



**HAL**  
open science

# A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme

Christian Bourdarias, Mehmet Ersoy, Stéphane Gerbi

## ► To cite this version:

Christian Bourdarias, Mehmet Ersoy, Stéphane Gerbi. A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme. 2008. hal-00342745v2

**HAL Id: hal-00342745**

**<https://hal.science/hal-00342745v2>**

Preprint submitted on 4 May 2009 (v2), last revised 6 Nov 2009 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme

Christian Bourdarias\*, Mehmet Ersoy† and Stéphane Gerbi‡

Laboratoire de Mathématiques, Université de Savoie  
73376 Le Bourget du Lac, France

## Abstract

We present the derivation of a new unidirectional model for unsteady mixed flows in non uniform closed water pipes. We introduce a local reference frame to take into account the local perturbation caused by the changes of section and slope. Then an asymptotic analysis is performed to obtain a model for free surface flows and another one for pressurised flows. By coupling these models through the transition points by the use of a common set of variables and a suitable pressure law, we obtain a simple formulation called PFS-model close to the shallow water equations with source terms. It takes into account the changes of section and the slope variation in a continuous way through transition points. Transition point between the two types of flows is treated as a free boundary associated to a discontinuity of the gradient of pressure. The numerical simulation is performed by making use of a Roe-like finite volume scheme that we adapted to take into account geometrical source terms in the convection matrix. Finally some numerical tests are presented.

**Keywords** : Shallow water equations, mixed flows, free surface flows, pressurised flows, curvilinear transformation, asymptotic analysis, VFRoe discretisation, well-balanced finite volume scheme, hyperbolic system with source terms.

## 1 Introduction

The presented work takes place in a more general framework: the modelisation of unsteady mixed flows in any kind of closed pipe taking into account the cavitation problem and air entrapment. We are interested in flows occurring in closed pipes with non uniform sections, where some parts of the flow can be free surface (it means that only a part of the pipe is filled) and other parts are pressurised (it means that the pipe is full). The transition phenomenon between the two types of flows occurs in many such as storm sewers, waste or supply pipes in hydroelectric installations. It can be induced by sudden change in the boundary conditions as failure pumping. During this process, the pressure can reach severe values and cause damages. The simulation of such a phenomenon is thus a major challenge and a great amount of works was devoted to it these last years (see [10],[11],[21],[25], for instance).

The classical shallow water equations are commonly used to describe free surface flows in open channels. They are also used in the study of mixed flows using the

---

\*email: Christian.Bourdarias@univ-savoie.fr,

†email: Mehmet.Ersoy@univ-savoie.fr,

‡email: Stephane.Gerbi@univ-savoie.fr

Preissman slot artefact (see for example [10, 25]). However, this technic does not take into account the depressurisation phenomenon which occurs during a water hammer except in recent works [17, 16, 18]. On the other hand the Allievi equations, commonly used to describe pressurised flows, are written in a non-conservative form which is not well adapted to a natural coupling with the shallow water equations.

A model for the unsteady mixed water flows in closed pipes and a finite volume discretisation have been previously studied by two of the authors [5] and a kinetic formulation has been proposed in [7]. We propose here the PFS-model which tends to extend naturally the work in [5] in the case of a closed pipe with non uniform section. For the sake of simplicity, we do not deal with the deformation of the domain induced by the change of pressure. We will consider only an infinitely rigid pipe.

The paper is organized as follows. The first section is devoted to the derivation of the free surface model from the 3D *incompressible* Euler equations which are written in a suitable local reference frame (following [3, 4]) in order to take into account the local effects produced by the changes of section and the slope variation. The construction of the free surface model is done by a formal asymptotic analysis. Seeking for an approximation at first order gives the model called FS-model. In Section 3, we adapt the derivation of the FS-model to derive the pressurised model, called P-model, from the 3D *compressible* Euler equations. Writing the source terms of these two models, P and FS-model, into a unified form and using the same couple of conservative unknowns as in [6], we propose a model for mixed flows, that we call PFS-model. We state some properties of this model. Section 5 is devoted to the extension of the Roe-like finite volume method described in [5] that was used for the case of uniform pipes. Finally some numerical tests are presented in Section 6.

## 2 Formal derivation of the FS-model for free surface flow

The classical shallow water equations are commonly used to describe physical situations like rivers, coastal domains, oceans and sedimentation problems. These equations are obtained from the incompressible Euler system (see e.g. [2, 19]) or from the incompressible Navier-Stokes system (see for instance [8, 9, 13, 20]) by several techniques (e.g. by direct integration or asymptotic analysis). We adapt here the derivation in [3, 4] to get a new unidirectional shallow water model. We start from the 3D incompressible Euler equations where we neglect the acceleration following the  $y$ -axis ensuring the existence of a privileged main flow axis. We write then the Euler equations in the local Serret-Frenet reference frame in order to take into account the local effects produced by the changes of section and the slope variation. Then we derive a shallow water model by a formal asymptotic analysis (done in Subsection 2.3).

### 2.1 Incompressible Euler equations and framework

Let us consider the cartesian reference frame  $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ . In the coordinate system  $(x, y, z)$ , the 3D incompressible Euler system writes:

$$\begin{cases} \operatorname{div} \mathbf{U} &= 0 \\ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P &= \mathbf{F} \end{cases} \quad (1)$$

where  $\mathbf{U}(t, x, y, z)$  denotes the velocity with components  $(u, v, w)$ ,  $P = p(t, x, y, z)I_3$  is the isotropic pressure tensor and  $\mathbf{F}$  is the exterior strength (including gravity). We define the domain  $\Omega_F(t)$  of the flow at time  $t$  as the union of sections  $\Omega(t, x)$  (assumed to be simply connected compact sets) orthogonal to some plane curve

lying in  $(O, \mathbf{i}, \mathbf{k})$  to follow the privileged main flow axis. As we neglect the y-axis acceleration, we can choose the parametrization  $(x, 0, b(x))$  in the cartesian reference frame  $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$  where  $\mathbf{k}$  follows the vertical direction;  $b(x)$  is then the elevation of the point  $\omega(x, 0, b(x))$  over the plane  $(O, \mathbf{i}, \mathbf{j})$ . Then, at each point  $\omega(x, 0, b(x))$ ,  $\Omega(t, x)$  is defined by the set:

$$\{(y, z) \in \mathbb{R}^2; z \in [-R(x), -R(x) + H(t, x)], y \in [\alpha(x, z), \beta(x, z)]\}$$

where  $R(x)$  denotes the radius,  $H(t, x)$  the water height at section  $\Omega(t, x)$ .  $\alpha(x, z)$ ,  $\beta(x, z)$  are respectively the left and right boundary point of the domain at altitude  $z$ ,  $-R(x) < z < R(x)$  (see FIG. 2).

We close classically System (1) using a kinematic law for the evolution of the free surface: *any free surface particle is advected by the fluid velocity  $\mathbf{U}$*  and on the wet boundary we assume the no-leak condition  $\mathbf{U} \cdot \mathbf{n}_{\text{wb}} = 0$  where  $\mathbf{n}_{\text{wb}}$  is the outward unit normal vector to the wet boundary (see FIG. 2). Denoting  $-R(x) + H(t, x)$  by  $h(t, x)$ , we set the atmospheric pressure  $P(z = h(t, x))$  to 0 at the free surface.

To define the local reference frame and to perform the curvilinear transformation, we introduce the curvilinear variable defined by:

$$X = \int_{x_0}^x \sqrt{1 + (b'(\xi))^2} d\xi$$

where  $x_0$  is an arbitrary abscissa. We set  $y = Y$  and we denote by  $Z$  the altitude of any fluid particle  $M$  in the Serret-Frenet reference frame  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$  at point  $\omega(x, 0, b(x))$  with  $\mathbf{T}$  the tangent vector  $\mathbf{N}$  the normal and  $\mathbf{B}$  the binormal vector at point  $w(x, 0, b(x))$  (see FIG. 1 and FIG. 2 for the notations).

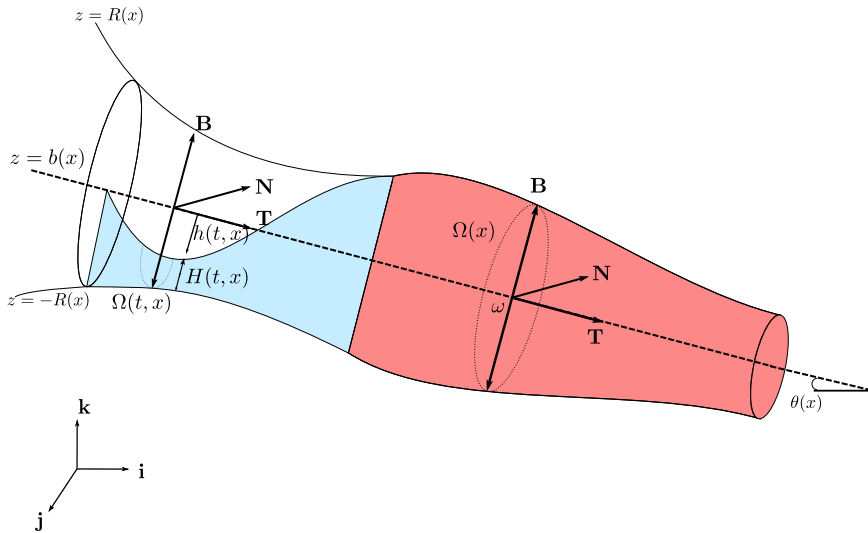


Figure 1: Geometric characteristics of the domain  
Mixed flow: free surface and pressurised

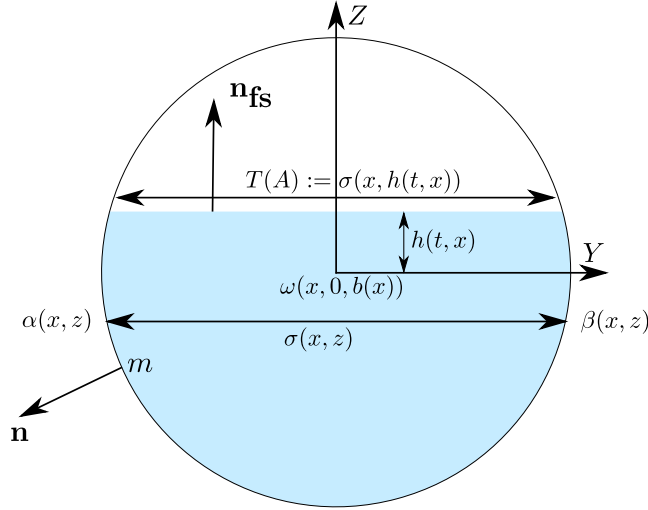


Figure 2: Cross-section of the domain at point  $\omega(x, 0, b(x))$

In what follows, we will use a curvilinear maps which will be an admissible transformation under the geometrical hypothesis on the domain:

(H) Let  $\mathcal{R}(x)$  be the algebraic curvature radius of the plane curve  $x \mapsto (x, 0, b(x))$  we assume that:

$$\forall x \in \Omega_F, |\mathcal{R}(x)| > R(x).$$

## 2.2 Incompressible Euler model in the curvilinear coordinates

Following the work in [3, 4], we write System (1) in the Serret-Frenet reference frame  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$  at point  $\omega(x, 0, b(x))$  by the transformation  $\mathcal{T} : (x, y, z) \rightarrow (X, Y, Z)$  by the use of the divergence chain rule lemma that we recall here:

**Lemma 2.1** *Let  $(x, y, z) \mapsto \mathcal{T}(x, y, z)$  be a  $C^1$  diffeomorphism and  $\mathcal{A}^{-1} = D_{(x,y,z)}\mathcal{T}$  the jacobian matrix of the transformation with determinant  $J$ . Then, for any vector field  $\Phi$  one has,*

$$JD_{(X,Y,Z)}\Phi = D_{(x,y,z)}(J\mathcal{A}\Phi).$$

In particular, for any scalar function  $f$ , one has

$$D_{(X,Y,Z)}f = \mathcal{A}^t D_{(x,y,z)}f.$$

Let  $(U, V, W)^t$  be the components of the velocity vector in the  $(X, Y, Z)$  coordinates defined as  $(U, V, W)^t = \Theta(u, v, w)^t$  where  $\Theta$  is the matrix

$$\Theta = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}.$$

We denote by  $\theta(x)$  the angle  $(\mathbf{i}, \mathbf{T})$ .

Using Lemma 2.1, the incompressible Euler system in the variables  $(X, Y, Z)$  reads:

$$\begin{cases} \partial_X U + \partial_Y (JV) + \partial_Z (JW) & = 0 \\ \partial_t (JU) + \partial_X (U^2) + \partial_Y (JUV) + \partial_Z (JUW) + \partial_X p & = G_1 \\ \partial_t (JV) + \partial_X (UV) + \partial_Y (JV^2) + \partial_Z (JVW) + \partial_Y (Jp) & = 0 \\ \partial_t (JW) + \partial_X (UW) + \partial_Y (JVW) + \partial_Z (JW^2) + J\partial_Z(p) & = G_2 \end{cases} \quad (2)$$

where  $J(X, Y, Z) = 1 - Z\theta'(X)$  is the determinant of the transformation and

$$G_1 = UW\theta'(X) - Jg \sin \theta \text{ and } G_2 = -U^2\theta'(X) - Jg \cos \theta.$$

The interested reader can be found the details of the calculus in [3]. We have denoted by  $f'$  the derivative with respect to the space variable  $X$  of any function  $f(X)$ .

On the wet boundary, the no-leak condition reads:

$$(U, V, W)^t \cdot \mathbf{n}_{\mathbf{wb}} = 0. \quad (3)$$

**Remark 2.1** Notice that  $\kappa(X) = \theta'(X)$  is the algebraic curvature of the axis at point  $\omega(X, 0, b(X))$  and the function  $J(X, Y, Z) = 1 - Z\kappa(X)$  depends only on the variables  $X, Z$ . Moreover, under the hypothesis (H), we have  $J > 0$  in  $\Omega_F$ . Consequently,  $\mathcal{T}$  defines a diffeomorphism and thus the performed transformation is admissible.

### 2.3 Formal derivation of the FS-model for free surface flows

In this section, we perform a formal asymptotic analysis on System (2). According to the work in [3, 13, 20], the shallow water equations can be obtained from the incompressible Navier-Stokes equations with particular boundary conditions. Here, we perform this analysis directly on the incompressible Euler system in order to get  $J = 1 + O(\epsilon)$ .

Let us introduce the usual small parameter  $\epsilon = H/L$  where  $H$  (the height) and  $L$  (the length) are two characteristics dimensions along the  $\mathbf{k}$  and  $\mathbf{i}$  axis respectively. Moreover, we assume that the characteristic dimension along the  $\mathbf{j}$  axis is the same as  $\mathbf{k}$ . We introduce the other characteristics dimensions  $T, P, \bar{U}, \bar{V}, \bar{W}$  for time, pressure and velocity respectively and the dimensionless quantities as follows:

$$\begin{aligned} \tilde{U} &= U/\bar{U}, \quad \tilde{V} = \epsilon V/\bar{U}, \quad \tilde{W} = \epsilon W/\bar{U}, \\ \tilde{X} &= X/L, \quad \tilde{Y} = Y/H, \quad \tilde{Z} = Z/H, \quad \tilde{p} = p/P, \quad \tilde{\theta} = \theta, \quad \tilde{\rho} = \rho. \end{aligned}$$

In the sequel, we set  $P = \bar{U}^2$  and  $L = T\bar{U}$  (i.e. we consider only laminar flows). Under these hypothesis, we have  $\tilde{J}(\tilde{X}, \tilde{Y}, \tilde{Z}) = 1 - \epsilon\tilde{Z}\tilde{\theta}'(\tilde{X})$ . Thus, the rescaled System (2) reads:

$$\left\{ \begin{array}{l} \partial_{\tilde{x}}\tilde{U} + \partial_{\tilde{y}}(\tilde{J}\tilde{V}) + \partial_{\tilde{z}}(\tilde{J}\tilde{W}) = 0 \\ \partial_{\tilde{t}}(\tilde{J}\tilde{U}) + \partial_{\tilde{x}}(\tilde{U}^2) + \partial_{\tilde{y}}(\tilde{J}\tilde{U}\tilde{V}) + \partial_{\tilde{z}}(\tilde{J}\tilde{U}\tilde{W}) + \partial_{\tilde{x}}\tilde{p} = G_1 \\ \epsilon^2 \left( \partial_{\tilde{t}}(\tilde{J}\tilde{V}) + \partial_{\tilde{x}}(\tilde{U}\tilde{V}) + \partial_{\tilde{y}}(\tilde{J}\tilde{V}^2) + \partial_{\tilde{z}}(\tilde{J}\tilde{V}\tilde{W}) \right) + \partial_{\tilde{y}}(\tilde{J}\tilde{p}) = 0 \\ \epsilon^2 \left( \partial_{\tilde{t}}(\tilde{J}\tilde{W}) + \partial_{\tilde{x}}(\tilde{U}\tilde{W}) + \partial_{\tilde{y}}(\tilde{J}\tilde{V}\tilde{W}) + \partial_{\tilde{z}}(\tilde{J}\tilde{W}^2) \right) \\ \quad + \tilde{J}\partial_{\tilde{z}}(\tilde{p}) = G_2 \end{array} \right. \quad (4)$$

where

$$\begin{aligned} G_1 &= \epsilon\tilde{U}\tilde{W}\tilde{\kappa}(\tilde{X}) - \frac{\sin \tilde{\theta}}{F_{r,L}^2} - \frac{\tilde{Z}}{F_{r,H}^2}(\cos \tilde{\theta})', \\ G_2 &= -\epsilon\tilde{U}^2\tilde{\rho}(\tilde{X}) - \frac{\cos \tilde{\theta}}{F_{r,H}^2} + \epsilon\kappa(X)\frac{\tilde{Z}\tilde{J}\cos \tilde{\theta}}{F_{r,H}^2}, \end{aligned}$$

$F_{r,M} = \frac{\bar{U}}{\sqrt{gM}}$  is the Froude number along the  $\mathbf{i}$  axis and the  $\mathbf{k}$  or  $\mathbf{j}$  axis where  $M$  is any generic variable equal to  $L$  or  $H$ .

Formally, when  $\epsilon$  vanishes, System (4) reduces to:

$$\left\{ \begin{array}{l} \partial_{\tilde{X}} \tilde{U} + \partial_{\tilde{Y}}(\tilde{V}) + \partial_{\tilde{Z}}(\tilde{W}) = 0 \\ \partial_{\tilde{t}}(\tilde{U}) + \partial_{\tilde{X}}(\tilde{U}^2) + \partial_{\tilde{Y}}(\tilde{U}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{U}\tilde{W}) + \partial_{\tilde{X}}\tilde{p} = -\frac{\sin\tilde{\theta}}{F_{r,L}^2} \\ \phantom{\partial_{\tilde{t}}(\tilde{U}) + \partial_{\tilde{X}}(\tilde{U}^2) + \partial_{\tilde{Y}}(\tilde{U}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{U}\tilde{W}) + \partial_{\tilde{X}}\tilde{p}} = -\frac{\tilde{Z}}{F_{r,H}^2}(\cos\tilde{\theta})' \\ \partial_{\tilde{Z}}(\tilde{p}) = -\frac{\cos\tilde{\theta}}{F_{r,H}^2} \end{array} \right. \quad (5)$$

Let us introduce the conservative variables  $A(t, X)$  and  $Q(t, X) = A(t, X)\bar{U}$  representing respectively the wet area and the discharge defined as:

$$A(t, X) = \int_{\Omega} dY dZ, \quad Q(t, X) = A(t, X)\bar{U}(t, X)$$

where  $\bar{U}$  is the mean value of the speed over the cross-section  $\Omega$ . Integrating the preceding system along the cross-section and returning to the physical variable, the free surface model that we call FS-model reads:

$$\left\{ \begin{array}{l} \partial_t A + \partial_X Q = 0 \\ \partial_t Q + \partial_X \left( \frac{Q^2}{A} + gI_1(X, A) \cos\theta \right) = gI_2(X, A) \cos\theta - gA \sin\theta \\ \phantom{\partial_t Q + \partial_X \left( \frac{Q^2}{A} + gI_1(X, A) \cos\theta \right) =} = -gA\bar{Z}(X, A)(\cos\theta)' \end{array} \right. \quad (6)$$

where  $I_1(X, A)$  and  $I_2(X, A)$  are respectively the classical term of hydrostatic pressure and the pressure source term and defined by:

$$I_1(X, A) = \int_{-R}^h (h - Z)\sigma dZ \quad \text{and} \quad I_2(X, A) = \int_{-R}^h (h - Z)\partial_X \sigma dZ.$$

In these formulas  $\sigma(X, Z)$  is the width of the cross-section at position  $X$  and at height  $Z$ . The additional term  $\bar{Z}(X, A)$  is defined by  $(h(A) - I_1(X, A)/A)$ . It is the  $Z$  coordinate of the center of mass.

### 3 Formal derivation of the P-model for pressurised flows

In this section, we present a new unidirectional shallow water-like equations to describe pressurised flows in closed non uniform water pipes. This model is constructed to be coupled in natural way with the obtained FS-model (6). Starting from the 3D compressible Euler equations in cartesian coordinates,

$$\partial_t \rho + \text{div}(\rho \mathbf{U}) = 0, \quad (7)$$

$$\partial_t(\rho \mathbf{U}) + \text{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \mathbf{F}, \quad (8)$$

where  $\mathbf{U}(t, x, y, z)$  and  $\rho(t, x, y, z)$  denotes the velocity with components  $(u, v, w)$  and the density respectively.  $p(t, x, y, z)$  is the scalar pressure and  $\mathbf{F}$  the exterior strength (of gravity).

We define the pressurised domain of the flow as the continuous extension of  $\Omega_F$  (see Subsection 2.1) defined by some plane curve with parametrization  $(x, 0, b(x))$  in the cartesian reference frame  $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ ; we recall that  $b(x)$  is then the elevation of the point  $\omega(x, 0, b(x))$  over the plane  $(O, \mathbf{i}, \mathbf{j})$  (see FIG. 1). The curve may be, for







Let us recall that  $S_{max}(X)$  denotes the area of the cross-section of the pipe at position  $X$ . We denote also  $S_{max}(X)$  by  $S(X)$  when we deal with pressurised state while  $S(t, X)$  depends on time when the type of flow is free surface and it is denoted simply by  $A$ : since the pressurised sections are connected continuously to free surface sections when the change of state occurs. Thus, we call  $S$  the *physical wet area* and  $A$  the wet equivalent area. In the same way, as we can express  $S$  by the fluid height  $H(t, X)$  and the radius  $R(X)$ , we define:

$$\mathcal{H}(t, X) = \mathbb{1}_{\{\rho=\rho_0\}} h(t, X) + \mathbb{1}_{\{\rho\neq\rho_0\}} R(X) \quad (16)$$

(also denoted by  $\mathcal{H}(S)$  with  $S$  defined as above) that also ensures the continuity of the following pressure law through the change of state:

$$p(X, A, S) = c^2(A - S) + gI_1(X, S) \cos \theta. \quad (17)$$

Indeed, we have simply  $gI_1(X, A) \cos \theta$  for the free surface pressure law when  $A = S$  (that is also  $\rho = \rho_0$ ) and  $c^2(A - S) + gI_1(X, S) \cos \theta$  for the pressurised case when  $A \neq S$  where the term  $gI_1(X, S) \cos \theta$  ensures the pressure law (17) to be continuous through transitions points. Let us also note that the flux gradient is discontinuous through the change of state since the sound of the speed of the free surface and the pressurised cannot be connected continuously.

The plane curve with parametrization  $(X, 0, b(X))$  is chosen as the main pipe axis in the axisymmetric case. Actually this choice is the more convenient for pressurised flows while the bottom line is adapted to free surface flows. Thus we must assume small variations of the section ( $S'_{max}$  small) or equivalently small angle  $\varphi$  as displayed on FIG. 3.

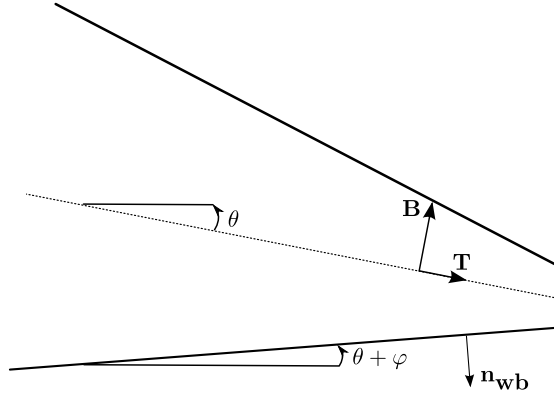


Figure 3: Some restriction concerning the geometric domain

Finally, the PFS-model for unsteady mixed flows can be simply expressed into a single formulation as:

$$\begin{cases} \partial_t(A) + \partial_X(Q) = 0 \\ \partial_t(Q) + \partial_X \left( \frac{Q^2}{A} + p(X, A, S) \right) = \begin{aligned} & -gAb' + Pr(X, A, S) \\ & -G(X, A, S) \\ & -K(X, S) \frac{Q|Q|}{A} \end{aligned} \end{cases} \quad (18)$$

where  $K$ ,  $Pr$ , and  $G$  denotes respectively the friction, the pressure source and the

geometry source term defined as above:

$$\begin{aligned} Pr(X, A, S) &= (c^2(A/S - 1)) S' + gI_2(X, S) \cos \theta, \\ G(X, A, S) &= gA \bar{Z}(X, S)(\cos \theta)', \\ K(X, S) &= \frac{1}{K_s^2 R_h(S)^{4/3}}. \end{aligned}$$

The PFS-model (18) satisfies the following properties:

**Theorem 4.1**

1. System (18) is strictly hyperbolic for  $A(t, X) > 0$ .
2. For smooth solutions, the mean velocity  $U = Q/A$  satisfies

$$\begin{aligned} \partial_t U + \partial_X \left( \frac{U^2}{2} + c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gb \right) \\ = -gK(X, S)U|U| \leq 0. \end{aligned} \quad (19)$$

3. The still water steady states for  $U = 0$  reads:

$$c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gb = 0. \quad (20)$$

4. It admits a mathematical entropy

$$E(A, Q, S) = \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + gA \bar{Z}(X, S) \cos \theta + gAb$$

which satisfies the entropy relation

$$\partial_t E + \partial_X ((E + p(X, A, S))U) = -gAK(X, S)U^2|U| \leq 0. \quad (21)$$

The quantity  $\frac{U^2}{2} + c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gb$  is called the total head. Notice that the total head and the energy are defined continuously through the transition points.

**Proof of Theorem 4.1:** The results (19) and (21) are obtained in a classical way. Indeed, Equation (19) is obtained by subtracting the result of the multiplication of the mass equation by  $U$  to the momentum equation. Then multiplying the mass equation by  $\left(\frac{U^2}{2} + c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gb\right)$  and adding the result of the multiplication of Equation (19) by  $Q$ , we get:

$$\begin{aligned} \partial_t \left( \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + gA \bar{Z}(X, S) \cos \theta + gAb \right) \\ + \partial_X \left( \left( \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + gA \bar{Z}(X, S) \cos \theta + gAb + p(X, A, S) \right) U \right) \\ + c^2 \left( \partial_t S \left( \frac{A}{S} - 1 \right) \right) = -gAK(X, S, E)U^2|U| \leq 0. \end{aligned}$$

We see that the term  $c^2 \left( \partial_t S \left( \frac{A}{S} - 1 \right) \right)$  is identically 0 since we have  $A = S$  when the flow is free surface whereas  $S = S(X)$  when the flow is pressurised. Moreover, from the last inequality, when  $A = S$ , we have the classical entropy inequality (see [5, 6]) with the energy  $E$ :

$$E(A, Q, S) = \frac{Q^2}{2A} + gA \bar{Z}(X, S) \cos \theta + gAb$$

while in the pressurised case, the energy is:

$$E(A, Q, S) = \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + gAb.$$

Finally, the energy for the PFS-model reads:

$$E(A, Q, S) = \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + gA\bar{Z}(X, S) \cos \theta + gAb.$$

Let us remark that the term  $c^2 S$  makes the energy  $E$  continuous through transition points and it permits also to write the entropy flux under the classical form  $(E+p)U$ .

□

## 5 Finite volume discretisation

In this section, we adapt the Roe-like explicit finite volume scheme described in [5]. The new terms appearing in the PFS-model related to the curvature and the section variation are upwinded in the same spirit of [5]. The numerical scheme is adapted to discontinuities of the flux gradient occurring in the treatment of the transitions between free-surface and pressurised flows.

### 5.1 Discretisation of the space domain

The spatial domain is a pipe of length  $L$ . The main axis of the pipe is divided in meshes  $m_i = [X_{i-1/2}, X_{i+1/2}]$ ,  $1 \leq i \leq N$ .  $\Delta t$  denotes the timestep at time  $t^n$  and we set  $t_{n+1} = t_n + \Delta t$ .

The discrete unknowns are  $U_i^n = \begin{pmatrix} A_i^n \\ Q_i^n \end{pmatrix}$ . For the sake of simplicity, the boundary conditions are not treated (the interested reader will find this treatment in details in [5]).

### 5.2 Explicit first order Roe scheme

We propose to extend the finite volume discretisation [5] to the PFS-model by adapting a well balanced scheme using the upwinding of the source terms: the curvature and section variation of the pipe.

First, following Leroux *et al.* [14, 22] we use piecewise constant functions to approximate  $b$  ( $b'(X) = \sin \theta(X)$ ) as well as the term  $\cos \theta$  and the cross section area  $S$ . Adding the equations  $\partial_t Z = 0$ ,  $\partial_t \cos \theta = 0$  and  $\partial_t S = 0$ , the PFS-model can be written under a non-conservative form with the variable  $\mathbf{W} = (b, \cos \theta, S, A, Q)^t$ :

$$\partial_t \mathbf{W} + \partial_X \mathbf{F}(X, \mathbf{W}) + B(X, \mathbf{W}) \cdot \partial_X \mathbf{W} = TS(\mathbf{W}) \quad (22)$$

where

$$\mathbf{F}(X, \mathbf{W}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \\ \frac{Q^2}{A} + p(X, A, S) \end{pmatrix}, \quad TS(\mathbf{W}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -gK(X, S) \frac{Q|Q|}{A} \end{pmatrix}$$

and

$$B(X, \mathbf{W}) = \begin{pmatrix} gA \\ gA\bar{Z} \\ -c^2(A/S - 1) - \mathcal{I}(X, \mathbf{W}) \\ 0 \\ 0 \end{pmatrix}$$

where we have written the pressure source term due to the geometry  $gI_2(X, S) \cos(\theta)$  as  $\mathcal{I}(X, \mathbf{W})S'$ . For instance, for a circular cross-section pipes we have:

$$\mathcal{I}(X, \mathbf{W}) = \frac{1}{2\sqrt{\pi S}} \left( \frac{h(S)\pi}{2} + h(S) \arcsin \left( \frac{h(S)}{R(X)} \right) + \sigma(X, h(S)) \right) .$$

Let  $W_i^n$  be an approximation of the mean value of  $\mathbf{W}$  on the mesh  $m_i$  at time  $t_n$ . Since the value of  $b, \cos \theta, S$  are known, integrating the above equations over  $]X_{i-1/2}, X_{i+1/2}[ \times ]t_n, t_{n+1}[$ , we can write a Finite Volume scheme as follows:

$$\begin{aligned} \mathbf{W}_i^{n+1} &= \mathbf{W}_i^n - \alpha_i \left( \mathbf{F}(\mathbf{W}_{i+1/2}^*(0^-, \mathbf{W}_i^n, \mathbf{W}_{i+1}^n)) - \mathbf{F}(\mathbf{W}_{i-1/2}^*(0^+, \mathbf{W}_{i-1}^n, \mathbf{W}_i^n)) \right) \\ &\quad + TS(\mathbf{W}_i^n) \end{aligned} \quad (23)$$

with  $\alpha_i = \frac{\Delta t}{h_i}$ .

$\mathbf{W}_{i+1/2}^*(\xi = x/t, \mathbf{W}_i, \mathbf{W}_{i+1})$  is then the exact or approximate solution to the Riemann problem at interface  $X_{i+1/2}$  associated to the left and right states  $\mathbf{W}_i$  and  $\mathbf{W}_{i+1}$ . Let us also remark that the term  $B(X, \mathbf{W})$  does not appear explicitly in this formulation since  $b', (\cos \theta)'$  and  $S'$  are null on  $]X_{i-1/2}, X_{i+1/2}[$  but contributes to the computation of the numerical flux.

The computation of the interface quantities  $\mathbf{W}_{i\pm 1/2}^*(0^\pm, \mathbf{W}_i, \mathbf{W}_{i+1})$  will depend on two types of interfaces located at the point  $X_{i+1/2}$ : the first one is a non transition point, that is when the flow on the left and on the right sides of the interface is of the same type. The second one is a transition point, that is when the flow changes of type through the interface. We recall the approach used in [5] and adapted it here to the new terms. According to the type of interface, we have to solve two different linearised Riemann problem :

### 5.2.1 The Case of a non transition point

Expanding the term  $\partial_X \mathbf{F}(X, \mathbf{W})$  in the non-conservative equations (22), the quasi-linear formulation of the PFS-model (18) reads:

$$\partial_t \mathbf{W} + D(\mathbf{W}) \partial_X \mathbf{W} = TS(\mathbf{W})$$

with  $D$  the convection matrix defined by

$$D(\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ gA & gA\mathcal{H}(S) & \Psi(\mathbf{W}) & c^2(\mathbf{W}) - u^2 & 2u \end{pmatrix}$$

where  $\Psi(\mathbf{W}) = gS\partial_S \mathcal{H}(S) \cos \theta - c^2(\mathbf{W})\frac{A}{S}$  and  $u = Q/A$  denotes the speed of the water.  $c(\mathbf{W})$  is then the sound speed equal to  $c$  for the pressurised flow or  $\sqrt{gA\partial_A \mathcal{H}(A) \cos \theta}$  for the free surface flow.

**Remark 5.1** Let us remark that, since  $\partial_X I_1(X, A) = I_2(X, A) + \partial_A I_1(A) \partial_X A$ , the pressure source term  $I_2$  does not appear in the convection matrix  $D$ .

To compute the interface quantities denoted by  $(AM, QM)$  for the left side and  $(AP, QP)$  for the right side (see Figure 4 below), we solve then the following linearised Riemann problem:

$$\begin{cases} \partial_t \mathbf{W} + \tilde{D} \partial_X \mathbf{W} & = 0 \\ \mathbf{W} = (b, C, S, A, Q) & = \begin{cases} W_l = (b_l, \cos \theta_l, S_l, A_l, Q_l) & \text{if } x < 0 \\ W_r = (b_r, \cos \theta_r, S_r, A_r, Q_r) & \text{if } x > 0 \end{cases} \end{cases} \quad (24)$$

with  $(\mathbf{W}_l, \mathbf{W}_r) = (\mathbf{W}_i, \mathbf{W}_{i+1})$  and  $\tilde{D} = \tilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\tilde{\mathbf{W}})$ .  $\tilde{\mathbf{W}}$  denotes then  $\frac{\mathbf{W}_l + \mathbf{W}_r}{2}$ . We have then  $W^*(0+, \mathbf{W}_l, \mathbf{W}_r) = (b_r, \cos \theta_r, S_r, AP, QP)$ .

The eigenvalues of the matrix  $\tilde{D}$  are  $\lambda_1 = 0$ ,  $\lambda_2 = 0$ ,  $\lambda_3 = 0$ ,  $\lambda_4 = \tilde{u} - c(\tilde{\mathbf{W}})$ ,  $\lambda_5 = \tilde{u} + c(\tilde{\mathbf{W}})$  and the associated right eigenvectors

$$r_1(\tilde{\mathbf{W}}) = \begin{pmatrix} c^2(\tilde{\mathbf{W}}) - \tilde{u}^2 \\ 0 \\ 0 \\ -g\tilde{A} \\ 0 \end{pmatrix}, \quad r_2(\tilde{\mathbf{W}}) = \begin{pmatrix} \Psi(\tilde{\mathbf{W}}) \\ 0 \\ -g\tilde{A} \\ 0 \\ 0 \end{pmatrix}, \quad r_3(\tilde{\mathbf{W}}) = \begin{pmatrix} \mathcal{H}(\tilde{S}) \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$r_4(\tilde{\mathbf{W}}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \tilde{u} - c(\tilde{\mathbf{W}}) \end{pmatrix}, \quad r_5(\tilde{\mathbf{W}}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \tilde{u} + c(\tilde{\mathbf{W}}) \end{pmatrix}.$$

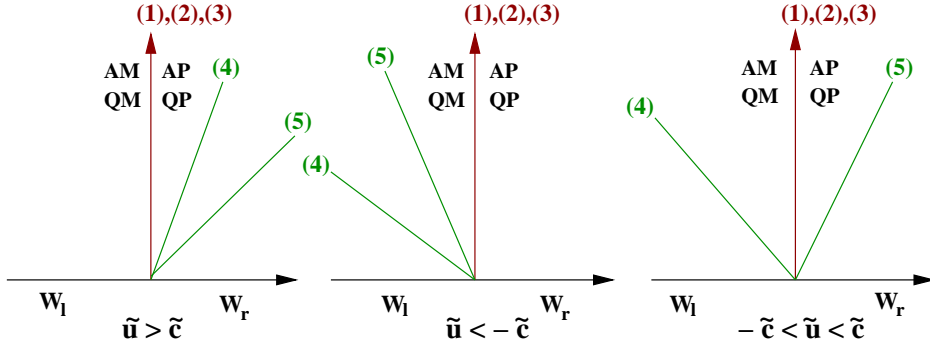


Figure 4: Solution of the Riemann problem (24). The number of the lines corresponds to the eigenvalues.

We denote  $P$  the transition matrix associated to the right eigenvectors of  $\tilde{D}$  and  $P^{-1}$  its inverse. Setting  $[\mathbf{W}] = \mathbf{W}_r - \mathbf{W}_l$ , the solution of the Riemann problem are connected by shocks propagating along the characteristic lines  $X/t = \lambda_i$ . The jump associated to the eigenvectors  $r_i$  is then equal to  $(P^{-1}[\mathbf{W}])_i r_i$ . In particular, the discharge is continuous through the line  $X/t = 0$  since the fifth component of vectors  $r_1, r_2$  and  $r_3$  are null. Thus, for instance in the subcritical case (that is when  $-c(\tilde{\mathbf{W}}) < \tilde{u} < c(\tilde{\mathbf{W}})$ ), we have:

$$AM = A_l + \frac{g\tilde{A}}{2c(\tilde{\mathbf{W}})(c(\tilde{\mathbf{W}}) - \tilde{u})} \psi_l^r + \frac{\tilde{u} + c(\tilde{\mathbf{W}})}{2c(\tilde{\mathbf{W}})} (A_r - A_l) - \frac{1}{2c(\tilde{\mathbf{W}})} (Q_r - Q_l)$$

$$QM = QP = Q_l - \frac{g\tilde{A}}{2c(\tilde{\mathbf{W}})} \psi_l^r + \frac{\tilde{u}^2 - c(\tilde{\mathbf{W}})^2}{2c(\tilde{\mathbf{W}})} (A_r - A_l) - \frac{\tilde{u} - c(\tilde{\mathbf{W}})}{2c(\tilde{\mathbf{W}})} (Q_r - Q_l)$$

$$AP = AM + \frac{g\tilde{A}}{\tilde{u}^2 - c(\tilde{\mathbf{W}})^2} \psi_l^r$$

where  $\psi_l^r$  is the upwinded source terms  $b_r - b_l + \mathcal{H}(\tilde{S})(\cos \theta_r - \cos \theta_l) + \Psi(\tilde{W})(S_r - S_l)$ .

### 5.2.2 Case of transition point

In the case of a transition point, we assume that the propagation of the interface (pressurised-free surface or free surface-pressurised) has a constant speed  $w$  during a time step, the half line  $x = wt$ , is then the discontinuity line of  $\tilde{D}(\mathbf{W}_l, \mathbf{W}_r)$ . Let us now consider  $U^- = (A^-, Q^-)$  and  $U^+ = (A^+, Q^+)$  the (unknown) states respectively on the left and on the right side of the line  $x = wt$ . Both states  $U_l$  and  $U^-$  (resp.  $U_r$  and  $U^+$ ) correspond to the same type of flow. Thus it makes sense to define the averaged matrices in each zone as follows:

- for  $x < wt$ , we set  $\tilde{D}_l = \tilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$  with  $\widetilde{\mathbf{W}}_l = \frac{\mathbf{W}_l + \mathbf{W}^-}{2}$ .
- for  $x > wt$ , we set  $\tilde{D}_r = \tilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$  with  $\widetilde{\mathbf{W}}_r = \frac{\mathbf{W}_r + \mathbf{W}^+}{2}$ .

Then we formally solve two Riemann problems and uses the Rankine-Hugoniot jump conditions through the line  $x = wt$  which writes:

$$Q^+ - Q^- = w(A^+ - A^-) \quad (25)$$

$$F_5(A^+, Q^+) - F_5(A^-, Q^-) = w(Q^+ - Q^-) \quad (26)$$

with  $F_5(A, Q) = \frac{Q^2}{A} + p(X, A)$ . The unknowns are  $U^-$ ,  $U^+$ ,  $UM$ ,  $UP$  and  $w$ . For all existing transition case, the upwinded altitude term  $b_r - b_l$  in [5] are replaced by  $\psi_l^r$ . Therefore, we recall just the case when the pressure state propagating downstream (Fig. 5). It is the case when on the left side of the line  $\xi = wt$  we have a pressurised flow and on the right side we have a free surface flow: the speed  $w$  of the transition point being positive. Following Song [24] (see also [12]), an equivalent stationary hydraulic jump must occur from a supercritical to a subcritical condition and thus the characteristics speed satisfies the inequalities:

$$\tilde{u}_r + c(\widetilde{\mathbf{W}})_r < w < \tilde{u}_l + c$$

where  $c$  is the sound speed for the pressure flow,  $\tilde{u}_l$ ,  $\tilde{u}_r$ , and  $c(\widetilde{\mathbf{W}})_r$  are defined by the same formula obtained in the case of a non transition point but according to  $\tilde{D}_l$  and  $\tilde{D}_r$ .

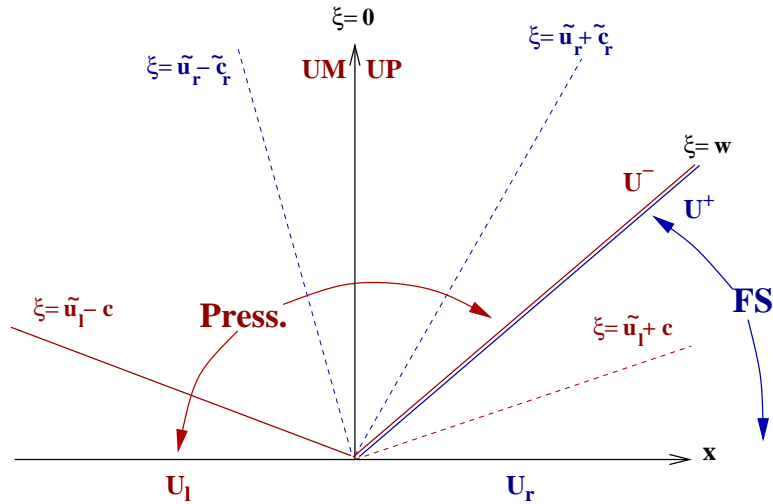


Figure 5: Pressure state propagating downstream

Therefore, only the characteristic lines drawn with solid lines are taken into account. Indeed they are related to incoming waves with respect to the corresponding space-time area  $-\infty < \xi < w$ . Conversely, the dotted line  $\xi = \tilde{u}_r - c(\tilde{\mathbf{W}})_r$ , for instance, related to the free surface zone but drawn in the area of pressurised flow is a "ghost wave" and is not considered. Thus  $U^+ = U_r$  and  $U_l, U^-$  are connected through the jumps across the characteristics  $\xi = 0$  and  $\xi = \tilde{u}_g - c$ . Eliminating  $w$  in the Rankine-Hugoniot jump relations (25)-(26), we get  $U^-$  as the solution to the nonlinear system:

$$(F_5(A_r, Q_r) - F_5(A^-, Q^-)) = \frac{(Q_r - Q^-)^2}{(A_r - A^-)} \quad (27)$$

$$Q^- - Q_l - (A^- - A_l)(\tilde{u}_l - c) + \frac{g\psi_l^r \tilde{A}_l}{c + \tilde{u}_l} = 0 \quad (28)$$

Finally, we obtain :

$$\begin{cases} AP & = A^- \\ (QM = QP) & = Q^- \\ AM & = AP - \frac{g\tilde{A}_l\psi_l^r}{\tilde{u}_l^2 - c^2}. \end{cases}$$

Using equations (23) we update the values of  $A_i^{n+1}, Q_i^{n+1}$  with a standard stability condition of Courant-Friedrich-Levy controlling the time step size  $\Delta t$ .

To update the state of the mesh, we introduce a state variable  $E$  equal to 1 for a pressurised flow and  $E = 0$  else. Following [5], after the computation of the "pseudo wet area"  $A_i^{n+1}$  we predict the state of the mesh  $m_i$  by the following criterion:

- if  $E_i^n = 0$  then :  
if  $A_i^{n+1} < A_{max}$  then  $E_i^{n+1} = 0$ , else  $E_i^{n+1} = 1$ ,
- if  $E_i^n = 1$  :  
if  $A_i^{n+1} \geq A_{max}$  then  $E_i^{n+1} = 1$ , else  $E_i^n = E_{i-1}^n \cdot E_{i+1}^n$ .

Indeed, if  $A_i^{n+1} \geq A_{max}$  it is clear that the mesh  $m_i$  becomes pressurised, on the other hand if  $A_i^{n+1} < S_{maxi}$  in a mesh previously pressurised, we do not know *a priori* if the new state is free surface ( $\rho = \rho_0$  and the value of the wetted area is less than  $S_{maxi}$ ) or pressurised (in depression, with  $\rho < \rho_0$  and the value of the wetted area is equal to  $A_{max}$ ).

So far as we do not take into account complex phenomena such that entrapment of air pockets or cavitation and keeping in mind that the CFL condition ensures that a transition point crosses at most one mesh at each time step, we postulate that:

1. if the mesh  $m_i$  is free surface at time  $t_n$ , its state at time  $t_{n+1}$  is only determined by the value of  $A_i^{n+1}$  and it cannot become in depression.
2. if the mesh  $m_i$  is pressurised at time  $t_n$  and if  $A_i^{n+1} < A_{max}$ , it becomes free surface if and only if at least one adjacent mesh was free surface at time  $t_n$ .

## 6 Numerical tests

The numerical validation for pipes with constant section and slope has been previously studied by two of the authors in [5, 6] and thus are not presented in this paper. Since experimental data for mixed flows in any pipes are not available, we focus on the behavior of our method for several circular cross-section contracting



and expanding pipe. Notice that, the equivalent pipe method is not relevant for the mixed flows as pointed out by [1, 25, 26] for instance.

The mixed flow case is numerically performed on a water hammer test. Starting from an horizontal free surface steady state, the water hammer occurs immediately after the increase of the upstream piezometric head while the downstream discharge is setted to 0. The prescribed hydrograph produces then a travelling wave which produces a pressurised state propagating from upstream to downstream end. Physically an trapped air pocket may appears: it is not taken into account in the PFS-model. Actually, the trapped air pockets vanish or move; some parts of these pockets undergo condensation/vaporisation and others parts move and lead to a two phase flow. Consequently the sound speed decreases. As our model does not take into account of these phenomena, the value of  $c$  is assumed to be constant. Moreover we should have to deal with the entrapment of air bubbles which have a non negligible effect (see [15, 23] for instance).

The numerical experiments are performed in the case of a 100 m long closed pipe circular pipe at altitude  $b_0 = 1$  m with 0 slope which corresponds to the elevation and slope of the main pipe axis (we have  $Z = b(X) = 0, \forall X$ ). The Manning roughness coefficient is  $1/K_s^2 = 0.012$  s/m<sup>1/3</sup>. The simulation starts from a steady state as a free surface flow with a discharge  $Q = 0$  m<sup>3</sup>/s. The upstream boundary condition is a prescribed hydrograph (see FIG. 7) while the downstream discharge is kept constant set to 0 m<sup>3</sup>/s (as displayed on FIG. 7). We compare then the results obtained for uniform, contracting an expanding pipes. For each test, the parameters are the same except the downstream diameter: the upstream diameter is kept constant to  $D = 1$  m. The contracting pipe is choosed for  $D = 0.6$  m and the expanding one for  $D = 1.4$  m (where  $D$  denotes the downstream diameter). Let us recall that the zero water level corresponds to the main pipe axis. The piezometric head is defined as:

$$piezo = z + p \quad \text{with} \quad \begin{cases} p = 2R + \frac{c^2(\rho - \rho_0)}{\rho_0 g} & \text{if the flow is pressurised} \\ p = h & \text{the water height if the flow is free surface} \end{cases}$$

Results are then represented on FIG. 8. The sudden elevation of the upstream piezometric level produces a pressurised state with a travelling wave. A water hammer is then observed since the downstream discharge is null. A careful analysis of the flow (which is performed by the variable  $E$  in the numerical code) shows that after this transition point, the flow is pressurised but in depression which starts approximatively at time 20s for the contracting pipe, 24s for the uniform pipe and 28s for the expanding one. We observe also a little smoothing effect and absorption due to the first order discretisation type.

## 7 Conclusion

We have derive a free surface and a pressurised model which have been coupled using a common set of variables and a suitable pressure law. We obtained then a mathematical model for unsteady mixed flows in non uniform water pipes, that we have called PFS-model. This model takes into account the local perturbation of the section and of the slope. Moreover it presents a simple form which is suitable for a numerical treatment as finite volume scheme or kinetic scheme. We choose to present the extension of the existing method in [5] for the simulation of the PFS-model by the use of a VFRoe-like method to treat the discontinuities occurring at the transition points. Moreover, as mentioned in [5] this numerical method reproduce correctly laboratory tests for uniform pipes and can deal with multiple points of transition between the two type of flows. As pointed out before, by the lack

of experimental data for non uniform pipes, we have only shown the behavior of the piezometric line which seems reasonable (at least no great difference are observed). Furthermore, the current adaptation to the PFS-model using a VFRoe like method are easily implemented. Unfortunately, this finite volume scheme has not a so good behavior when stationary hydraulic jump is present. This is due to the upwinding of the source term. Moreover, it does not deal with drying or flooding area. We are, at the present time, interested in a mathematical kinetic formulation of the PFS and the construction of a numerical kinetic scheme that avoids all these inconvenients.

The next step is to take into account the air entrapment which may have non negligible effects on the behaviour of the piezometric head. A first approach has been derived in the case of perfect fluid and perfect gas seen has a bilayer model based on the PFS-model.

## References

- [1] A. Adamkowski. Analysis of transient flow in pipes with expanding or contracting sections. *ASME J. of Fluid Engineering*, 125:716–722, 2003.
- [2] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.*, 171(3):485–541, 2008.
- [3] F. Bouchut, E.D. Fernández-Nieto, A. Mangeney, and P.-Y. Lagrée. On new erosion models of savage-hutter type for avalanches. *Acta Mech.*, 199:181–208, 2008.
- [4] F. Bouchut, A. Mangeney-Castelnau, B. Perthame, and J.-P. Vilotte. A new model of Saint Venant and Savage-Hutter type for gravity driven shallow water flows. *C. R. Math. Acad. Sci. Paris*, 336(6):531–536, 2003.
- [5] C. Bourdarias and S. Gerbi. A finite volume scheme for a model coupling free surface and pressurised flows in pipes. *J. Comp. Appl. Math.*, 209(1):109–131, 2007.
- [6] C. Bourdarias and S. Gerbi. A conservative model for unsteady flows in deformable closed pipes and its implicit second order finite volume discretisation. *Computers & Fluids*, 37:1225–1237, 2008.
- [7] C. Bourdarias, S. Gerbi, and M. Gisclon. A kinetic formulation for a model coupling free surface and pressurised flows in closed pipes. *J. Comp. Appl. Math.*, 218(2):522–531, 2008.
- [8] M. Boutounet, L. Chupin, P. Noble, and J-P. Vila. Shallow water viscous flows for arbitrary topopgraphy. *Commun. Math. Sci.*, 6(1):29–55, 2008.
- [9] D. Bresch and P. Noble. Mathematical justification of a shallow water model. *Methods Appl. Anal.*, 14(2):87–117, 2007.
- [10] H. Capart, X. Sillen, and Y. Zech. Numerical and experimental water transients in sewer pipes. *Journal of Hydraulic Research*, 35(5):659–672, 1997.
- [11] Nguyen Trieu Dong. Sur une méthode numérique de calcul des écoulements non permanents soit à surface libre, soit en charge, soit partiellement à surface libre et partiellement en charge. *La Houille Blanche*, 2:149–158, 1990.
- [12] Musandji Fuamba. Contribution on transient flow modelling in storm sewers. *Journal of Hydraulic Research*, 40(6):685–693, 2002.

- [13] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Cont. Dyn. Syst. Ser. B*, 1(1):89–102, 2001.
- [14] J.M. Greenberg and A.Y. LeRoux. A well balanced scheme for the numerical processing of source terms in hyperbolic equation. *SIAM J. Numer. Anal.*, 33(1):1–16, 1996.
- [15] M.A. Hamam and A. McCorquodale. Transient conditions in the transition from gravity to surcharged sewer flow. *Can. J. Civ. Eng.*, (9):189–196, 1982.
- [16] F. Kerger, P. Archambeau, S. Erpicum, B. J. Dewals, and M. Pirotton. Improved one-dimensional numerical simulation of transient mixed flow in water pipe. In *Proceedings of 4th Int. Conf. on Advanced Computational Methods in Engineering Liege, Belgium.*, 2008.
- [17] F. Kerger, P. Archambeau, S. Erpicum, B. J. Dewals, and M. Pirotton. Numerical simulation of highly transient mixed flow in sewer system. *La Houille Blanche*, 2009. to appear.
- [18] F. Kerger, S. Detrembleur, P. Archambeau, S. Erpicum, B. J. Dewals, and M. Pirotton. An experimental analysis of effects induced by moving bodies in shallow water. In *Proceedings of 2nd Int. Junior Researcher and Engineer Workshop on Hydraulic Structures, Pisa, Italy.*, 2008.
- [19] C-D. Levermore, M. Oliver, and Edriss S. Titi. Global well-posedness for models of shallow water in a basin with a varying bottom. *Indiana University Mathematics Journal.*, 45(2), 1996.
- [20] F. Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic. B, Fluids*, 26(1):49–63, 2007.
- [21] P.L. Roe. Some contributions to the modelling of discontinuous flow. In B. E. Engquist, S. Osher, and R. C. J. Somerville, editors, *Large-scale computations in fluid mechanics. Part 2. Proceedings of the fifteenth AMS-SIAM summer seminar on applied mathematics held at Scripps Institution of Oceanography, La Jolla, Calif., June 27-July 8, 1983*, volume 22 of *Lectures in Applied Mathematics*, pages 163–193. American Mathematical Society, 1985.
- [22] A. Y. Le Roux and M. N. Le Roux. Convergence d’un schéma à profils stationnaires pour les équations quasi linéaires du premier ordre avec termes sources. *C. R. Acad. Sci., Sér. I, Math.* 333(7):703–706, 2001.
- [23] C.S.S. Song. Two-phase flow hydraulic transient model for storm sewer systems. In *Second international conference on pressure surges*, pages 17–34, Bedford, England, 1976. BHRA Fluid engineering.
- [24] C.S.S. Song, J.A. Cardle, and K.S. Leung. Transient mixed-flow models for storm sewers. *Journal of Hydraulic Engineering, ASCE*, 109(11):1487–1503, 1983.
- [25] V.L. Streeter, E.B. Wylie, and K.W. Bedford. *Fluid Mechanics*. McGraw-Hill, 1998.
- [26] E.B. Wylie and V.L. Streeter. *Fluid Transients*. McGraw-Hill, New York, 1978.

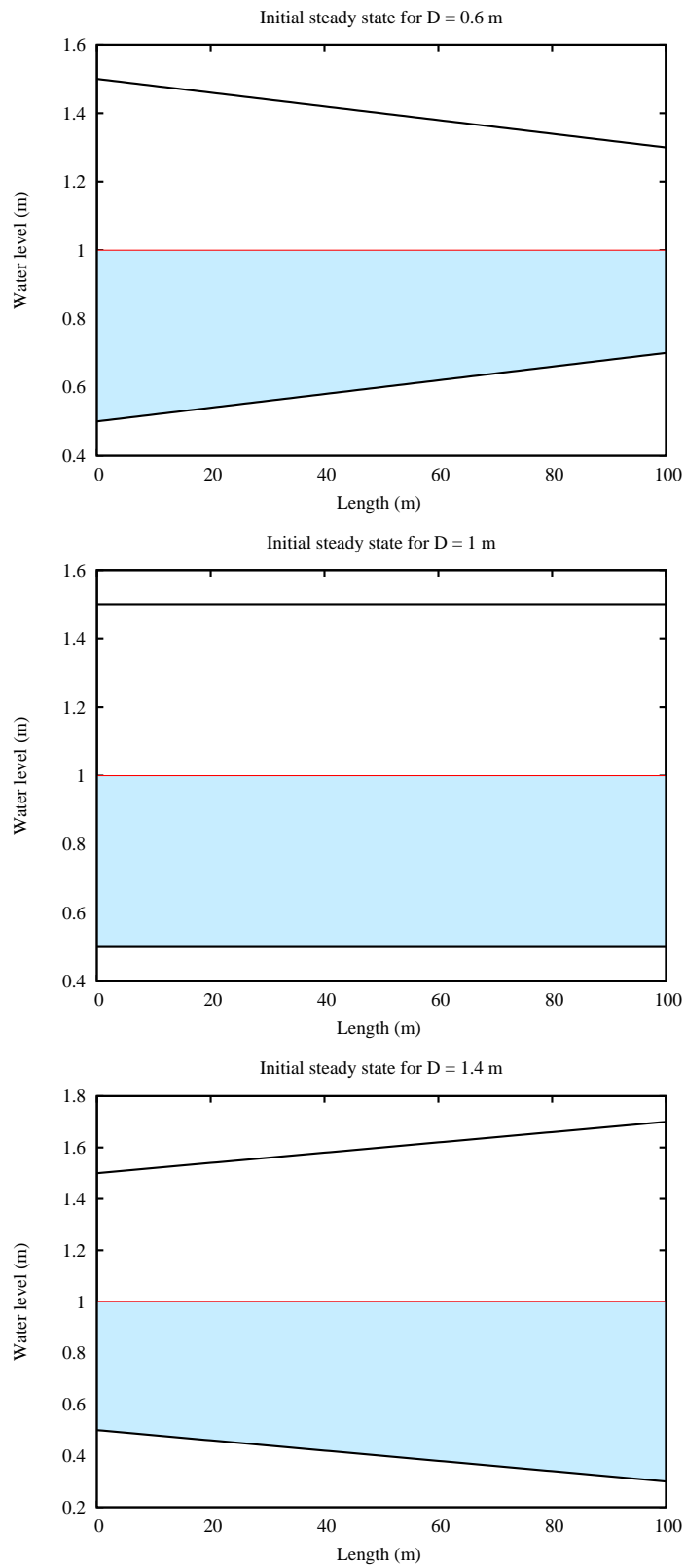


Figure 6: Initial still water steady state for contracting, uniform and expanding pipes

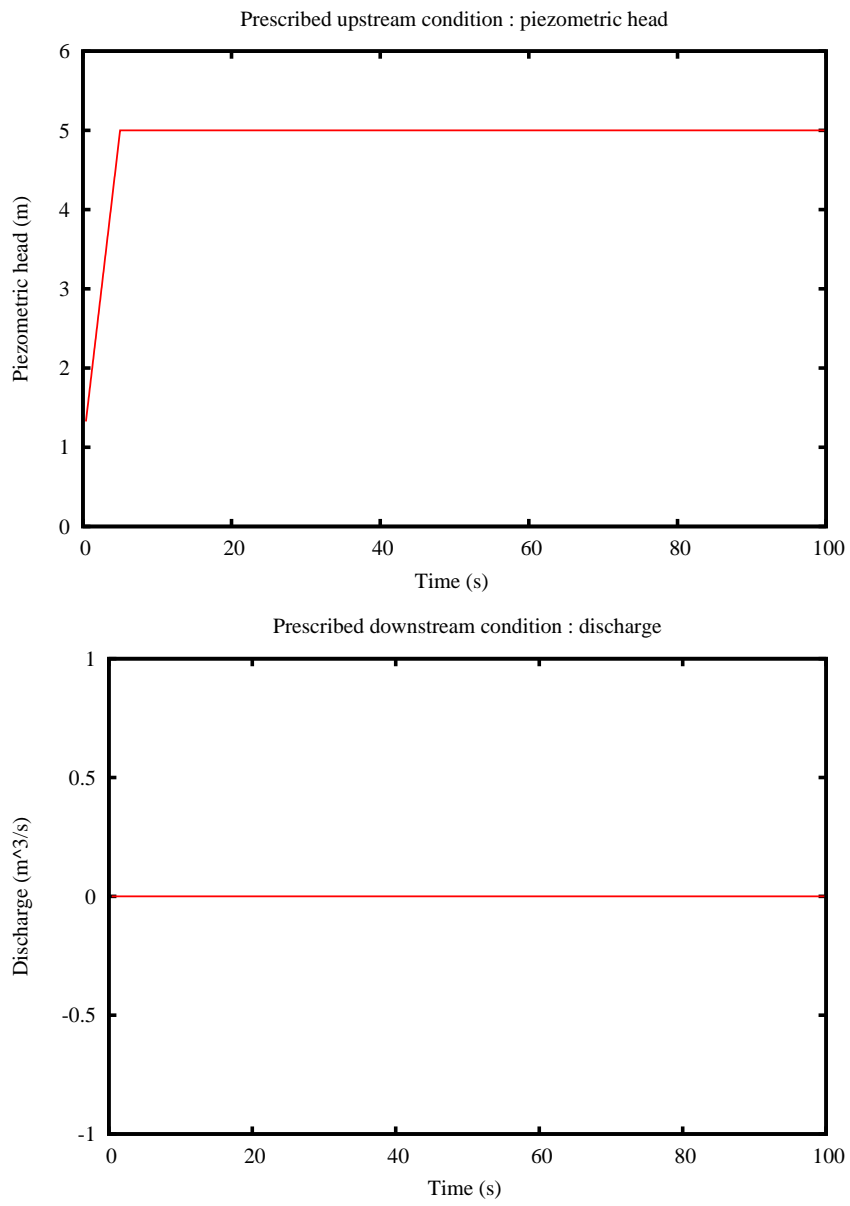


Figure 7: Boundary conditions

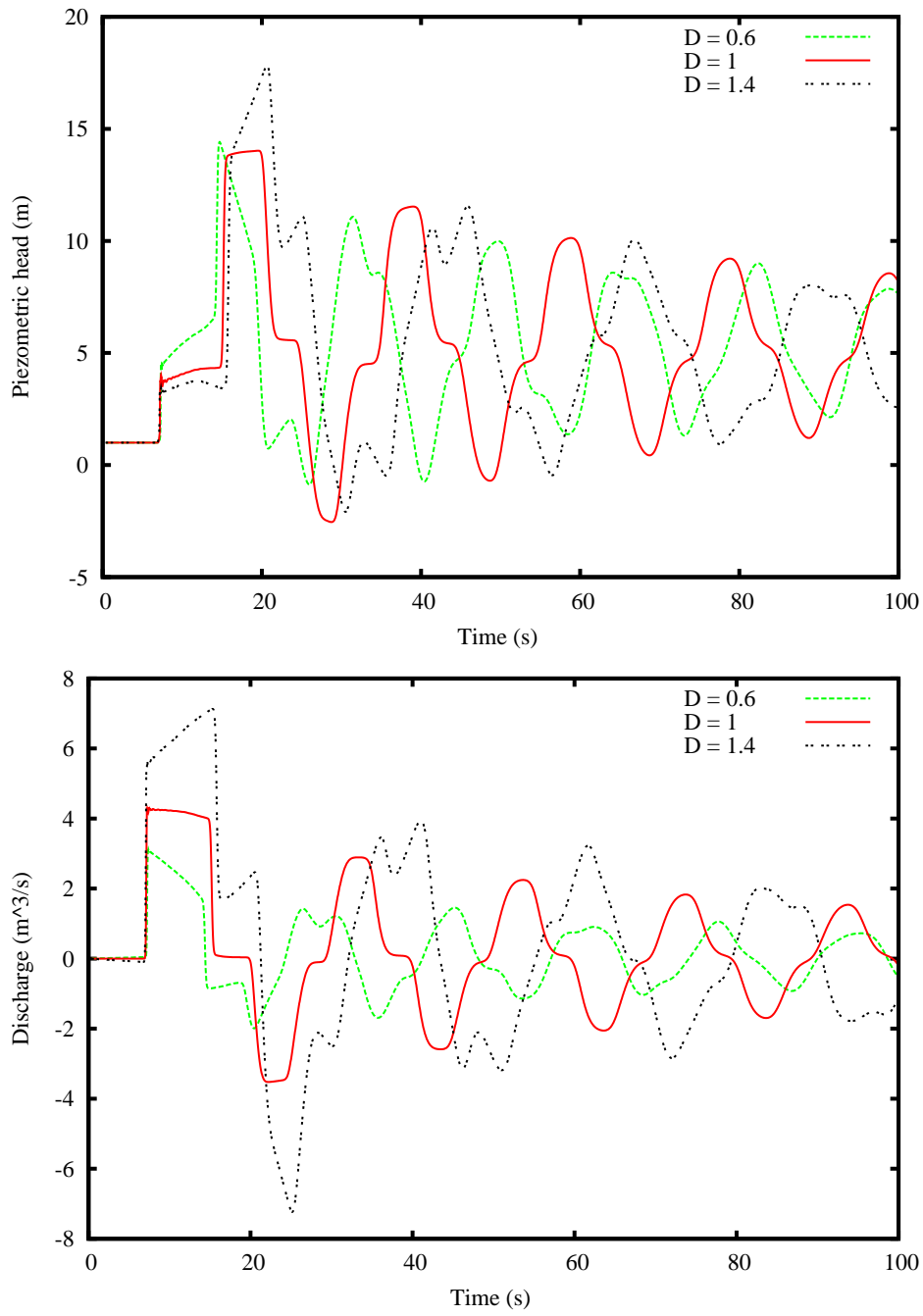


Figure 8: Piezometric head and discharge at  $X = 50m$

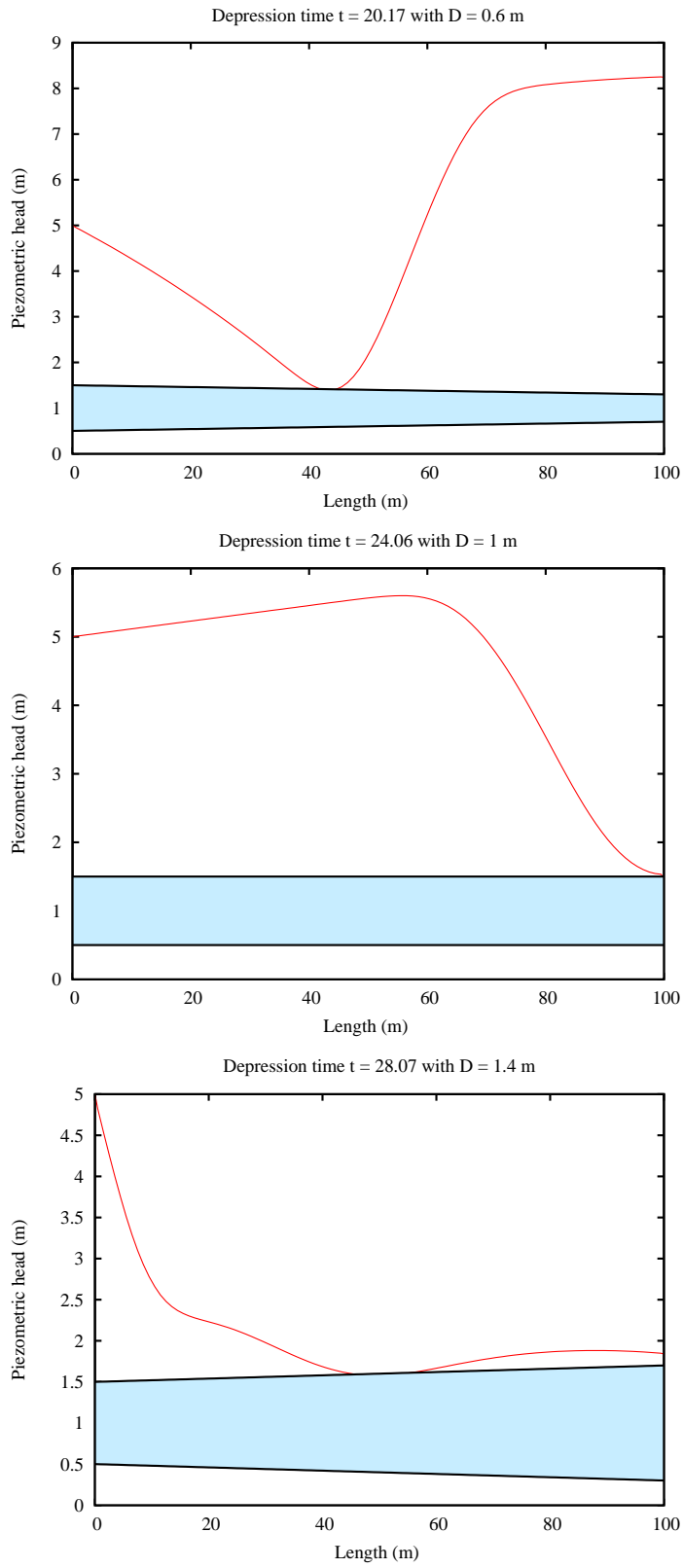


Figure 9: First depression times for contracting, uniform and expanding pipes