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# A mathematical model for unsteady mixed flows in closed water pipes

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## Abstract

We present the derivation of a new unidirectional model for unsteady mixed flows in non uniform closed domains. We introduce a local reference frame to take into account the local perturbation caused by the changes of section and slope. Then an asymptotic analysis is performed to obtain a model for the free surface flow and another for the pressurised flow. By coupling these models through the transition points by the use of a common set of variables and a suitable pressure law, we obtain a simple formulation called PFS-model close to the shallow water equations with source terms. It takes into account the changes of section and the slope variation in a continuous way through transition points.

**Keywords** : Shallow water equations, mixed flows, free surface flows, pressurised flows, curvilinear transformation, asymptotic analysis.

## 1 Introduction

The presented work takes place in a more general framework: the modelisation of unsteady mixed flows in any kind of closed domain taking into account the cavitation problem and air entrapment. We are interested in flows occurring in closed pipe with non uniform sections, where some parts of the flow can be free surface (it means that only a part of the pipe is filled) and other parts are pressurised (it means that the pipe is full). The transition phenomenon between the two types of flows occurs in many situations such as storm sewers, waste or supply pipes in hydroelectric installations. It can be induced by sudden change in the boundary conditions as failure pumping. During this process, the pressure can reach severe values and cause damages. The simulation of such a phenomenon is thus a major challenge and a great amount of works was devoted to it these last years (see [9],[15],[16],[8] for instance). Recently Fuamba [10] proposed a model for the transition from a free surface flow to a pressurised one in a way very close to ours.

The classical shallow water equations are commonly used to describe free surface flows in open channels. They are also used in the study of mixed flows using the Preissman slot artefact (see for example [8, 16]). However, this technic does not take into account the depressurisation phenomenon which occurs during a water hammer. On the other hand the Allievi equations, commonly used to describe pressurised flows, are written in a non-conservative form which is not well adapted to a natural coupling with the shallow water equations.

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A model for the unsteady mixed water flows in closed pipes and a finite volume discretization have been previously studied by two of the authors [3] and a kinetic formulation has been proposed in [5]. We propose here the PFS-model which tends to extend naturally the work in [3] in the case of a closed pipe with non uniform section. For the sake of simplicity, we do not deal with the deformation of the domain induced by the change of pressure. We will consider only an infinitely rigid pipe.

The paper is organized as follows. The first section is devoted to the derivation of the free surface model from the 3D incompressible Euler equations which are written in a suitable local reference frame in order to take into account the local effects produced by the changes of section and the slope variation. To this end, we present two models derived by two techniques inspired from the works in [2] and [12]. The first one consists in taking the mean value in the Euler equations along the normal section to the main axis. The obtained model provides a description taking in account the geometry of the domain, namely the changes of section and also the inertia strength produced by the slope variation. The second one is a formal asymptotic analysis. In this approach, we seek for an approximation at the first order and, by comparison with the previous model, the term related to the inertia strength vanishes since it is a term of second order. We obtain the FS-model. However, the system is more convenient for the coupling process done in Section 4. In Section 3, we follow the derivation of the FS-model and we derive the model for pressurised flows, called P-model, from the 3D compressible Euler equations by a formal asymptotic analysis. Writing the source terms into a unified form and using the same couple of conservative unknowns as in [4], we propose a natural model for mixed flows, that we call PFS-model, which ensures the continuity of the unknowns and the source terms.

## 2 Formal derivation of the FS-model for free surface flow

The classical shallow water equations are commonly used to describe physical situations like rivers, coastal domains, oceans and sedimentation problems. These equations are obtained from the incompressible Euler system (see e.g. [1, 13]) or from the incompressible Navier-Stokes system (see for instance [6, 7, 12, 14]) by several techniques (e.g. by direct integration or asymptotic analysis). We present here the derivation of a new unidirectional shallow water model from the incompressible Euler equations for non uniform closed pipe using a section-averaging and a formal asymptotic analysis. We write the Euler equations in the local Serret-Frenet reference frame in order to take into account the local effects produced by the changes of section and the slope variation.

The 3D incompressible Euler system writes:

$$\begin{cases} \operatorname{div} \vec{U} &= 0 \\ \partial_t \vec{U} + \vec{U} \cdot \nabla \vec{U} + \nabla \cdot P &= \vec{F} \end{cases} \quad (1)$$

where  $\vec{U}(t, x, y, z)$  denotes the velocity with components  $(u, v, w)$ ,  $P = p(t, x, y, z)I_3$  the isotropic pressure tensor and  $\vec{F}$  the exterior strength (including gravity).

We define the domain  $\Omega_F(t)$  of the flow at time  $t$  as the union of sections  $\Omega(t, x)$  (assumed to be simply connected compact sets) orthogonal to some plane curve with parametrization  $(x, 0, b(x))$  in a convenient cartesian reference frame  $(O, \vec{i}, \vec{j}, \vec{k})$  where  $\vec{k}$  follows the vertical direction;  $b(x)$  is then the elevation of

the point  $\omega(x, 0, b(x))$  over the plane  $(O, \vec{i}, \vec{j})$ . Then, at each point  $\omega(x, 0, b(x))$ ,  $\Omega(t, x)$  is defined by the set

$$\{(y, z) \in \mathbb{R}^2; z \in [-R(x), -R(x) + H(t, x)], y \in [a(x, z), b(x, z)]\}.$$

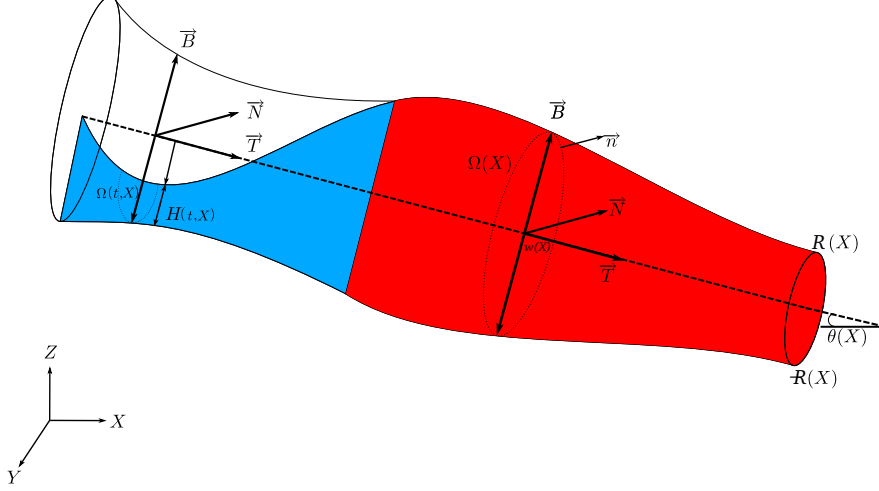


Figure 1: Geometric characteristics of the domain  
Mixed flow: free surface and pressurised

We close classically System (1) using a kinematic law for the evolution of the free surface: *any free surface particle is advected by the fluid velocity  $\vec{U}$*  and on the wet boundary we assume the no-leak condition  $\vec{U} \cdot \vec{n}_{wp} = 0$  where  $\vec{n}_{wp}$  is the outward unit normal vector to the wet boundary (see FIG. 2) defined by  $\frac{1}{\sqrt{1 + b'(x)^2}}(-b'(x), 0, 1)^t = (-\sin \theta(x), 0, \cos \theta(x))^t$ . We set the atmospheric pressure  $P(z = h(t, x))$  to 0 at the free surface.

To define the local reference frame and to perform the curvilinear transformation, let us introduce the curvilinear variable defined by

$$X = \int_{x_0}^x \sqrt{1 + (b'(\xi))^2} d\xi$$

where  $x_0$  is an arbitrary abscissa. We set  $y = Y$  and we denote by  $Z$  the altitude of any fluid particle  $M$  in the Serret-Frenet reference frame  $(\vec{T}, \vec{N}, \vec{B})$  at point  $\omega(x, 0, b(x))$ :  $\vec{T}$  is the tangent vector,  $\vec{N}$  the normal vector and  $\vec{B}$  the binormal vector (see FIG. 1). We denote by  $\vec{OM}$  the vector position of any fluid particle at the height  $Z$  along the binormal axis  $\vec{B}$ .

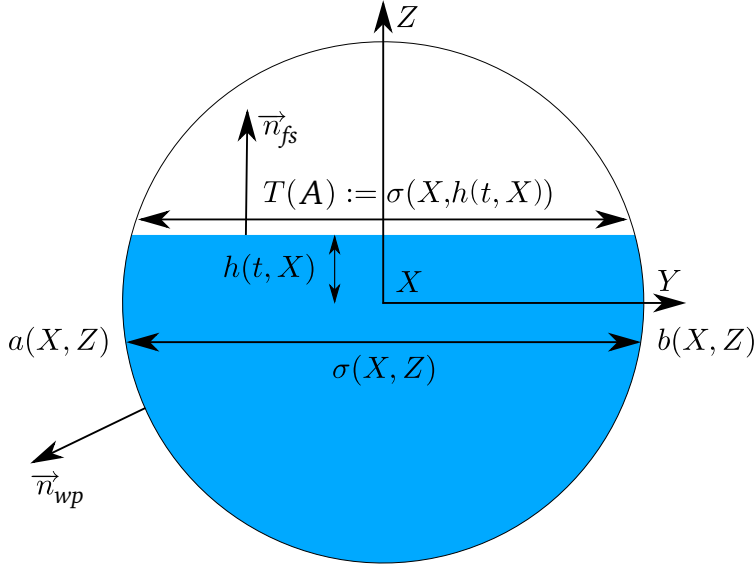


Figure 2: Cross-section of the domain

In the following, we assume that:

- (H) the algebraic curvature radius  $\mathcal{R}$  of the curve  $x \mapsto (x, 0, b(x))$  is such that  $\forall X, |\mathcal{R}(X)| > R(X)$ ,

## 2.1 Incompressible Euler model in the curvilinear coordinates

To write Sytem (1) in the Serret-Frenet reference frame  $(\vec{T}, \vec{N}, \vec{B})$  at point  $\omega(x, 0, b(x))$ , we perform, following [2], the transformation  $\mathcal{T} : (x, y, z) \rightarrow (X, Y, Z)$  and we recall a useful lemma:

**Lemma 2.1** *Let  $(x, y, z) \mapsto \mathcal{T}(x, y, z)$  be a  $C^1$  diffeomorphism and  $\mathcal{A}^{-1} = D_{(x,y,z)}\mathcal{T}$  the jacobian matrix of the transformation with determinant  $J$ .*

*Then, for any vector field  $\Phi$  one has,*

$$JD_{(X,Y,Z)}\Phi = D_{(x,y,z)}(J\mathcal{A}\Phi).$$

*In particular, for any scalar function  $f$ , one has*

$$D_{(X,Y,Z)}f = \mathcal{A}^t D_{(x,y,z)}f.$$

Let  $(U, V, W)^t$  be the components of the velocity vector in the  $(X, Y, Z)$  coordinates defined as  $(U, V, W)^t = \Theta(u, v, w)^t$  where  $\Theta$  is the matrix

$$\Theta = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}.$$

To write the equation  $\text{div} \vec{U} = 0$  in the curvilinear coordinates, we apply Lemma 2.1 and by a straightforward computation, we obtain the incompressibility condition in the  $(X, Y, Z)$  coordinates as:

$$J\nabla \cdot \vec{U} = \partial_X U + \partial_Y (JV) + \partial_Z (JW) = 0 \quad (2)$$

where  $J(X, Y, Z) = 1 - Z \frac{d\theta}{dX}$  is the determinant of the transformation.

**Remark 2.1** Notice that  $\kappa(X) = \frac{d\theta}{dX}$  is the algebraic curvature of the axis at point  $\omega(x)$  and the function  $J(X, Y, Z) = 1 - Z\kappa(X)$  depends only on the variables  $X, Z$ . Moreover, we have  $J > 0$  in  $\Omega_F$  which corresponds to the reasonable geometric hypothesis (H). Consequently,  $\mathcal{T}$  defines a diffeomorphism and thus the performed transformation is admissible.

Now, to perform the curvilinear transformation for the momentum equations, we can remark that the identity  $(\vec{U} \cdot \nabla) f = \nabla \cdot (f \vec{U})$  holds (thanks to  $\text{div}(\vec{U}) = 0$ ). Then for any scalar field  $f$ , the term  $(\partial_t + \vec{U} \cdot \nabla) f$  can be written as

$$\nabla_{(t,x,y,z)} \cdot \begin{pmatrix} f \\ f \vec{U} \end{pmatrix}.$$

Applying Lemma 2.1 with  $\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \nabla_{(X,Y,Z)} \overrightarrow{OM} \end{pmatrix}$ , we get for all scalar  $C^1$  function  $f$ :

$$J(\partial_t + \vec{U} \cdot \nabla) f = \partial_t(Jf) + \partial_X(fU) + \partial_Y(JfV) + \partial_Z(JfW). \quad (3)$$

Next, to obtain the equation for  $(U, V, W)^t$ , we multiply the conservation equation (1) on the left by the matrix  $J\Theta$  and we get the following incompressible Euler system in the variables  $(X, Y, Z)$ :

$$\begin{cases} \partial_X U + \partial_Y(JV) + \partial_Z(JW) & = 0 \\ \partial_t(JU) + \partial_X(U^2) + \partial_Y(JUV) + \partial_Z(JUW) + \partial_X p & = G_1 \\ \partial_t(JV) + \partial_X(UV) + \partial_Y(JV^2) + \partial_Z(JVW) + \partial_Y(Jp) & = 0 \\ \partial_t(JW) + \partial_X(UW) + \partial_Y(JVW) + \partial_Z(JW^2) + J\partial_Z(p) & = G_2 \end{cases} \quad (4)$$

where  $G_1 = UW\kappa(X) - Jg \sin \theta$  and  $G_2 = -U^2\kappa(X) - Jg \cos \theta$ .

On the wet boundary, the no-leak condition reads:

$$(U, V, W)^t \cdot \vec{n}_{wp} = 0, \quad (5)$$

and the condition on the pressure at the free surface reads:

$$P(Z = h(t, X)) = 0, \quad (6)$$

where  $\vec{n}_{wp} = (-\sin \theta(X), 0, \cos \theta(X))^t$  and .

## 2.2 Shallow water equations for free surface flows by orthogonal averaging

In this section, we derive the shallow water equations by taking the mean value of the incompressible Euler system (4) along the cross-section  $\Omega(t, X)$ . To this end, let us introduce the conservative variables  $A(t, X)$  and  $Q(t, X) = A(t, X)\bar{U}$  representing respectively the wet area and the discharge defined as:

$$A(t, X) = \int_{\Omega} dY dZ, \quad Q(t, X) = A(t, X)\bar{U}(t, X)$$

where  $\bar{U}$  is the mean value of the speed

$$\bar{U}(t, X) = \frac{1}{A(t, X)} \int_{\Omega} U(t, X, Y, Z) dY dZ.$$

The shallow water assumption consists in neglecting the binormal and normal accelerations. Consequently, this hypothesis leads to the classical shallow water equations with the hydrostatic pressure law  $\partial_z p = -g$ . Here, due to the source term  $-U^2\kappa(X)$  in System (4), we get a *modified hydrostatic pressure*.

Indeed, if we write the shallow water assumption in the variables  $(X, Y, Z)$  by applying the identity (3) successively with  $f = X, Y$  and  $Z$ , the following equalities hold (for the sake of simplicity, we do not write the dependency of  $X(t), Y(t), Z(t)$  on  $x, y, z$ ):

$$\begin{cases} J \frac{d}{dt} X &= J \left( \partial_t X + (\vec{U} \cdot \nabla)(X) \right) &= U \\ J \frac{d}{dt} Y &= J \left( \partial_t Y + (\vec{U} \cdot \nabla)(Y) \right) &= JV \\ J \frac{d}{dt} Z &= J \left( \partial_t Z + (\vec{U} \cdot \nabla)(Z) \right) &= JW \end{cases} \quad (7)$$

This yields

$$\begin{cases} J \frac{d}{dt} V(t, X(t), Y(t), Z(t)) &= 0 \\ J \frac{d}{dt} W(t, X(t), Y(t), Z(t)) &= 0 \end{cases}$$

We have then

$$\begin{cases} \partial_t V + \frac{d}{dt} X(t) \partial_X V + \frac{d}{dt} Y(t) \partial_Y (JV) + \frac{d}{dt} Z(t) \partial_Z V &= 0 \\ \partial_t (JW) + \frac{d}{dt} X(t) \partial_X W + \frac{d}{dt} Y(t) \partial_Y W + \frac{d}{dt} Z(t) \partial_Z W &= 0 \end{cases}$$

So we finally obtain:

$$\begin{cases} \partial_t (JV) + U \partial_X V + JV \partial_Y V + JW \partial_Z V &= 0 \\ \partial_t (JW) + U \partial_X W + JV \partial_Y W + JW \partial_Z W &= 0 \end{cases} \quad (8)$$

Now, according to System (8) and the equations for  $V$  and  $W$  in System (4), we obtain:

$$\begin{cases} \partial_Y p &= 0 \\ \partial_Z p &= -\frac{U^2}{J} \kappa(X) - g \cos \theta \end{cases}$$

Then the modified hydrostatic pressure reads:

$$p(t, X, Y, Z) = P_{hs}(t, X, Y, Z) + P_c(t, X, Y, Z) \quad (9)$$

where  $P_{hs}(t, X, Y, Z) = (h(t, X) - Z)g \cos \theta$  denotes the usual hydrostatic pressure and  $P_c(t, X, Y, Z) = -\int_Z^{h(t, X)} U(t, X, Y, \xi)^2 \frac{\kappa(X)}{1 - \xi \kappa(X)} d\xi$  the term due to the inertia effect.

**Remark 2.2** We can use the particular form  $U(t, X, Y, Z) = \frac{u(t, X, Y)}{J(X, Z)}$  (described in [2]) when  $P_c$  derives from a potential. Thus, one obtains:

$$P_c(t, X, Y, Z) = \frac{u^2(t, X, Y)}{2} \left( \frac{1}{(1 - h(t, X)\kappa(X))^2} - \frac{1}{(1 - Z\kappa(X))^2} \right).$$

To close System (4), it remains to define the free surface condition in curvilinear coordinates. Let us consider the boundary  $\partial\Omega(t, X)$  as the union of  $\gamma_{fs}(t, X)$  and  $\gamma_{wp}(t, X)$  where  $\gamma_{fs}(t, X)$  and  $\gamma_{wp}(t, X)$  represents the free surface and the wet perimeter on the  $OYZ$ -plane. We assume the no-leak condition (5) on the wet boundary  $\gamma_{wp}$  and the kinematic boundary condition on  $\gamma_{fs}$ . Denoting any fluid particle  $M(Z = h)$  by  $M_{fs}$  and from the boundary conditions, we have:

$$\frac{d}{dt}A(t, X(t)) = \partial_t A(t, X(t)) + \frac{d}{dt}X(t) \partial_X A(t, X(t)) = \frac{d}{dt} \int_{\Omega} dY dZ.$$

As System (7) holds, one has:

$$\frac{d}{dt}X(t) = \frac{U}{J} \text{ and } \frac{d}{dt} \int_{\Omega} dY dZ = \int_{\gamma_{fs}} \partial_t M \cdot \vec{n}_{fs} ds + \int_{\gamma_{wp}} \partial_t M \cdot \vec{n}_{wp} ds$$

where  $\vec{n}_{fs}$  denotes the outward unit normal vector to the free surface (see on FIG. 2). Since  $\partial_t M$  represents the fluid particle velocity  $(U, V, W)^t$  on the wet boundary, the integral  $\int_{\gamma_{wp}} \partial_t M \cdot \vec{n}_{wp} ds$  vanishes thanks to the no-leak condition.

Next, we have  $\int_{\gamma_{fs}} \partial_t M \cdot \vec{n}_{fs} ds = T(X)W_{fs}$ . Then, we get the following free surface boundary condition:

$$\partial_t A(t, X) + \frac{U_{fs}(t, X)}{J_{fs}(t, X)} \partial_X A(t, X) = T(X)W_{fs}(t, X) \quad (10)$$

where we have used the generic notation  $F_{fs}$  to denote  $F(Z = h)$ .

Now to get the shallow water equations, we integrate Sytem (4) along the normal section  $\Omega(t, X)$ . Thus,

$$\begin{aligned} \int_{\Omega} \partial_X U dY dZ + \int_{\Omega} \operatorname{div}_{Y,Z} \left( \begin{array}{c} JV \\ JW \end{array} \right) dY dZ &= \partial_X \int_{\Omega} U dY dZ - \int_{\partial\Omega} U \partial_X M \cdot \vec{n} ds \\ &+ \int_{\partial\Omega} \left( \begin{array}{c} JV \\ JW \end{array} \right) \cdot \vec{n} ds \\ &= \partial_X(Q) - \int_{\gamma_{fs}} U \partial_X M \cdot \vec{n}_{fs} ds \\ &+ \int_{\gamma_{fs}} \left( \begin{array}{c} JV \\ JW \end{array} \right) \cdot \vec{n}_{fs} ds \end{aligned}$$

$$\text{since } \int_{\gamma_{wp}} \left( \left( \begin{array}{c} JV \\ JW \end{array} \right) - U \partial_X M \right) \cdot \vec{n}_{wp} ds = 0$$

Moreover,

$$\int_{\gamma_{fs}} U \partial_X M \cdot \vec{n} ds = U_{fs} \partial_X A \quad \text{and} \quad \int_{\gamma_{fs}} \left( \begin{array}{c} JV \\ JW \end{array} \right) \cdot \vec{n}_{fs} ds = J_{fs} W_{fs} T.$$

Next, from the free surface condition (10), we get

$$J_{fs} W_{fs} T = J_{fs} \partial_t A + U_{fs} \partial_X A.$$

Hence the mass conservation equation finally reads:

$$J(X, Z = h) \partial_t (A) + \partial_X Q = 0 \quad (11)$$

To get the momentum conservation equation, we integrate System (4) for  $U$  along the cross-section  $\Omega(t, X)$  with the usual approximation  $\overline{UV} \approx \overline{U} \overline{V}$  and  $\overline{UV^2} \approx$



$\bar{U}\bar{V}^2$ . Finally, the shallow water equations for free surface flows reads:

$$\begin{cases} J(X, Z = h)\partial_t(A) + \partial_X Q & = 0 \\ \partial_t(\bar{J}Q) + \partial_X \left( \frac{Q^2}{A} + A\bar{P}_c + gI_1(X, A) \cos \theta \right) & = G \end{cases} \quad (12)$$

where  $G = gI_2(X, A) \cos \theta - gA \sin \theta - gA(h(A) - I_1(X, A)/A) \frac{d}{dX} \cos \theta(X)$ ,

$\bar{J}(t, X) = J(X, Z = h) + \kappa(X) \frac{I_1(X, A)}{A}$ . The terms  $I_1(X, A)$  and  $I_2(X, A)$  are defined by

$$I_1(X, A) = \int_{-R}^h (h - Z)\sigma dZ \quad \text{and} \quad I_2(X, A) = \int_{-R}^h (h - Z)\partial_X \sigma dZ$$

and represent respectively the classical term of hydrostatic pressure and the pressure source term induced by the change of geometry. In these formulas  $\sigma(X, Z)$  is the width of the cross-section at position  $X$  and at height  $Z$ . In addition, by a straightforward computation, we get:

$$\partial_A I_1(X, A) = A \partial_A h(A) = \frac{A}{T}$$

where  $T(X, A) := \sigma(X, Z = h(A))$  is the width of the free surface.

**Remark 2.3**

- The term  $(h(A) - I_1(X, A)/A) := \bar{Z}(X, A)$  is the  $Z$  coordinate of the center of mass.
- Generally, we cannot compute explicitly the term  $\bar{P}_C(A, Q)$ . Nevertheless, thanks to Remark 2.2, if  $U(t, X, Y, Z) = u(t, X, Y)/J(X, Z)$  then,

$$P_c(t, X, Y, Z) = \frac{J^2(Z)U^2(t, X, Y)}{2} \left( \frac{1}{(1 - h(A)\kappa(X))^2} - \frac{1}{(1 - Z\kappa(X))^2} \right),$$

$$\bar{P}_C = \frac{\bar{J}^2 Q^2}{2A^2} \left( \frac{1}{J_{fs}^2} - \frac{1}{\bar{J}^2} \right)$$

Then System (12) can be written explicitly as:

$$\begin{cases} \partial_t A + \partial_X Q & = G_1 \\ \partial_t(Q) + \partial_X \left( \frac{Q^2}{A} + \frac{\bar{J}^2 Q^2}{2} \left( \frac{1}{J_{fs}^2} - \frac{1}{\bar{J}^2} \right) + gI_1(X, A) \cos \theta \right) & = G_2 \end{cases} \quad (13)$$

where

$$G_1 = h(A)\kappa(X)\partial_t A$$

and

$$G_2 = \kappa(X)\partial_t(h(A)Q - I_1(X, A)\bar{U}) + gI_2(X, A) \cos \theta - gA \sin \theta - gA\bar{Z}(X, A) \frac{d}{dX}(\cos \theta).$$

- If we neglect  $\kappa(X)$  in System (12) or if we assume  $\theta$  to be constant then we get the classical shallow water equations with source terms (see e.g. [3, 11]):

$$\begin{cases} \partial_t A + \partial_X Q & = 0 \\ \partial_t(Q) + \partial_X \left( \frac{Q^2}{A} + gI_1(X, A) \cos \theta \right) & = gI_2(X, A) \cos \theta - gA \sin \theta \end{cases} \quad (14)$$

Now, omitting the overlined notation for  $\bar{U}$  except  $\bar{Z}$  and  $\bar{P}_c$ , let us present some classical properties of the frictionless System (12).

**Theorem 2.1**

1. The System (12) is strictly hyperbolic for  $A(t, X) > 0$  under the condition

$$c^2 + 4\frac{Q}{A}\partial_Q(A\bar{P}_c) + (\partial_Q(A\bar{P}_c))^2 + 4\partial_A(A\bar{P}_c) > 0$$

where  $c(X, A) = \sqrt{\frac{gA \cos \theta}{T(A)}}$  is the sound speed.

2. For smooth solutions, the velocity  $U$  satisfies

$$\bar{J}(A)\partial_t(U) + \partial_X\left(\frac{U^2}{2} + \Psi(A, Q) + gh(A) \cos \theta + gb(X)\right) = 0 \quad (15)$$

where  $\Psi$  is such that  $\partial_A\Psi = \frac{\bar{P}_c}{A} + \partial_A\bar{P}_c$ .

3. The still water steady state for  $U = 0$  is given by

$$h(A) \cos \theta + b(X) = 0 \quad (16)$$

4. It admits a mathematical entropy

$$E(A, Q) = \frac{Q^2}{2A} + A(\Psi(A, Q) - \bar{P}_c) + gA\bar{Z}(X, A) \cos \theta + gb$$

which satisfies the entropy inequality

$$\bar{J}(A)\partial_t E + \partial_X((E + A\bar{P}_c + gI_1(X, A) \cos \theta)U) \leq 0$$

We present in the next section another approach to get a simpler model.

### 2.3 Shallow water-like equations for free surface flow by asymptotic analysis

In this section, we perform a formal asymptotic analysis on System (4). According to the work in [12, 14], the shallow water equations can be obtained from the incompressible Navier-Stokes equations with particular boundary conditions. Here, we perform this analysis directly on the incompressible Euler system in order to get  $J = 1 + O(\epsilon)$ , i.e. it is a way to seek for uniform solution in  $Y$  and  $Z$  variable.

Let us introduce the usual small parameter  $\epsilon = H/L$  where  $H$  (the height) and  $L$  (the length) are two characteristics dimensions along the  $\vec{k}$  and  $\vec{i}$  axis respectively. We assume that the characteristic dimension along the  $\vec{j}$  axis is the same as  $\vec{k}$ . We introduce the others characteristics dimensions  $T, P, \bar{U}, \bar{V}, \bar{W}$  for time, pressure and velocity respectively and the dimensionless quantities as follows:

$$\tilde{U} = U/\bar{U}, \tilde{V} = \epsilon V/\bar{U}, \tilde{W} = \epsilon W/\bar{U},$$

$$\tilde{X} = X/L, \tilde{Y} = Y/H, \tilde{Z} = Z/H, \tilde{p} = p/P, \tilde{\theta} = \theta, \tilde{\rho} = \rho.$$

In the sequel, we set  $P = \bar{U}^2$  and  $L = T\bar{U}$  (i.e. we consider only laminar flow).

Under these hypothesis, we have  $\tilde{J}(\tilde{X}, \tilde{Y}, \tilde{Z}) = 1 - \epsilon \tilde{Z} \frac{d\tilde{\theta}}{d\tilde{X}}$ . Thus, the rescaled System (4) reads:

$$\left\{ \begin{array}{l} \partial_{\tilde{X}} \tilde{U} + \partial_{\tilde{Y}}(\tilde{J}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{J}\tilde{W}) = 0 \\ \partial_{\tilde{t}}(\tilde{J}\tilde{U}) + \partial_{\tilde{X}}(\tilde{U}^2) + \partial_{\tilde{Y}}(\tilde{J}\tilde{U}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{J}\tilde{U}\tilde{W}) + \partial_{\tilde{X}}\tilde{p} = G_1 \\ \epsilon^2 \left( \partial_{\tilde{t}}(\tilde{J}\tilde{V}) + \partial_{\tilde{X}}(\tilde{U}\tilde{V}) + \partial_{\tilde{Y}}(\tilde{J}\tilde{V}^2) + \partial_{\tilde{Z}}(\tilde{J}\tilde{V}\tilde{W}) \right) + \partial_{\tilde{Y}}(\tilde{J}\tilde{p}) = 0 \\ \epsilon^2 \left( \partial_{\tilde{t}}(\tilde{J}\tilde{W}) + \partial_{\tilde{X}}(\tilde{U}\tilde{W}) + \partial_{\tilde{Y}}(\tilde{J}\tilde{V}\tilde{W}) + \partial_{\tilde{Z}}(\tilde{J}\tilde{W}^2) \right) \\ \quad + \tilde{J}\partial_{\tilde{Z}}(\tilde{p}) = G_2 \end{array} \right. \quad (17)$$

where  $G_1 = \epsilon \tilde{U}\tilde{W}\tilde{\kappa}(\tilde{X}) - \frac{\sin \tilde{\theta}}{F_{r,L}^2} - \frac{\tilde{Z}}{F_{r,H}^2} \frac{d}{d\tilde{X}}(\cos \tilde{\theta})$ ,

$G_2 = -\epsilon \tilde{U}^2 \tilde{\rho}(\tilde{X}) - \frac{\cos \tilde{\theta}}{F_{r,H}^2} + \epsilon \kappa(X) \frac{\tilde{Z}\tilde{J} \cos \tilde{\theta}}{F_{r,H}^2}$ ,  $F_{r,M} = \frac{\bar{U}}{\sqrt{gM}}$  is the Froude number along the  $\vec{i}$  axis and the  $\vec{k}$  or  $\vec{j}$  axis where  $M$  is any generic variable equal to  $L$  or  $H$ .

Formally, when  $\epsilon$  vanishes, the system reduces to:

$$\left\{ \begin{array}{l} \partial_{\tilde{X}} \tilde{U} + \partial_{\tilde{Y}}(\tilde{V}) + \partial_{\tilde{Z}}(\tilde{W}) = 0 \\ \partial_{\tilde{t}}(\tilde{U}) + \partial_{\tilde{X}}(\tilde{U}^2) + \partial_{\tilde{Y}}(\tilde{U}\tilde{V}) + \partial_{\tilde{Z}}(\tilde{U}\tilde{W}) + \partial_{\tilde{X}}\tilde{p} = -\frac{\sin \tilde{\theta}}{F_{r,L}^2} \\ \quad - \frac{\tilde{Z}}{F_{r,H}^2} \frac{d}{d\tilde{X}}(\cos \tilde{\theta}) \\ \partial_{\tilde{Z}}(\tilde{p}) = -\frac{\cos \tilde{\theta}}{F_{r,H}^2} \end{array} \right. \quad (18)$$

Now, integrating the preceding system as in Subsection 2.2, we get the following free surface model that we call *FS-model*:

$$\left\{ \begin{array}{l} \partial_t A + \partial_X Q = 0 \\ \partial_t Q + \partial_X \left( \frac{Q^2}{A} + gI_1(X, A) \cos \theta \right) = gI_2(X, A) \cos \theta - gA \sin \theta \\ \quad - gA\bar{Z}(X, A) \frac{d}{dX}(\cos \theta) \end{array} \right. \quad (19)$$

Now, omitting the overlined notation for  $\bar{U}$  except  $\bar{Z}$ , we get some classical properties of Model (19):

### Theorem 2.2

1. The System (19) is strictly hyperbolic for  $A(t, X) > 0$
2. For smooth solutions, the velocity  $U$  satisfies

$$\partial_t(U) + \partial_X \left( \frac{U^2}{2} + gh(A) \cos \theta + gb(X) \right) = 0. \quad (20)$$

3. The still water steady state for  $U = 0$  is given by

$$h(A) \cos \theta + b(X) = 0. \quad (21)$$

4. It admits a mathematical entropy

$$E(A, Q) = \frac{Q^2}{2A} + gA\bar{Z}(X, A) \cos \theta + gb$$

which satisfies the entropy inequality

$$\partial_t E + \partial_X ((E + gI_1(X, A) \cos \theta)U) \leq 0.$$

**Remark 2.4**

- We have already pointed out (see Remark 2.3) that System (12) is not suitable for a numerical computation, since the terms  $\overline{P}_c$  and  $\overline{J}$  are not explicitly expressed. Moreover, as we will expose in Section 4, System (12) is not convenient for a “natural” coupling with the pressurised model (see Section 3). Thus, we will preferably use the simple model (19) which takes into account the domain and the slope effects through the term  $A\overline{Z}(X, A)\frac{d}{dX}(\cos \theta)$  which was not present in the model derived in [3].
- Let us also note that we recover the model obtained by two of the authors in [3] from System (19) if we assume  $\frac{d}{dX}(\cos \theta) = o(1)$ .

### 3 Formal derivation of the P-model for pressurised flows

In this section, we present a new unidirectional shallow water-like equations to describe pressurised flows in closed non uniform domains to be coupled in natural way with the obtained FS-model (19). We will derive it from the compressible Euler equations following the analysis used to obtain the pressurised flow model that we call *P-model*.

#### 3.1 The compressible Euler system in curvilinear coordinates

The 3D compressible Euler system in the cartesian coordinates is written as follows

$$\partial_t \rho + \operatorname{div}(\rho \vec{U}) = 0, \tag{22}$$

$$\partial_t(\rho \vec{U}) + \operatorname{div}(\rho \vec{U} \otimes \vec{U}) + \nabla p = \vec{F}, \tag{23}$$

where  $\vec{U}(t, x, y, z)$  and  $\rho(t, x, y, z)$  denotes the velocity with components  $(u, v, w)$  and the density respectively.  $p(t, x, y, z)$  is the scalar pressure and  $\vec{F}$  the exterior strenght (of gravity).

We define the pressurised domain of the flow as the continuous extension of  $\Omega_F$  (see Section 2) defined by some plane curve with parametrization  $(x, 0, b(x))$  in a convenient cartesian reference frame  $(O, \vec{i}, \vec{j}, \vec{k})$  where  $\vec{k}$  follows the vertical direction; we recall that  $b(x)$  is then the elevation of the point  $\omega(x, 0, b(x))$  over the plane  $(O, \vec{i}, \vec{j})$  (see FIG. 1). The curve may be, for instance, the axis spanned by the center of mass of each orthogonal section  $\Omega(x)$  to the main mean flow axis, particularly in the case of a piecewise cone-shaped pipe. Notice that we consider only the case of infinitely rigid pipes: the sections are only  $x$ -dependent (since the domain is full-filled).

As previously, to see the local effect induced by the geometry due to the changes of sections and/or slope, we write the 3D compressible Euler system in the curvilinear coordinates. To this end, we perform the same change of variables using the transformation  $\mathcal{T} : (x, y, z) \rightarrow (X, Y, Z)$  where  $X, Y, Z$  denote respectively the curvilinear abscissa, the width variable and the altitude of any fluid particle  $M$  in the Serret-Frenet reference frame  $(\vec{T}, \vec{N}, \vec{B})$  at point  $\omega(x, 0, b(x))$ .

Let  $(U, V, W)^t$  be the components of the velocity vector in the  $(X, Y, Z)$  coordinates defined by:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \Theta \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where  $\Theta$  is the rotation matrix (see Section 2).

Applying Lemma 2.1 to the mass conservation equation, we get

$$J(\partial_t \rho + \operatorname{div}(\rho \vec{U})) = 0 \text{ or also}$$

$$\partial_t(J\rho) + \partial_X(\rho U) + \partial_Y(\rho J V) + \partial_Z(\rho J W) = 0 \quad (24)$$

where

$$J = \det \begin{pmatrix} \left(1 - Z \frac{d\theta}{dX}\right) \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \left(1 - Z \frac{d\theta}{dX}\right) \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (25)$$

To get the unidirectional model, we suppose that the mean flow follows the  $X$ -axis. Hence, we neglect the second and third equation for the conservation of the momentum. Therefore, we only perform the curvilinear transformation for the first conservation equation. To this end, multiplying the conservation of the momentum

equation of System (23) by  $J \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix}$  and using Lemma 2.1, we get:

$$J \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix} \left( \partial_t(\rho \vec{U}) + \operatorname{div}(\rho \vec{U} \otimes \vec{U}) + \nabla p = -\rho \nabla(\vec{g} \cdot \overrightarrow{OM}) \right).$$

It may be rewritten as:

$$\begin{aligned} \partial_t(J\rho U) + \partial_X(\rho U^2) + \partial_Y(\rho JUV^2) + \partial_Z(\rho JUW) + \partial_X p \\ = -\rho Jg \sin \theta + \rho UW \frac{d}{dX}(\cos \theta) \end{aligned} \quad (26)$$

where  $\overrightarrow{OM}$  denotes the position of any particule  $M$  in the local reference frame  $(\vec{T}, \vec{N}, \vec{B})$  at point  $\omega(x, 0, b(x))$ .

Finally, in the  $(X, Y, Z)$  coordinates the system reads:

$$\begin{cases} \partial_t(J\rho) + \partial_X(\rho U) + \partial_Y(\rho J V) + \partial_Z(\rho J W) = 0 \\ \partial_t(J\rho U) + \partial_X(\rho U^2) + \partial_Y(\rho JUV^2) + \partial_Z(\rho JUW) + \partial_X p \\ = -\rho Jg \sin \theta + \rho UW \frac{d}{dX}(\cos \theta) \end{cases} \quad (27)$$

**Remark 3.1** As in Remark 2.1,  $\mathcal{T}$  defines a diffeomorphism and thus the performed transformation is admissible.

We recall that the main objective is to obtain a formulation close to the shallow water equation in order to couple the two models in a natural way (in a close manner described in [3]). The direct integration of Equations (27) over  $\Omega(x)$  gives a model which is not useful (as the model for free surface flow derived in Section 2) due to the term  $J$ , to perform a natural coupling with the FS-model for non uniform pipes.

We perform again the same rescaling as for the free surface model and the system in variables  $(X, Y, Z)$  describing the slope variation and the section variation in a closed pipe reads:

$$\left\{ \begin{array}{l} \partial_t(\rho) + \partial_X(\rho U) + \partial_Y(\rho V) + \partial_Z(\rho W) = 0 \\ \partial_t(U\rho) + \partial_X(\rho U^2) + \partial_Y(\rho UV) + \partial_Z(\rho UW) + \partial_X p = -\rho g \sin \theta \\ -gZ \frac{d}{dX}(\cos \theta) \end{array} \right. \quad (28)$$

### 3.2 Shallow water-like equations for pressurised flows in a closed pipe

In the following, we use the linearized pressure law  $p = p_a + \frac{\rho - \rho_0}{\beta \rho_0}$  (see e.g. [16, 17]) in which  $\rho_0$  represents the density of the fluid at atmospheric pressure  $p_a$  and  $\beta$  the water compressibility coefficient equal to  $5.0 \cdot 10^{-10} m^2 \cdot N^{-1}$  in practice. The sonic speed is then given by  $c = 1/\sqrt{\beta \rho_0}$  and thus  $c \approx 1400 m \cdot s^{-1}$ .

System (28) is integrated over the cross-section  $\Omega$ . In the following, overlined letters represents the averaged quantities over  $\Omega$ . For  $m \in \partial\Omega$ ,  $\vec{n} = \frac{\vec{m}}{|\vec{m}|}$  is the outward unit vector at the point  $m$  in the  $\Omega$ -plane and  $\vec{m}$  stands for the vector  $\overline{\omega \vec{m}}$  (as displayed on FIG. 1).

Following the work in [3], using the approximations  $\overline{\rho U} \approx \overline{\rho} \overline{U}$ ,  $\overline{\rho U^2} \approx \overline{\rho} \overline{U^2}$  and Lebesgue integral formulas, the mass conservation equation becomes:

$$\partial_t(\overline{\rho S}) + \partial_X(\overline{\rho q}) = \int_{\partial\Omega} \rho \left( U \partial_X \vec{m} - \vec{V} \right) \cdot \vec{n} ds, \quad (29)$$

where  $q = S\overline{U}$  is the discharge of the flow and the velocity  $\vec{V} = (V, W)^t$  in the  $(\vec{N}, \vec{B})$ -plane. We denote by  $S$  the area of the cross-section  $S_{max}(X)$  of the pipe at position  $X$ .

The equation of the conservation of the momentum becomes

$$\begin{aligned} \partial_t(\overline{\rho q}) + \partial_X \left( \frac{\overline{\rho q^2}}{S} + c^2 \overline{\rho S} \right) &= -g \overline{\rho S} \sin \theta + c^2 \overline{\rho} \frac{dS}{dX} \\ &- g \overline{\rho S} \overline{Z} \frac{d}{dX}(\cos \theta) \\ &+ \int_{\partial\Omega} \rho U \left( U \partial_X \vec{m} - \vec{V} \right) \cdot \vec{n} ds \end{aligned} \quad (30)$$

The integral terms appearing in (29) and (30) vanish, as the pipe is infinitely rigid, i.e.  $\Omega = \Omega(X)$  (see [3] for the dilatible case). It follows the non-penetration condition:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} \cdot \vec{N} = 0.$$

Finally, omitting the overlined notation except for  $\overline{Z}$ , we obtain the equations for pressurised flows under the form

$$\left\{ \begin{array}{l} \partial_t(\rho S) + \partial_X(\rho q) = 0 \\ \partial_t(\rho q) + \partial_X \left( \frac{\rho q^2}{S} + c^2 \rho S \right) = -g \rho S \sin \theta - g \rho S \overline{Z} \frac{d}{dX}(\cos \theta) + c^2 \rho \frac{dS}{dX} \end{array} \right. \quad (31)$$

where the quantity  $\overline{Z} = \overline{Z}(X, S)$  is the  $Z$  coordinate of the center of mass.

**Remark 3.2** In the case of a circular section pipe, we choose the plane curve  $(x, 0, b(x))$  as the mean axis and we get obviously  $\bar{Z} = 0$ .

Now, following [3], let us introduce the conservative variables  $A = \frac{\rho S}{\rho_0}$  the *equivalent wet area* and the *equivalent discharge*  $Q = AU$ . Then dividing System (31) by  $\rho_0$  we get:

$$\begin{cases} \partial_t(A) + \partial_X(Q) = 0 \\ \partial_t(Q) + \partial_X\left(\frac{Q^2}{A} + c^2 A\right) = -gA \sin \theta - gA \bar{Z}(X, S) \frac{d}{dX}(\cos \theta) + \\ c^2 A \frac{d}{dX} \ln(S) \end{cases} \quad (32)$$

**Remark 3.3** This choice of variables is motivated by the fact that this system is formally closed to the FS-model (19) where the terms  $gI_1(X, A) \cos \theta$ ,  $gI_2(X, A) \cos \theta$ ,  $\bar{Z}(X, A)$  are respectively the equivalent terms to  $c^2 A$ ,

$c^2 A \frac{d}{dX} \ln(S)$ ,  $\bar{Z}(X, S)$  in System (32). Finally, the choice of these unknowns leads to a “natural” coupling between the pressurised and free surface model as we will see in Section 4.

To close this section, let us give the classical properties of System (32):

**Theorem 3.1**

1. The system (32) is strictly hyperbolic for  $A(t, X) > 0$ .
2. For smooth solutions, the mean velocity  $U = Q/A$  satisfies

$$\partial_t U + \partial_X \left( \frac{U^2}{2} + c^2 \ln(A/S) + g\Phi_\theta + gZ \right) = 0 \quad (33)$$

where  $\Phi_\theta(X) = \int_{X_0}^X \bar{Z}(\xi) \frac{d}{dX} \cos \theta(\xi) d\xi$  for any arbitrary  $x_0$  and  $Z$  the elevation term defined by  $\partial_X Z = \sin \theta$ .

3. The still water steady states for  $U = 0$  is given by

$$c^2 \ln(A/S) + g\Phi_\theta + gZ = 0. \quad (34)$$

4. It admits a mathematical entropy

$$E(A, Q) = \frac{Q^2}{2A} + c^2 A \ln(A/S) + gA\Phi_\theta + gAZ$$

which satisfies the entropy inequality

$$\partial_t E + \partial_X ((E + c^2 A)U) \leq 0$$

The quantity  $\frac{U^2}{2} + c^2 \ln(A/S) + g\Phi_\theta + gZ$  is also called the total head.

## 4 The PFS-model for mixed flows

The formulation of the FS-model (19) and P-model (32) are very close to each other. The main difference comes from the pressure law. In order to build a coupling between the two types of flows, we have to build a pressure that ensure its continuity through transition points in the same spirit of [3]. And as pointed out in Remark 3.3, we will use a common couple of unknowns  $(A, Q)$  to get a continuous model for mixed flows.

Let us first introduce in both model the exterior strength of friction  $-\rho g S_f \vec{N}$  given by the Manning Strickler law (see e.g. [16]):

$$S_f(A) = K(A)U|U|$$

where  $K(A)$  is defined as follows:

$$K(A) = \frac{1}{K_s^2 R_h(A)^{4/3}}.$$

$K_s > 0$  is the Strickler coefficient of the roughness and  $R_h = A/P_m$  is the hydraulic radius where  $P_m$  is the perimeter of the wet surface area  $A$ . We rewrite the FS-model (19) and P-model (32) (resp.) with the friction term:

$$\left\{ \begin{array}{l} \partial_t A + \partial_X Q = 0 \\ \partial_t Q + \partial_X \left( \frac{Q^2}{A} + gI_1(X, A) \cos \theta \right) = -gA \sin \theta + gI_2(X, A) \cos \theta \\ \qquad \qquad \qquad -gA \bar{Z}(X, A) \frac{d}{dX}(\cos \theta) \\ \qquad \qquad \qquad -gAS_f(A) \end{array} \right. \quad (35)$$

$$\left\{ \begin{array}{l} \partial_t A + \partial_X Q = 0 \\ \partial_t Q + \partial_X \left( \frac{Q^2}{A} + c^2 A \right) = -gA \sin \theta + c^2 A \partial_X \ln(S) \\ \qquad \qquad \qquad -gA \bar{Z}(X, S) \frac{d}{dX}(\cos \theta) \\ \qquad \qquad \qquad -gAS_f(S) \end{array} \right. \quad (36)$$

Let us recall that  $S_{max}(X)$  denotes the area of the cross-section of the pipe at position  $X$ . We denote also  $S_{max}(X)$  by  $S(X)$  when we deal with pressurised state while  $S(t, X)$  depend on time when the type of flow is free surface and it is denoted simply by  $A$ . Thus, we call  $S$  the physical wet area and  $A$  the wet equivalent area.

In order to ensure the continuity of the pressure through the change of state we define the water height:

$$\mathcal{H}(t, X) = \mathbb{1}_{\{\rho=\rho_0\}} h(t, X) + \mathbb{1}_{\{\rho \neq \rho_0\}} R(X). \quad (37)$$

(also denoted by  $\mathcal{H}(A)$ ). We set

$$p(X, A) = c^2(A - S) + gI_1(X, S) \cos \theta. \quad (38)$$

and we choose the same plane curve with parametrization  $(x, 0, \sin \theta)$ , namely the main pipe axis. Actually this choice is the more convenient for pressurised flows while the bottom line is adapted to free surface flows. Thus we must assume small variations of the section ( $\frac{d}{dX} S_{max}$  small) or equivalently small angle  $\varphi$  (see FIG. 3).



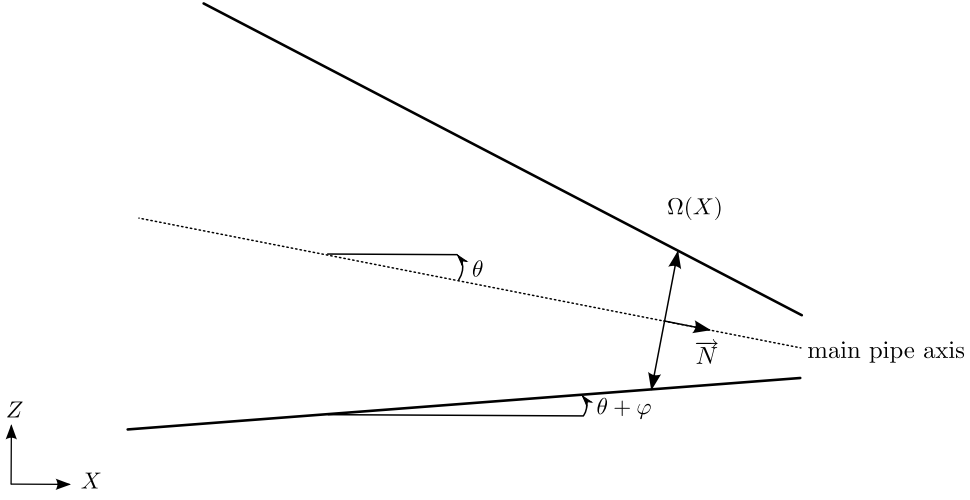


Figure 3: Some restriction concerning the geometric domain

This modified pressure law permits to recover both cases. Indeed, if  $E$  denotes the state variable, we have:

$$p(X, A, E) = \begin{cases} gI_1(X, A) \cos \theta & \text{if } E = FS \quad (\text{i.e. } \rho = \rho_0) \\ gI_1(X, S) \cos \theta + c^2(A - S) & \text{if } E = P \quad (\text{i.e. } \rho \neq \rho_0) \end{cases} \quad (39)$$

Finally, the PFS-model for unsteady mixed flows reads:

$$\begin{cases} \partial_t(A) + \partial_X(Q) = 0 \\ \partial_t(Q) + \partial_X \left( \frac{Q^2}{A} + p(X, A, E) \right) = -gA \frac{dZ}{dX} + Pr(X, A, E) \\ \quad -G(X, A, E) \\ \quad -K(X, S, E) \frac{Q|Q|}{A} \end{cases} \quad (40)$$

where  $K$ ,  $Pr$ , and  $G$  denotes respectively the friction, the pressure source and the geometry source term defined as above:

$$Pr(X, A, E) = (c^2(A/S - 1)) \frac{dS}{dX} + gI_2(X, S) \cos \theta,$$

$$G(X, A, E) = gA \bar{Z}(X, S) \frac{d}{dX} \cos \theta$$

$$K(X, S, E) = \frac{1}{K_s^2 R_h(S)^{4/3}}.$$

The PFS system (40) satisfies the following properties:

#### Theorem 4.1

1. The PFS system (40) is strictly hyperbolic for  $A(t, X) > 0$ .
2. For smooth solutions, the mean velocity  $U = Q/A$  satisfies

$$\begin{aligned} \partial_t U + \partial_X \left( \frac{U^2}{2} + c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gZ \right) \\ = -gK(X, S, E)U|U|. \end{aligned} \quad (41)$$

3. The still water steady states for  $U = 0$  reads:

$$c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gZ = 0. \quad (42)$$

4. It admits a mathematical entropy

$$E(A, Q) = \frac{Q^2}{2A} + c^2 A \ln(A/S + S/A) + gA\bar{Z}(X, S) \cos \theta + gAZ$$

which satisfies the entropy inequality

$$\partial_t E + \partial_X((E + p(X, A, E))U) = -gAK(X, S, E)U^2|U| \leq 0$$

The quantity  $\frac{U^2}{2} + c^2 \ln(A/S) + g\mathcal{H}(S) \cos \theta + gZ$  is called the total head. Notice that the total head and the energy are defined continuously through the transition points.

**Proof of Theorem 4.1:** The results (41) and (42) are obtained in a classical way. We compute also the entropy inequality by classical means and we obtain the following equation:

$$\begin{aligned} & \partial_t \left( \frac{Q^2}{2A} + c^2 A \ln(A/S + S/A) + gA\bar{Z}(X, S) \cos \theta + gAZ \right) \\ & + \partial_X \left( \left( \frac{Q^2}{2A} + c^2 A \ln(A/S + S/A) + gA\bar{Z}(X, S) \cos \theta + gAZ + p \right) U \right) \\ & + c^2 \left( \partial_t S \left( \frac{A}{S} - 1 \right) \right) = -gAK(X, S, E)U^2|U| \end{aligned}$$

We see that the term  $c^2 \left( \partial_t S \left( \frac{A}{S} - 1 \right) \right)$  is equal to 0 since we have  $A = S$  when the flow is free surface whereas  $S = S(X)$  when the flow is pressurised. Moreover, from the last inequality, when  $A = S$ , we have the classical entropy inequality (see [3, 4]) with the energy  $E$ :

$$E(A, Q) = \frac{Q^2}{2A} + gA\bar{Z}(X, S) \cos \theta + gAZ$$

while the energy is:

$$E(A, Q) = \frac{Q^2}{2A} + c^2 A (\ln(A/S) + S/A) + gAZ$$

for the pressurised case. Finally, the energy for the PFS-model reads:

$$E(A, Q) = \frac{Q^2}{2A} + c^2 A \ln(A/S + S/A) + gA\bar{Z}(X, S) \cos \theta + gAZ.$$

Let us remark that the term  $c^2 S$  makes the energy  $E$  continuous through transition points and it permits also to write the entropy flux under the classical form  $(E+p)U$ .

□

## 5 Conclusion

We have used two different approaches to derive both a free surface flow model and a pressurised flow model which have been coupled using a common set of variables [3] and a suitable pressure law. We obtained a mixed model, that we call PFS-model. This model takes into account the local perturbation of the section and of the slope, in particular when the domain is a piecewise cone-shaped pipe. Moreover it presents a simple form which is suitable for a numerical treatment as a finite volume method or a kinetic scheme extending [3, 4, 5].

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