X, Y\} \geq 0.99 Z^{n/2} \min\{|\Lambda|, 0.001\},$$

where  $\Lambda$  is a linear form in two logarithms of algebraic numbers.

**3.1. A first application of linear forms.** In a number field  $K$  embedded in the complex field, containing a root of unity  $\zeta = e^{i\pi/m}$ , where  $m$  is maximal, and a number  $\alpha$  of modulus one which is not a root of unity, a linear form

$$\Lambda = n \log \alpha - ik\pi$$

as above can be written as

$$\Lambda = n \log \alpha - mk \log \zeta.$$

We remark that changing  $\alpha$  into a suitable  $\alpha\zeta^\ell$  if necessary we can assume that  $|\log \alpha| \leq \pi/(2m)$ . We may work under this hypothesis without changing the notation because  $h(\zeta^\ell \alpha) = h(\alpha)$ .

On using the main result of Laurent-Mignotte-Nesterenko [12], it is possible to prove that

$$(5) \quad z > 55000 \quad \text{implies} \quad a > c^{z/(2\sqrt{3})}.$$

On using relations  $a^2 + b^y = c^z$  and  $c = u_1^2 + v_1^2$ , by a computation of a suitable continued fraction we verify that

$$(6) \quad \text{for } 85 \leq c < 4 \cdot 10^{10}, z > 10 \quad \text{implies} \quad a > c^{z/(2\sqrt{3})}.$$

Similarly, from  $a^2 + b^2 = c^r$  and  $c = u^2 + v^2$  for some integers  $u, v$  which may be different from  $u_1, v_1$ , we obtain

$$(7) \quad \text{for } 85 \leq c < 4 \cdot 10^{10}, r > 10 \quad \text{implies} \quad b > c^{r/(2\sqrt{3})}.$$

This information is exploited in conjunction with the following remarks.

**Lemma 3.4.** *Assume both conditions (2) and (3) are fulfilled. Then:*

- a) *If for some  $\mu > 0$  one has  $a \geq c^{z/\mu}$  then  $2z < \mu r$ .*
- b) *If for some  $\lambda > 0$  one has  $b \geq c^{r/\lambda}$  then  $yr < \lambda z$ .*
- c) *If  $\mu_1 > 0, \mu_2 > 0$  are such that  $\mu_1 \mu_2 \leq 2y$  then  $a \geq c^{z/\mu_1}$  and  $b \geq c^{r/\mu_2}$  cannot simultaneously hold. In particular,*

$$a < c^{z/(2\sqrt{3})} \quad \text{or} \quad b < c^{r/(2\sqrt{3})}.$$

*Proof.* a) From  $a \geq c^{z/\mu}$  and  $a^2 + b^2 = c^r$  it readily follows that  $c^{2z/\mu} < c^r$ .

b) If  $b \geq c^{r/\lambda}$  then  $c^{yr/\lambda} \leq b^y < c^z$ .

c) The first assertion follows directly from a) and b). The last part follows from this because  $y \geq 6$  by Lemma 3.1.  $\square$

Using this lemma, we could rule out the small values of  $r$  and  $z$  (precisely, those with  $2 < r < z < 10$ ) and prove that  $c$  cannot be comparatively small. After around two weeks of computation we could verify the following result.

**Lemma 3.5.** *Assume the system of equations (2) has solutions satisfying (3). Then  $c > 4 \cdot 10^{10}$ .*

**3.2. A second application of linear forms.** From now on we consider  $c > 4 \cdot 10^{10}$  without further explicitly mentioning it. In order to improve the bounds on  $r$  and  $z$  obtained in the previous subsection, we apply a very recent result of Laurent [11].

**Lemma 3.6.** *Consider a nonzero linear form*

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2,$$

*where  $\alpha_1$  and  $\alpha_2$  are nonzero algebraic numbers, both different from 1, and  $b_1$  and  $b_2$  are positive integers. Put*

$$D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}] / [\mathbf{R}(\alpha_1, \alpha_2) : \mathbf{R}].$$

*Let  $K$  be an integer  $\geq 3$ ,  $L$  an integer  $\geq 2$ ,  $R_1, R_2, S_1, S_2$  positive integers. Let  $\rho$  and  $\mu$  be real numbers with  $\rho > 1$  and  $1/3 \leq \mu \leq 1$ . Put  $R = R_1 + R_2 - 1$ ,  $S = S_1 + S_2 - 1$ ,  $N = KL$ ,*

$$g = \frac{1}{4} - \frac{N}{12RS}, \quad \sigma = \frac{1 + 2\mu - \mu^2}{2},$$

$$b = \frac{((R-1)b_2 + (S-1)b_1)}{2} \left( \prod_{k=1}^{K-1} k! \right)^{-2/(K^2-K)}.$$

Let  $a_1, a_2$  be positive real numbers such that

$$a_i \geq \rho |\log \alpha_i| - \log |\alpha_i| + 2D h(\alpha_i),$$

for  $i = 1, 2$ . Suppose that:

$$(I) \quad \text{Card}\{\alpha_1^r \alpha_2^s; 0 \leq r < R_1, 0 \leq s < S_1\} \geq L,$$

$$(II) \quad \text{Card}\{rb_2 + sb_1; 0 \leq r < R_2, 0 \leq s < S_2\} > (K - 1)L$$

and also that

(III)

$$K(\sigma L - 1) \log \rho - (D + 1) \log N - D(K - 1) \log b - gL(Ra_1 + Sa_2) > c(N),$$

where

$$c(N) = \frac{2}{N} \log(N! N^{-N+1}(e^N + (e - 1)^N)).$$

Then

$$|\Lambda'| \geq \rho^{-\mu KL},$$

where

$$\Lambda' = \Lambda \cdot \max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

In our case  $\alpha_1 = \alpha$  (up to a power of  $\sqrt{-1}$ ),  $\alpha_2 = \sqrt{-1}$ ,  $b_1 = r$  or  $z$ , and  $b_2 = k$ . (To work with the linear form associated to the relation  $a^2 + b^y = c^z$  we only need to take above  $b_1 = z$  instead of  $b_1 = r$ .) For  $c = 4 \cdot 10^{10} + 5$ , we choose the parameters as follows:  $L = 8$ ,  $\rho = 7.7$ ,  $\mu = 0.56$ ,  $K = \lceil mL a_1 a_2 \rceil$ ,  $R_1 = 4$ ,  $S_1 = 2$ ,  $R_2 = \lceil \sqrt{m} L a_2 \rceil$ , and  $S_2 = \lceil (1 + (K - 1)L)/R_2 \rceil$ , where  $m = 0.1166$ , and we get

$$|\Lambda| > c^{-0.2113r} \quad \text{for } r \geq 771,$$

which implies

$$a > c^{z/(2\sqrt{3})} \quad \text{and } b > c^{r/(2\sqrt{3})} \quad \text{for } r \geq 771.$$

Taking into account Lemma 3.4b), one concludes that  $r \leq 769$ .

Now, combining Lemma 3.4a) and Lemma 3.4c), we see that if the system has a solution then  $r \leq 769$  and  $z \leq 983$ . The detailed argument is the following: we apply Laurent's result twice, a first computation for  $z \geq 985$  gives an upper bound for  $\mu_2$  which combined with part a) implies  $r \geq 641$ ; then a second computation for  $r \geq 641$  gives an upper bound for  $\mu_1$  with  $\mu_1 \mu_2 < 12$ , and part c) leads to a contradiction. Thus  $z \leq 983$ . Moreover, it is easy to check that the greater  $c$ , the better our estimates, so that the conclusion holds for all  $c > 4 \cdot 10^{10}$ .

Arguing in the same way, we can establish tighter bounds for  $r$  and  $z$ , provided a higher lower bound on  $y$  is available.

**Lemma 3.7.** *If the Diophantine system (2) has solutions satisfying (3) then in all cases*

$$r \leq 769 \quad \text{and} \quad z \leq 983.$$



Moreover

$$y \geq 10 \implies r \leq 539 \quad \text{and} \quad z \leq 759,$$

$$y \geq 14 \implies r \leq 461 \quad \text{and} \quad z \leq 681,$$

$$y \geq 18 \implies r \leq 419 \quad \text{and} \quad z \leq 647,$$

$$y \geq 22 \implies r \leq 395 \quad \text{and} \quad z \leq 627,$$

and

$$y \geq 602 \implies r \leq 263 \quad \text{and} \quad z \leq 539.$$

**3.3. Elementary lower bounds on  $b$ .** Let  $\varepsilon = u + iv = |\varepsilon|e^{i\xi}$ , where  $c = u^2 + v^2$ , with  $u$  even, and  $|\varepsilon| = \sqrt{c}$ . Then  $\tan \xi = v/u$  and

$$b = \frac{1}{2}|\varepsilon^r - \bar{\varepsilon}^r| = c^{r/2}|\sin(r\xi)|,$$

with  $r \geq 3$ . In this subsection we derive lower bounds on  $b$  from lower bounds on  $v$ .

**Lemma 3.8.** *With the above notation, one has*

$$r \leq \pi/\xi - 1 \implies b \geq v c^{(r-1)/2} \geq v c.$$

*In particular, one gets  $b \geq v c^{(r-1)/2}$  whenever  $r \leq u\pi/v - 1$ .*

*Proof.* The hypothesis  $3 \leq r \leq \pi/\xi - 1$  implies that  $\xi \leq \pi/4$  and  $3\xi \leq r\theta \leq \pi - \xi$ , and therefore  $\sin(r\xi) \geq \sin \xi = v/\sqrt{c}$ .

For the last part, note that the hypothesis  $r \leq \pi/\xi - 1$  holds if  $r \leq u\pi/v - 1$  because  $0 < \xi < \tan \xi = v/u$ .  $\square$

Despite its innocuous appearance, lemma just proved plays an important role in subsequent reasonings. Thus,  $v \leq 925$  implies  $u/v > 216$  (recall our standing hypothesis  $c > 4 \cdot 10^{10}$ ) and then the previous lemma gives  $b \geq v c^{(r-1)/2} \geq c^{r/3}$  (since  $3 \leq r$ ). Having in view Lemma 2.1, it follows that one always has  $b \geq v$ . Therefore,  $b \geq 925$ .

More importantly, with the help of Lemma 3.8 we shall derive a strikingly sharp bound for the quotient  $y/z$ .

**Lemma 3.9.** *We always have*

$$b \geq \frac{\pi}{r+1} \left( 1 + \frac{\pi^2}{(r+1)^2} \right)^{-1/2} \sqrt{c}$$

and

$$y < z \left( 2 + \frac{9.982}{\log b} \right).$$

Moreover, if  $y > 600$  then

$$y < z \left( 2 + \frac{8.863}{\log b} \right).$$

*In particular, it always holds*

$$y < 1778.$$

*Proof.* From our previous study we know that

$$(8) \quad b \geq \begin{cases} c^{(r-1)/2} \geq c & \text{if } (r+1)v < \pi u, \\ v & \text{otherwise.} \end{cases}$$

Notice that  $(r+1)v \geq \pi u$  implies

$$c \leq \left(1 + \frac{(r+1)^2}{\pi^2}\right) v^2,$$

so that in all cases  $b$  satisfies

$$b \geq \frac{\pi}{r+1} \left(1 + \frac{\pi^2}{(r+1)^2}\right)^{-1/2} \sqrt{c}.$$

Now we consider the upper bounds for  $y$ . From 8 we get

$$c \leq \begin{cases} b & \text{if } (r+1)v < \pi u, \\ v^2 + \frac{(r+1)^2}{\pi^2} v^2 & \text{otherwise.} \end{cases}$$

Hence,

$$c \leq \left(1 + \frac{(r+1)^2}{\pi^2}\right) b^2.$$

Using the inequality  $b^y < c^z$  one gets

$$(9) \quad y < z \left(2 + \frac{\log(1 + (r+1)^2/\pi^2)}{\log b}\right).$$

If  $y \leq 10$  the second estimate of the lemma is trivial, hence we suppose  $y \geq 14$ . Then  $r < 462$ , and after a simple computation we get the stated inequality.

When  $y$  is greater than 600 we know from Lemma 3.7 that  $r \leq 263$  and the third estimate follows. The last one is deduced by using the fact that  $b$  is at least 925 and  $z$  is less than 540 whenever  $y$  is at least 600.  $\square$

**3.4. Estimates on  $a$ .** Our next goal is to obtain some estimates on  $a$ . Put  $b = c^\lambda$ . The information we have up to know allows us to conclude that  $1/2 - (\log 1800)/\log c < \lambda < r/2$ . We use this knowledge to prove the following.

**Lemma 3.10.** *Put  $a = b^{(y-\lambda')/2}$ . Then  $\lambda'$  is positive and satisfies*

$$\lambda' > \frac{\log c}{\log b} (z - r - 10^{-22}) > \frac{2}{r} (2 - 10^{-22}).$$

*Proof.* From the second equation in (2) we get

$$c^z (1 - c^{-z+r}) < b^y < c^z$$

and since  $z \geq r + 2$  this implies

$$z \log c + \log(1 - c^{-2}) < y \log b < z \log c,$$

while the first equation in (2) and the definition of  $\lambda'$  imply

$$(y - \lambda') \log b < r \log c.$$

Hence

$$z \log c - 10^{-21} < r \log c + \lambda' \log b,$$

and therefore

$$2 \leq z - r < 10^{-22} + \lambda' \frac{\log b}{\log c} 10^{-22} + \lambda' \lambda,$$

by the definition of  $\lambda$ . In other words

$$\lambda' > \frac{1}{\lambda} (z - r - 10^{-22}),$$

and in particular

$$\lambda' > \frac{2}{r} (2 - 10^{-22}) > 0.$$

□

#### 4. MAIN RESULTS

Recall the result of Corollary 3.2: we have seen that  $c = u^2 + v^2 = u_1^2 + v_1^2$  for some positive integers, with  $u, u_1$  even and  $v, v_1$  odd, and that

$$a = \frac{1}{2} |\varepsilon^r + \bar{\varepsilon}^r| = \frac{1}{2} |\varepsilon_1^z + \bar{\varepsilon}_1^z|, \quad b = \frac{1}{2} |\varepsilon^r - \bar{\varepsilon}^r|, \quad b^{y/2} = \frac{1}{2} |\varepsilon_1^z - \bar{\varepsilon}_1^z|,$$

where  $\varepsilon = u + v\sqrt{-1}$  and  $\varepsilon_1 = u_1 + v_1\sqrt{-1}$ . It follows that, up to a sign,  $a$ ,  $b$  and  $b^{y/2}$  are values of binary linear recursive sequences. If  $(u, v) = (u_1, v_1)$  then the term  $b^{y/2}$  has no primitive divisors, so that on checking tables of binary Lucas sequences having terms without primitive divisors given in [3] and [1] we recover Cao's result [4] mentioned in Introduction.

**Theorem 4.1.** *If  $c$  is a prime power then the system (2) has no solutions subject to restrictions from (3).*

Now we are in a position to prove that the conjecture holds perhaps with the exception of finitely many pairs  $(c, r)$ .

Subtracting the two equations from (2) results in the Diophantine equation

$$(10) \quad b^y - b^2 = c^z - c^r.$$

Since  $6 \leq y$  and  $5 \leq z$ , for fixed exponents  $(y, r, z)$  one gets an algebraic curve of positive genus. The absolute irreducibility and the genus of the curve defined by Eq. (10) are given by a theorem of Davenport, Lewis and Schinzel [9].

**Lemma 4.2.** *Let  $f(X)$  and  $g(Y)$  be polynomials with integral coefficients of degree  $n > 1$  and respectively  $m > 1$ . Let  $D(\lambda) = \text{disc}(f(X) + \lambda)$  and  $E(\lambda) = \text{disc}(g(Y) + \lambda)$ . Suppose there are at least  $n/2$  distinct roots of  $D(\lambda) = 0$  for which  $E(\lambda) \neq 0$ . Then  $f(X) - g(Y)$  is irreducible over the complex field. Further, the genus of the curve  $f(x) - g(y) = 0$  is positive except possibly when  $m = 2$  or  $m = n = 3$ . Apart from these possible exceptions, the equation  $f(x) - g(y) = 0$  has at most finitely many integral solutions.*

Stickelberger's formula [21] (cf. [22]) for the discriminant of a trinomial gives

$$\begin{aligned} \text{disc}(b^y - b^2 + \lambda) &= -\lambda (y^{y/2} \lambda^{y/2-1} - 2(y-2)^{y/2-1})^2, \\ \text{disc}(c^z - c^r + \lambda) &= (-1)^{z(z-1)/2} \lambda^{r-1} (z^z \lambda^{z-r} - (z-r)^{z-r} r^r), \end{aligned}$$

so that the last quoted result applies.

Combining these classical facts with some of our results in the previous sections, we obtain the main result of the paper.

**Theorem 4.3.** *If the Diophantine equation  $X^x + Y^y = Z^z$  has a solution with  $X = a \equiv 2 \pmod{4}$ ,  $Y = b \equiv 3 \pmod{4}$ ,  $Z = c$ ,  $x = 2$ ,  $y = 2$  and  $z = r$  odd, where  $\gcd(a, b) = 1$ , then this is the only solution in positive integers, with the possible exception of finitely many values  $(c, r)$ .*

*Proof.* For each fixed pair of odd numbers  $(r, z)$ ,  $1 < r < z$ , any solution to the system (2) subject to (3) corresponds to an integer point on a curve (10) of positive genus. By Siegel's seminal paper [19], such an equation has only finitely many integral solutions. According to Lemma 3.7, in any compatible system (2) one has  $r < 770$  and  $z \leq 983$ . Moreover,  $y$  is bounded from above by 1800 (see Lemma 3.9). Therefore, a compatible system (2) gives rise to finitely many nonrational plane curves, each of which can have only finitely many integer points.  $\square$

The case when  $c$  is the successor of a perfect square has received a lot of attention by people working on Terai's conjecture (cf. [6] and the references therein). Our next result improves on all published results on this case.

**Theorem 4.4.** *If in the representation for  $c$  derived from Lemma 3.2 one has  $v = 1$ , then the system (2) has no solutions subject to restrictions from (3).*

*Proof.* We argue by reduction to absurd. Assume that  $c = u^2 + 1$ , and consequently  $b = \pm \sum_{j=0}^{(r-1)/2} c_{r,j} c^j (-4)^{(r-1)/2-j}$ . Suppose that  $(x, y, z)$  is a solution to the simultaneous equations (2) satisfying all the conditions from (3). From Lemma 3.2 applied for  $n = r$  we know that  $a + ib = \eta_1(u + \eta_2 i)^r$  with  $\eta_1, \eta_2 \in \{\pm 1\}$ , thus

$$a \equiv \pm r u \pmod{u^3}, \quad b \equiv \pm \left(1 - \binom{r}{2} u^2\right) \pmod{u^4},$$

and it follows that

$$c^z = a^2 + b^y \equiv r^2 u^2 + \left(1 - \frac{1}{2} r(r-1) y u^2\right) \equiv 1 + z u^2 \pmod{u^4},$$

that is,  $\frac{1}{2} r(r-1) y + z \equiv r^2 \pmod{u^2}$ . On noting that the left-hand side of this relation is greater than the right-hand side (because  $y \geq 6$ ), one obtains the first inequality from the chain

$$(11) \quad u^2 + r^2 \leq \frac{1}{2} r(r-1) y + z < \frac{1}{2} r^2 y.$$

The second inequality holds since  $z < ry/2$ . Indeed,  $c^{ry/2} = (a^2 + b^2)^{y/2} > a^2 + b^y = c^z$ . Since in this case  $u^2 \geq 4 \cdot 10^{10}$ , Eq. (11) readily contradicts the bounds  $r < 770$  and  $y < 1800$  already obtained.  $\square$

We are now in a position to prove Terai's conjecture when  $b$  is a prime power. The proof relies on the observation that  $b$  is of the form  $\pm vU_r$ , where

$$U_r = U_r(\alpha, \beta) = \frac{\alpha^r - \beta^r}{\alpha - \beta}$$

is the  $r$ th Lucas number associated to the pair  $(\alpha, \beta) = (u + v\sqrt{-1}, u - v\sqrt{-1})$ . In a subsequent proof we shall use the fact that  $a = \pm u\tilde{U}_r$ , with

$$\tilde{U}_r = \tilde{U}_r(\tilde{\alpha}, \tilde{\beta}) = \frac{\tilde{\alpha}^r - \tilde{\beta}^r}{\tilde{\alpha} - \tilde{\beta}}$$

the  $r$ th Lehmer number associated to the pair  $(\tilde{\alpha}, \tilde{\beta}) = (u + v\sqrt{-1}, -u + v\sqrt{-1})$ . Recall that a prime divisor of  $U_r$ , respectively  $\tilde{U}_r$ , is called primitive if it does not divide

$$(12) \quad (\alpha - \beta)^2 U_1 \cdots U_{r-1} = -4v^2 U_1 \cdots U_{r-1},$$

respectively

$$(13) \quad (\tilde{\alpha}^2 - \tilde{\beta}^2)^2 \tilde{U}_1 \cdots \tilde{U}_{r-1} = -16u^2 v^2 \tilde{U}_1 \cdots \tilde{U}_{r-1}.$$

Bilu, Hanrot and Voutier [3] showed that for  $n > 30$ , every  $n$ th Lucas and Lehmer number has a primitive divisor. Moreover, they and Abouzaid [1] have given the complete list of  $n$  and  $(\alpha, \beta)$ , respectively  $(\tilde{\alpha}, \tilde{\beta})$ , for which  $U_r(\alpha, \beta)$  or  $\tilde{U}_r(\tilde{\alpha}, \tilde{\beta})$  does not have a primitive divisor.

**Theorem 4.5.** *If  $b$  is a prime power then the system (2) has no solutions subject to restrictions from (3).*

*Proof.* Let  $p$  be an odd prime and  $s$  a positive integer such that  $b = p^s$ . Having in view the result just proved, we conclude that if the system (2) has a solution satisfying (3), then  $p$  divides  $v$ . Therefore, either  $U_r = 1$  or its only prime divisor  $p$  is not primitive (see Eq. (12)). Checking the relevant tables from [3] and [1], one finds that one necessarily has  $r = 3, 5, 7$  or  $13$ . Moreover, when  $r = 3$ ,  $c$  would result even, in contradiction to (3). For  $r = 5$ , all the candidates for  $(\alpha, \beta)$  do not yield an integer value for  $v$ , while for  $r = 7$  or  $13$  the resulting value for  $u$  is not integer.  $\square$

To the best of our knowledge, the literature contains nothing of the kind of our next result.

**Theorem 4.6.** *If  $a$  is a prime power then the system (2) has no solutions subject to restrictions from (3).*

*Proof.* As explained before, we use the equality  $a = \pm u\tilde{U}_r$ , with  $u \geq 2$ . We proceed as in the previous proof, reasoning about the Lehmer pair  $(\tilde{\alpha}, \tilde{\beta})$  instead of the Lucas pair  $(\alpha, \beta)$ . Since the differences are insignificant, the details can be safely left to the reader.  $\square$

## 5. FURTHER RESULTS

In subsequent reasonings we shall need to know that  $v_1 \neq 1$ . This fact follows from the following.

**Lemma 5.1.** *With the notation of the previous section we have the two following results:*

$$\min\{u_1/v_1, v_1/u_1\} \leq 0.01 \implies r \leq 659 \quad \text{and} \quad z \leq 845$$

and

$$\min\{u_1/v_1, v_1/u_1\} \geq 0.001856.$$

In particular,

$$\min\{u_1, v_1\} \geq 372.$$

*Proof.* With the notation  $\varepsilon_1 = u_1 + iv_1 = |\varepsilon|e^{i\xi_1}$  and  $\xi'_1 = \pi/2 - \xi_1$ , the corresponding linear form is

$$\Lambda = z(2i\xi_1) - k(i\pi/2) = z(-2i\xi'_1) - k'(i\pi/2)$$

and when  $\xi_1$  or  $\xi'_1$  is small we can get much better estimates in the application of Laurent's lower bound. Technically: we can take a much larger radius of interpolation and we obtain the above upper bounds for  $r$  and  $z$ .

The proof of the second result is elementary. We have  $a = c^{z/2}|\cos(z\xi_1)| = c^{z/2}|\sin(z\xi'_1)|$ . Hence the condition  $(z+1)\xi_1 < \pi/2$  implies

$$|\cos(z\xi_1)| \geq \cos(\pi/2 - \xi_1) = \sin \xi_1 = \frac{v_1}{\sqrt{c}},$$

where  $0 < \xi_1 < \tan \xi_1 = v_1/u_1$ . It follows that

$$\frac{v_1}{u_1} < \frac{\pi}{2 \times 846} = 0.001856733\dots \implies a > c^{z/2-1} \geq c^{r/2}.$$

Since  $a^2 + b^2 = c^r$ , this is a contradiction that proves the lower bound  $v_1/u_1 \geq 0.001856$ . A similar reasoning leads to the inequality  $u_1/v_1 \geq 0.001856$ .

Now, since  $u_1^2 + v_1^2 > 4 \cdot 10^{10}$ , a simple computation gives  $\min\{u_1, v_1\} \geq 372$ .  $\square$

In a similar way we can prove partially analogous results concerning the pair  $(u, v)$ .

**Lemma 5.2.** *The following implication holds*

$$\min\{u/v, v/u\} \leq 0.01 \implies r \leq 553 \quad \text{and} \quad z \leq 705.$$

*If the Diophantine system (2) has solutions satisfying (3) with*

$$b \geq c^{(r-1)/2}$$

*(which is true if  $v(r+1) < \pi u$ ) then*

$$y \geq 6 \implies r \leq 101 \quad \text{and} \quad z \leq 299.$$

*Moreover, again under the hypothesis  $b \geq c^{(r-1)/2}$ ,*

$$y \geq 10 \implies r \leq 47 \quad \text{and} \quad z \leq 227, \quad y \geq 14 \implies r \leq 31 \quad \text{and} \quad z \leq 209,$$

$y \geq 18 \implies r \leq 23$  and  $z \leq 197$ ,  $y \geq 22 \implies r \leq 19$  and  $z \leq 189$ ,  
 $y \geq 30 \implies r \leq 13$  and  $z \leq 185$ ,  $y \geq 50 \implies r \leq 7$  and  $z \leq 161$ ,  
 $y \geq 70 \implies r \leq 5$  and  $z \leq 155$ ,  $y \geq 98 \implies r \leq 3$  and  $z \leq 147$ ,  
 and there is no solution for  $y \geq 142$ .

We add some other estimates related to  $b$ .

**Lemma 5.3.** *If the Diophantine system (2) has solutions satisfying (3) then*

$$ry/2 = z + 2t, \quad \text{with } t \geq 1,$$

and

$$b < c^{\frac{r}{2} - \frac{2}{y}}.$$

Moreover, if

$$b \geq (1 + 10^{-20}) c^{\frac{r}{2} - \frac{4}{y}}$$

then

$$ry/2 = z + 2.$$

When it holds  $ry/2 = z + 2$  then

$$\begin{aligned}
 y \geq 6 &\implies r \leq 101, & y \geq 10 &\implies r \leq 47, & y \geq 14 &\implies r \leq 29, \\
 y \geq 18 &\implies r \leq 19, & y \geq 22 &\implies r \leq 17, & y \geq 26 &\implies r \leq 13, \\
 y \geq 30 &\implies r \leq 11, & y \geq 38 &\implies r \leq 9, & y \geq 42 &\implies r \leq 7, \\
 y \geq 50 &\implies r \leq 5, & y \geq 66 &\implies r = 3,
 \end{aligned}$$

and there is no solution for  $y \geq 102$ .

*Proof.* We give a proof just for the first two assertions. From the relations

$$(a^2 + b^2)^{y/2} > a^2 + b^y = c^z$$

we deduce  $ry/2 > z$  and the first assertion follows since  $ry/2$  and  $z$  are both odd.

If  $b \geq (1 + 10^{-20})c^{\frac{r}{2} - \frac{4}{y}}$  then, since  $b^y > (1 - 10^{-21})c^z$ , we see that  $z > ry/2 - 4$  and the relation  $z = ry/2 - 2$  follows from the first assertion.

The remaining estimates result from computation with the help of lower bounds on linear forms.  $\square$

It is very likely that actually there are no solutions to (2) under the conditions stated in (3). This is the case under the hypothesis of the next result.

**Theorem 5.4.** *The system (2) has no solutions  $(r, y, z)$  subject to restrictions (3) in which  $z$  is divisible by 3 and  $y \neq 6, 10, 14, 18, 30, 42, 50, 54, 62, 70, 90, 98, 126, 150, 162, 186, 210, 250, 270, 294, 310, 350, 378, 434, 450, 486, 490, 558, 630$ .*

*Proof.* I. Chen [7] very recently proved that for any prime satisfying the restrictions  $7 < p < 10^7$  and  $p \neq 31$  there are no coprime integers  $A, B, C$  satisfying

$$A^2 + B^{2p} = C^3.$$

This confirms Terai's conjecture in case  $z$  is multiple of 3 and  $y$  has a prime divisor  $p > 7$ ,  $p \neq 31$ . The only values of the  $y$ -component in a solution of Eqs. (2)–(3) not covered by Chen's result are listed having in view Proposition 5.5.  $\square$

The following remarks are helpful when trying to further reduce the number of candidate pairs  $(y, z)$ .

**Remark 1.** When  $r$  divides  $z$ , we may remove the multiples of 3 from this list because Mignotte and Pethő [16] have proved that if there are points with both coordinates greater than 1 on the curve  $X^m - X = Y^n - Y$ , then  $\gcd(m, n) = 1$ .

**Remark 2.** A deep result of Darmon and Méréel [8], according to which the equation  $X^n + Y^n = Z^2$  has no solutions in nonzero integers when  $n \geq 4$ , implies that  $\gcd(y, z) \leq 3$  always holds.

Our last result is a bit surprising because it shows that the hypothesis  $b > a$  from the main results of [25], [5] (see  $(\alpha)$  and respectively  $(\gamma)$  in Introduction) and [13] is never fulfilled (the reader is warned that in Le's paper  $b$  denotes the unique even number among  $a$  and  $b$ ).

**Proposition 5.5.** *If system (2) has solutions subject to restrictions from (3) then*

$$a > 4.608b, \quad c > 3^{y-10} \quad \text{and} \quad y \leq 2z + 4.$$

Moreover

$$y \leq 2z - 4 \quad \text{for} \quad y \geq 34$$

and

$$y \geq 602 \implies r \leq 149 \quad \text{and} \quad z \leq 319.$$

In particular,

$$y \leq 634.$$

*Proof.* When  $y \leq 10$  one has  $c > 10^y$  because  $c > 4 \cdot 10^{10}$ . For the same reason,  $c > 2.2^y$  when  $y$  is between 14 and 30. It is much harder to obtain similar inequalities for higher values of  $y$ . We now prove that it always holds  $c > 2.1716^y$ .

As seen above,  $v_1$  has a prime divisor  $p$ . Recall that in Lagrange's formula given in Lemma 2.1 the coefficients for  $n$  odd are

$$c_{n,j} = \frac{(n-j-1)!n}{(n-2j)!j!},$$



where  $0 \leq j \leq (n-1)/2$ , and the quotient  $(n-j-1)!/j!$  is an integer. It follows that we have

$$v_p(c_{n,j}) \geq v_p(n) - v_p((n-2j)!) > v_p(n) - \frac{n-2j}{p-1} \geq v_p(n) - \frac{n-2j}{2}.$$

As  $p$  divides  $v_1$ , it does not divide  $c$  and therefore

$$\begin{aligned} v_p(c_{z,j}c(-4v_1^2)^{(z-1)/2-j}) &= v_p(c_{z,j}) + (z-1-2j)v_p(v_1) \\ &\geq v_p(z) - \frac{1}{2}(z-1-2j) + (z-1-2j)v_p(v_1) \\ &\geq v_p(z) + \frac{1}{2}(z-1-2j)v_p(v_1) > v_p(z) \end{aligned}$$

for  $0 \leq j < (z-1)/2$ .

Corollary 2.2 yields

$$y v_p(b) = 2(v_p(z) + v_p(v_1)).$$

Having in view the upper bounds for  $z$  given in Lemma 3.7, we see that for  $y \geq 34$  it holds  $v_3(z) \leq 5$ , and  $v_p(z) \leq 3$  for  $p \geq 5$ . Consequently, for  $p = 3$  one obtains

$$c > v_1^2 \geq 3^{y-10} \geq 3^{y(1-5/17)} > 2.1716^y.$$

For  $p \geq 5$  one has  $c > p^{y-6} \geq 5^{y-6} > 3^{y-10}$ , so that the claim that  $c > 2.1716^y$  is true for any solution of the system (2) satisfying conditions (3).

Denote provisionally  $\mu = b^2/c^r$ . Then  $c^z > b^y$  implies that

$$\mu^{y/2} < c^{z-ry/2} \leq c^{-2} < 2.1716^{-2y}.$$

Hence,

$$a = b \sqrt{\mu^{-1} - 1} > b \sqrt{2.1716^4 - 1} > 4.608 b.$$

Since  $a^2 < c^{z-2}$  and  $c > 10^{10}$  we have  $b^y > (1-10^{-20})c^z$ , and the inequality  $c > 2.1716^y$  implies

$$b > 2.171^z.$$

The inequalities relating  $y$  and  $z$  are proved in three steps. First, we show that we always have  $y \leq 2z+12$ . Next, we disprove the equalities  $y = 2z+8$  and  $y = 2z+12$  by combining information already available with some more computations. Similar arguments are employed to show that one can not have  $y = 2z+4$  for  $y \geq 34$ , while  $y \neq 2z$  follows from the result of Darmon and M erel mentioned in Remark 2. Here are the details.

The upper bound

$$y < z \left( 2 + \frac{\log(1 + (r+1)^2/\pi^2)}{\log b} \right),$$

combined with the lower bound  $b > 2.171^z$ , leads to

$$y < 2z + \frac{\log(1 + (r+1)^2/\pi^2)}{\log 2.171}.$$

The bound  $y \leq 2z + 12$  is trivially satisfied for  $y \leq 22$ , and for  $y \geq 22$  we have seen that  $r < 396$ , so that

$$y < 2z + \frac{\log(1 + 396^2/\pi^2)}{\log 2.171} < 2z + 12.5,$$

which implies  $y \leq 2z + 12$ . To show that the equality in this relation never holds, one argues similarly to the case  $y = 2z + 4$  for  $y \geq 34$  detailed below. Then one repeats the reasoning to show that  $y \neq 2z + 8$ , so that we always have

$$y \leq 2z + 4.$$

Suppose that  $y = 2z + 4$  for some  $y \geq 34$ . Theorem 5.4 implies  $y \geq 38$  and we verify by a computation with linear form estimates that

$$y \geq 38 \implies r \leq 239.$$

From the relation  $b^y < c^z$ , we get

$$y = 2z + 4 \implies b < c^{1/2-2/y}.$$

Besides we know that

$$b < c^{(r-1)/2} \implies u < (r+1)v/\pi \implies c < (1 + (r+1)^2/\pi^2)v^2.$$

Put  $b = vb'$ —then  $b'$  is a positive integer. The above facts imply

$$y = 2z + 4 \implies c < \left( \frac{1}{b'} \sqrt{1 + \frac{(r+1)^2}{\pi^2}} \right)^{y/2}.$$

Now we consider  $v_1$ . We have  $v_1 < u_1\pi/(2(z+1))$ , thus

$$v_1^2 < \left( 1 + \frac{\pi^2}{4(z+1)^2} \right)^{-1} c.$$

Moreover, we can write

$$v_1 = w_1^{y/2}/w_0, \quad \text{where } w_0 \mid \gcd(z, v_1^2)$$

and  $b = w_1 b''$ , where  $b''$  is a positive integer.

If  $b' = 1$ , a short computer verification shows that  $w_1 < 9$  for  $y \geq 38$ . Since  $w_1$  is an odd integer greater than 1, one has  $w_1 \in \{3, 5, 7\}$ . But we know that  $b$  is not a power of a prime, hence  $b' \geq 3$ , with  $b' \geq 5$  when  $w_1 = 3$ . Using now  $b' \geq 3$  another computer verification leads to  $w_1 \in \{3, 5\}$  and  $w_1 = 3$  for  $y \geq 102$ .

Now we apply again Laurent's result but with the much better lower bound  $c > 3^{y-10}$  (better for  $y \geq 34$  than  $c > 4 \cdot 10^{10}$ ) and we get for example

$$\begin{aligned} y \geq 102 &\implies r \leq 181 \quad \text{and} \quad z \leq 373, \\ y \geq 302 &\implies r \leq 157 \quad \text{and} \quad z \leq 329, \\ y \geq 602 &\implies r \leq 149 \quad \text{and} \quad z \leq 319. \end{aligned}$$

Comparing the previous estimates we conclude that we always have  $y \leq 634$ .

□

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