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# From quantum electrodynamics to posets of planar binary trees

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## Abstract

This paper is a brief mathematical excursion which starts from quantum electrodynamics and leads to the Möbius function of the Tamari lattice of planar binary trees, within the framework of groups of tree-expanded series.

First we recall Brouder's expansion of the photon and the electron Green's functions on planar binary trees, before and after the renormalization. Then we recall the structure of Connes and Kreimer's Hopf algebra of renormalization in the context of planar binary trees, and of their dual group of tree-expanded series. Finally we show that the Möbius function of the Tamari posets of planar binary trees gives rise to a particular series in this group.

## Introduction

Planar binary trees are among the most classical objects in combinatorics. Being counted by the Catalan numbers, they are in bijection with more than one hundred other combinatorial objects, cf. [Sta99], such as noncrossing partitions and Dyck paths. In the last fifteen years, they have begun to play a key role in some algebraic structures, ranging from groups to Hopf algebras and to operads, cf. [Lod01, HNT05, AS06]. In this article, we would like to show how a problem in quantum electrodynamics can lead to such algebraic structures.

As shown by Christian Brouder in [Bro00], in quantum electrodynamics the Green functions can be described perturbatively with an expansion over planar binary trees. In the quantum field theory describing a scalar self-interacting field, the Green functions are naturally expanded over rooted trees (non-planar and non-binary). In contrast to this, in quantum electrodynamics Brouder adopted binary trees to describe the coupling of two interacting fields, the left path for the electron field and the right path for the photon field, and adopted planar trees to respect the non-commutativity of the product among the Green functions' coefficients, which are  $4 \times 4$  matrices.

The expansion of the Green functions over planar binary trees requires some simple grafting operations among trees, [Bro00]. Instead, the renormalization of these Green functions gives rise to some Hopf algebras on planar binary trees, cf. [BF01], analogue to the famous Connes-Kreimer Hopf algebra encoding the renormalization of the scalar  $\Phi^3$  theory, cf. [Kre98, CK00a]. These Hopf algebras can naturally be seen as coordinate rings of some pro-algebraic groups of formal series expanded over trees. One of them turns out to be related to an operad on planar binary trees, cf. [Fra08].

There are in fact two operads on planar binary trees in the literature. In the present context we are concerned with the duplicial operad, cf. [Lod06]. The other one is the dendriform operad [Lod01], which has been the subject of a lot of attention recently. In particular, it has been shown that the dendriform operad is deeply related to the family of posets called Tamari lattices, cf. [HT72]. In this article, we show that Tamari lattices are also directly related to the duplicial operad. To do it, we compute the inverse of some series in the group of tree-expanded series associated with the duplicial operad.

The paper is organized as follows. In the first section we recall Brouder’s expansion of the photon and the electron Green’s functions on planar binary trees, before and after the renormalization. In the second section we review the Hopf algebras of renormalization for quantum electrodynamics in the context of planar binary trees, and their dual groups of tree-expanded series. In the last section we show that the Möbius function of the Tamari posets of planar binary trees gives rise to a particular series in the group of tree-expanded series which generalises the group of formal diffeomorphisms on a line.

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## Contents

<b>1</b>	<b>Quantum electrodynamics and tree-expanded series</b>	<b>2</b>
1.1	Dyson-Schwinger equations for the electron and the photon propagators . . . . .	2
1.2	Renormalization factors and Feynman graphs . . . . .	4
1.3	Tree-expansion of QED propagators . . . . .	5
<b>2</b>	<b>Renormalization Hopf algebras and groups of tree-expanded series</b>	<b>7</b>
2.1	QED renormalization Hopf algebras on trees . . . . .	7
2.2	Groups of tree-expanded series . . . . .	9
<b>3</b>	<b>Tree-expanded series in relation with Tamari posets</b>	<b>10</b>

## 1 Quantum electrodynamics and tree-expanded series

Quantum electrodynamics (QED) is the quantisation of classical electromagnetism, that is the field theory describing the attraction/repulsion between electrons, by means of the photons as mediators. In this section we review the perturbative expansion of the 2-point Green’s functions of the electron and the photon on planar binary trees, as proposed by C. Brouder in [Bro00].

### 1.1 Dyson-Schwinger equations for the electron and the photon propagators

Denote by  $\mathbb{R}^{1,3}$  the Minkowski space, with flat diagonal metric  $g_{\mu\nu} = (1, -1, -1, -1)$ , relativistic space-time coordinates  $x = x^\mu \in \mathbb{R}^{1,3}$  and momenta  $p = p_\mu$ .

Let  $\psi : \mathbb{R}^{1,3} \longrightarrow \mathbb{C}^4$  be the fermionic field describing the electron, with mass  $m$  and electric charge  $e$ , and let  $A = (A^\mu) : \mathbb{R}^{1,3} \longrightarrow \mathbb{C}^4$  be the massless bosonic field describing the electromagnetic potential (the photon). The dynamic of the system of interacting electrons and photons is described by the Lagrangian density

$$\mathbb{L}_{QED}(\psi, A; e, m) = \mathbb{L}_{Dirac}(\psi; m) + \mathbb{L}_{Maxwell}^\xi(A^\mu) + \mathbb{L}_{int}(\psi, A^\mu; e)$$

where

$$\mathbb{L}_{Dirac}(\psi; m) = \bar{\psi}(x^\mu)(i\gamma^\mu\partial_\mu - m)\psi(x^\mu),$$

is the Lagrangian describing the free electrons ( $\gamma^\mu$  are the Dirac 4x4 matrices and  $\bar{\psi} = \psi^*\gamma^0$  is the anti-fermionic field, e.g. the positron),

$$\mathbb{L}_{Maxwell}^\xi(A) = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)^2 - \frac{\xi}{2}(\partial_\nu A^\mu)^2,$$

is the Lagrangian describing the free photons ( $\xi$  is a gauge parameter introduced as a fictive mass to avoid the problems due to the absence of the photon mass), and

$$\mathbb{L}_{int}(\psi, A^\mu; e) = -e\bar{\psi}\gamma^\mu\psi A^\mu$$

is the term describing the interaction (of order 3), with coupling constant given by the electric charge  $e$ .

In this paper we are concerned with the the connected 2-point Green functions

$$\left\{ \begin{array}{l} S(x, y) = \langle \psi(x)\psi(y) \rangle, \quad \text{for the electron} \\ D_{\mu\nu}(x, y) = \langle A_\mu(x)A_\nu(y) \rangle, \quad \text{for the photon} \end{array} \right.$$

whose square module represent the probability that the quantum field moves from the point  $y$  to the point  $x$  in space-time. (We suppose that the 1-point Green's functions are zero.) If the fields are free, that is, the Lagrangian of the system is the sum  $L_{Dirac}(\psi; m) + L_{Maxwell}^\xi(A^\mu)$  without the interaction part, the 2-point Green functions coincide with the free propagators. These are  $4 \times 4$  matrix-valued distributions on  $\mathbb{R}^{1,3}$ , whose Fourier transforms on the momenta space are

$$\left\{ \begin{array}{l} S_0(p) = \frac{1}{\gamma^\mu p_\mu - m + i\epsilon}, \\ D_{0,\mu\nu}(p) = -\frac{g_{\mu\nu}}{p^2 + i\epsilon} + \left(1 - \frac{1}{\xi}\right) \frac{p_\mu p_\nu}{(p^2 + i\epsilon)^2}. \end{array} \right.$$

If the fields interact according to the Lagrangian  $L_{QED}$ , the 2-point Green functions (that we shall keep calling propagators) satisfy a system of two functional equations called *Dyson-Schwinger equations*, analogue to the equation of motion in the case of classical fields. For an isolated system, C. Brouder presented these two equations in the following form, cf. [Bro00]:

$$\left\{ \begin{array}{l} S(p) = S_0(p) + i e^2 S_0(p) \int \frac{d^4 q}{(2\pi)^4} \gamma^\lambda D_{\lambda\lambda'}(q) \frac{\delta S(p-q)}{e \delta A_{\lambda'}(q)}, \\ D_{\mu\nu}(p) = D_{0,\mu\nu}(p) - i e^2 D_{0,\mu\lambda}(p) \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left[ \gamma^\lambda \frac{\delta S(q)}{e \delta A_{\lambda'}(-p)} \right] D_{\lambda'\nu}(p). \end{array} \right. \quad (1.1)$$

In this equations, the electron propagator  $S(p)$  depends *a priori* on an external electromagnetic field  $A_\lambda(q)$ , and the functional derivative  $\frac{\delta S(p)}{\delta A_\lambda(q)}$  detects this dependence. Then, since the equations are valid for an isolated system, we imply that  $S(p)$  and its functional derivatives are evaluated at  $A_\lambda(q) = 0$ .

The solution of the equations (1.1) is a perturbative series in the powers of the *fine structure constant*  $\alpha = \frac{e^2}{4\pi}$ ,

$$\left\{ \begin{array}{l} S(p) = S_0(p) + \sum_{n \geq 1} \alpha^n S_n(p), \\ D_{\mu\nu}(p) = D_{0,\mu\nu}(p) + \sum_{n \geq 1} \alpha^n D_{n,\mu\lambda}(p), \end{array} \right. \quad (1.2)$$

obtained recursively from the initial datas, which are the free propagators  $S_0(p)$ ,  $D_{0,\mu\nu}(p)$  and all the functional derivatives  $\frac{\delta S_0(p)}{\delta A_\lambda(q)}$ ,  $\frac{\delta^2 S_0(p)}{\delta A_\lambda(q_1) \delta A_\lambda(q_2)}$ , etc. At each new order of perturbation, the coefficient of the series is an analytical expression involving the previous coefficients. In fact, for the series (1.2), the system (1.1) is equivalent to

$$\left\{ \begin{array}{l} S_n(p) = i S_0(p) \sum_{k+l=n-1} \int \frac{d^4 q}{(2\pi)^4} \gamma^\lambda D_{k,\lambda\lambda'}(q) \frac{\delta S_l(p-q)}{e \delta A_{\lambda'}(q)}, \\ D_{n,\mu\nu}(p) = -i D_{0,\mu\lambda}(p) \sum_{k+l=n-1} \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left[ \gamma^\lambda \frac{\delta S_k(q)}{e \delta A_{\lambda'}(-p)} \right] D_{l,\lambda'\nu}(p). \end{array} \right. \quad (1.3)$$

## 1.2 Renormalization factors and Feynman graphs

Equations (1.3) can be formally solved, but they present a major problem of quantum field theory: at any order of perturbation  $n > 0$ , even for  $n = 1$ , the perturbative coefficients  $S_n(p)$  and  $D_{n,\mu\nu}(p)$  contain divergent integrals. The physical explanation is that the quantum fields  $\psi$  and  $A$ , and the measurable parameters  $e$  and  $m$ , have an energy level which is far too high with respect to the classical ones, and the classical Lagrangian  $\mathbb{L}_{QED}(\psi, A; e, m)$  is not sufficient to describe them. As a consequence, the Lagrangian must be modified into the so-called the *renormalized Lagrangian*

$$\begin{aligned} \mathbb{L}_{QED}^{ren}(\psi, A; e, m) &= Z_2(e) \bar{\psi}(x^\mu) (i\gamma^\mu \partial_\mu - (m + \delta m(e))) \psi(x^\mu) \\ &\quad - Z_3(e) \frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)^2 - \frac{\xi}{2} (\partial_\nu A^\mu)^2 \\ &\quad - Z_1(e) e \bar{\psi} \gamma^\mu \psi A^\mu, \end{aligned}$$

where the so-called *renormalization factors*  $Z_1(e)$ ,  $Z_2(e)$ ,  $Z_3(e)$  and  $\delta m(e)$  are series in the powers of  $\alpha$  which compensate the divergences and allow to produce finite quantities. If we call *bare parameters* the rescaled parameters

$$\begin{cases} m_0 &= m + \delta m(e), \\ e_0 &= \frac{e Z_1(e)}{Z_2(e) \sqrt{Z_3(e)}}, \end{cases} \quad (1.4)$$

and *bare fields* the rescaled fields

$$\begin{cases} \psi_0 &= \sqrt{Z_2(e)} \psi, \\ A_0^\mu &= \sqrt{Z_3(e)} A^\mu, \end{cases}$$

it turns out that

$$\mathbb{L}_{QED}^{ren}(\psi, A; e, m) = \mathbb{L}_{QED}(\psi_0, A_0; e_0, m_0).$$

Then, using the Dyson-Schwinger equations (1.1) one can compute the *bare propagators*  $S(p; e_0, m_0)$  and  $D_{\mu\nu}(p; e_0, m_0)$  relative to the bare fields, and F. Dyson showed in [Dys49] that the *renormalized propagators* can be found simply as

$$\begin{cases} S^{ren}(p; e, m) &= Z_2(e)^{-1} S(p; e_0, m_0), \\ (D_{\mu\nu}^T)^{ren}(p; e, m) &= Z_3(e)^{-1} D_{\mu\nu}^T(p; e_0, m_0), \end{cases} \quad (1.5)$$

where  $D_{\mu\nu}^T(p) = D_{\mu\nu}(p) - \frac{1}{\xi} \frac{p_\lambda p_\mu}{(p^2 + i\epsilon)^2}$  is the transversal part of the propagator, while the longitudinal part  $\frac{1}{\xi} \frac{p_\lambda p_\mu}{(p^2 + i\epsilon)^2}$  is not affected by the renormalization.

One way to compute the renormalization factors is to expand the propagators over all possible connected *Feynman diagrams* with 2 external legs. The amplitude of the graphs are the simplest integrals appearing in the perturbative coefficients of the integral expansion of the propagators. Those which diverge must be renormalized. The divergence extracted from a graph forms the *counterterm* of the graph, and the renormalization factors are constructed by reassembling the counterterms of all concerned graphs.

Feynman graphs describe all possible virtual interactions between electrons and photons in the indeterminate quantum fluctuation. They are constructed on some fixed types of edges, which represent the free propagators, and with some fixed types of vertices, which represent the allowed interactions, and therefore are completely determined by the Lagrangian. If we denote by  $\Gamma$  the connected Feynman graphs with 2 external legs, and we distinguish the electron graphs ( $e$ ) from the photon graphs ( $\gamma$ ), the

perturbative expansion of the QED propagators is

$$\left\{ \begin{array}{l} S(p) = S_0(p) + \sum_{\Gamma^e} \alpha^{L(\Gamma)} S_{\Gamma}(p), \\ D_{\mu\nu}(p) = D_{0,\mu\nu}(p) + \sum_{\Gamma^\gamma} \alpha^{L(\Gamma)} D_{\Gamma,\mu\lambda}(p), \end{array} \right. \quad (1.6)$$

where  $L(\Gamma)$  denotes the number of loops of the graph. The coefficients  $S_{\Gamma}(p)$  and  $D_{\Gamma,\mu\lambda}(p)$  are some analytical expressions called the *amplitude* of the graph. The integral expansion (1.2) and the diagrammatic expansion (1.6) of the solution of Eqs. (1.1) are of course related: at each order of perturbation we have

$$\left\{ \begin{array}{l} S_n(p) = \sum_{L(\Gamma)=n} S_{\Gamma}(p), \\ D_{n,\mu\nu}(p) = \sum_{L(\Gamma)=n} D_{\Gamma,\mu\lambda}(p). \end{array} \right.$$

The system (1.1), applied to the series expanded as (1.6), gives an algorithm to construct the amplitudes, called the *Feynman's rules*, cf. for instance Appendix A in [IZ80]. Roughly speaking, Feynman's rules are a dictionary between graphical signs (edges, vertices, loops) and analytical expressions (free propagators, factors of the coupling constant, integrals, overall symmetry factors). Feynman's rules have many advantages: they are intuitive and easy to memorize, they allow to compute the coefficients of the propagators each independently of any other, and finally they allow to compute the counterterms of the graphs and therefore the coefficients of the renormalization factors. Based on this method, one can prove for instance the *Ward-Takahashi identity*  $Z_2(e) = Z_1(e)$ , which imply that the charge is renormalized as

$$e_0 = e Z_3(e)^{-\frac{1}{2}}. \quad (1.7)$$

The computation of the counterterms requires a specific algorithm, the so-called *BPHZ formula*, which takes into account the subdivergences of the complicated graphs. The intricate combinatorics of this algorithm is nowadays completely clear, since A. Connes and D. Kreimer proved in [CK00a, CK00b] that it is equivalent to a Hopf algebra structure on the set of Feynman graphs.

However the method based on Feynman graphs has also two main disadvantages. On one side, the number of Feynman diagrams increases exponentially with the number of vertices, making difficult to keep control of the summability of the perturbative expansions on graphs. On the other side, the amplitude of a graph with many vertices turns out to be a very complicated analytical expression. The computation of the propagators requires hundreds of integrals even at low orders of perturbation, and the computation of the counterterms, performed through the BPHZ formula, makes the computation of each renormalized integral even longer.

### 1.3 Tree-expansion of QED propagators

In the paper [Bro00], C. Brouder proposed an alternative perturbative solution of the Dyson-Schwinger equations (1.1), which has the main advantage of reducing drastically the number of computations required at each order of perturbation. To do it, he based the expansion of the perturbative series on the set of *planar binary rooted trees*, instead of Feynman graphs. These trees are planar graphs without loops, and a preferred external edge called the root. For a tree  $t$ , we denote by  $|t|$  the number of its internal vertices. The tree  $\downarrow$ , with no internal vertices, is called the *root tree*. The tree  $\vee$ , with one internal vertex, is called the *vertex tree*. Other examples of trees, those with number of internal vertices equal to two and three, are

$$\vee\vee, \quad \vee\vee\vee, \quad \vee\vee\vee\vee, \quad \vee\vee\vee\vee\vee, \quad \vee\vee\vee\vee\vee\vee, \quad \vee\vee\vee\vee\vee\vee\vee.$$

Let us denote by  $Y_n$  the set of planar binary trees with  $n$  internal vertices, by  $Y = \bigcup_{n=0}^{\infty} Y_n$  the set of all trees, and by  $\overline{Y} = \bigcup_{n=1}^{\infty} Y_n$  the set of all trees with at least one internal vertex. Finally, let us denote by

$\vee : Y_n \times Y_m \longrightarrow Y_{n+m+1}$  the operation which grafts two trees on a new root, that is

$$s \vee t = \begin{array}{c} s \\ \vee \\ t \end{array}.$$

If we suppose that the bare 2-point Green's functions of QED are formal series of the form

$$\left\{ \begin{array}{l} S(p; e_0, m_0) = S_0(p) + \sum_{t \in \overline{Y}} e_0^{2|t|} S_t(p), \\ D_{\mu\nu}(p; e_0, m_0) = D_{0,\mu\nu}(p) + \sum_{t \in \overline{Y}} e_0^{2|t|} D_{t,\mu\lambda}(p), \end{array} \right. \quad (1.8)$$

then the system (1.1) determines the coefficients of these series as

$$\left\{ \begin{array}{l} S_t(p) = iS_0(p) \int \frac{d^4q}{(2\pi)^4} \gamma^\lambda D_{t^l, \lambda\lambda'}(q) \frac{\delta S_{t^r}(p-q)}{e_0 \delta(A_0)_{\lambda'}(q)}, \\ D_{t,\mu\nu}(p) = -iD_{0,\mu\lambda}(p) \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[ \gamma^\lambda \frac{\delta S_{t^l}(q)}{e_0 \delta(A_0)_{\lambda'}(-p)} \right] D_{t^r, \lambda'\nu}(p), \end{array} \right. \quad (1.9)$$

where, for any tree  $t \neq \mid$ , the trees  $t^l$  and  $t^r$  are such that  $t = t^l \vee t^r$ . The expansion on trees (1.8) is related to the integral expansion (1.2) because the coefficients of the integral series at the perturbative order  $n$  are the sum of the corresponding coefficients over the trees  $t$  such that  $2|t| = n$ . It is also related to the expansion (1.6) on Feynman graphs because the coefficients depending on a tree  $t$  are the sum of the coefficients depending on some Feynman graphs which can be found with an algorithm given in [BF01]. Roughly speaking, a tree corresponds to a finite sum of Feynman diagrams.

*A priori*, a BPHZ formula for the amplitude of the trees is not known, nor its equivalent algebraic version based on a Hopf algebra of planar binary trees. Therefore, to find the tree-expansion for the renormalized propagators, it is necessary to solve directly the Dyson-Schwinger equations to which they are constrained. These are best given on the inverse of the photon propagator, that is, on the *vacuum polarization*

$$\Pi_{\lambda\mu}(p) = (p_\lambda p_\mu - p^2 g_{\lambda\mu}) - \xi p_\lambda p_\mu - [D^{-1}]_{\lambda\mu}(p).$$

Then, the *renormalized Dyson-Schwinger equations* are given by

$$\left\{ \begin{array}{l} S^{ren}(p; e, m) = Z_2^{-1} S_0(p) - \delta m S_0(p) S^{ren}(p) + ie^2 S_0(p) \int \frac{d^4q}{(2\pi)^4} \gamma^\lambda D^{ren}_{\lambda\lambda'}(q) \frac{\delta S^{ren}(p-q)}{e \delta A_{\lambda'}(q)}, \\ \Pi_{\lambda\mu}^{ren}(p; e, m) = (1 - Z_3)(p_\lambda p_\mu - p^2 g_{\lambda\mu}) - i e^2 Z_2 \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[ \gamma^\lambda \frac{\delta S^{ren}(q)}{e \delta A^\mu(-p)} \right]. \end{array} \right. \quad (1.10)$$

Assuming that the propagators are expanded on planar binary trees according to

$$\left\{ \begin{array}{l} S^{ren}(p; e, m) = S_0(p) + \sum_{t \in \overline{Y}} e^{2|t|} S_t^{ren}(p), \\ D_{\mu\nu}^{ren}(p; e, m) = D_{0,\mu\nu}(p) + \sum_{t \in \overline{Y}} e^{2|t|} D_{t,\mu\lambda}^{ren}(p), \end{array} \right. \quad (1.11)$$

and that the renormalization factors are also expanded on planar binary trees, that is,

$$\begin{aligned} Z_2(e) &= 1 + \sum_{|t|>0} e^{2|t|} C_2(t), \quad \text{with } C_2(\mid) = 1, \\ Z_3(e) &= 1 - \sum_{|t|>0} e^{2|t|} C_3(t), \quad \text{with } C_3(\mid) = 1, \\ \delta m(e) &= \sum_{|t|>0} e^{2|t|} C_m(t), \quad \text{with } C_m(\mid) = 0, \end{aligned} \quad (1.12)$$

then the system (1.10) determines the coefficients of the renormalized propagators as

$$\left\{ \begin{array}{l} S_t^{ren}(p) = C_2(I(t)) - S_0(p) (C_m * S^{ren})(t) + iS_0(p) \int \frac{d^4q}{(2\pi)^4} \gamma^\lambda D_{t^i, \lambda \lambda'}^{ren}(q) \frac{\delta S_{t^r}^{ren}(p-q)}{e \delta A_{\lambda'}(q)}, \\ D_{t, \mu\nu}^{ren}(p) = D_{|\vee t_r, \mu\nu}^{ren}(p) (p_\mu p_\nu - p^2 g_{\mu\nu}) D_{t_l, \mu\nu}^{ren}(p), \quad \text{if } t = t_l \vee t_r \text{ with } t_l \neq |, \\ D_{|\vee t, \mu\nu}^{ren}(p) = C_3(| \vee t) D_{0, \mu\lambda}^T(p) - i D_{0, \mu\lambda}^T(p) \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[ \gamma^\lambda \frac{\delta(S^{ren}(q) * C_2)(t)}{e \delta A_{\lambda'}(-p)} \right] D_{0, \lambda' \nu}^T(p). \end{array} \right. \quad (1.13)$$

In these equations we make use of a convolution  $*$  dual to the pruning coproduct on trees defined recursively by  $\Delta_e^{\text{inv}}(|) = | \otimes |$  and

$$\Delta_e^{\text{inv}}(t) = | \otimes t + \sum_{\Delta_e^{\text{inv}}(t_r)} (t_l \vee (t_r)_{(1)}) \otimes (t_r)_{(2)}, \quad \text{for } t = t_l \vee t_r. \quad (1.14)$$

The map  $I$  appearing in the first equation is the antipode associated to this coproduct.

The coproduct  $\Delta_e^{\text{inv}}$  turns out to be related to the operation *under* among trees, introduced by J.-L. Loday in [Lod02]: given two planar binary trees  $u, v \neq |$ , the tree  $u$  under  $v$ , denoted by  $u \setminus v$ , is the tree obtained by grafting the right-most leaf of  $u$  under the root of  $v$ , that is,

$$u \setminus v = u \begin{array}{c} \nearrow \\ v \end{array}.$$

Then, Eq. (1.14) can be simply rewritten as

$$\Delta_e^{\text{inv}}(t) = \sum_{t=u \setminus v} u \otimes v,$$

where we also suppose that  $\Delta_e^{\text{inv}}$  is a multiplicative map which respects the decomposition of a tree  $t = t_l \vee t_r$  into its two multiplicative factors  $t = (t_l \vee |) \setminus t_r$ .

In fact, observing carefully the equations for the photon propagator, one can see that another operation among trees is implicitly used: the decomposition of a tree  $t$  into two multiplicative factors  $t^l$  and  $| \vee t^r$ . The product among trees which corresponds to this decomposition is the operation *over*, cf. [Lod02]: given two planar binary trees  $u, v \neq |$ , the tree  $u$  over  $v$ , denoted by  $u/v$ , is the tree obtained by grafting the root of  $u$  over the left-most leaf of  $v$ , that is,

$$u/v = \begin{array}{c} u \\ \searrow \\ v \end{array}.$$

Associated to this product there is of course a coproduct

$$\Delta_\gamma^{\text{inv}}(t) = \sum_{t=u/v} u \otimes v.$$

It turns out that the relationship between the bare propagators  $S(p; e_0, m_0)$ ,  $D_{\mu\nu}(p; e_0, m_0)$  and the renormalized propagators  $S^{ren}(p; e, m)$ ,  $D_{\mu\nu}^{ren}(p; e, m)$ , all expanded on planar binary trees, is governed by two Hopf algebra structures on the set of trees based on the two coproducts  $\Delta_e^{\text{inv}}$ ,  $\Delta_\gamma^{\text{inv}}$  and also on another coproduct  $\Delta^\alpha$  analogue to the Connes-Kreimer coproduct on Feynman diagrams.

## 2 Renormalization Hopf algebras and groups of tree-expanded series

### 2.1 QED renormalization Hopf algebras on trees

In [BF01] and [BF03] it is shown that the relationship between the QED propagators before and after the renormalization, excluding the mass renormalization and considering the expansion over planar binary



trees, is described by a semidirect coproduct of Hopf algebras on trees, involving a Hopf algebra  $\mathcal{H}^e$  for the electron propagator, one for the photon propagator  $\mathcal{H}^\gamma$ , and one for the electric charge  $\mathcal{H}^\alpha$ .

As algebras,  $\mathcal{H}^e$  and  $\mathcal{H}^\gamma$  are both isomorphic to the free non-commutative algebra generated by all trees, where we identify the root tree  $|$  to the formal unit. Therefore  $\mathcal{H}^e = \mathcal{H}^\gamma = \mathbb{Q}\langle \overline{Y} \rangle \cong \mathbb{Q}\langle Y \rangle / (| - 1)$ . The coalgebra structures are given by the pruning coproducts  $\Delta_\gamma^{\text{inv}} : \mathcal{H}^\gamma \longrightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\gamma$  and  $\Delta_e^{\text{inv}} : \mathcal{H}^e \longrightarrow \mathcal{H}^e \otimes \mathcal{H}^e$  defined as the dual operations of the two grafting products *over* and *under* on trees:

$$\begin{aligned} \text{over:} \quad t/s &= \begin{array}{c} t \\ \diagdown \\ s \end{array}, & \Delta_\gamma^{\text{inv}}(u) &= \sum_{t/s=u} t \otimes s, \\ \text{under:} \quad t \setminus s &= \begin{array}{c} t \\ \diagup \\ s \end{array}, & \Delta_e^{\text{inv}}(u) &= \sum_{t \setminus s=u} t \otimes s. \end{aligned}$$

The counits  $\varepsilon : \mathcal{H}^\gamma \longrightarrow \mathbb{Q}$  and  $\varepsilon : \mathcal{H}^e \longrightarrow \mathbb{Q}$  are dual to the unit  $|$ , that is  $\varepsilon(t) = \delta_{t,|}$ . Note that the Hopf algebras  $\mathcal{H}^e$  and  $\mathcal{H}^\gamma$  are neither commutative nor cocommutative.

On the other side,  $\mathcal{H}^\alpha$  is the abelian quotient of the algebra  $\mathbb{Q}Y$  of all trees endowed with the over product (which is not commutative). Thus the root tree  $|$  is the unit, and if we set  $V(t) = \Upsilon \setminus t$ , the algebra  $\mathcal{H}^\alpha$  is in fact isomorphic to the polynomial algebra  $\mathbb{Q}[V(t), t \in Y]$ . The coproduct  $\Delta^\alpha : \mathcal{H}^\alpha \longrightarrow \mathcal{H}^\alpha \otimes \mathcal{H}^\alpha$  is defined on the generators by the assignment

$$\Delta^\alpha(V(t)) = 1 \otimes V(t) + \delta^\alpha(V(t)), \quad (2.1)$$

where  $\delta^\alpha : \mathcal{H}^\alpha \longrightarrow \mathcal{H}^\alpha \otimes \mathcal{H}^\alpha$  is a right coaction of  $\mathcal{H}^\alpha$  on itself (w.r.t. the coproduct  $\Delta^\alpha$ ), defined recursively as

$$\delta^\alpha(V(t)) = (V \otimes \text{Id}) [\Delta^\alpha(t^l) / \delta^\alpha(V(t^r))], \quad (2.2)$$

where  $t = t^l / V(t^r)$ . The counit  $\varepsilon : \mathcal{H}^\alpha \longrightarrow \mathbb{Q}$  is of course the unital algebra morphism with value  $\varepsilon(V(t)) = 0$  on the generators.

According to [BF01] and [BF03], the massless renormalization of the electron propagator expanded over planar binary trees is described by the semidirect (or smash) coproduct Hopf algebra  $\mathcal{H}^{\text{qed}} = \mathcal{H}^\alpha \ltimes \mathcal{H}^e$ , while the massless renormalization of the photon propagator is described by the Hopf algebra  $\mathcal{H}^\alpha$  itself, seen as a Hopf subalgebra of the semidirect coproduct  $\mathcal{H}^\alpha \ltimes \mathcal{H}^\gamma$ . These semidirect coproduct Hopf algebras can be defined, according to [Mol77], because there is a coaction of  $\mathcal{H}^\alpha$  on  $\mathcal{H}^e$  and another one on  $\mathcal{H}^\gamma$ , namely two variations of  $\delta^\alpha$ , which preserve the algebra structures.

The results can be summarized as follows:

- the bare and renormalized amplitudes  $S(p; e_0, m_0)$ ,  $S^{\text{ren}}(p; e_0, m_0)$  and the renormalization factor  $Z_2(e)$  can be seen as characters of the Hopf algebra  $\mathcal{H}^e$  with values in an algebra of regularized amplitudes  $\mathcal{A}_{\text{reg}}$ ;
- the bare and renormalized amplitudes  $D_{\mu\nu}(p; e, m)$ ,  $D_{\mu\nu}^{\text{ren}}(p; e, m)$  and the renormalization factor  $Z_3(e)$  can be seen as characters of the Hopf algebra  $\mathcal{H}^\gamma$ , with values in the same algebra  $\mathcal{A}_{\text{reg}}$ ;
- the bare charge  $e_0(e)$  can be seen as a character of the Hopf algebra  $\mathcal{H}^\alpha$ , with values in the same algebra  $\mathcal{A}_{\text{reg}}$ ;
- the relationship between the tree-expanded coefficients of the propagators before and after the renormalization for massless QED is given by the convolution associated to the coproducts of the Hopf algebras  $\mathcal{H}^{\text{qed}} = \mathcal{H}^\alpha \ltimes \mathcal{H}^e$ , for the electron, and  $\mathcal{H}^\alpha \subset \mathcal{H}^\alpha \ltimes \mathcal{H}^\gamma$  for the photon.

It is well known that the characters of a commutative Hopf algebra form a group. If the algebra is the inductive limit of finite-dimensional graded algebras, as is the case here, the group is pro-algebraic (and even pro-unipotent). In the next section we describe the pro-algebraic groups formed by the characters of the Hopf algebras related to the QED renormalization, abelianized when necessary.

## 2.2 Groups of tree-expanded series

The abelian quotients  $\mathcal{H}_{ab}^\gamma$  and  $\mathcal{H}_{ab}^e$  of the two Hopf algebras  $\mathcal{H}^\gamma$  and  $\mathcal{H}^e$  are the coordinate rings of two pro-algebraic groups, denoted by  $G^\gamma$  and  $G^e$  respectively, cf. [Fra08]. For a given unital, associative and commutative algebra  $\mathcal{A}$ , consider the set

$$G_Y^{\text{inv}} = \left\{ \mathbf{A} = \sum_{t \in Y} \mathbf{A}_t t, \mathbf{A}_\perp = 1 \right\},$$

of tree-expanded series with coefficients  $\mathbf{A}_t$  in  $\mathcal{A}$  and invertible constant term. The groups  $G^\gamma$  and  $G^e$  are given on  $G^{\text{inv}}$  respectively by the over and the under products on series, induced by the analogous operations on trees:

$$\begin{aligned} \text{over} \quad \mathbf{A}/\mathbf{B} &= \sum_{s,t \in Y} \mathbf{A}_s \mathbf{B}_t s/t, & \text{for } G^e, \\ \text{under} \quad \mathbf{A} \setminus \mathbf{B} &= \sum_{s,t \in Y} \mathbf{A}_s \mathbf{B}_t s \setminus t, & \text{for } G^\gamma. \end{aligned}$$

In both cases, the unit is given by the series  $\perp$ .

We can identify a tree-expanded series  $\mathbf{A} = \sum_{t \in Y} \mathbf{A}_t t$  with a “generalized” series on a variable  $x$ , by setting  $\mathbf{A}(x) = \sum_{t \in Y} \mathbf{A}_t x^t$ , where the monomial  $x^t$  is a formal symbol. Then, it is easy to see that the groups  $G^e$  and  $G^\gamma$  are non-abelian generalizations of the abelian group

$$G^{\text{inv}} = \left\{ \mathbf{A}(x) = \sum_{n \geq 0} \mathbf{A}_n x^n, \mathbf{A}_0 = 1 \right\},$$

of usual invertible series in one variable, with coefficients  $\mathbf{A}_n$  in  $\mathcal{A}$ , and considered with the multiplication.

Similarly, the Hopf algebra  $\mathcal{H}^\alpha$  can be realized as a group of tree-expanded series, endowed, this time, with an operation which generalizes the composition (substitution) of formal series. To do this, we first consider the set

$$G_Y^{\text{dif}} = \left\{ \mathbf{A} = \sum_{t \in \bar{Y}} \mathbf{A}_t t, \mathbf{A}_\gamma = 1 \right\}$$

of tree-expanded formal diffeomorphisms with coefficients  $\mathbf{A}_t$  in  $\mathcal{A}$ . The composition of two tree-expanded series  $\mathbf{A} = \sum_t \mathbf{A}_t t$  and  $\mathbf{B} = \sum_s \mathbf{B}_s s$  is given by

$$\mathbf{A} \circ \mathbf{B} = \sum_{\substack{t \in \bar{Y} \\ s_1, s_2, \dots, s_{|t|} \in \bar{Y}}} \mathbf{A}_t \mathbf{B}_{s_1} \mathbf{B}_{s_2} \dots \mathbf{B}_{s_{|t|}} \mu_t(s_1, s_2, \dots, s_{|t|}),$$

where  $\mu_t(s_1, s_2, \dots, s_{|t|})$  is the tree obtained by inserting the trees  $s_1, \dots, s_{|t|}$  in the vertices of  $t$ , ordered from left to right and from the leaves to the root of  $t$ . In fact, the operation  $\mu$  is the operadic composition of the Duplicial operad, also called OverUnder operad, cf. [Lod06, AL07, van03], and the group  $G_Y^{\text{dif}}$  is an example of a general construction which works for any connected operad, cf. [Cha, van04, Fra08]. The unit of the group  $G_Y^{\text{dif}}$  is the series  $\gamma$ .

As before, we can identify a tree-expanded series  $\mathbf{A} = \sum_{t \in \bar{Y}} \mathbf{A}_t t$  with a “generalized” series in one variable,  $\mathbf{A}(x) = \sum_{t \in \bar{Y}} \mathbf{A}_t x^t$ . Then, if we define the power of the series  $\mathbf{B}(x)$  by a tree  $t \neq \perp$  as  $\mathbf{B}(x)^t = \mu_t(\mathbf{B}(x), \mathbf{B}(x), \dots, \mathbf{B}(x))$ , the composition of tree-expanded series can also be seen as a substitution, that is  $(\mathbf{A} \circ \mathbf{B})(x) = \mathbf{A}(\mathbf{B}(x)) = \sum_{t \in \bar{Y}} \mathbf{A}_t \mathbf{B}(x)^t$ . In [Fra08] it is then shown that  $G_Y^{\text{dif}}$  is a group which projects onto the group

$$G^{\text{dif}} = \left\{ \mathbf{A}(x) = \sum_{n \geq 1} \mathbf{A}_n x^n, \mathbf{A}_1 = 1 \right\}$$

of usual formal diffeomorphisms on a line (tangent to the identity).

The group  $G_Y^{\text{dif}}$  contains two proper subgroups

$$\begin{aligned} G_Y^\rho &= G^\gamma / \Upsilon = \left\{ \rho_{\mathbf{A}} = \sum_{t \in Y} \mathbf{A}_t \overset{t}{\Upsilon}, \mathbf{A}_{\mid} = 1 \right\}, \\ G_Y^\lambda &= \Upsilon \backslash G^e = \left\{ \lambda_{\mathbf{A}} = \sum_{t \in Y} \mathbf{A}_t \Upsilon^t, \mathbf{A}_{\mid} = 1 \right\}, \end{aligned}$$

whose projection onto usual series is the product  $xG^{\text{inv}}$ , which is isomorphic to  $G^{\text{dif}}$  itself.

In [Fra08] it was proved that the group  $G^\alpha$  is a proper subgroup of  $G_Y^\rho$  of the form

$$G^\alpha = \left\{ \alpha_{\mathbf{A}} = (\mid - \Upsilon \backslash \mathbf{A})^{-1} / \Upsilon, \mathbf{A} \in \mathcal{A}[[Y]] \right\}, \quad (2.3)$$

where  $\mathcal{A}[[Y]]$  denotes the set of tree-expanded formal series  $\mathbf{A} = \sum_{t \in Y} \mathbf{A}_t \overset{t}{\Upsilon}$  with coefficients  $\mathbf{A}_t$  in the algebra  $\mathcal{A}$ . In fact, for any tree-expanded formal series  $\mathbf{A} = \sum_{t \in Y} \mathbf{A}_t \overset{t}{\Upsilon} \in \mathcal{A}[[Y]]$ , the series  $\mid - \Upsilon \backslash \mathbf{A}$  belongs to the set  $G_Y^{\text{inv}}$ , and therefore to the group  $G^\gamma$  of tree-expanded invertible series with respect to the product  $/$ . The projection of  $G^\alpha$  onto usual series is the group  $x(1 - x\mathcal{A}[[x]])^{-1}$  which coincides, again, with the whole group  $G^{\text{dif}}$ .

### 3 Tree-expanded series in relation with Tamari posets

In this section, we will consider some examples of tree-expanded series in  $G_Y^{\text{dif}}$ . We will compute the inverse of some of them with respect to composition and show that some others are related to the Tamari lattices.

We will need the following involution on tree-expanded series. Let  $\mathbf{S} = \sum_{n \geq 1} \mathbf{S}_n$  be a tree-expanded series in  $G_Y^{\text{dif}}$ , decomposed with respect to the order of the trees. The *suspension* of  $\mathbf{S}$  is the series  $\tilde{\mathbf{S}} = \sum_{n \geq 1} (-1)^{n-1} \mathbf{S}_n$ . The suspension of a composition  $\widetilde{\mathbf{A} \circ \mathbf{B}}$  is the composition of the suspensions  $\tilde{\mathbf{A}} \circ \tilde{\mathbf{B}}$ .

We will also use the following properties of the composition:

$$\forall \mathbf{A}, \mathbf{C}, \mathbf{D} \quad (\mathbf{A}/\mathbf{D}) \circ \mathbf{C} = (\mathbf{A} \circ \mathbf{C}) / (\mathbf{D} \circ \mathbf{C}) \quad \text{and} \quad (\mathbf{A} \backslash \mathbf{D}) \circ \mathbf{C} = (\mathbf{A} \circ \mathbf{C}) \backslash (\mathbf{D} \circ \mathbf{C}). \quad (3.1)$$

These properties follow from the relation of  $G_Y^{\text{dif}}$  with the Duplicial operad.

Let us consider the tree-expanded series  $\mathbf{A}$  defined inductively by

$$\mathbf{A} = \Upsilon + w \mathbf{A} / \Upsilon + \Upsilon \backslash \mathbf{A} + w \mathbf{A} / \Upsilon \backslash \mathbf{A}, \quad (3.2)$$

where  $w$  is a formal parameter. This is in fact the sum over all trees where each tree  $t$  appears with coefficient  $w$  to the power the number of right-oriented leaves of  $t$  minus 1:

$$\mathbf{A} = \Upsilon + w \Upsilon \backslash \Upsilon + \Upsilon \backslash \Upsilon + w^2 \Upsilon \backslash \Upsilon \backslash \Upsilon + w \Upsilon \backslash \Upsilon \backslash \Upsilon + w \Upsilon \backslash \Upsilon \backslash \Upsilon + w \Upsilon \backslash \Upsilon \backslash \Upsilon + \Upsilon \backslash \Upsilon \backslash \Upsilon + \dots \quad (3.3)$$

Let us consider also the tree-expanded series  $\mathbf{B}$  defined by

$$\mathbf{B} = \Upsilon - w \Upsilon / \mathbf{B} - \mathbf{B} \backslash \Upsilon - w \Upsilon / \mathbf{B} \backslash \Upsilon, \quad (3.4)$$

where  $w$  is a formal parameter. This is in fact the sum over  $p \geq 0$  and  $q \geq 0$  of  $(-1)^{p+q} w^p c_p / \Upsilon \backslash d_q$  where  $c_p$  is the left comb with  $p$  vertices and  $d_q$  is the right comb with  $q$  vertices:

$$\mathbf{B} = \Upsilon - w \Upsilon \backslash \Upsilon - \Upsilon \backslash \Upsilon + w^2 \Upsilon \backslash \Upsilon \backslash \Upsilon + w \Upsilon \backslash \Upsilon \backslash \Upsilon + \Upsilon \backslash \Upsilon \backslash \Upsilon + \dots \quad (3.5)$$

**Proposition 3.1** *In the group  $G_Y^{\text{dif}}$ , one has  $\mathbf{A} \circ \mathbf{B} = \Upsilon$ , that is  $\mathbf{A}$  is the inverse of  $\mathbf{B}$ . This identity can be specialized to any complex value of  $w$ .*

*Proof.* Let us compute the composition  $A \circ B$  starting from the definition of  $A$ . By properties of the composition, one gets, by multiplication on the right by  $B$ , that

$$A \circ B = B + w(A \circ B)/B + B \setminus (A \circ B) + w(A \circ B)/B \setminus (A \circ B).$$

Comparing this to the following form of the definition of  $B$ :

$$\Upsilon = B + w\Upsilon/B + B \setminus \Upsilon + w\Upsilon/B \setminus \Upsilon, \quad (3.6)$$

one can see that  $\Upsilon$  and  $A \circ B$  both satisfy the same induction and have the same initial term, hence they are equal.  $\square$

Let us now consider the tree-expanded series  $C$  and  $D$  defined by

$$C = \Upsilon + C/\Upsilon = \Upsilon + \Upsilon/C = \Upsilon + \begin{array}{c} \diagdown \\ \Upsilon \end{array} + \begin{array}{c} \diagdown \diagdown \\ \Upsilon \end{array} + \dots \quad (3.7)$$

and

$$D = \Upsilon + D \setminus \Upsilon = \Upsilon + \Upsilon \setminus D = \Upsilon + \begin{array}{c} \diagup \\ \Upsilon \end{array} + \begin{array}{c} \diagup \diagup \\ \Upsilon \end{array} + \dots \quad (3.8)$$

So  $C$  is in fact the sum over all left combs and  $D$  is the sum over all right combs. The series  $C$  (resp.  $D$ ) belongs to the subgroup of  $G_Y^{\text{dif}}$  formed by series indexed by left combs only (resp. right combs only). It is quite clear (by inversion in these subgroups) that  $C^{-1} = \tilde{C}$  and  $D^{-1} = \tilde{D}$ .

**Proposition 3.2** *One has*

$$C + C \setminus D = D + C/D. \quad (3.9)$$

*The series  $E = C \circ D^{-1}$  is characterized by the equation*

$$E = \Upsilon + E/\Upsilon - E \setminus \Upsilon. \quad (3.10)$$

*Proof.* It is enough to see that the left hand side of (3.9) is exactly the sum of all trees of the shape  $c^p/\Upsilon \setminus d^q$  where  $c^p$  is a left comb (possibly empty) and  $d^q$  is a right comb (idem). By symmetry, this is also true for the right-hand side, hence they are equal. The equation for  $C \circ D^{-1}$  is obtained from this by composition on the right by  $D^{-1}$ .  $\square$

The series  $E$  appears surprisingly to be related to the Möbius function of the Tamari lattice. Recall that the Tamari lattice is a classical partial order [HT72] on the set  $Y_n$  of trees of order  $n$ , where the left comb  $c_n$  is the unique minimal element and the right comb  $d_n$  is the unique maximal element. The partial order is defined by transitive closure of the following relation:  $t \leq t'$  if  $t'$  is obtained from  $t$  by replacing a local configuration  $\begin{array}{c} \diagdown \\ \Upsilon \end{array}$  by a local configuration  $\begin{array}{c} \diagup \\ \Upsilon \end{array}$ .

**Proposition 3.3** *One has the following description of the series  $E$  and  $D^{-1}$ :*

$$E = \sum_{n \geq 1} \sum_{t \in Y_n} \mu(c_n, t) t \quad (3.11)$$

and

$$D^{-1} = \Upsilon + \sum_{n \geq 1} \sum_{t \in Y_n} \mu(c_n, \Upsilon \setminus t) \Upsilon \setminus t, \quad (3.12)$$

where  $\mu$  is the Möbius function of the Tamari lattice.

*Proof.* The first statement is equivalent to the second one, by using the following property of Tamari lattices, see for instance [BW97, Lemma 2.1]: any interval  $[c_{p+q}, t'/t'']$  in a Tamari lattice is isomorphic to the product of the intervals  $[c_p, t']$  and  $[c_q, t'']$  in smaller Tamari lattices. This shows that the generating series of Möbius number for all trees is the composition of  $C$  by the generating series of Möbius number for trees of the shape  $\Upsilon \setminus t$ .

The second statement can be deduced from the computation of the Möbius function of the Tamari lattice by Björner and Wachs: see [BW97, Corollary 9.5]. In their notations,  $c_n$  corresponds to the word  $(0, \dots, 0)$  and a tree of shape  $\Upsilon \setminus t$  to a word  $w_t$  beginning with the letter  $n - 1$ . One gets that the Möbius

number  $\mu((0, \dots, 0), w_t)$  is non zero if and only if the word  $w_t$  is  $(n-1, n-2, \dots, 2, 1)$ , which corresponds to the tree  $t = d_n$  in our notations.  $\square$

Remark: By using the projection morphism from  $G_Y^{\text{dif}}$  to  $G^{\text{dif}}$ , one can see that the sum of coefficients in  $\mathbf{E}$  of all trees of a fixed order  $n > 1$  is zero, which also follows from the proposition.

Let us then introduce the tree-expanded series  $\mathbf{R} = \Upsilon + \Upsilon \backslash \mathbf{A}$  and  $\mathbf{L} = \Upsilon + \mathbf{A}/\Upsilon$ . In fact,  $\mathbf{R}$  is the sum of all trees of the shape  $\Upsilon \backslash t$  and  $\mathbf{L}$  is the sum of all trees of the shape  $t/\Upsilon$ . The series  $\mathbf{R}$  (resp.  $\mathbf{L}$ ) belongs to the subgroup of  $G_Y^{\text{dif}}$  formed by series indexed by trees of the shape  $\Upsilon \backslash t$  only (resp. by trees of the shape  $t/\Upsilon$  only).

**Proposition 3.4** *One has*

$$\mathbf{R} = \Upsilon + \mathbf{R} \backslash \mathbf{L}, \quad (3.13)$$

$$\mathbf{L} = \Upsilon + \mathbf{R}/\mathbf{L}. \quad (3.14)$$

*The composition  $\mathbf{R} \circ \mathbf{L}^{-1}$  is equal to the suspension of  $\mathbf{E}$ .*

*Proof.* The first two formulas follows directly from a standard combinatorial argument using a decomposition of trees. For instance, the first formula can be deduced from the existence of an unique maximal decomposition of a tree as an iterated  $\backslash$  product.

By multiplication on the right by  $\mathbf{L}^{-1}$ , one has

$$\mathbf{R} \circ \mathbf{L}^{-1} = \mathbf{L}^{-1} + (\mathbf{R} \circ \mathbf{L}^{-1}) \backslash \Upsilon, \quad (3.15)$$

$$\Upsilon = \mathbf{L}^{-1} + (\mathbf{R} \circ \mathbf{L}^{-1})/\Upsilon. \quad (3.16)$$

Hence by elimination of  $\mathbf{L}^{-1}$  one has

$$\mathbf{R} \circ \mathbf{L}^{-1} = \Upsilon + (\mathbf{R} \circ \mathbf{L}^{-1}) \backslash \Upsilon - (\mathbf{R} \circ \mathbf{L}^{-1})/\Upsilon. \quad (3.17)$$

By definition of the suspension, the suspension of  $\mathbf{E}$  and  $\mathbf{R} \circ \mathbf{L}^{-1}$  satisfy the same induction, hence they are equal.  $\square$

Remark: one can also deduce from this a similar description of  $\mathbf{L}^{-1}$ .

## References

- [AL07] M. Aguiar and M. Livernet. The associative operad and the weak order on the symmetric groups. *J. Homotopy Relat. Struct.*, 2(1):57–84 (electronic), 2007.
- [AS06] M. Aguiar and F. Sottile. Structure of the Loday-Ronco Hopf algebra of trees. *J. Algebra*, 295(2):473–511, 2006.
- [BF01] C. Brouder and A. Frabetti. Renormalization of QED with planar binary trees. *Eur. Phys. J.*, C19:715–741, 2001.
- [BF03] C. Brouder and A. Frabetti. QED Hopf algebras on planar binary trees. *J. Algebra*, 267(1):298–322, 2003.
- [Bro00] C. Brouder. On the trees of quantum fields. *Eur. Phys. J.*, C12:535–549, 2000.
- [BW97] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.
- [Cha] F. Chapoton. Rooted trees and an exponential-like series. arXiv:math.QA/0209104.
- [CK00a] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem. *Commun. Math. Phys.*, 210:249–273, 2000.

- [CK00b] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem II: the  $\beta$  function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 2000. hep-th/0003188.
- [Dys49] F. J. Dyson. The S matrix in quantum electrodynamics. *Phys. Rev.*, 75:1736–55, 1949.
- [Fra08] A. Frabetti. Groups of tree-expanded series. *J. Algebra*, 319(1):377–413, 2008.
- [HNT05] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. The algebra of binary search trees. *Theoret. Comput. Sci.*, 339(1):129–165, 2005.
- [HT72] S. Huang and D. Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *J. Combinatorial Theory Ser. A*, 13:7–13, 1972.
- [IZ80] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*. McGraw-Hill, New York, 1980.
- [Kre98] Dirk Kreimer. On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.*, 2(2):303–334, 1998.
- [Lod01] Jean-Louis Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.
- [Lod02] J.-L. Loday. Arithmetree. *J. Algebra*, 258(1):275–309, 2002. Special issue in celebration of Claudio Procesi’s 60th birthday.
- [Lod06] J.-L. Loday. Generalized bialgebras and triples of operads, 2006.
- [Mol77] R. K. Molnar. Semi-direct products of Hopf algebras. *J. Algebra*, 47(1):29–51, 1977.
- [Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [van03] P. van der Laan. Operads and the Hopf algebras of renormalisation, 2003.
- [van04] P. van der Laan. *Operads. Hopf algebras and coloured Koszul duality*. PhD thesis, Universiteit Utrecht, 2004.