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A kinetic scheme for pressurized flows in non uniform pipes

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Abstract

The aim of this paper is to present a kinetic numerical scheme for the computations of transient pressurised flows in closed water pipes with variable sections. Firstly, we detail the derivation of the mathematical model in curvilinear coordinates under some hypothesis and we performe a formal asymptotic analysis. Then the obtained system is written as a conservative hyperbolic partial differential system of equations, and we recall how to obtain the corresponding kinetic formulation based on an upwinding of the source term due to the “pseudo topography” performed in a close manner described by Perthame and al. [2, 8, 3]. The validation is lastly performed in the case of a water hammer in a uniform pipe where we compare the numerical results provided by an industrial code used at EDF-CIH (France), which solves the Allievi equation (the commonly used equation for pressurised flows in pipes) by the method of characteristics, with those of the kinetic scheme. To validate the contracting or expanding case, we compare the presented technique to the equivalent pipe in the case of an immediate flow shut down in a frictionless cone-shaped.

Key words : Curvilinear transformation, asymptotic analysis, pressurized flows, kinetic scheme.

1 Introduction

The presented work in this paper is the second step in a more general project : the modelization of unsteady mixed flows in any kind of closed domain taking into account the cavitation problem and air entrapment. We are interested in flows occuring in closed pipes of variable sections, since some parts of the flow can be free surface and other parts are pressurized. The transition phenomenon occurs in many situation such as storm sewers, waste or supply pipes in hydroelectric installation. It can be induced by sudden change in the boundary conditions as failure pumping and to father more or less damage. During this process, the pressure can be reach

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severe values and cause the scrap of materials. Moreover, the excess pressure can be also induced by the geometry of the domain. Therefore, it is important to take into account the change of section, even if the change is small.

The classical Saint-Venant equations are commonly used to describe physical situations like free surface flows in open channel. They are also used in the study of mixed flows using the Preissman slot artefact (see for example [7, 10]). However, this technics does not take into account depressuration phenomenon which occurs during a waterhammer. We can also cite the Allievi equations which are commonly used to describe pressurized flows. Nonetheless, the non conservative form is not well adapted to a natural coupling with the Saint-Venant equations (see PFFS-model).

The model for the unsteady mixed water flows in closed water pipes and a finite volume discretization has been previously studied by the authors [5] and a kinetic formulation has been proposed in [6]. This paper tends to extend naturally the work in [6] in the case of closed pipes with variable sections.

We establish in section 2 the model for pressurized flows in curvilinear coordinates and recall some classical properties of this model. Rewriting the appearing source term into a *pseudo-altitude* term, we get a system closed to the presented by authors in [8]. Applying the generalized characteristic method we get the so-called kinetic formulation with *pseudo-reflection*. Then we present the main results and the properties of this kinetic formulation. The rest of this section is devoted to the construction of the kinetic scheme. The upwinding of the source term due to the pseudo topography is performed in a close manner described by Perthame and al. [8] using an energetic balance at microscopic level for the Shallow Water equations.

Finally, we present in section 4 a numerical validation of this study in the uniform case by the comparison between the resolution of this model and the resolution of the Allievi equation solved by the research code `belier` used at Center in Hydraulics Engineering of Electricité De France (EDF) [11] for the case of critical waterhammer tests. The validation in the non uniform pipes is performed in the case of an immediate flow shut down in a frictionless cone-shaped where the results are compared to the equivalent pipes method.

2 Formal Derivation of the model

2.1 The Euler system in curvilinear coordinates

The presented model is derived from the 3D Euler system of compressible flow written in curvilinear coordinates, then integrated over sections orthogonal to the mean flow axis where we have neglected the second and third equation of the conservation of the momentum. The 3D Euler system in the cartesian coordinates is written as follows

$$\partial_t \rho + \mathbf{div}(\rho \vec{U}) = 0 \quad (1)$$

$$\partial_t \vec{U} + \mathbf{div}(\rho \vec{U} \otimes \vec{U}) + \nabla p = F \quad (2)$$

where $(\vec{U}(t, x, y, z), \rho(t, x, y, z))$ denotes the velocity and the density of the equations, $p(t, x, y, z)$ is the scalar pressure and F the exterior strenght of gravity.

To see the local effect induced by the geometry due to the change of sections and/or slope effect we write the 3D compressible Euler system in the curvilinear

coordinates. To this end, let us introduce the curvilinear variable defined by $X = \int_{x_0}^x \sqrt{1 + (b'(\xi))^2} d\xi$ where $b(x)$ denotes the mean flow axis and x_0 an arbitrary abscissa. Let us also suppose that there is no variation in the direction \vec{j} , i.e. we have $y = Y$ and we denote by Z the altitude of any fluid particle $M(X)$ (attached to the mean flow axis) in the Serret-Frénét basis formed by $(M(X), T, N, B)$: T is the tangent vector, N the normal vector and B the binormal vector (see FIGURE. 1). Then we perform the following transformation $(x, y, z) \rightarrow \overline{OM}(x, y, z)$ by the following lemma (using the same notation in [4])

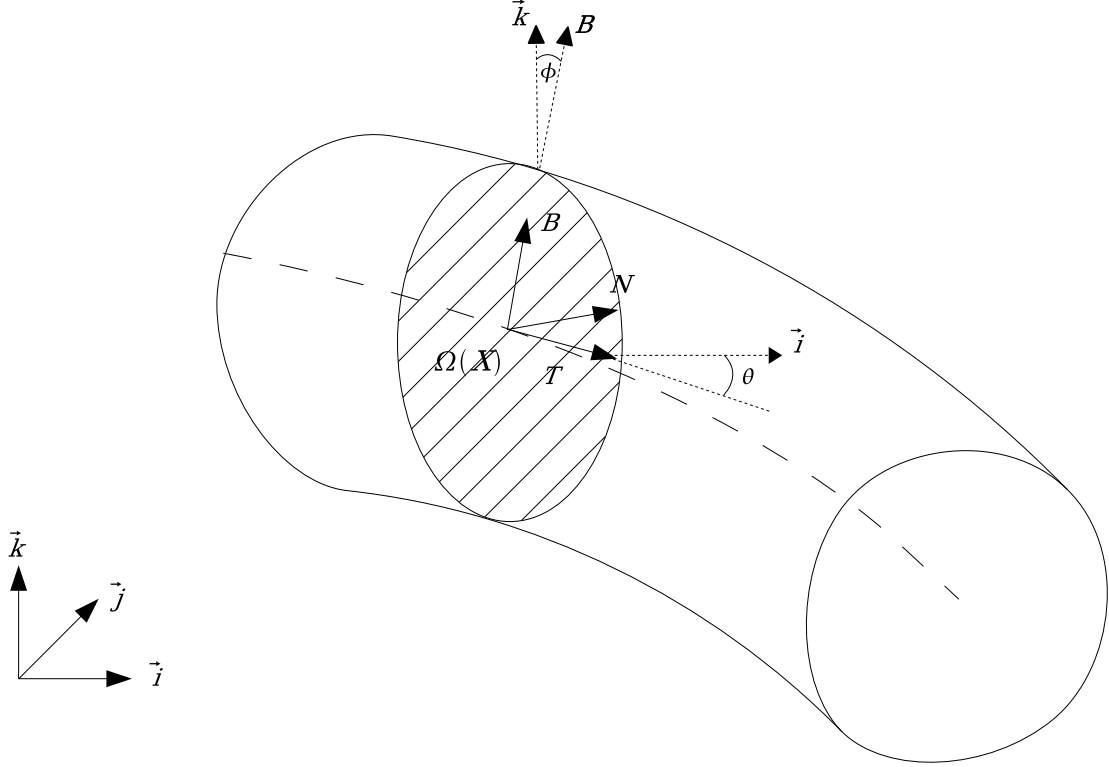


Figure 1: Geometric characteristics of the pipe

Lemma 2.1 Let $\vec{\xi} \mapsto \vec{Y}(\vec{\xi})$ and $\mathcal{A}^{-1} = \nabla_{\vec{\xi}} \vec{Y}$ the jacobian matrix of the transformation where J denotes its determinant.

Then, for any vector field $\vec{\Phi}$ one has,

$$J \nabla_{\vec{Y}} \cdot \vec{\Phi} = \nabla_{\vec{\xi}} \cdot (J \mathcal{A} \Phi)$$

In particular, for any scalar function f , one has

$$\nabla_{\vec{Y}} f = \mathcal{A}^t \nabla_{\vec{\xi}} f$$

For the proof, we refer to [4].

The velocity fluid is reoriented and denoted by $(U, V, W)^t$ in such a way that the flow will be orthogonal to the cross-section along the binormal axis B . Let R

be the matrix defined by $R = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$ then the velocity attached to

the new basis is given by $\begin{pmatrix} U \\ V \\ W \end{pmatrix} = R\vec{U}$.

Applying the lemma 2.1 to the mass conservation, we get

$$J(\partial_t \rho + \mathbf{div}(\rho \vec{U})) = 0$$

$$\iff$$

$$\partial_t(J\rho) + \partial_X(\rho U) + \partial_X(\rho J V) + \partial_Z(\rho J W) = 0 \quad (3)$$

where

$$J = \det \begin{pmatrix} \partial_X x - Z \partial_X \theta \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \partial_X b - Z \partial_X \theta \sin \theta & 0 & \cos \theta \end{pmatrix} \quad (4)$$

$$= \det \begin{pmatrix} (1 - Z \partial_X \theta) \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ (1 - Z \partial_X \theta) \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (5)$$

To get the unidirectionnal Saint-Venant like equations we neglect the second and third equation for the conservation of the momentum. Therefore, we only perform the curvilinear transformation for the first conservation equation. To this end, multiplying the Euler system (2) by $J \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix}$ and using the lemma (2.1), this yields

$$J \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix} \left(\partial_t \vec{U} + \mathbf{div}(\rho \vec{U} \otimes \vec{U}) + \nabla . P = -\rho \nabla(\vec{g} \cdot \vec{OM}) \right)$$

$$\iff$$

$$\partial_t(J\rho U) + \partial_X(\rho U^2) + \partial_Y(\rho JUV^2) + \partial_Z(\rho JUW) + \partial_X p = -\rho J g \sin \theta + \rho U W \partial_X \theta \quad (6)$$

Finally, in variables (X, Y, Z) the system reads

$$\begin{cases} \partial_t(J\rho) + \partial_X(\rho U) + \partial_X(\rho J V) + \partial_Z(\rho J W) & = & 0 \\ \partial_t(J\rho U) + \partial_X(\rho U^2) + \partial_Y(\rho JUV^2) + \partial_Z(\rho JUW) + \partial_X p & = & -\rho J g \sin \theta \\ & & + \rho U W \partial_X \theta \end{cases} \quad (7)$$

Remark 2.1 Notice that $\rho(X) = \partial_X \theta$ is the algebraic curvature and the function $J(X, Y, Z) = 1 - Z \partial_X \theta(X)$ is a function only depend on variables X, Z . Moreover, J is always positive since the curvature radius is greater than Z .

We recall that the main objectif is to obtain a formulation closed to the Saint-Venant equation in order to couple the models in a natural way (in a closed manner described in [5]). To this end, let us introduce a small parameter $\epsilon = H/L$ where H and L are two characteristics dimensions along \vec{k} and \vec{i} axis respectively. We suppose that the characteristic dimension along the \vec{j} axis is the same as \vec{k} in order to obtain a unidirectionnal model. We introduce some characteristics dimensions $T, P, \bar{U}, \bar{V}, \bar{W}$ for time, pressure and velocity repectively and the dimensionless quantities $\tilde{U} = U/\bar{U}, \tilde{V} = \epsilon V/\bar{U}, \tilde{W} = \epsilon W/\bar{U}, \tilde{X} = X/L, \tilde{Y} = Y/H, \tilde{Z} = Z/H, \tilde{p} = p/P, \tilde{\theta} = \theta, \tilde{\rho} = \rho$. In the sequel, we write $P = \bar{U}^2$ and $L = T\bar{U}$ (i.e. we consider only laminar flow).

Under these hypothesis $J(X, Y, Z) = \tilde{J}(\tilde{X}, \tilde{Y}, \tilde{Z}) = 1 - \epsilon \tilde{Z} \partial_{\tilde{X}} \theta$. So, the rescaled system (7) is given by

$$\left\{ \begin{array}{l} \partial_{\tilde{t}}(\tilde{J}\tilde{\rho}) + \partial_{\tilde{X}}(\tilde{\rho}\tilde{U}) + \partial_{\tilde{Y}}(\tilde{J}\tilde{\rho}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{J}\tilde{\rho}\tilde{W}) = 0 \\ \partial_{\tilde{t}}(\tilde{J}\tilde{U}\tilde{\rho}) + \partial_{\tilde{X}}(\tilde{\rho}\tilde{U}^2) + \partial_{\tilde{Y}}(\tilde{J}\tilde{\rho}\tilde{U}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{J}\tilde{\rho}\tilde{U}\tilde{W}) + \partial_{\tilde{X}}\tilde{p} = \epsilon\tilde{\rho}\tilde{U}\tilde{W}\tilde{\rho}(\tilde{X}) \\ \qquad \qquad \qquad - \tilde{\rho}\frac{\sin\tilde{\theta}}{F_{r,L}^2} \\ \qquad \qquad \qquad - \frac{\tilde{Z}\partial_{\tilde{X}}(\cos\tilde{\theta})}{F_{r,H}^2} \end{array} \right. \quad (8)$$

with $F_{r,M} = \frac{\bar{U}}{\sqrt{gM}}$ the Froude number along the \vec{i} axis and the \vec{k} or \vec{j} axis where M is any generic variable.

Formally, when ϵ tends to 0 the system converges to

$$\left\{ \begin{array}{l} \partial_{\tilde{t}}(\tilde{\rho}_\epsilon) + \partial_{\tilde{X}}(\tilde{\rho}_\epsilon\tilde{U}_\epsilon) + \partial_{\tilde{Y}}(\tilde{\rho}_\epsilon\tilde{V}_\epsilon) + \partial_{\tilde{Z}}(\tilde{\rho}_\epsilon\tilde{W}_\epsilon) = 0 \\ \partial_{\tilde{t}}(\tilde{U}_\epsilon\tilde{\rho}_\epsilon) + \partial_{\tilde{X}}(\tilde{\rho}_\epsilon\tilde{U}_\epsilon^2) + \partial_{\tilde{Y}}(\tilde{\rho}_\epsilon\tilde{U}_\epsilon\tilde{V}_\epsilon) + \partial_{\tilde{Z}}(\tilde{\rho}_\epsilon\tilde{U}_\epsilon\tilde{W}_\epsilon) + \partial_{\tilde{X}}\tilde{p}_\epsilon = -\tilde{\rho}\frac{\sin\tilde{\theta}}{F_{r,L}^2} \\ \qquad \qquad \qquad - \frac{\tilde{Z}\partial_{\tilde{X}}(\cos\tilde{\theta})}{F_{r,H}^2} \end{array} \right. \quad (9)$$

Finally, the system in variables (X, Y, Z) reads

$$\left\{ \begin{array}{l} \partial_t(\rho) + \partial_X(\rho U) + \partial_Y(\rho V) + \partial_Z(\rho W) = 0 \\ \partial_t(\tilde{U}\rho) + \partial_X(\rho U^2) + \partial_Y(\rho UV) + \partial_Z(\rho UW) + \partial_X p = -\rho g \sin\theta \\ \qquad \qquad \qquad - Z\partial_X(g \cos\theta) \end{array} \right. \quad (10)$$

To take into account the friction, we add the source term $-\rho g S_f T$ and we obtain the physical model that describe the slope variation and the section variation in a closed geometry :

$$\left\{ \begin{array}{l} \partial_t(\rho) + \partial_X(\rho U) + \partial_Y(\rho V) + \partial_Z(\rho W) = 0 \\ \partial_t(\tilde{U}\rho) + \partial_X(\rho U^2) + \partial_Y(\rho UV) + \partial_Z(\rho UW) + \partial_X p = -\rho g(\sin\theta + S_f) \\ \qquad \qquad \qquad - Z\partial_X(g \cos\theta) \end{array} \right. \quad (11)$$

2.2 Saint-Venant like equations on a closed pipes

In what follows, we use the linearized pressure law $p = p_a + \frac{\rho - \rho_0}{\beta\rho_0}$ in which ρ_0 represents the density of the fluid at atmospheric pressure p_a and β the water

compressibility coefficient equal to $5.0 \cdot 10^{-10} m^2 \cdot N^{-1}$ in practice. The sonic speed is then given by $c = 1/\sqrt{\beta \rho_0}$ and thus $c \approx 1400 m \cdot s^{-1}$. The friction term is given by the Manning-Strickler law (see [10]),

$$S_f = K(A)U|U| \text{ with } K(A) = \frac{1}{K_s^2 R_h(A)^{4/3}}$$

where $A = A(X)$ is the surface area of the section $\Omega(X)$ normal to the pipe mean axis (see FIGURE. 1 for the notations). K_s is the coefficient of roughness and $R_h(A) = A/P_m$ is the hydraulic radius where P_m is the perimeter of Ω .

The system (11) is integrated over the cross-section $\Omega(X)$. In the following, overlined letters represents the averaged quantities over $\Omega(X)$, $\overline{m} \in \partial\Omega$, $\vec{n} = \frac{\overline{m}}{|\overline{m}|}$ the outward unit vector at the point \vec{m} in the Ω -plane (as displayed on FIGURE. 1).

Following the work in [5], using the approximation $\overline{\rho U} \approx \overline{\rho} \overline{U}$, $\overline{\rho U^2} \approx \overline{\rho} \overline{U}^2$ and Lebesgues integral formulas, the mass conservation equations becomes

$$\partial_t(\overline{\rho A}) + \partial_X(\overline{\rho Q}) = \int_{\partial\Omega(X,t)} \rho \left(\partial_t \vec{M} + U \partial_X \vec{M} - \vec{V} \right) \cdot \vec{n} ds, \quad (12)$$

where $Q = A\overline{U}$ is the discharge of the flow and the velocity $\vec{V} = (V, W)^t$ in the (N, B) -plane.

The equation of the conservation of the momentum becomes

$$\begin{aligned} \partial_t(\overline{\rho Q}) + \partial_X \left(\frac{\overline{\rho Q^2}}{A} + c^2 \overline{\rho A} \right) &= -g \overline{\rho A} (\sin \theta + S_f) + c^2 \overline{\rho} \frac{dA}{dX} \\ &- \overline{\rho A} \bar{Z} \partial_X (g \cos \theta) \\ &+ \int_{\partial\Omega(X,t)} \rho U \left(\partial_t \vec{M} + U \partial_X \vec{M} - \vec{V} \right) \cdot \vec{n} ds \end{aligned} \quad (13)$$

As the pipe is infinitely rigid (since $\Omega = \Omega(X)$; see [5] for the dilatable case), integral terms appearing in (12) and (13) vanishes where the system is closed by a non penetration condition given by

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} \cdot \vec{N} = 0,$$

Indeed, we have

$$\int_{\partial\Omega(t,X)} \rho U \left(\partial_t \vec{m} + U \partial_X \vec{m} - \vec{V} \right) \cdot \vec{n} ds = 0$$

since $\vec{n} = \cos \phi \vec{B} \kappa$ for some constant κ where \vec{B} denotes the outward unit vector at the point m .

Finally, we obtain the Saint-Venant like equations for pressurized flows

$$\begin{cases} \partial_t(\overline{\rho A}) + \partial_X(\overline{\rho Q}) &= 0 \\ \partial_t(\overline{\rho Q}) + \partial_X \left(\frac{\overline{\rho Q^2}}{A} + c^2 \overline{\rho A} \right) &= -\overline{\rho A} g \sin \theta - \overline{\rho A} \bar{Z} \partial_X (g \cos \theta) + c^2 \overline{\rho} \frac{dA}{dX} \end{cases} \quad (14)$$

where the quantities \bar{Z} is the water column of water above the center of the mass. This means that one has $\bar{Z} = R(X)$ the radius of the section.

Now let us introduce the conservative variables corresponding to the wet area $M := \bar{\rho}A$ and the discharge $D := \bar{\rho}Q$. Then the system (14) reads

$$\begin{cases} \partial_t(M) + \partial_X(D) &= 0 \\ \partial_t(D) + \partial_X\left(\frac{D^2}{M} + c^2 M\right) &= -Mg \sin \theta - MR\partial_X(g \cos \theta) + c^2 M \frac{d}{dX} \log(A) \end{cases} \quad (15)$$

To close this section, let us give classical properties of the frictionless system.

Theorem 2.1

1. The system (15) is strictly hyperbolic for $A(X) > 0$.
2. For smooth solutions, the mean velocity $\bar{U} = D/M$ satisfies

$$\partial_t \bar{U} + \partial_X \left(\frac{\bar{U}^2}{2} + c^2 \log(M/A) + g\Phi_\theta + gZ \right) = 0 \quad (16)$$

The still steady states is given by

$$c^2 \log(M/A) + g\Phi_\theta + gZ = 0 \quad (17)$$

where $\Phi_\theta(t, X) = \int_{X_0}^X R(\xi) \partial_X \cos \theta(t, \xi) d\xi$ for any arbitrary x_0 and Z the altitude term (defined by $\partial_X Z = \sin \theta$). The quantity $\frac{\bar{U}^2}{2} + c^2 \log(M/A) + g\Phi_\theta + gZ$ is also called the total head.

3. It admits a mathematical entropy

$$E(M, D) = \frac{D^2}{2M} + Mc^2 \log M/A + gM\Phi_\theta + gMZ$$

which satisfies the entropy equality

$$\partial_t E + \partial_X((E + c^2 M)\bar{U}) = 0$$

Remark 2.2

- If we consider the friction term, we have : for smooth solutions,

$$\partial_t \bar{U} + \partial_X \left(\frac{\bar{U}^2}{2} + c^2 \log(M/A) + g\Phi_\theta + gZ \right) = -gK\bar{U}|\bar{U}|$$

and the previous entropy equality becomes an inequality and reads

$$\partial_t E + \partial_X((E + c^2 M)\bar{U}) \leq -gMK(A)\bar{U}^2|\bar{U}|$$

- If we introduce \tilde{Z} the so-called *pseudo altitude* source term given by

$$\tilde{Z} = Z + \Phi_\theta - c^2/g \log(A)$$

(where Φ_θ is defined in Theorem 2.1) then we can rewrite the system (15) in a closed manner of the classical Saint-Venant formulation,

$$\begin{cases} \partial_t(M) + \partial_X(D) &= 0 \\ \partial_t(D) + \partial_X\left(\frac{D^2}{M} + c^2 M\right) + g\partial_X \tilde{Z} &= 0 \end{cases} \quad (18)$$

3 The kinetic scheme with pseudo-reflection

We present in this section the kinetic formulation for pressurised flows in water pipes modeled by the system (15). To this end, we introduce a smooth real function χ such that

$$\chi(w) = \chi(-w) \geq 0, \quad \int_{\mathbb{R}} \chi(w) dw = 1, \quad \int_{\mathbb{R}} w^2 \chi(w) dw = 1$$

and defines the Gibbs equilibrium as follows

$$\mathcal{M}(t, x, \xi) = \frac{M}{c} \mathcal{M} \left(\frac{\xi - \bar{U}}{c} \right)$$

which represents the density of particles. Then these definitions allows us to get the following kinetic formulation :

Theorem 3.1 *The couple of functions (M, D) is a strong solution of the Saint-Venant likesystem (18) if and only if $\mathcal{M}(M, \xi - U)$ satisfies the kinetic transport equation*

$$\partial_t \mathcal{M} + \xi \partial_X \mathcal{M} - g \partial_X \tilde{Z} \partial_\xi \mathcal{M} = K(t, x, \xi) \quad (19)$$

for some collision kernel $K(t, x, \xi)$ which admits a vanishing moments up to order 1 for a.e (t, x) . Furthermore, the solution (M, D) is an entropic solution of (18) if and only if

$$\int_{\mathbb{R}} \xi^2 K d\xi \leq 0, \quad a.e.(t, x)$$

Proof of Theorem 3.1. We get easily the above results since the following macro-microscopic relations holds

$$M = \int_{\mathbb{R}} \mathcal{M}(\xi) d\xi \quad (20)$$

$$D = \int_{\mathbb{R}} \xi \mathcal{M}(\xi) d\xi \quad (21)$$

$$\frac{D^2}{M} + c^2 M = \int_{\mathbb{R}} \xi^2 \mathcal{M}(\xi) d\xi \quad (22)$$

The proof of the entropy condition on K is direct computation.

□

The reformulation of the equation (15) and the above theorem has the advantage to get only one equation for \mathcal{M} which it is easier to find simple numerical scheme (see for instance [8, 9]). In fact,

Theorem 3.2 *Let us consider the minimization problem $\min \epsilon(f)$ under the constraints*

$$f > 0, \quad \int_{\mathbb{R}} f(\xi) d\xi = M, \quad \int_{\mathbb{R}} \xi f(\xi) d\xi = D$$

where the kinetic functional energy is defined by

$$\epsilon(f) = \int_{\mathbb{R}} \frac{\xi^2}{2} f(\xi) + c^2 f(\xi) \log(f(\xi)) + c^2 f(\xi) \log(c\sqrt{2\pi}) + g \tilde{Z} f(\xi) d\xi.$$

Then the minimum is attained by the function $\mathcal{M}(t, x, \xi) = \frac{M}{c} \chi\left(\frac{\xi - \bar{U}}{c}\right)$ where

$$\chi(w) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right) \text{ a.e.}$$

Moreover, the minimal energy is

$$\epsilon(\mathcal{M}) = E(A, Q, \tilde{Z}) = \frac{D^2}{2M} + Mc^2 \log M + gM\tilde{Z}$$

and \mathcal{M} satisfies the steady state for $\bar{U} = 0$, that is,

$$\xi \partial_X \mathcal{M} - g \partial_X \tilde{Z} \partial_\xi \mathcal{M} = 0.$$

Proof of Theorem 3.2 One may easily verify that $f = \mathcal{M}$ is a solution of the minimization problem. Moreover, under the hypothesis $f > 0$ the functional $\epsilon(f)$ is strictly convex which ensures the unicity of the minimum. Moreover, by a direct computation, one has $\epsilon(\mathcal{M}) = E$.

The minimum \mathcal{M} of the functional $\epsilon(f)$ satisfies the steady state for $\bar{U} = 0$,

$$\xi \partial_X \mathcal{M} - g \partial_X \tilde{Z} \partial_\xi \mathcal{M} = 0.$$

Since $\partial_X \mathcal{M} = \frac{\partial_X M}{c} \chi\left(\frac{\xi}{c}\right)$, $\partial_\xi \mathcal{M} = \frac{M}{c^2} \chi'\left(\frac{\xi}{c}\right)$, denoting $w = \xi/c$, we get

$$w \partial_X M \chi(w) - g \partial_X \tilde{Z} \frac{M}{c} \chi'(w) = 0.$$

On the other hand, the steady state is given by

$$c^2 \log(M) + g\tilde{Z} = cst,$$

thus one has $g \partial_X \tilde{Z} = -c^2 \partial_X (\log M)$. Remarking that $\partial_X M \neq 0$ a.e., we obtain the following ordinary differential equation

$$w \chi(w) + \chi'(w) = 0.$$

which gives the result. □

3.1 Kinetic scheme

This section is devoted to the construction of the numerical kinetic scheme and properties using a flux splitting method on the previous kinetic formulation with a the upwinding of the source term \tilde{Z} .

Let us consider the uniform discretization of closed pipes with variable section by $X_i = i\Delta X$ being the center of the cell $m_i = [X_{i-1/2}, X_{i+1/2}]$ where $\Delta X = X_{i+1/2} - X_{i-1/2}$ is the spacestep for $i \in \mathbb{Z}$. Let $t_n = n\Delta t$, $n \in \mathbb{N}$ be the discretization in time where Δt is the timestep.

Let us introduce $\mathcal{U}_i^n = (M_i^n, D_i^n)$ the mean approximation of the wetted area and discharge. Let U_i^n, F_i^n be the approximations of the mean speed and the flux of the system (18) respectively, where $F(\mathcal{U}) = (D, D^2/M + c^2 M)^t$.

Let $\mathcal{M}_i^n(\cdot, \xi) = \frac{M_i^n}{c} \chi\left(\frac{\xi - U_i^n}{c}\right)$ be the approximation of the microscopic quantities and $\tilde{Z}_i \mathbb{1}_{m_i}(X)$ be the piecewise constant representation of the pseudo-altitude \tilde{Z} . Then, integrating equations (18) over $m_i \times [t_n, t_{n+1}]$, we get

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta X} \left(F_{i+1/2}^- - F_{i-1/2}^+ \right)$$

where $F_{i\pm 1/2}^\pm = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F\left(\mathcal{U}(t, X_{i\pm 1/2}^\pm)\right) dt$.

To obtain a finite volume scheme it remains to find an approximation $F_{i\pm 1/2}^\pm$ of the flux on the interface at the points $X_{i\pm 1/2}$. To this end, we use the previous kinetic formulation.

Supposing that we know the solution \mathcal{M}_i^n at time t_n for each node $X_{i+1/2}$, we deduce \mathcal{U}_i^n by the integral relation

$$\mathcal{U}_i^n = \int_{\mathbb{R}} \left(\frac{1}{\xi} \right) \mathcal{M}_i^n(\xi) d\xi.$$

Then considering the following relaxed problem

$$\begin{aligned} \partial_t f + \xi \partial_X \mathcal{M} - g \partial_X \tilde{Z} \partial_\xi \mathcal{M} &= 0 & (t, X, \xi) \in [t_n, t_{n+1}] \times m_i \times \mathbb{R} \\ f(t_n, X, \xi) &= \mathcal{M}(t_n, X, \xi) & (X, \xi) \in m_i \times \mathbb{R} \end{aligned} \quad (23)$$

which is discretized as follows

$$\forall i \in \mathbb{Z}, \forall n \in \mathbb{N}, \quad f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) - \xi \frac{\Delta t}{\Delta X} \left\{ \mathcal{M}_{i+1/2}^-(\xi) - \mathcal{M}_{i-1/2}^+(\xi) \right\} \quad (24)$$

where $\mathcal{M}_{i\pm 1/2}^\pm$ denotes the interface density equilibrium (computed in section 3.1.1). Finally, we set

$$\mathcal{U}_i^{n+1} = \int_{\mathbb{R}} \left(\frac{1}{\xi} \right) f_i^{n+1}(\xi) d\xi$$

and

$$\mathcal{M}_i^{n+1} = \frac{M_i^{n+1}}{c} \chi\left(\frac{\xi - U_i^{n+1}}{c}\right).$$

To conclude this section, let us do the following

Remark 3.1 This method avoids to compute the collision kernel K . Indeed, subtracting the kinetic equation (19) to equation (23), we get

$$\partial_t(\mathcal{M} - f)(\xi) = K(t, x, \xi).$$

Integrating in time t and ξ we obtain

$$\int_{\mathbb{R}} \left(\frac{1}{\xi} \right) f(\xi) d\xi = \mathcal{U}.$$

In other words, replacing \mathcal{M}_i^{n+1} by f_i^{n+1} at each time step is a manner to perform all collisions at once.

Now to complete the numerical kinetic scheme, it remains to define the microscopic fluxes $\mathcal{M}_{i\pm 1/2}^\pm$ appearing in equation (24) introduce by the choice of the constant piecewise representation of the pseudo-altitude term \tilde{Z} .

3.1.1 Interface equilibrium densities

To define the interface equilibrium densities, we use the generalized characteristics method. Let $s \in (t_n, t_{n+1})$ be a time variable and \mathcal{M} the solution of the kinetic equation (19). Let $\xi_i, i \in \mathbb{Z}$ be the discretization of the kinetic velocity.

Let Ξ and X be the characteristics curves

$$\begin{aligned} \frac{dX}{ds} &= \Xi(s) \\ X(t_{n+1}) &= X_{i+1/2} \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{d\Xi}{ds} &= -g\partial_x \tilde{Z}(X(s)) \\ \Xi(t_{n+1}) &= \xi_{i+1/2}^- := \xi_i \end{aligned} \quad (26)$$

If X is monotone on (t_n, t_{n+1}) then we can rewrite Ξ as a function of X and we get the following mechanic law

$$\frac{d}{dX} \left(\frac{\Xi^2}{2} + g\tilde{Z} \right) = 0, \quad (27)$$

then the equality holds,

$$\forall i \in \mathbb{Z}, \frac{\Xi^2}{2}(X_i) + g\tilde{Z}(X_i) = \frac{\Xi^2}{2}(X_{i+1}) + g\tilde{Z}(X_{i+1})$$

Writing down, $\Delta\tilde{Z}_{i+1/2} = \tilde{Z}(X_{i+1}) - \tilde{Z}(X_i)$, we obtain

$$\frac{\Xi^2}{2}(X_i) - \frac{\Xi^2}{2}(X_{i+1}) = g\Delta\tilde{Z}_{i+1/2} \quad (28)$$

To justify such a transformation, let us consider \tilde{Z} approximated by a piecewise constant function as in section 3.1 and we get

$$\tilde{Z}_{i+1} - \tilde{Z}_i = \Delta\tilde{Z}_{i+1/2}\delta_{X_{i+1/2}}$$

where δ_a is the Dirac mass at the point a . Solving the equations (25) and (26) on the interval (t_n, t_{n+1}) on each side of the abscissa $X_{i+1/2}$ gives $\Xi(s) = \xi_i$ and $\frac{dX}{ds} = \xi_i$ and we deduce X is strictly monotonous.

On other hand, the equality (28) becomes

$$\frac{\xi_i^2}{2} - \frac{\xi_{i+1}^2}{2} = g\Delta\tilde{Z}_{i+1/2} \quad (29)$$

In what follows, from a physical point of view, the quantities $\Delta\tilde{Z}_{i+1/2}$ will denote a *potential bareer*. Now according to the characteristic method and in particular to the equation (29), we can define the interface equilibrium densities $\mathcal{M}_{i\pm 1/2}^\pm$.

In order to derive these formulas, we proceed case by case. We suppose that there exists a CFL condition. Since the computation of $\mathcal{M}_{i+1/2}^-$ and $\mathcal{M}_{i+1/2}^+$ present the same difficulty, we just consider for $\mathcal{M}_{i+1/2}^-$ (see FIGURE. 2).

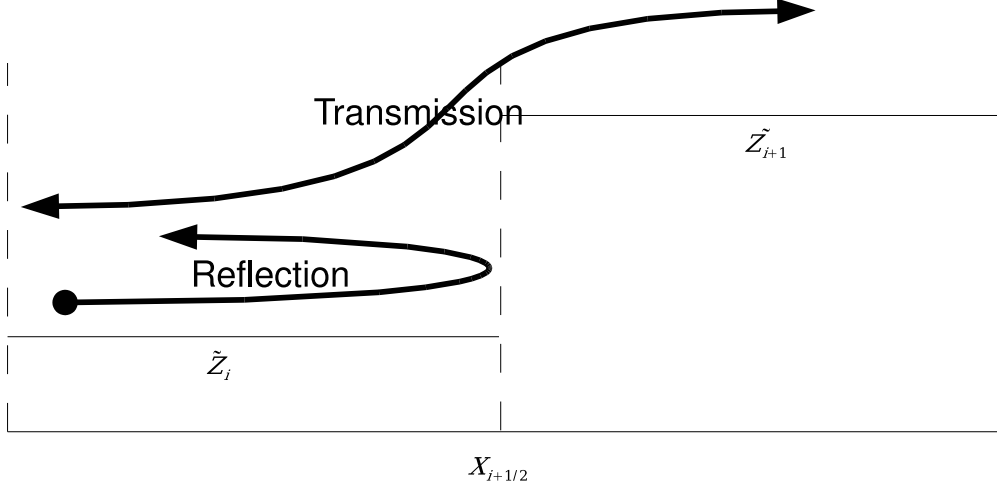


Figure 2: The potential bareer : transmission and reflection of particle

1. If $\xi_i > 0$ then the characteristic curve X strictly increase, so under a CFL condition the interface density depends of the left state, i.e. $\mathcal{M}_i^n(\xi)$, $\forall \xi > 0$. Under these assumptions, any particle coming from the left to the right overpass the potential bareer.
2. If $\xi_i < 0$ we distinguish two cases.
 - (a) The case $\xi_i^2 - 2g\Delta\tilde{Z}_{i+1/2} < 0$ means the particle comes from the left without enough kinetical energy to overpass the bareer, so it is reflected with a kinetic speed $-\xi_i$. Moreover, under a CFL condition, X decrease and the density depends of the left state, i.e. $\mathcal{M}_i^n(-\xi)$, $\forall \xi \leq 0$
 - (b) Otherwise, the particle comes from the right with a kinetic speed $\xi_{i+1} = \sqrt{\xi_i^2 - 2g\Delta\tilde{Z}_{i+1/2}} > 0$, therefore, X increase and the density depends of the right state $\mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2 - 2g\Delta\tilde{Z}_{i+1/2}}\right)$, $\forall \xi^2 - 2g\Delta\tilde{Z}_{i+1/2} > 0$, $\xi < 0$. This simply means that any particle comes from the right always pass the bareer.

Finally, the interface density on the left and right side of $X_{i+1/2}$ are given by :

$$\begin{aligned}
\mathcal{M}_{i+1/2}^-(\xi) &= \mathbb{1}_{\xi > 0} \mathcal{M}_i^n(\xi) + \mathbb{1}_{\xi < 0, \xi^2 - 2g\Delta\tilde{Z}_{i+1/2} < 0} \mathcal{M}_i^n(-\xi) \\
&\quad + \mathbb{1}_{\xi < 0, \xi^2 - 2g\Delta\tilde{Z}_{i+1/2} > 0} \mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2 - 2g\Delta\tilde{Z}_{i+1/2}}\right) \\
\mathcal{M}_{i+1/2}^+(\xi) &= \mathbb{1}_{\xi < 0} \mathcal{M}_{i+1}^n(\xi) + \mathbb{1}_{\xi > 0, \xi^2 + 2g\Delta\tilde{Z}_{i+1/2} < 0} \mathcal{M}_{i+1}^n(-\xi) \\
&\quad + \mathbb{1}_{\xi > 0, \xi^2 + 2g\Delta\tilde{Z}_{i+1/2} > 0} \mathcal{M}_i^n\left(\sqrt{\xi^2 + 2g\Delta\tilde{Z}_{i+1/2}}\right)
\end{aligned} \tag{30}$$

3.1.2 Numerical properties

In this part, we establish some numerical properties of the kinetic scheme (23)-(30).

Let us do the following

Remark 3.2 From the CFL condition $\sigma|\xi| \leq 1, \forall \xi$, since the support of the maxwellian (3.2) is not compact, this function cannot be used in numerical experiments. Therefore, in what follows, we will consider the particular Gibbs equilibrium $\chi(w) = \frac{1}{2\sqrt{3}}\mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(w)$ introduced by the authors in [2] and used in [6] in the case of pressurized flows in uniform closed pipes.

Let us announce the numerical properties of the scheme (23),

Theorem 3.3

1. Assuming the CFL condition

$$\sigma \max_{i \in \mathbb{Z}} \left(|\bar{U}_i^n| + \sqrt{3}c \right) \leq 1,$$

the numerical scheme (23,30) keeps the wet area positive.

2. The steady state is preserved $\bar{U}_i^n = 0, \frac{e^2}{g} \ln(\rho_i^n) + \tilde{Z}_i = cst$

Proof of Theorem 3.3. Let us suppose that at time t_n and at each node, $M_i^{n+1} > 0$. Let $\xi_{\pm} = \max(0, \pm\xi)$ be the positive and negative part of any real and $\sigma = \frac{\Delta t}{\Delta X}$, then the equation (23) reads

$$\begin{aligned} f_i^{n+1}(\xi) &= (1 - \sigma|\xi|)\mathcal{M}_i^n(\xi) \\ &+ \sigma\xi_+ \left(\mathbb{1}_{\xi^2 + 2g\Delta\tilde{Z}_{i+1/2} < 0} \mathcal{M}_i^n(-\xi) \right. \\ &\quad \left. + \mathbb{1}_{\xi^2 + 2g\Delta\tilde{Z}_{i-1/2} > 0} \mathcal{M}_{i-1}^n \left(\sqrt{\xi^2 + 2g\Delta\tilde{Z}_{i+1/2}} \right) \right) \\ &+ \sigma\xi_- \left(\mathbb{1}_{\xi^2 - 2g\Delta\tilde{Z}_{i+1/2} < 0} \mathcal{M}_i^n(-\xi) \right. \\ &\quad \left. + \mathbb{1}_{\xi^2 - 2g\Delta\tilde{Z}_{i-1/2} > 0} \mathcal{M}_{i+1}^n \left(-\sqrt{\xi^2 - 2g\Delta\tilde{Z}_{i+1/2}} \right) \right) \end{aligned}$$

Since the support of the χ function is compact, we get

$$f_i^{n+1}(\xi) > 0 \text{ if } |\xi - \bar{U}_j^n| < \sqrt{3}c, \forall j \in \mathbb{Z}$$

which implies $|\xi| < |\bar{U}_j^n| + \sqrt{3}c$. Using the CFL condition $\sigma|\xi| \leq 1$, we get the result. Moreover, since f_i^{n+1} is a convex combination of positive term, we obtain $f_i^{n+1} > 0$, hence

$$M_i^{n+1} = \int_{\mathbb{R}} f_i^{n+1}(\xi) d\xi > 0.$$

To proof the second point of the theorem, we consider the case $\xi > 0$ and $\xi > 0$ in order to get $\mathcal{M}_{i+1/2} = \mathcal{M}_{i-1/2}$ and thus $f_i^{n+1} = \mathcal{M}_i^n$ (see the mechanical law (27)).

□

Now let us also remark that the kinetic scheme (23)-(30) is conservative wet area. Indeed, let us denote the first component of the discrete fluxes $(F_M)_{i+1/2}^{\pm}$:

$$(F_M)_{i+1/2}^{\pm} := \int_{\mathbb{R}} \xi \mathcal{M}_{i+1/2}^{\pm}(\xi) d\xi$$

An easy computation using the change of variables $w^2 = \xi^2 - 2g\Delta\tilde{Z}_{i+1/2}$ in the interface densities formulas defining the kinetic fluxes $\mathcal{M}_{i+1/2}^\pm$ allows us to show that:

$$(F_M)_{i+\frac{1}{2}}^+ = (F_M)_{i+\frac{1}{2}}^-$$

In the case where the friction S_f is present, from the macroscopic equation (18) defining the state M_i^{n+1} and D_i^{n+1} , it is easy to construct it.

4 Numerical Validation

4.1 The uniform case

We present now numerical results of a water hammer test. The pipe of circular cross-section of 2 m^2 and thickness 20 cm is 2000 m long. The altitude of the upstream end of the pipe is 250 m and the slope is 5° . The Young modulus is $23 \cdot 10^9$ Pa since the pipe is supposed to be built in concrete. The total upstream head is 300 m. The initial downstream discharge is $10\text{ m}^3/\text{s}$ and we cut the flow in 10 seconds for the first test case and in 5 seconds for the other.

We present a validation of the proposed scheme by comparing numerical results of the proposed model solved by the kinetic scheme with the ones obtained by solving Allievi equations by the method of characteristics with the so-called `belier` code: an industrial code used by the engineers of the Center in Hydraulics Engineering of Electricité De France (EDF) [11].

A simulation of the water hammer test was done for a CFL coefficient equal to 0.8 and a spatial discretisation of 1000 mesh points. In the figures 3-4, we present a comparison between the results obtained by our kinetic scheme and the ones obtained by the “belier” code : the behavior of the discharge at the middle of the pipe. One can observe that the results for the proposed model are in very good agreement with the solution of Allievi equations. A little smoothing effect and absorption may be probably due to the first order discretisation type. A second order scheme may be implemented naturally and will produce a better approximation.

4.2 The non uniform case

We present a validation of the proposed kinetic scheme in the case of an contracting-expanding frictionless circular pipes of length $L = 1000\text{ m}$. The upstream radius is equal to $R_0 = 1\text{ m}$ and the downstream radius varies from $R_1 = 0.25\text{ m}$ to 2 m by step 0.25. The upstream condition is an immediate flow shut-down (3 seconds) while the upstream condition is constant (e.g. the total head is constant in time). We assume also that the pipe is rigid (like steel pipe). Then for each radius R_1 , we compute the waterhammer pressure rise at the middle of the pipe and we compare the one obtained by the equivalent pipe (see [1]). The results are presented in FIGURE. 5 and shows a very good agreement with the equivalent pipe theory. The others paramaters are $N = 100$ mesh point, CFL= 0.8, the downstream discharge before the shut-down is fixed to $1\text{ m}^3.\text{s}^{-1}$.

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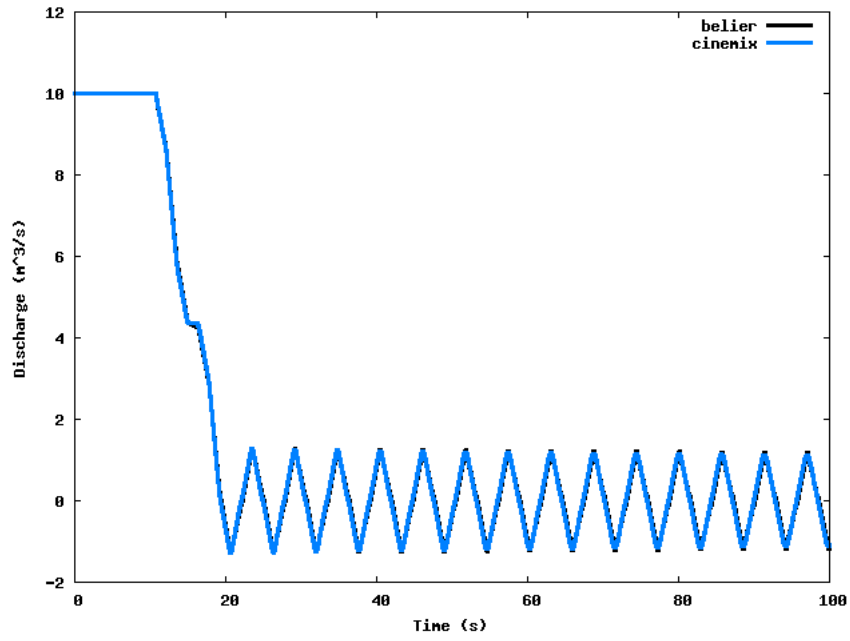


Figure 3: Comparison between the kinetic scheme and the industrial code belier
 First case : discharge at the middle of the pipe

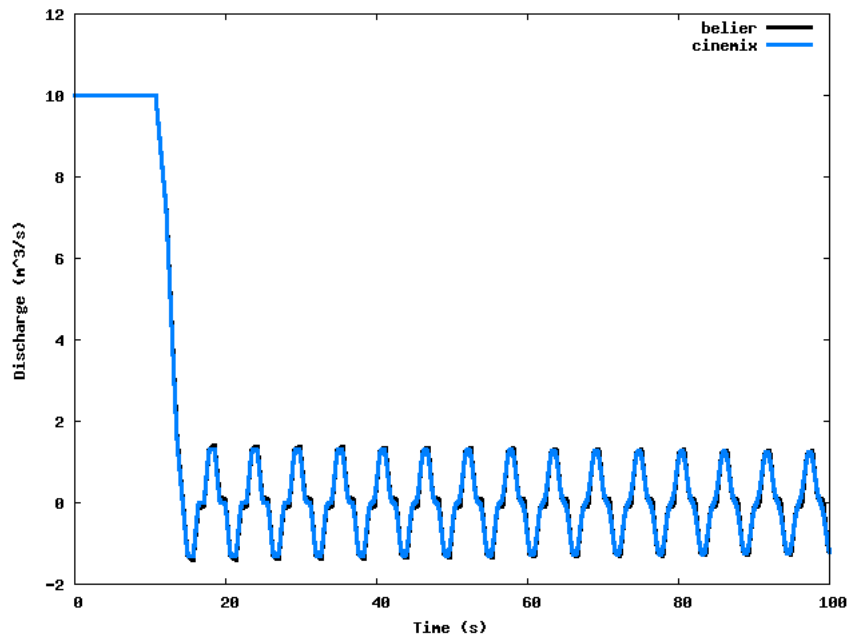


Figure 4: Comparison between the kinetic scheme and the industrial code belier
 Second case : discharge at the middle of the pipe

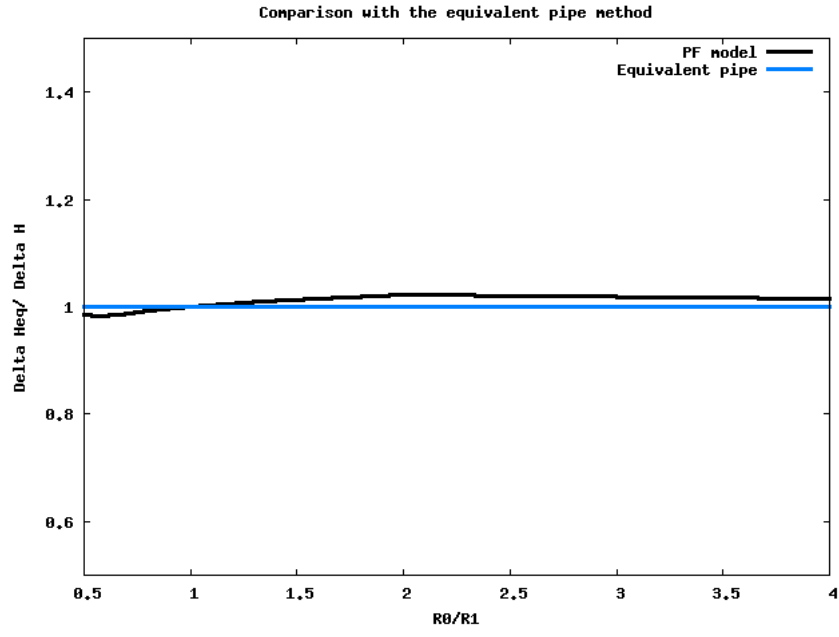


Figure 5: ΔH : computed with the present scheme, ΔH_{eq} :computed with an equivalent pipe method