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ANALYTIC EQUIVALENCE OF NORMAL CROSSING FUNCTIONS ON A REAL ANALYTIC MANIFOLD

GOULWEN FICHOU AND MASAHIRO SHIOTA

ABSTRACT. By Hironaka Desingularization Theorem, any real analytic function has only normal crossing singularities after a suitable modification. We focus on the analytic equivalence of such functions with only normal crossing singularities. We prove that for such functions C^∞ right equivalence implies analytic equivalence. We prove moreover that the cardinality of the set of equivalence classes is zero or countable. We apply these results to study the cardinality of the set of equivalence classes for almost blow-analytic equivalence.

In the study of real *analytic* function singularities, the choice of a relevant equivalence relation is a crucial but difficult topic. After Hironaka Desingularization Theorem [Hi], which enables to produce functions with only normal crossing singularities after a finite sequence of blowings-up along smooth analytic centers, it seems natural to expect that equivalent real analytic functions should admit similar resolution of their singularities. In that spirit, we propose in this paper to study the equivalence relation obtained by requiring that two real analytic functions, defined on a compact real analytic manifold, are equivalent if there exist two Hironaka desingularizations such that the modified functions with only normal crossing singularities, obtained after the desingularizations, become analytically equivalent.

This definition is weaker than that of blow-analytic equivalence introduced by T. C. Kuo [Ku, K-W], which seems to be, up to now, the best candidate to be the real counterpart of the topological equivalence for complex analytic functions (cf [F-P] for a recent survey). The blow-analytic equivalence imposes to the analytic isomorphism that realizes the equivalence, the additional condition to induce a topological equivalence between the functions. Here, we release this strong condition which imposes drastic restrictions that are often hopeless for many concrete applications, typically when we only know that after some simplification process, two given objects are similar in some sense.

The equivalence relation we propose to study in this paper is called almost blow-analytic equivalence in the language of blow-analytic theory. Our main result (corollary 3.3) states that the cardinality of the set of equivalence classes for almost blow-analytic equivalence is zero or countable (corollary 3.3). Therefore it is reasonable to hope for a classification!

To prove the result, we reduce the problem to the analogous study for Nash functions, i.e. real analytic functions whose graph is a semi-algebraic set (described by a set of polynomial equalities and inequalities with real coefficient polynomials).

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The great advantage then is that we can use complexity arguments to deal with cardinality issues. Actually, the study of almost blow-Nash equivalence for Nash functions presents a great interest by itself (cf. [Ko,Fi] for notions on blow-Nash equivalence). Notably the cardinality of the set of equivalence classes remains zero or countable even if the underlying Nash manifold is no longer compact.

We give now more details on the strategy we use to tackle these issues. First, thanks to Hironaka Desingularization Theorem, it is of great importance to focus on the analytic (respectively Nash) equivalence between real analytic (resp. Nash) functions with only normal crossing singularities. The first main result in this direction is theorem 3.1,(1) which asserts that C^∞ right equivalent real analytic functions with only normal crossing singularities are automatically analytically right equivalent. Theorem 3.1,(1) is a crucial tool in order to deal with cardinality issues, in particular in view to make a reduction from the analytic case to the Nash one. Its proof consists in a careful use of Cartan Theorems A and B and Oka Theorem in order to use integration along analytic vector fields to produce analytic isomorphisms.

The second main result (theorem 3.2) deals with the cardinality of the set of equivalence classes of real analytic (respectively Nash) functions with only normal crossing singularities on a compact analytic manifold (resp. on a non necessarily compact Nash manifold) with respect to the analytic (resp. Nash) equivalence. To prove that this cardinality is zero or countable, we first reduce the study to the Nash case by theorem 3.1,(1), then from the non compact to the compact case via Nash sheaf theory, a Nash version of Hironaka Desingularization Theorem and a finer analysis of the normal crossing property on a Nash manifold with corners. Finally Hardt triviality [Hd], Artin-Mazur Theorem [S₂] and Nash Approximation Theorem (via Popescu Approximation Theorem [Sp]) enable to achieve the proof. Note that the techniques developed there can be used to offer (as theorem 3.1,(3)) a C^2 plus semi-algebraic version of theorem 3.1,(1), namely semialgebraically C^2 right equivalent Nash functions with only normal crossing singularities on a Nash manifold are Nash right equivalent (see also theorem 3.1,(2) for a C^2 version).

At this point, the nullity or countableness of the cardinality of the set of classes for almost blow-analytic equivalence, or almost blow-Nash equivalence (corollary 3.3), becomes a direct consequence of Hironaka Desingularization Theorem and theorem 3.2.

The paper is organized as follows. In the first section we state the definitions of almost blow-analytic and blow-Nash equivalence relations, and try to give the flavour of these relations. We devote the second section to some preliminaries about real analytic and Nash sheaf theory. In particular we state Cartan Theorems A and B and Oka Theorem in the real analytic case, and prove some more precise corollaries that will be intensively used in the paper. We need also stronger versions of classical Cartan Theorems A and B and Oka Theorem in the Nash case (compare with [C-S₁]), and write down a proof of Hironaka Desingularization Theorem in the Nash setting. The last section is dedicated to the statement and proofs of the main results.

In this paper a manifold means a manifold without boundary, analytic manifolds and maps mean real analytic ones unless otherwise specified, and id stands for the identity map.

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1 ALMOST BLOW-ANALYTIC EQUIVALENCE

Definition 1.1. Let M be an analytic manifold and $f, g : M \rightarrow \mathbf{R}$ be analytic functions on M . Then f and g are said to be *almost blow-analytically equivalent* if there exist two compositions of finite sequences of blowings-up along smooth analytic centers $\pi_f : N \rightarrow M$ and $\pi_g : L \rightarrow M$ and an analytic diffeomorphism $h : N \rightarrow L$ so that $f \circ \pi_f = g \circ \pi_g \circ h$. In the case where there exist $\pi_f : N \rightarrow M$ and $\pi_g : L \rightarrow M$ with the above property and analytic diffeomorphisms $h : N \rightarrow L$ and $\tau : \mathbf{R} \rightarrow \mathbf{R}$ such that $\tau \circ f \circ \pi_f = g \circ \pi_g \circ h$, f and g are called *almost blow-analytically R-L (=right-left) equivalent*. We define also *almost blow-analytic (R-L) equivalence* of germs of analytic functions (see definition 1.2 for the details).

Here and from now on we treat without mention only the case where the images of the centers of the blowings-up of π_f and π_g are contained in their singular point sets $\text{Sing } f$ and $\text{Sing } g$, respectively, and the center C of each blowing-up is of codimension > 1 and normal crossing with the union D of the inverse images of the previous centers, i.e. there exists an analytic local coordinate system (x_1, \dots, x_n) at each point of C such that $C = \{x_1 = \dots = x_k\}$ and $D = \{x_{i_1} \cdots x_{i_l} = 0\}$ for some $0 < k \in \mathbf{N}$ and $1 \leq i_1 < \dots < i_l \leq n \in \mathbf{N}$, where $\mathbf{N} = \{0, 1, \dots\}$.

Remark. There does not exist large difference between almost blow-analytic equivalence and almost blow-analytic R-L equivalence in the following sense. If the germs of f and g on a connected component of $\text{Sing } f$ in M are almost blow-analytically R-L equivalent then the germs of f and $g + \text{const}$ or of f and $-g + \text{const}$ are almost blow-analytically equivalent by lemma 3.10 and proposition 3.11,(i) because we can reduce the problem to the case where $f \circ \pi_f$ and $g \circ \pi_g$ have only normal crossing singularities (see the section 3.1 for the definitions of normal crossing singularities and equivalence of function germs). The reason why we nevertheless introduce the concept of almost blow-analytic R-L equivalence is that the cardinality of the set of classes of analytic functions on a compact analytic manifold, classified by almost blow-analytic equivalence, is of the continuum even if $\dim M = 0$, and, on the other hand, the one classified by almost blow-analytic R-L equivalence is countable (corollary 3.3)

We define also *blow-analytic equivalence* (cf. [Ku]) by requiring above h to induce a homeomorphism of M . However, we treat almost blow-analytic equivalence more than blow-analytic equivalence for the following reason. First we can construct a theory on almost blow-analytic and blow-Nash equivalences (see definition 1.2 below for the definition of blow-Nash equivalence) and secondly we do not need to require (at least in this paper) above f and g to be defined globally on M for almost blow-analytic equivalence but we need only that $f \circ \pi_f$ and $g \circ \pi_g$ are extensible to analytic functions on N and L , respectively.

(i) However, we do not know whether the almost blow-analytical (R-L) equivalence and the blow-analytical (R-L) equivalence give equivalence relations. (See [F-K-K].) This is the case if we admit blowings-up along non-smooth analytic center.

(ii) Even in the case of germs of functions, almost blow-analytically equivalent function germs are not necessarily blow-analytically equivalent as follows.

Proof of (i). Let \mathcal{O}_{M_1} denote the sheaf of analytic function germs on an analytic space M_1 . For a morphism $g : M_1 \rightarrow M_2$ of analytic spaces, as locally ringed spaces, and for a sheaf of \mathcal{O}_{M_2} -ideals \mathcal{I} , let $g^{-1}\mathcal{I} \cdot \mathcal{O}_{M_1}$ denote the inverse image ideal sheaf, i.e., the sheaf of \mathcal{O}_{M_1} -ideals generated by the image of the inverse

image $g^{-1}\mathcal{I}$ of \mathcal{I} (see [Ht]). Let f_i , $i = 1, \dots, 4$, be analytic functions on an analytic manifold M with $f_2 = f_3$, let $N_{i,k_i} \xrightarrow{\pi_{i,k_i}} \dots \longrightarrow N_{i,1} \xrightarrow{\pi_{i,1}} N_{i,0} = M$, $i = 1, \dots, 4$, be sequences of blowings-up with respect to coherent sheaves of non-zero $\mathcal{O}_{N_{i,j}}$ -ideals, and let $\tau_2 : N_{1,k_1} \rightarrow N_{2,k_2}$ and $\tau_4 : N_{3,k_3} \rightarrow N_{4,k_4}$ be isomorphisms of locally ringed spaces such that $f_i \circ \pi_{i,1} \circ \dots \circ \pi_{i,k_i} \circ \tau_i = f_{i-1} \circ \pi_{i-1,1} \circ \dots \circ \pi_{i-1,k_{i-1}}$, $i = 2, 4$. If $N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \dots \longrightarrow M$ and $N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \dots \longrightarrow M$ coincide each other, then $\tau_4 \circ \tau_2$ is an isomorphism from N_{1,k_1} to N_{4,k_4} and $f_4 \circ \pi_{4,1} \circ \dots \circ \pi_{4,k_4} \circ \tau_4 \circ \tau_2 = f_1 \circ \pi_{1,1} \circ \dots \circ \pi_{1,k_1}$. Hence it suffices to reduce the problem to the case where $N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \dots \longrightarrow M$ and $N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \dots \longrightarrow M$ coincide.

Remember a fact. Let \mathcal{J}_1 and \mathcal{J}_2 be coherent sheaves of non-zero \mathcal{O}_{M_1} -ideals on a reduced and irreducible analytic space M_1 , and let $g_1 : L_1 \rightarrow M_1$, $g_2 : L_2 \rightarrow M_1$, $h_1 : N_1 \rightarrow L_1$ and $h_2 : N_2 \rightarrow L_2$ denote the blowings-up with respect to \mathcal{J}_1 , \mathcal{J}_2 , $g_1^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{L_1}$ and $g_2^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_2}$, respectively. Then there exists a unique isomorphism $\tau : N_1 \rightarrow N_2$ such that $g_1 \circ h_1 = g_2 \circ h_2 \circ \tau$ for the following reason.

$$\begin{array}{ccc} N_1 \xrightarrow{\tau} N_2 & \xrightarrow{h_2} & L_2 \\ h_1 \downarrow & & g_2 \downarrow \\ L_1 & \xrightarrow{g_1} & M_1 \end{array}$$

Apply the universal property theorem of blowing-up (see [Ht] in the algebraic case) to the blowing-up $g_2 : L_2 \rightarrow M_1$ and the morphism $g_1 \circ h_1 : N_1 \rightarrow M_1$. Then there exists a unique morphism $\pi : N_1 \rightarrow L_2$ such that $g_2 \circ \pi = g_1 \circ h_1$ since $(g_1 \circ h_1)^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{N_1} (= h_1^{-1}(g_1^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{L_1}) \cdot \mathcal{O}_{N_1})$ is invertible. Next, considering the blowing-up $h_2 : N_2 \rightarrow L_2$ and the morphism $\pi : N_1 \rightarrow L_2$, we obtain a unique morphism $\tau : N_1 \rightarrow N_2$ such that $\pi = h_2 \circ \tau$ since $\pi^{-1}(g_2^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_2}) \cdot \mathcal{O}_{N_2} (= (g_2 \circ \pi)^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{N_2} = (g_1 \circ h_1)^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{N_2} = h_1^{-1}(g_1^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_1}) \cdot \mathcal{O}_{N_2})$ is invertible. Then $g_1 \circ h_1 = g_2 \circ \pi = g_2 \circ h_2 \circ \tau$. By the same reason we have a unique morphism $\tau' : N_2 \rightarrow N_1$ such that $g_1 \circ h_1 \circ \tau' = g_2 \circ h_2$. Hence τ is an isomorphism.

By this fact we have a commutative diagram of blowings-up with respect to coherent sheaves of $\mathcal{O}_{M_{i,j}}$ -ideals:

$$\begin{array}{ccccccc} M_{k_2,k_3} & \xrightarrow{\nu_{k_2,k_3}} & M_{k_2-1,k_3} & \longrightarrow & \dots & \longrightarrow & M_{0,k_3} \\ \mu_{k_2,k_3} \downarrow & & \mu_{k_2,-1,k_3} \downarrow & & & & \mu_{0,k_3} \downarrow \\ M_{k_2,k_3-1} & \xrightarrow{\nu_{k_2,k_3-1}} & M_{k_2-1,k_3} & \longrightarrow & \dots & \longrightarrow & M_{0,k_3-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow \\ M_{k_2,0} & \xrightarrow{\nu_{k_2,0}} & M_{k_2-1,0} & \longrightarrow & \dots & \longrightarrow & M_{0,0} \end{array}$$

such that $M_{k_2,0} \xrightarrow{\nu_{k_2,0}} \dots \xrightarrow{\nu_{1,0}} M_{0,0}$ and $M_{0,k_3} \xrightarrow{\mu_{0,k_3}} \dots \xrightarrow{\mu_{0,1}} M_{0,0}$ coincide with $N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \dots \xrightarrow{\pi_{2,1}} M$ and $N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \dots \xrightarrow{\pi_{3,1}} M$, respectively. In particular,

$$\pi_{3,1} \circ \dots \circ \pi_{3,k_3} \circ \nu_{1,k_3} \circ \dots \circ \nu_{k_2,k_3} = \pi_{2,1} \circ \dots \circ \pi_{2,k_2} \circ \mu_{k_2,1} \circ \dots \circ \mu_{k_2,k_3}.$$

Since $\tau_2 : N_{1,k_1} \rightarrow M_{k_2,0}$ is an isomorphism we have a sequence of blowings-up $N_{1,k_1+k_3} \xrightarrow{\pi_{1,k_1+k_3}} \dots \xrightarrow{\pi_{1,k_1+1}} N_{1,k_1}$ and isomorphisms $\tau_{2,k_3} : N_{1,k_1+k_3} \rightarrow M_{k_2,k_3}, \dots, \tau_{2,0} = \tau_2 : N_{1,k_1} \rightarrow M_{k_2,0}$ such that the following diagram is commutative.

$$\begin{array}{ccccccc} N_{1,k_1+k_3} & \xrightarrow{\pi_{1,k_1+k_3}} & N_{1,k_1+k_3-1} & \longrightarrow & \dots & \xrightarrow{\pi_{1,k_1+1}} & N_{1,k_1} \\ \tau_{2,k_3} \downarrow & & \tau_{2,k_3-1} \downarrow & & & & \tau_{2,0} \downarrow \\ M_{k_2,k_3} & \xrightarrow{\mu_{k_2,k_3}} & M_{k_2,k_3-1} & \longrightarrow & \dots & \xrightarrow{\mu_{k_2,1}} & M_{k_2,0}. \end{array}$$

Then

$$\begin{aligned} f_2 \circ \pi_{2,1} \circ \dots \circ \pi_{2,k_2} \circ \mu_{k_2,1} \circ \dots \circ \mu_{k_2,k_3} \circ \tau_{2,k_3} &= \dots = \\ f_2 \circ \pi_{2,1} \circ \dots \circ \pi_{2,k_2} \circ \tau_{2,0} \circ \pi_{1,k_1+1} \circ \dots \circ \pi_{1,k_1+k_3} &= \\ f_1 \circ \pi_{1,1} \circ \dots \circ \pi_{1,k_1} \circ \pi_{1,k_1+1} \circ \dots \circ \pi_{1,k_1+k_3}. & \end{aligned}$$

In the same way we obtain a commutative diagram :

$$\begin{array}{ccccccc} M_{k_2,k_3} & \xrightarrow{\nu_{k_2,k_3}} & M_{k_2-1,k_3} & \longrightarrow & \dots & \xrightarrow{\nu_{1,k_3}} & M_{0,k_3} \\ \tau_{4,k_2} \downarrow & & \tau_{4,k_2-1} \downarrow & & & & \tau_{4,0} \downarrow \\ N_{4,k_2+k_4} & \xrightarrow{\pi_{4,k_2+k_4}} & N_{4,k_2+k_4-1} & \longrightarrow & \dots & \xrightarrow{\pi_{4,k_4+1}} & N_{4,k_4}, \end{array}$$

where the horizontal morphisms are blowings-up and the vertical ones are isomorphisms, with $\tau_{4,0} = \tau_4$, and

$$\begin{aligned} f_4 \circ \pi_{4,1} \circ \dots \circ \pi_{4,k_4} \circ \pi_{4,k_4+1} \circ \dots \circ \pi_{4,k_2+k_4} \circ \tau_{4,k_2} &= \\ f_3 \circ \pi_{3,1} \circ \dots \circ \pi_{3,k_3} \circ \nu_{1,k_3} \circ \dots \circ \nu_{k_2,k_3}. & \end{aligned}$$

Hence

$$\begin{aligned} f_4 \circ \pi_{4,1} \circ \dots \circ \pi_{4,k_2+k_4} \circ \tau_{4,k_2} \circ \tau_{2,k_3} &= f_3 \circ \pi_{3,1} \circ \dots \circ \nu_{k_2,k_3} \circ \tau_{2,k_3} = \\ f_2 \circ \pi_{2,1} \circ \dots \circ \mu_{k_2,k_3} \circ \tau_{2,k_3} &= f_1 \circ \pi_{1,1} \circ \dots \circ \pi_{1,k_1+k_3}. \end{aligned}$$

If τ_2 and τ_4 induce homeomorphisms of M , then $\tau_{4,k_2} \circ \tau_{2,k_3}$ induces clearly a homeomorphism of M . \square

Proof of (ii). Let f and g be C^ω functions on \mathbf{R}^4 in variables (u, v, w, x) defined by

$$\begin{aligned} f &= \phi\psi\xi, \quad g = \phi\psi\eta, \\ \phi &= u^2 + v^2, \quad \psi = u^4 + v^2 + u^2w^2, \\ \xi &= u^4 + v^2 + u^2(w-x)^2, \quad \eta = u^4 + (v-xu)^2 + u^2w^2. \end{aligned}$$

Then f and g are almost blow-analytically equivalent but their germs at 0 are not blow-analytically equivalent (i.e. there do not exist open neighborhoods U and V of 0 in \mathbf{R}^4 such that $f|_U$ and $g|_V$ are blow-analytically equivalent and the induced homeomorphism from U to V carries 0 to 0).

Proof of almost blow-analytic equivalence.

Set $X = \{0\} \times \mathbf{R}^2 \subset \mathbf{R}^4$. Let Λ denote the half-lines in \mathbf{R}^4 starting from points in X and orthogonal to X , and let $e(\lambda)$ denote $\lambda \cap X$ —the end of $\lambda \in \Lambda$. It is easy to see that for each $\lambda \in \Lambda$, $f|_\lambda$ and $g|_\lambda$ have singularities only at $e(\lambda)$ and that $f(\lambda) = g(\lambda) = [0, \infty)$. We will find almost blow-analytic equivalence so that the diffeomorphism induces a diffeomorphism of $\mathbf{R}^4 - X$ carrying any $\lambda - e(\lambda)$, $\lambda \in \Lambda$, to some $\lambda' - e(\lambda')$, $\lambda' \in \Lambda$. Let $\pi : M \rightarrow \mathbf{R}^4$ denote the blowing-up along center X . Then

$$\begin{aligned} M &= \{(s : t, u, v, w, x) \in \mathbf{P}(1) \times \mathbf{R}^4 : sv = tu\}, \\ f \circ \pi(1 : t, u, v, w, x) &= u^6(1 + t^2)(u^2 + t^2 + w^2)(u^2 + t^2 + (w - x)^2), \\ g \circ \pi(1 : t, u, v, w, x) &= u^6(1 + t^2)(u^2 + t^2 + w^2)(u^2 + (t - x)^2 + w^2), \\ f \circ \pi(s : 1, u, v, w, x) &= v^6(1 + s^2)(s^4v^2 + 1 + s^2v^2w^2)(s^4v^2 + 1 + s^2v^2(w - x)^2), \\ g \circ \pi(s : 1, u, v, w, x) &= v^6(1 + s^2)(s^4v^2 + 1 + s^2v^2w^2)(s^4v^2 + (1 - xs)^2 + s^2w^2), \end{aligned}$$

for each $\lambda \in \Lambda$

$$\begin{aligned} (*) \quad \pi^{-1}(\lambda - e(\lambda)) &= \{(1 : t_0, u, t_0u, w_0, x_0) : u \in (0, \infty)\} \\ &\text{or} = \{(s_0 : 1, s_0v, v, w_0, x_0) : v \in (0, \infty)\} \\ &\text{for some } s_0, t_0, w_0, x_0 \in \mathbf{R}. \end{aligned}$$

Hence $\text{Sing } f \circ \pi = \text{Sing } g \circ \pi = \pi^{-1}(X)$,

and the germs of $f \circ \pi$ and $g \circ \pi$ at points of $\pi^{-1}(X) - M_1 - M_2$ and $\pi^{-1}(X) - M_1 - M_3$, respectively, are 6th powers of regular function germs, where

$$\begin{aligned} M_1 &= \{(1 : 0, 0, 0, 0, x) \in M\}, \\ M_2 &= \{(1 : 0, 0, 0, w, x) \in M : w = x\}, \\ M_3 &= \{(1 : t, 0, 0, 0, x) \in M : t = x\}. \end{aligned}$$

Therefore, if we have a C^∞ diffeomorphism h of M such that $f \circ \pi = g \circ \pi \circ h$, $h(M_1 \cup M_2) = M_1 \cup M_3$ and h is of class C^ω on a neighborhood of $M_1 \cup M_2$, then by Hironaka Desingularization Theorem (Main Theorem II, [Hi]) there exist compositions of finite sequences of blowings-up $\pi_f : N \rightarrow M$ and $\pi_g : L \rightarrow M$ along smooth analytic center and a C^∞ diffeomorphism $\tilde{h} : N \rightarrow L$ such that $f \circ \pi \circ \pi_f = g \circ \pi \circ \pi_g \circ \tilde{h}$ and $f \circ \pi \circ \pi_f$ and $g \circ \pi \circ \pi_g$ have only normal crossing singularities, and hence by theorem 3.1,(1) below $f \circ \pi \circ \pi_f$ and $g \circ \pi \circ \pi_g$ are C^ω right equivalent, i.e., f and g are almost blow-analytically equivalent.

Let the variables of \mathbf{R}^2 be (t, w) , and for each $x \in \mathbf{R}$, let $B_{|x|}$ denote the ball in \mathbf{R}^2 with center 0 and radius $|x|$. Let h_x be a C^∞ diffeomorphism of \mathbf{R}^2 such that $h_x = \text{id}$ outside of $B_{2+|x|}$, $h_x(t, w) = (w, -t)$ on $B_{1+|x|}$ and the map $h : \mathbf{R}^4 \rightarrow \mathbf{R}^4$, defined by $h(u, t, w, x) = (u, h_x(t, w), x)$, is a C^∞ diffeomorphism. Regard \mathbf{R}^4 as a subset of M by the map $(u, t, w, x) \rightarrow (1 : t, u, tu, w, x)$. Then h can be extended to a C^∞ diffeomorphism \tilde{h} of M , with $\tilde{h}(\pi^{-1}(X)) = \pi^{-1}(X)$, the image U of $\mathbf{R} \times \cup_{x \in \mathbf{R}} B_{1+|x|} \times \{x\}$ in M under the inclusion map is a neighborhood of $M_1 \cup M_2 \cup M_3$, $\tilde{h}(M_1) = M_1$, $\tilde{h}(M_2) = M_3$, for any $\lambda \in \Lambda$ there exists λ' by (*) such

that $\tilde{h}(\pi^{-1}(\lambda - e(\lambda))) = \pi^{-1}(\lambda' - e(\lambda'))$, \tilde{h} is of class C^ω on U , and $f \circ \pi = g \circ \pi \circ \tilde{h}$ on U .

We need to modify \tilde{h} outside of U so that $f \circ \pi = g \circ \pi \circ \tilde{h}$ everywhere. That is easy. For each $a \in M$, replace $\tilde{h}(a)$ with $\overline{\pi^{-1}(\lambda - e(\lambda))} \cap (g \circ \pi)^{-1}(f \circ \pi(a))$ for some $\lambda \in \Lambda$ with $\tilde{h}(a) \in \overline{\pi^{-1}(\lambda - e(\lambda))}$. Then the new \tilde{h} is a well-defined C^∞ diffeomorphism of M because $\tilde{h}(a) = \overline{\pi^{-1}(\lambda - e(\lambda))} \cap (g \circ \pi)^{-1}(f \circ \pi(a))$ for $a \in U$, U is a union of some $\overline{\pi^{-1}(\lambda - e(\lambda))}$, $\lambda \in \Lambda$, and because the restrictions of $f \circ \pi$ and $g \circ \pi$ to each $\overline{\pi^{-1}(\lambda - e(\lambda))}$ outside of U are C^∞ right equivalent to the function $[0, \infty) \ni z \rightarrow z^6 \in \mathbf{R}$. Clearly $f \circ \pi = g \circ \pi \circ \tilde{h}$ is satisfied for the new \tilde{h} . \square

We can prove, moreover, that f and g are almost blow-Nash equivalent (definition 1.2) as follows, though we omit the details. In the same way as the following arguments we find compositions of finite sequences of blowings-up of \mathbf{R}^4 along smooth Nash center $\pi_f : N \rightarrow \mathbf{R}^4$ and $\pi_g : L \rightarrow \mathbf{R}^4$ and we see that $f \circ \pi_f$ and $g \circ \pi_g$ are semialgebraically C^m right equivalent for any $m \in \mathbf{N}$ by using a partition of unity of class semialgebraic C^m , § II.2, [S₂]. Then by theorem 3.1,(3) $f \circ \pi_f$ and $g \circ \pi_g$ are Nash right equivalent.

Proof of non- blow-analytic equivalence. We introduce an invariant of blow-analytic equivalence, which is a generalization of one in [Fu]. Let C denote the set of analytic curves germs $c : [0, \epsilon) \rightarrow \mathbf{R}^4$ at 0, $\epsilon > 0$, and give a topology on C by identifying C with 4-product of the one-variable convergent power series ring $\mathbf{R}\langle\langle t \rangle\rangle^4$ and choosing the product topology on $\mathbf{R}\langle\langle t \rangle\rangle$, i.e., $\mathbf{R}\langle\langle t \rangle\rangle \ni \sum_{n=0}^{\infty} a_{n,k} t^n \rightarrow 0$ as $k \rightarrow \infty$ if for each n , $\mathbf{R} \ni a_{n,k} \rightarrow 0$ as $k \rightarrow \infty$. Then $p : C \rightarrow \mathbf{R}^4$, defined by $p(c) = c(0)$, is a topological fibre bundle. Let $\pi : M \rightarrow \mathbf{R}^4$ be the composite of a finite sequence of blowings-up of \mathbf{R}^4 along smooth analytic centers. Then π naturally induces a surjective C^0 map $\pi_* : C_M \rightarrow C$ such that $p \circ \pi_* = \pi \circ p_M$, where C_M is the analytic curve germs in M and $p_M : C_M \rightarrow M$ is defined by $p_M(c) = c(0)$. Hence if the germs of f and g at 0 are blow-analytically equivalent, there exist open neighborhoods U and V of 0 in \mathbf{R}^4 and an isomorphism $\tau : p^{-1}(U) \rightarrow p^{-1}(V)$ as topological fibre bundles such that $f \circ c = g \circ \tau(c)$ as analytic function germs in a variable t for $c \in p^{-1}(U)$. Let $\tau_0 : U \rightarrow V$ denote the homeomorphism such that $p \circ \tau = \tau_0 \circ p$. Assuming so, we will arrive at a contradiction.

Define a map $o_f : C \rightarrow \mathbf{N} \cup \{\infty\}$ by $o_f(c) =$ the order of $f \circ c$ at 0. Set $C_a = p^{-1}(a)$, $a \in U$, and consider a family $\{C_a \cap o_f^{-1}(i) : i \in \mathbf{N}\}$. We stratify U by the type of the family. Clearly

$$X_1 \stackrel{\text{def}}{=} \{a \in U : o_f = 0 \text{ on } C_a\} = \{(u, v, w, x) \in U : u \neq 0 \text{ or } v \neq 0\}.$$

Set

$$X_2 = \{u = v = 0, w \neq 0, w \neq x\},$$

$$X_3 = \{u = v = w = 0, x \neq 0\} \cup \{u = v = 0, w = x \neq 0\},$$

$$X_4 = \{u = v = w = x = 0\}.$$

Then $\{X_1, X_2, X_3, X_4\}$ is a stratification of U and we see the following property by easy calculations. If $a \in X_2, X_3$ or X_4 , $(C_a, C_a \cap o_f^{-1}(6))$ has the same homotopy type as (B^2, S^1) , $(B^2, S^1 - S^0)$ or $(B^2, S^1 - S^0)$, respectively, because

$$C_a \cap o_f^{-1}(6) = \{c = (c_1, \dots, c_4) : [0, \epsilon) \rightarrow \mathbf{R}^4 \in C_a : \frac{dc_1}{dt}(0) \neq 0 \text{ or } \frac{dc_2}{dt}(0) \neq 0\}$$

$$\text{or } = \{\frac{dc_2}{dt}(0) \neq 0\} \text{ or } = \{\frac{dc_2}{dt}(0) \neq 0\}, \text{ respectively,}$$

and if $a \in X_3$ or X_4 , $(C_a, C_a \cap o_f^{-1}(8))$ has the same one as (B^2, S^0) or (B^2, \emptyset) , respectively, where B^2 is a 2-ball, S^1 is its boundary and S^0 is two points of S^1 . Thus $\{X_1, X_2, X_3, X_4\}$ is a stratification of U by type of $C_a \cap o_f^{-1}(i)$, $i = 6, 8$.

Consider g . Define o_g in the same way and set

$$Y_1 = \{a \in V : o_g = 0 \text{ on } C_a\} = \{(u, v, w, x) \in V : u \neq 0 \text{ or } v \neq 0\},$$

$$Y_2 = \{u = v = 0, w \neq 0\}, Y_3 = \{u = v = w = 0, x \neq 0\}, Y_4 = \{u = v = w = x = 0\}.$$

Then, if $a \in Y_2, Y_3$ or Y_4 , $(C_a, C_a \cap o_g^{-1}(6))$ has the same homotopy type as (B^2, S^1) , $(B^2, S^1 - 4 \text{ points})$ or $(B^2, S^1 - S^0)$, respectively. Hence $\{Y_1, Y_2, Y_3, Y_4\}$ is a stratification of V by type of $C_a \cap o_g^{-1}(6)$.

However, $o_f(c) = o_g(\tau(c))$ for $c \in C$ with $p(c) \in U$ by the definition of blow-analytic equivalence, and hence for $a \in U$, $o_f : C_a \rightarrow \mathbf{N} \cup \{\infty\}$ and $o_g : C_{\tau_0(a)} \rightarrow \mathbf{N} \cup \{\infty\}$ are C^0 right equivalent. Therefore, the stratifications of U by $C_a \cap o_f^{-1}(i)$ and V by $C_a \cap o_g^{-1}(i)$ coincide up to the homeomorphism τ_0 . That is a contradiction. \square

Definition 1.2. A *semialgebraic* set is a subset of a Euclidean space which is described by finitely many equalities and inequalities of polynomial functions. A *Nash manifold* is a C^ω submanifold of a Euclidean space which is semialgebraic. A *Nash function* on a Nash manifold is a C^ω function with semialgebraic graph. A *Nash subset* is the zero set of a Nash function on a Nash manifold. Let M be a Nash manifold, $X \subset M$ be a semialgebraic subset and f, g be Nash function germs on X in M . (We call a germ *on* but not *at* X in M to distinguish the case where X is a set from the case of a point.) Then f and g are said to be *almost blow-Nash equivalent* if there exist open semialgebraic neighborhoods U and V of X in M , two compositions of finite sequences of blowings-up along smooth Nash centers $\pi_f : N \rightarrow U$ and $\pi_g : L \rightarrow V$ and a Nash diffeomorphism h from an open semialgebraic neighborhood of $\pi_f^{-1}(X)$ in N to one of $\pi_g^{-1}(X)$ in L so that f and g are supposed to be defined on U and V , respectively, $f \circ \pi_f = g \circ \pi_g \circ h$ and $h(\pi_f^{-1}(X)) = \pi_g^{-1}(X)$. We naturally define also *almost blow-Nash R-L equivalence*. We define the *semialgebraic C^r topology* on spaces of semialgebraic C^r maps (see [S₂] for it and for elementary properties of Nash manifolds, Nash functions and the topology, e.g., a semialgebraic C^1 map between semialgebraic C^1 manifolds close to a semialgebraic C^1 diffeomorphism is a diffeomorphism).

2. REAL ANALYTIC AND NASH SHEAF THEORY

2.1. Real analytic sheaves.

In this section, we deal with the real analytic and Nash case of Cartan Theorems A and B, and Oka Theorem. A crucial tool consists in the complexification of real analytic manifolds and coherent sheaves.

Let \mathcal{O} , \mathcal{N} and $N(M)$ always denote, respectively, the sheaves of analytic and Nash function germs on an analytic and Nash manifold and the ring of Nash functions on a Nash manifold M . We write \mathcal{O}_M and \mathcal{N}_M when we emphasize the domain M . For a function f on an analytic (Nash) manifold M , a subset X of M , a vector field v on M and for a sheaf of \mathcal{O} - (\mathcal{N} -) modules \mathcal{M} on M , let f_x , X_x , v_x and \mathcal{M}_x denote the germs of f and X at a point x of M , the tangent vector assigned to x by v and the stalk of \mathcal{M} at x , respectively. For a compact semialgebraic subset X of a Nash manifold M , let $\mathcal{N}(X)$ denote the germs of Nash functions on X in M

with the topology of the inductive limit space of the topological spaces $N(U)$ with the compact-open C^∞ topology where U runs through the family of open semialgebraic neighborhoods of X in M . In the same way we define $\mathcal{O}(X)$ for a compact semianalytic subset X of an analytic manifold M . Here a *semianalytic* subset is a subset whose germ at each point of M is described by finitely many equalities and inequalities of analytic function germs.

Theorem 2.1. (*Cartan Theorem A*) *Let \mathcal{M} be a coherent sheaf of \mathcal{O} -modules on an analytic manifold M . Then for any $x \in M$*

$$\mathcal{M}_x = H^0(M, \mathcal{M})\mathcal{O}_x.$$

See [G-R] for Cartan Theorems A and B in the complex case and [C] for the real case. Next corollary will be useful in this paper. It deals with the case where the number of local generators is uniformly bounded.

Corollary 2.2. *In theorem 2.1, assume that \mathcal{M}_x is generated by a uniform number of elements for any x . Then $H^0(M, \mathcal{M})$ is finitely generated as a $H^0(M, \mathcal{O})$ -module.*

Theorem 2.3. (*Cartan Theorem B*) *Let \mathcal{M} be a coherent sheaf of \mathcal{O} -modules on an analytic manifold M . Then*

$$H^1(M, \mathcal{M}) = 0$$

Corollary 2.4. *Let M be an analytic manifold and $X \subset M$ be a global analytic set—the zero set of an analytic function. Let \mathcal{I} be a coherent sheaf of \mathcal{O} -ideals on M such that any element of \mathcal{I} vanishes on X . Then any $f \in H^0(M, \mathcal{O}/\mathcal{I})$ can be extended to some $F \in C^\omega(M)$, i.e., f is the image of F under the natural map $H^0(M, \mathcal{O}) \rightarrow H^0(M, \mathcal{O}/\mathcal{I})$.*

If X is normal crossing (see the definition of a normal crossing analytic set in the section 3.1), we can choose \mathcal{I} so as to be the function germs vanishing on X . Then $H^0(M, \mathcal{O}/\mathcal{I})$ consists of functions on X whose germs at each point of X are extensible to analytic function germs on M .

Consider the exact sequence $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}$. Then corollary 2.4 follows from theorem 2.3.

Theorem 2.5. (*Oka Theorem*) *Let \mathcal{M}_1 and \mathcal{M}_2 be coherent sheaves of \mathcal{O} -modules on an analytic manifold M , and $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be an \mathcal{O} -homomorphism. Then $\text{Ker } h$ is a coherent sheaf of \mathcal{O} -modules.*

See [G-R] in the complex case. The real case follows from it since we can complexify $M, \mathcal{M}_1, \mathcal{M}_2$ and h , in the same way as in the proof of corollary 2.2 below.

Proof of corollary 2.2. Let l be a uniform number of generators. We define complexifications $M^{\mathbf{C}}$ and $\mathcal{M}^{\mathbf{C}}$ as follows. By Grauert Theorem, let M be a C^ω submanifold of \mathbf{R}^n . Hence a complexification of M is naturally defined. Let $M^{\mathbf{C}}$ be the intersection of the complexification and a small Stein open neighborhood of \mathbf{R}^n in \mathbf{C}^n , which is automatically a Stein manifold if we choose the neighborhood so small that the intersection is closed in the neighborhood. Existence of such a neighborhood follows from by the fact that any open subset of \mathbf{C} , a finite product of Stein manifolds and a finite intersection of Stein submanifolds of \mathbf{C}^n are

Stein manifolds. Indeed, any neighborhood of \mathbf{R}^n in \mathbf{C}^n contains a neighborhood of the form $\bigcap_{k=0}^n \pi_k(E)$, where $E = \{x + iy \in \mathbf{C} : |y| < \epsilon(|x|)\}^n$ for some small positive C^0 function ϵ on \mathbf{R} , $\pi_0 = \text{id}$ and $\pi_k(z_1, \dots, z_n) = (z_1 + z_k, \dots, z_n + z_k)$ for $(z_1, \dots, z_n) \in \mathbf{C}^n$, $k = 1, \dots, n$, (see [G-R] for the fact). Hence when \mathcal{M} is a sheaf of \mathcal{O} -ideals or, moreover, a subsheaf of \mathcal{O}^p , shrinking the open neighborhood if necessary, we define a complexification $\mathcal{M}^{\mathbf{C}}$ of \mathcal{M} on $M^{\mathbf{C}}$ and can prove Cartan Theorems A and B for \mathcal{M} by Cartan Theorems A and B in the complex case because $H^0(M, \mathcal{M})$ contains the real valued elements of $H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})|_M$ and $H^1(M, \mathcal{M})$ is the inductive limit of the real valued elements of $H^1(M^{\mathbf{C}}, \mathcal{M})|_M$ as $M^{\mathbf{C}}$ is shrunk to M (Cartan Theorem B).

Consider general \mathcal{M} . Let $\{U_i\}$ be a locally finite open covering of M such that there are exact sequences $(\mathcal{O}|_{U_i})^{p_i} \xrightarrow{\phi_i} (\mathcal{O}|_{U_i})^{q_i} \xrightarrow{\psi_i} \mathcal{M}|_{U_i} \rightarrow 0$. Then by Cartan Theorem B for a coherent subsheaf of \mathcal{O}^p we find homomorphisms $g_{i,j} : (\mathcal{O}|_{U_i \cap U_j})^{p_i} \rightarrow (\mathcal{O}|_{U_i \cap U_j})^{p_j}$ and $h_{i,j} : (\mathcal{O}|_{U_i \cap U_j})^{q_i} \rightarrow (\mathcal{O}|_{U_i \cap U_j})^{q_j}$ for each i and j such that the following diagram is commutative:

$$\begin{array}{ccccc} (\mathcal{O}|_{U_i \cap U_j})^{p_i} & \xrightarrow{\phi_i|_{U_i \cap U_j}} & (\mathcal{O}|_{U_i \cap U_j})^{q_i} & \xrightarrow{\psi_i|_{U_i \cap U_j}} & \mathcal{M}|_{U_i \cap U_j} \\ g_{i,j} \downarrow & & h_{i,j} \downarrow & & \text{id} \downarrow \\ (\mathcal{O}|_{U_i \cap U_j})^{p_j} & \xrightarrow{\phi_j|_{U_i \cap U_j}} & (\mathcal{O}|_{U_i \cap U_j})^{q_j} & \xrightarrow{\psi_j|_{U_i \cap U_j}} & \mathcal{M}|_{U_i \cap U_j} \end{array}$$

Shrink, moreover, the above Stein open neighborhood so that we can complexify ϕ_i , ψ_i , $g_{i,j}$ and $h_{i,j}$ to $\phi_i^{\mathbf{C}}$, $\psi_i^{\mathbf{C}}$, $g_{i,j}^{\mathbf{C}}$ and $h_{i,j}^{\mathbf{C}}$, and define $\mathcal{M}^{\mathbf{C}}$ to be the union of Coker $\phi_i^{\mathbf{C}}$. Then $\mathcal{M}^{\mathbf{C}}$ is a well-defined coherent sheaf of $\mathcal{O}^{\mathbf{C}}$ -modules by existence of $g_{i,j}^{\mathbf{C}}$ and $h_{i,j}^{\mathbf{C}}$ such that each $\mathcal{M}_x^{\mathbf{C}}$ is generated by l -elements, where $\mathcal{O}^{\mathbf{C}}$ denotes the sheaf of complex analytic function germs. Note that $\mathcal{M}^{\mathbf{C}}|_M \supset \mathcal{M}$, $\mathcal{M}^{\mathbf{C}}|_M = \mathcal{M} \oplus i\mathcal{M}$ and Cartan Theorems A and B for \mathcal{M} hold by the same arguments as before. We see the following statement.

Let $x \in M^{\mathbf{C}}$, $f_1, \dots, f_l, g_{1,i}, \dots, g_{l,i} \in H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$, $i = 1, 2, \dots$, such that $f_{1,x}, \dots, f_{l,x}$ are generators of $\mathcal{M}_x^{\mathbf{C}}$. Then there exist arbitrarily small positive numbers $\epsilon_1, \epsilon_2, \dots$ such that for any non-negative numbers $\epsilon'_1 \leq \epsilon_1, \epsilon'_2 \leq \epsilon_2, \dots$, $f_1 + \sum_{i=1}^{\infty} \epsilon'_i g_{1,i}, \dots, f_l + \sum_{i=1}^{\infty} \epsilon'_i g_{l,i}$ are elements of $H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$, their germs at x are generators of $\mathcal{M}_x^{\mathbf{C}}$, and the germs of $g_{1,1} + \epsilon_1 f_1, \dots, g_{l,1} + \epsilon_l f_l$ at x also are generators of $\mathcal{M}_x^{\mathbf{C}}$.

The first condition follows from the fact that $H^0(M^{\mathbf{C}}, \mathcal{O}^{\mathbf{C}})$ is a Fréchet space in the compact-open C^0 topology, [G-R], and the second and third are easy consequences of elementary linear algebra.

For each irreducible analytic component $M_i^{\mathbf{C}}$ of $\text{supp } \mathcal{M}^{\mathbf{C}}$, let $f_{1,i}, \dots, f_{l,i} \in H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$ such that $f_{1,i x_i}, \dots, f_{l,i x_i}$ are generators of $\mathcal{M}_{x_i}^{\mathbf{C}}$ for some point x_i of $M_i^{\mathbf{C}}$. Then by the above statement there exist non-negative numbers $\epsilon_1, \epsilon_2, \dots$ such that $\sum_{i=1}^{\infty} \epsilon_i f_{1,i}, \dots, \sum_{i=1}^{\infty} \epsilon_i f_{l,i}$ are elements of $H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$ and their germs at any x_j are generators of $\mathcal{M}_{x_j}^{\mathbf{C}}$. Set $f_j = \sum_{i=1}^{\infty} \epsilon_i f_{j,i}$, $j = 1, \dots, l$, $\mathcal{M}_1^{\mathbf{C}} = \sum_{j=1}^l \mathcal{O}^{\mathbf{C}} f_j$, $\mathcal{M}_1 = \mathcal{M}_1^{\mathbf{C}}|_M$, $\mathcal{M}_2^{\mathbf{C}} = \mathcal{M}^{\mathbf{C}}/\mathcal{M}_1^{\mathbf{C}}$ and $\mathcal{M}_2 = \mathcal{M}/\mathcal{M}_1$. Then \mathcal{M}_1 and \mathcal{M}_2 are coherent sheaves of \mathcal{O} -modules by Oka Theorem, $\mathcal{M}_1^{\mathbf{C}}$ and $\mathcal{M}_2^{\mathbf{C}}$ are their complexifications, $\mathcal{M}_{2x}^{\mathbf{C}}$ is generated by l elements for any $x \in M^{\mathbf{C}}$, and $\dim(\text{supp } \mathcal{M}_2^{\mathbf{C}}) < \dim(\text{supp } \mathcal{M}^{\mathbf{C}})$. Choose one point x'_i in each irreducible analytic component of $\text{supp } \mathcal{M}_2^{\mathbf{C}}$ and repeat the same arguments. Then we have $f_{l+1}, \dots, f_{2l} \in H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$ such that $\dim(\text{supp}(\mathcal{M}^{\mathbf{C}}/\sum_{i=1}^{2l} \mathcal{O}^{\mathbf{C}} f_i)) < \dim(\text{supp } \mathcal{M}_2^{\mathbf{C}})$. In this

way we obtain generators of $H^0(M^{\mathbf{C}}, \mathcal{M}^{\mathbf{C}})$ of numbers $(n'+1)l$, where $n' = \dim M$. Let $g_1, \dots, g_{(n'+1)l}$ be their restrictions to M . Then each g_j is of the form $g'_j + ig''_j$ for $g'_j, g''_j \in H^0(M, \mathcal{M})$. Hence $g'_1, g''_1, \dots, g'_{(n'+1)l}, g''_{(n'+1)l}$ are generators of $H^0(M, \mathcal{M})$. \square

2.2. Nash sheaves.

In this section M stands for a Nash manifold. A sheaf of \mathcal{N} -modules \mathcal{M} on M is called *finite* if for some finite open semialgebraic covering $\{U_i\}$ of M and for each i there exists an exact sequence $\mathcal{N}^{m_i}|_{U_i} \rightarrow \mathcal{N}^{n_i}|_{U_i} \rightarrow \mathcal{M}|_{U_i} \rightarrow 0$ of \mathcal{N} -homomorphisms, $m_i, n_i \in \mathbf{N}$. Non-finite examples are the sheaf of \mathcal{N} -ideals \mathcal{I} on \mathbf{R} of germs vanishing on \mathbf{Z} and \mathcal{N}/\mathcal{I} .

Remark. (i) If \mathcal{M}_1 is a sheaf of \mathcal{N} -submodules of a finite sheaf of \mathcal{N} -modules \mathcal{M} on a Nash manifold M , and if there exists a finite open semialgebraic covering $\{U_i\}$ of M such that for each i , $\mathcal{M}_1|_{U_i}$ is generated by its global cross-sections $H^0(U_i, \mathcal{M}_1|_{U_i})$, then \mathcal{M}_1 is finite.

(ii) Let \mathcal{M}_1 and \mathcal{M}_2 be finite sheaves of \mathcal{N} -submodules of a finite sheaf of \mathcal{N} -modules. Then $\mathcal{M}_1 + \mathcal{M}_2$ and $\mathcal{M}_1 \cap \mathcal{M}_2$ are also finite sheaves of \mathcal{N} -modules.

(iii) For a finite sheaf of \mathcal{N} -modules \mathcal{M} on a Nash manifold M , $H^0(M, \mathcal{M})$ is a finitely generated $N(M)$ -module.

?We will prove these remarks after theorem 2.8.

?The following three theorems are slight generalizations of results in [S₂], [C-R-S₁] and [C-S₃], and theorem 2.7 for $n = 1$ and theorem 2.8 for $\mathcal{M}_1 = \mathcal{N}$ were shown there. Hence, assuming that these cases hold, we give the proofs of the theorems.

Theorem 2.6. (*Nash case of Oka Theorem*) *Let h be an \mathcal{N} -homomorphism between finite sheaves of \mathcal{N} -modules on a Nash manifold. Then $\text{Ker } h$ is finite.*

Proof. Let $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be the homomorphism on a Nash manifold M in the theorem. There exists a finite open semialgebraic covering $\{U_i\}$ of M such that $\mathcal{M}_j|_{U_i}$, $j = 1, 2$, satisfy the condition of exact sequence in the definition of a finite sheaf, and it suffices to prove the theorem on each U_i . Hence we assume that \mathcal{M}_j , $j = 1, 2$, are generated by global cross-sections $\alpha_1, \dots, \alpha_{n_1}$ and $\beta_1, \dots, \beta_{n_2}$, respectively, and there are Nash maps $\gamma_1, \dots, \gamma_{n_3} \in N(M)^{n_2}$ which are generators of the kernel of the surjective \mathcal{N} -homomorphism $p : \mathcal{N}^{n_2} \supset \mathcal{N}_x^{n_2} \ni (\phi_1, \dots, \phi_{n_2}) \rightarrow \sum_{i=1}^{n_2} \phi_i \beta_{ix} \in \mathcal{M}_{2x} \subset \mathcal{M}_2$, $x \in M$. Let $\bar{\alpha}_1, \dots, \bar{\alpha}_{n_1}$ denote the images of $\alpha_1, \dots, \alpha_{n_1}$ in $H^0(M, \mathcal{M}_2)$ under the induced homomorphism $h_* : H^0(M, \mathcal{M}_1) \rightarrow H^0(M, \mathcal{M}_2)$ of h .

?We prove the theorem by induction on n_2 . Let $n_2 = 1$. Then there exist $\hat{\alpha}_1, \dots, \hat{\alpha}_{n_1} \in H^0(M, \mathcal{N})$ such that $p_*(\hat{\alpha}_i) = \bar{\alpha}_i$, $i = 1, \dots, n_1$ because by [C-R-S₁] and [C-S₃] $p_* : H^0(M, \mathcal{N}) \rightarrow H^0(M, \mathcal{M}_2)$ is surjective by theorem 2.8 for $\mathcal{M}_1 = \mathcal{N}$. Let $\delta_1, \dots, \delta_{n_4} \in N(M)^{n_1}$ be generators of the kernel of the surjective homomorphism $\mathcal{N}^{n_1} \supset \mathcal{N}_x^{n_1} \ni (\phi_1, \dots, \phi_{n_1}) \rightarrow \sum_{i=1}^{n_1} \phi_i \alpha_{ix} \in \mathcal{M}_{1x} \subset \mathcal{M}_x$, $x \in M$ (we choose the above $\{U_i\}$ so that $\delta_1, \dots, \delta_{n_4}$ exist). Multiplying one small positive Nash function on $\alpha_i, \bar{\alpha}_i, \hat{\alpha}_i$, γ_i and δ_i we can assume by the Łojasiewicz inequality that the Nash maps $\hat{\alpha}_i$, γ_i and δ_i are bounded. Then by Proposition VI.2.8, [S₂] we can regard \tilde{M} and the maps as the interior of a compact Nash manifold possibly with corners \tilde{M} and the restrictions to M of Nash maps $\tilde{\alpha}_i$, $\tilde{\gamma}_i$ and $\tilde{\delta}_i$ on \tilde{M} . Replace \mathcal{M}_1 and \mathcal{M}_2 by the sheaves of \mathcal{N} -modules on \tilde{M} $\mathcal{N}^{n_1}/(\tilde{\delta}_1, \dots, \tilde{\delta}_{n_4})\mathcal{N}^{n_1}$ and $\mathcal{N}/(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{n_3})\mathcal{N}$, respectively, and replace $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ with the \mathcal{N} -homomorphism $\tilde{h} : \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2$

defined by

$$\tilde{h}(0, \dots, 0, \overset{i}{1}, 0, \dots, 0) = \tilde{\alpha}_i \pmod{(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{n_3})\mathcal{N}}, \quad i = 1, \dots, n_1.$$

Then it suffices to see that $\text{Ker } \tilde{h}$ is finite. Hence we assume from the beginning that M is a compact Nash manifold possibly with corners. Then $\text{Ker } h$ is isomorphic to $\mathcal{N} \otimes_{N(M)} \text{Ker } h_*$ by Theorem 5.2, [C-R-S₁]. Hence $\text{Ker } h$ is finite.

Let $n_2 > 1$ and assume that the theorem holds for $n_2 - 1$. Let \mathcal{M}_0 denote the sheaf of \mathcal{N} -ideals with $\mathcal{M}_{0x} = \{0\}$. Set $\mathcal{M}_3 = \mathcal{M}_2/p(\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$ and let $h_3 : \mathcal{M}_1 \rightarrow \mathcal{M}_3$ denote the composite of h with the projection $h_3 : \mathcal{M}_2 \rightarrow \mathcal{M}_3$. Then \mathcal{M}_3 is generated by the images $\bar{\beta}_2, \dots, \bar{\beta}_{n_2}$ of $\beta_2, \dots, \beta_{n_2}$, and $\gamma'_1, \dots, \gamma'_{n_3} \in N(M)^{n_2-1}$ are generators of the kernel of the \mathcal{N} -homomorphism $h_3 : \mathcal{N}^{n_2-1} \supset \mathcal{N}_x^{n_2-1} \ni (\phi_1, \dots, \phi_{n_2-1}) \rightarrow \sum_{i=1}^{n_2-1} \phi_i \bar{\beta}_{i+1x} \in \mathcal{M}_{3x} \subset \mathcal{M}_3$, $x \in M$, where $\gamma_i = (\gamma_{i,1}, \dots, \gamma_{i,n_2}) = (\gamma_{i,1}, \gamma'_i)$, $i = 1, \dots, n_3$. Hence \mathcal{M}_3 is finite, and by induction hypothesis $\text{Ker } h_3$ is finite. Consider $h|_{\text{Ker } h_3} : \text{Ker } h_3 \rightarrow \mathcal{M}_2$. The image is contained in $p(\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$ which is isomorphic to $(\text{Ker } p \cup \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)/\text{Ker } p$ and then to $\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0/(\text{Ker } p \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$. Hence we can regard $h|_{\text{Ker } h_3}$ as an \mathcal{N} -homomorphism from $\text{Ker } h_3$ to $\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0/(\text{Ker } p \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$. Now $\text{Ker } p \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$ is finite for the following reason.

Define a sheaf of \mathcal{N} -submodules \mathcal{M} of \mathcal{N}^{n_3} on M by

$$\mathcal{M}_x = \{(\phi_1, \dots, \phi_{n_3}) \in \mathcal{N}_x^{n_3} : \sum_{i=1}^{n_3} \phi_i \gamma_{i,jx} = 0, \quad j = 2, \dots, n_2\}.$$

Then it suffices to see that \mathcal{M} is finite because $\text{Ker } p \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$ is the image of \mathcal{M} under the \mathcal{N} -homomorphism $h_3 : \mathcal{N}^{n_3} \supset \mathcal{N}_x^{n_3} \ni (\phi_1, \dots, \phi_{n_3}) \rightarrow (\sum_{i=1}^{n_3} \phi_i \gamma_{i,1x}, 0, \dots, 0) \in \mathcal{N}_x \times \{0\} \times \dots \times \{0\} \subset \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$, $x \in M$. On the other hand, if we define an \mathcal{N} -homomorphism $r : \mathcal{N}^{n_3} \rightarrow \mathcal{N}^{n_2-1}$ by $r(\phi_1, \dots, \phi_{n_3}) = (\sum_{i=1}^{n_3} \phi_i \gamma_{i,2x}, \dots, \sum_{i=1}^{n_3} \phi_i \gamma_{i,n_2x})$ for $(\phi_1, \dots, \phi_{n_3}) \in \mathcal{N}_x^{n_3}$, $x \in M$, then $\text{Ker } r = \mathcal{M}$. As in the case of $n_2 = 1$ we reduce the problem to the case where $\gamma_{i,j}$ are bounded and then M is a compact Nash manifold possibly with corners. Then $\text{Ker } r$ is finite by Theorem 5.2, [C-R-S₁].

Thus $\text{Ker } p \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$ is finite. We can regard it as a sheaf of \mathcal{N} -ideals. Hence by the result in case of $n_2 = 1$, $\text{Ker}(h|_{\text{Ker } h_3}) = \text{Ker } h$ is finite. \square

The following two theorems do not hold for general sheaves of \mathcal{N} -modules, [Hu], [B-C-R] and VI.2.10, [S₂]. However, our case is sufficient for applications.

Theorem 2.7. *(Nash case of Cartan Theorem A) Let \mathcal{M} be a finite sheaf of \mathcal{N} -submodules of \mathcal{N}^n on a Nash manifold M for $n > 0 \in \mathbf{N}$. Then \mathcal{M} is finitely generated by its global cross-sections.*

Proof. We assume that $n > 1$ and proceed by induction on n . Let $p : \mathcal{N}^n \rightarrow \mathcal{N}^{n-1}$ denote the projection forgetting the first factor, and set $\mathcal{M}_1 = \text{Ker } p|_{\mathcal{M}}$ and $\mathcal{M}_2 = \text{Im } p|_{\mathcal{M}}$. Then the sequence $0 \rightarrow \mathcal{M}_1 \xrightarrow{p_1} \mathcal{M} \xrightarrow{p_2} \mathcal{M}_2 \rightarrow 0$ is exact, we can regard \mathcal{M}_1 as a sheaf of \mathcal{N} -ideals, which is finite by theorem 2.6, and \mathcal{M}_2 is clearly a finite sheaf of \mathcal{N} -submodules of \mathcal{N}^{n-1} . By induction hypothesis we have global generators h_1, \dots, h_l of \mathcal{M}_1 and g_1, \dots, g_k of \mathcal{M}_2 . Then it suffices to find

$f_1, \dots, f_k \in H^0(M, \mathcal{M})$ such that $p_{2*}(f_i) = g_i$, $i = 1, \dots, k$ because $f_1, \dots, f_k, h_1, \dots, h_l$ are generators of \mathcal{M} if f_i do exist.

Fix i . Since $H^0(M, \mathcal{M}) \subset N(M)^n$ and $H^0(M, \mathcal{M}_2) \subset N(M)^{n-1}$, setting $g_i = (g_{i,2}, \dots, g_{i,n})$ we construct $g_{i,1} \in N(M)$ such that $(g_{i,1}, \dots, g_{i,n}) \in H^0(M, \mathcal{M})$. For each $x \in M$, the set $\Phi_x = \{\phi \in \mathcal{N}_x : (\phi, g_{ix}) \in \mathcal{M}_x\}$ is a residue class of \mathcal{N}_x modulo \mathcal{M}_{1x} , and the correspondence $\Phi : x \rightarrow \Phi_x$ is a global cross-section of $\mathcal{N}/\mathcal{M}_1$ for the following reason.

Since it suffices to see it on each member of a finite open semialgebraic covering of M , we assume that \mathcal{M} is generated by global cross-sections $\alpha_1 = (\alpha_{1,1}, \dots, \alpha_{1,n}), \dots, \alpha_{k'} = (\alpha_{k',1}, \dots, \alpha_{k',n}) \in N(M)^n$. Then $\alpha'_1 = (\alpha_{1,2}, \dots, \alpha_{1,n}), \dots, \alpha'_{k'} = (\alpha_{k',2}, \dots, \alpha_{k',n})$ are also generators of \mathcal{M}_2 . Let \mathcal{M}_3 denote the kernel of the \mathcal{N} -homomorphism $\mathcal{N}^{k'+1} \supset \mathcal{N}_x^{k'+1} \ni (\phi_1, \dots, \phi_{k'+1}) \rightarrow \sum_{j=1}^{k'} \phi_j \alpha'_{jx} - \phi_{k'+1} g_{ix} \in \mathcal{N}_x^{n-1} \subset \mathcal{N}^{n-1}$, $x \in M$. Then \mathcal{M}_3 is finite by theorem 2.6, and each stalk \mathcal{M}_{3x} contains a germ of the form $(\phi_1, \dots, \phi_{k'}, 1)$. Hence refining the covering if necessary, we assume that \mathcal{M}_3 is generated by a finite number of global cross-sections. Then we have $\beta_1, \dots, \beta_{k'} \in N(M)$ such that $g_i = \sum_{j=1}^{k'} \beta_j \alpha'_j$. It follows $\Phi = \sum_{j=1}^{k'} \beta_j \alpha_{j,1} \pmod{\mathcal{M}_1}$. Thus Φ is a global cross-section.

Apply the next theorem to the projection $\mathcal{N} \rightarrow \mathcal{N}/\mathcal{M}_1$ and Φ . Then there exists $g_{i,1} \in N(M)$ such that $g_{i,1x} = \Phi_x \pmod{\mathcal{M}_{1x}}$ for $x \in M$ and hence $(g_{i,1}, \dots, g_{i,n}) \in H^0(M, \mathcal{M})$. \square

Theorem 2.8. (*Nash case of Cartan Theorem B*) *Let $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a surjective \mathcal{N} -homomorphism between finite sheaves of \mathcal{N} -modules on a Nash manifold M . Assume that \mathcal{M}_1 is finitely generated by its global cross-sections. Then the natural map $h_* : H^0(M, \mathcal{M}_1) \rightarrow H^0(M, \mathcal{M}_2)$ is surjective.*

Proof. We can assume that $\mathcal{M}_1 = \mathcal{N}^n$ for some $n > 0 \in \mathbf{N}$ because there exist global generators g_1, \dots, g_n of \mathcal{M}_1 and then we can replace h with the surjective homomorphism $\mathcal{N}^n \supset \mathcal{N}_x^n \ni (\phi_1, \dots, \phi_n) \rightarrow h(\sum_{i=1}^n \phi_i g_{ix}) \in \mathcal{M}_{2x} \subset \mathcal{M}_2$, $x \in M$. Set $\mathcal{M} = \text{Ker } h$. Then by theorem 2.6, \mathcal{M} is a finite sheaf of \mathcal{N} -submodules of \mathcal{N}^n , and $h : \mathcal{N}^n \rightarrow \mathcal{M}_2$ coincides with the projection $p : \mathcal{N}^n \rightarrow \mathcal{N}^n/\mathcal{M}$. Hence we consider p in place of h . As in the last proof, assume that $n > 1$ and the theorem holds for smaller n .

Let $f \in H^0(M, \mathcal{N}^n/\mathcal{M})$. We need to find $g \in H^0(M, \mathcal{N}^n) = N(M)^n$ such that $p_*(g) = f$. Let \mathcal{M}_0 denote the sheaf of \mathcal{N} -ideals with $\mathcal{M}_{0x} = \{0\}$ for $x \in M$. Then the homomorphism $\mathcal{M}_0 \times \mathcal{N}^{n-1} \rightarrow \mathcal{N}^n/(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$ is surjective and we can regard it as the projection $\mathcal{N}^{n-1} \rightarrow \mathcal{N}^{n-1}/\mathcal{L}$ for some finite sheaf of \mathcal{N} -submodules \mathcal{L} of \mathcal{N}^{n-1} . Hence by induction hypothesis there exists $(0, g_2, \dots, g_n) \in H^0(M, \mathcal{M}_0 \times \mathcal{N}^{n-1})$ whose image in $H^0(M, \mathcal{N}^n/(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0))$ coincides with the image of f there. Replace f with the difference of f and the image of $(0, g_2, \dots, g_n)$ in $H^0(M, \mathcal{N}^n/\mathcal{M})$. Then we can assume from the beginning that $f \in H^0(M, (\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)/\mathcal{M})$. Hence we regard f as a global cross-section of $\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$ since $(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)/\mathcal{M}$ is naturally isomorphic to $\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0)$. It was shown in the proof of theorem 2.6 that $\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$ is finite. Hence f is the image of some global cross-section g of $\mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0$ under the projection $H^0(M, \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0) \rightarrow H^0(M, \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \dots \times \mathcal{M}_0))$ because this is the case of $\mathcal{M}_1 = \mathcal{N}$ in the theorem. Then $p_*(g) = f$. \square

Proof of Remark (i). We can assume that there exists an exact sequence $\mathcal{N}^m \xrightarrow{q}$

$\mathcal{N}^n \xrightarrow{p} \mathcal{M} \longrightarrow 0$ and \mathcal{M}_1 is generated by global cross-sections. Then \mathcal{M}_1 is finitely generated, say by f_1, \dots, f_k , because $H^0(M, \mathcal{M}_1)$ is a $N(M)$ -submodule of $H^0(M, \mathcal{N}^n) = N(M)^n$ and $N(M)$ is Noetherian ([E] and [Ri]). Then, by theorem 2.8, there exist $g_i \in H^0(M, \mathcal{N}^n)$ such that $p_*(g_i) = f_i$, $i = 1, \dots, k$. Let an \mathcal{N} -homomorphism $r : \mathcal{N}^k \rightarrow \mathcal{N}^n$ be defined by $r(\phi_1, \dots, \phi_k) = \sum_{i=1}^k \phi_i g_{ix}$ for $(\phi_1, \dots, \phi_k) \in \mathcal{N}_x^k$, $x \in M$. Then $\text{Im } p \circ r = \mathcal{M}_1$ and $\text{Ker } p \circ r = r^{-1}(\text{Im } q)$. Apply, once more, theorem 2.8 to r and $q_*(1, 0, \dots, 0), \dots, q_*(0, \dots, 0, 1)$. Then there exist global cross-sections of \mathcal{N}^k which generate $\text{Ker } p \circ r$. Hence \mathcal{M}_1 is finite. \square

Proof of Remark (ii). As usual we assume that \mathcal{M}_1 and \mathcal{M}_2 are generated by global cross-sections $\alpha_1, \dots, \alpha_k$ and β_1, \dots, β_l , respectively. Then $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$ generate $\mathcal{M}_1 + \mathcal{M}_2$, and hence by (i) $\mathcal{M}_1 + \mathcal{M}_2$ is finite.

Consider $\mathcal{M}_1 \cap \mathcal{M}_2$. Set $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$, and let $p : \mathcal{N}^{k+l} \rightarrow \mathcal{M}$ be an \mathcal{N} -homomorphism defined by

$$p(\phi_1, \dots, \phi_k, \psi_1, \dots, \psi_l) = \sum_{i=1}^k \phi_i \alpha_{ix} - \sum_{i=1}^l \psi_i \beta_{ix} \quad \text{for } (\phi_1, \dots, \psi_l) \in \mathcal{N}_x^{k+l}, \quad x \in M.$$

Then $\mathcal{M}_1 \cap \mathcal{M}_2$ is the image of $\text{Ker } p$ under the homomorphism $\mathcal{N}^{k+l} \supset \mathcal{N}_x^{k+l} \ni (\phi_1, \dots, \psi_l) \rightarrow \sum_{i=1}^k \phi_i \alpha_{ix} \in \mathcal{M}_x \subset \mathcal{M}$, $x \in M$. By theorem 2.6 $\text{Ker } p$ is finite, and hence we can assume that $\text{Ker } p$ is generated by global cross-sections. Then its image— $\mathcal{M}_1 \cap \mathcal{M}_2$ also is generated by global cross-sections. Therefore, by (i) $\mathcal{M}_1 \cap \mathcal{M}_2$ is finite. \square

Proof of Remark (iii). Let $\{U_i\}$ be a finite open semialgebraic covering of M such that \mathcal{M} satisfy the condition of exact sequence in the definition of a finite sheaf of \mathcal{N} -modules. Set $H^0(M, \mathcal{M}) \otimes_{N(M)} \mathcal{N} = \mathcal{M}_1$. Then \mathcal{M}_1 is generated by its global cross-sections and, moreover, finite because for each i , $\mathcal{M}_1|_{U_i}$ is an $N(M)$ -submodule of $H^0(U_i, \mathcal{M}|_{U_i})$ and $H^0(U_i, \mathcal{M}|_{U_i})$ is a finitely generated $N(U_i)$ -module. Hence we can assume from the beginning that \mathcal{M} is generated by its global cross-sections, say f_α , $\alpha \in A$, because $H^0(M, \mathcal{M}) = H^0(M, \mathcal{M}_1)$. For each i choose a finite subset A_i of A so that $\{f_\alpha|_{U_i} : \alpha \in A_i\}$ generate $\mathcal{M}|_{U_i}$. Then $\{f_\alpha : \alpha \in \cup_i A_i\}$ generate \mathcal{M} . \square

Remark. (iv) The above arguments show that the family of finite sheaves of \mathcal{N} -modules on a Nash manifold which are generated by global cross-sections is closed under the following operations if the operations are well-defined: to take a finite sheaf of \mathcal{N} -submodules, $+$, \cap , \otimes and $/$.

Let X be a Nash subset of \mathbf{R}^n and f_1, \dots, f_k be generators of the ideal of $N(\mathbf{R}^n)$ of functions vanishing on X . Let $\text{Sing } X$ denote the subset of X where the Jacobian matrix rank of f_1, \dots, f_k is smaller than $\text{codim } X$. Let a complexification $X^{\mathbf{C}}$ of X in \mathbf{C}^n be defined to be the common zero set of some complexifications $f_1^{\mathbf{C}}, \dots, f_k^{\mathbf{C}}$ of f_1, \dots, f_k . Then by Lemma 1.9 and Theorem 1.10 in [C-R-S₂] and theorem 2.7, we obtain the next remark.

Remark. (v) $\text{Sing } X$ is the smallest Nash subset of X whose complement is a Nash manifold but not the subset of X of points where the germ of X is not a Nash manifold germ of $\dim X$. Moreover $\text{Sing } X$ is also $X \cap \text{Sing } X^{\mathbf{C}}$, where $\text{Sing } X^{\mathbf{C}}$ denotes the C^ω singular point set of $X^{\mathbf{C}}$.

Remark. (vi) If X is irreducible as a Nash subset of \mathbf{R}^n then X is irreducible as a global analytic subset of \mathbf{R}^n .

Proof of remark (vi). Assume that X is irreducible as a Nash subset of \mathbf{R}^n . Let I denote the ideal of $N(\mathbf{R}^n)$ generated by f_1, \dots, f_n , and let \mathcal{I} the sheaf of \mathcal{O} -ideals on \mathbf{R}^n of germs whose complexifications vanish on $X^{\mathbf{C}}$. Then I is a prime ideal of $N(\mathbf{R}^n)$, and hence, by I.6.9, [S₂], $I\mathcal{O}_x$ is an intersection of a finite number of prime ideals of \mathcal{O}_x for each $x \in \mathbf{R}^n$. Therefore, by Hilbert Nullstellensatz, $\mathcal{I}_x = I\mathcal{O}_x$ for each $x \in \mathbf{R}^n$, i.e., $\mathcal{I} = I\mathcal{O}$. Then by theorem 2.3 we see that $IC^\omega(\mathbf{R}^n) = H^0(\mathbf{R}^n, \mathcal{I})$. On the other hand, $IC^\omega(\mathbf{R}^n)$ is a prime ideal of $C^\omega(\mathbf{R}^n)$ by Proposition 0.5, [C-R-S₁] and Corollary 2, [C-S₃]. Hence \mathcal{I} is irreducible as a coherent sheaf of \mathcal{O} -ideals. Therefore, the germ of $X^{\mathbf{C}}$ on \mathbf{R}^n is irreducible as the germ on \mathbf{R}^n of complex conjugation invariant complex analytic set. Hence X is irreducible as a global analytic subset of \mathbf{R}^n . \square

We state and prove a Nash version of Hironaka Desingularization Theorem, [Hi]. The result may be known to some specialists of singularities. However, the authors do not know its references. Hence we give a proof.

Theorem 2.9. (*Nash case of Main Theorem I of [Hi]*) *Let X be a Nash subset of \mathbf{R}^n . Then there exists a finite sequence of blowings-up $X_r \xrightarrow{\pi_r} \dots \xrightarrow{\pi_1} X_0 = X$ along smooth Nash centers $C_i \subset X_i$, $i = 1, \dots, r - 1$, such that X_r is smooth and $C_i \subset \text{Sing } X_i$.*

Main Theorems I and II of [Hi] state some additional conditions. Those are satisfied in the Nash case also, which is clear by the following proofs.

Proof. Let f_1, \dots, f_k be generators of the ideal of $N(\mathbf{R}^n)$ of functions vanishing on X . Set $F = (f_1, \dots, f_k)$, which is a Nash map from \mathbf{R}^n to \mathbf{R}^k , and $Y = \text{graph } F$. Let $Y^{\mathbf{Z}}$ denote the Zariski closure of \widetilde{Y} in $\mathbf{R}^n \times \mathbf{R}^k$ and let $\widetilde{Y^{\mathbf{Z}}} \subset \mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'}$ be an algebraic set such that $p : \widetilde{Y^{\mathbf{Z}}} \rightarrow \mathbf{R}^n \times \mathbf{R}^k$ —the restriction to $\widetilde{Y^{\mathbf{Z}}}$ of the projection $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^n \times \mathbf{R}^k$ —is the normalization of $Y^{\mathbf{Z}}$ (we simply call $\widetilde{Y^{\mathbf{Z}}}$ the *normalization* of $Y^{\mathbf{Z}}$). Then by Artin-Mazur Theorem (see Theorem I.5.1, [S₂]) there exists a connected component L of $\widetilde{Y^{\mathbf{Z}}}$ consisting of only regular points such that $p(L) = Y$ and $p|_L : L \rightarrow Y$ is a Nash diffeomorphism. Let $q_1 : \widetilde{Y^{\mathbf{Z}}} \rightarrow \mathbf{R}^n$ and $q_2 : \widetilde{Y^{\mathbf{Z}}} \rightarrow \mathbf{R}^k$ denote the restrictions to $\widetilde{Y^{\mathbf{Z}}}$ of the projections $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^n$ and $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^k$, respectively. Then $q_1|_L$ is a Nash diffeomorphism onto \mathbf{R}^n , $q_2^{-1}(0)$ is an algebraic set, $(q_1|_L)^{-1}(X) = (q_2|_L)^{-1}(0)$, and $(q_1|_L)^{-1}(\text{Sing } X)$ is just the intersection of L with the algebraic singular point set of $q_2^{-1}(0)$. Indeed, $(q_1|_L)^{-1}(\text{Sing } X)$ is contained in the intersection by remark (v) that $(q_1|_L)^{-1}(\text{Sing } X)$ is the smallest Nash subset of $(q_1|_L)^{-1}(X) (= (q_2|_L)^{-1}(0))$ whose complement is a Nash manifold, and the converse inclusion follows from the equality $q_2 = F \circ q_1$ on L . Hence we can replace X by $L \cap q_2^{-1}(0)$ —a union of some connected components of $q_2^{-1}(0)$. By Main Theorem I there exists a finite sequence of blowings-up $\tilde{X}_r \xrightarrow{\tilde{\pi}_r} \dots \xrightarrow{\tilde{\pi}_1} \tilde{X}_0 = q_2^{-1}(0)$ along smooth algebraic centers $\tilde{C}_i \subset \tilde{X}_i$, $i = 0, \dots, r - 1$, such that \tilde{X}_r is smooth and $\tilde{C}_i \subset \text{Sing } \tilde{X}_i$. Then $\tilde{X}_r \cap (\tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_r)^{-1}(L) \rightarrow \dots \rightarrow \tilde{X}_0 \cap L$ fulfills the requirements. \square

A sheaf of \mathcal{N} -(\mathcal{O} -)ideals on a Nash (analytic) manifold M is called *normal crossing* if there exists a local Nash (analytic) coordinate system (x_1, \dots, x_n) of M at

each point such that the stalk of the sheaf is generated by $\prod_{i=1}^n x_i^{\alpha_i}$ for some $(\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$.

Theorem 2.10. *(Nash case of Main Theorem II of [Hi]) Let M be a Nash manifold and let \mathcal{I}_1 and \mathcal{I}_2 be finite sheaves of non-zero \mathcal{N} -ideals on M . Assume that \mathcal{I}_2 is normal crossing. Then there exists a finite sequence of blowings-up $M_r \xrightarrow{\pi_r} \dots \xrightarrow{\pi_1} M_0 = M$ along smooth Nash centers $C_i \subset M_i$, $i = 1, \dots, r-1$, such that $(\pi_1 \circ \dots \circ \pi_r)^{-1} \mathcal{I}_1 \mathcal{I}_2 \mathcal{N}_{M_r}$ is normal crossing, each C_i is normal crossing with $(\pi_1 \circ \dots \circ \pi_i)^{-1}(\text{supp } \mathcal{N}_M / \mathcal{I}_2) \cup \cup_{j=1}^i (\pi_j \circ \dots \circ \pi_i)^{-1}(C_{j-1})$ and $\pi_1 \circ \dots \circ \pi_i(C_i)$ is contained in $\{x \in M : \mathcal{I}_{1x} \text{ is not generated by any power of one regular function germ or } \mathcal{I}_{1x} + \mathcal{I}_{2x} \neq \mathcal{N}_x\}$.*

Note that $(\pi_1 \circ \dots \circ \pi_i)^{-1} \mathcal{I}_2 \mathcal{N}_{M_i}$, $i = 1, \dots, r$, are normal crossing.

Proof. Let $f_1, \dots, f_{k'} \in N(\mathbf{R}^n)$ and $f_{k'+1}, \dots, f_k \in N(\mathbf{R}^n)$ be global generators of \mathcal{I}_1 and \mathcal{I}_2 , respectively, (theorem 2.7), and define $F, Y, Y^Z, \widetilde{Y^Z}, L, q_1 : \widetilde{Y^Z} \rightarrow \mathbf{R}^n$ and $q_2 : \widetilde{Y^Z} \rightarrow \mathbf{R}^k$ as in the last proof. Let W be the subset of $\widetilde{Y^Z}$ of the points where $f_{k'+1} \circ q_1, \dots, f_k \circ q_1$ do not generate a normal crossing sheaf of \mathcal{N} -ideals. Consider the algebraic \mathbf{R} -scheme of the topological underlying space $\widetilde{Y^Z} - \text{Sing } \widetilde{Y^Z} - W$, and let \mathcal{J}_1 and \mathcal{J}_2 denote the sheaf of ideals of the scheme generated by $f_1 \circ q_1, \dots, f_{k'} \circ q_1$ and by $f_{k'+1} \circ q_1, \dots, f_k \circ q_1$, respectively. Then we can replace M, \mathcal{I}_1 and \mathcal{I}_2 with the scheme, \mathcal{J}_1 and \mathcal{J}_2 . Hence the theorem follows from Main Theorem II. \square

3. EQUIVALENCE OF NORMAL CROSSING FUNCTIONS

3.1. On C^∞ equivalence of analytic functions with only normal crossing singularities.

If we compare C^ω and C^∞ right equivalences of two analytic functions on an analytic manifold, the latter is easier to check, and the former implies the latter. Hence we are interested to know whether the converse holds. We will show that this is the case for analytic functions with only normal crossing singularities, and apply the fact to the proof of the main theorem 3.2 and hence corollary 3.3. Here an analytic function with *only normal crossing singularities* at a point x of the manifold is a function whose germ at x is of the form $\pm x^\alpha (= \pm \prod_{i=1}^n x_i^{\alpha_i}) + \text{const}$, $\alpha = (\alpha_1, \dots, \alpha_n) \neq 0 \in \mathbf{N}^n$ for some local analytic coordinate system (x_1, \dots, x_n) at x . If the function has only normal crossing singularities everywhere we say the function has *only normal crossing singularities*. Remember that by Hironaka Desingularization Theorem an analytic function becomes one with only normal crossing singularities after a finite sequence of blowings-up along smooth center. An analytic subset of an analytic manifold is called *normal crossing* if it is the zero set of an analytic function with only normal crossing singularities. This analytic function is called *defined by* the analytic set. It is not unique. However, the sheaf of \mathcal{O} -ideals *defined by* the analytic set is naturally defined and unique. We can naturally stratify a normal crossing analytic subset X into analytic manifolds X_i of dimension i . We call $\{X_i\}$ the *canonical stratification* of X . We extend this definition to the boundary of an analytic manifold with corners.

Let M be a C^∞ manifold. To prove the following theorem, we introduce a topology on $C^\infty(M)$, called the *strong Whitney C^∞ topology*, as follows. Let $\{M_l\}$ be a family of compact C^∞ submanifolds of M possibly with boundary such that $\{\text{Int } M_l\}$ is a locally finite covering of M . Regard $C^\infty(M)$ as a subset of $\prod_l C^\infty(M_l)$

by the injective map $C^\infty(M) \ni f \rightarrow \prod_l f|_{M_l} \in \prod_l C^\infty(M_l)$. Then $\{C^\infty(M) \cap \prod_l O_l : O_l \text{ open subsets of } C^\infty(M_l) \text{ in the } C^\infty \text{ topology}\}$ is a system of open sets of $C^\infty(M)$. We apply this topology unless otherwise specified. Note that this topology is stronger than the usual Whitney C^∞ topology, and Whitney Approximation Theorem—any C^∞ function on an analytic manifold is approximated by a C^ω function—holds also in this topology (see [W]).

The main theorem of this section is

Theorem 3.1. (1) *Let M be a C^ω manifold and $f, g \in C^\omega(M)$. Assume that f and g admit only normal crossing singularities. If f is C^∞ right equivalent to g , f is C^ω right equivalent to g .*

(2) *If C^∞ functions f and g on a C^∞ manifold M admit only normal crossing singularities and are proper and C^2 right equivalent, then they are C^∞ right equivalent.*

(3) *If f and g are semialgebraically C^2 right equivalent Nash functions on a Nash manifold M with only normal crossing singularities then they are Nash right equivalent.*

Remark. (i) We are interested also in the germ case. Let M , f and g be the same as in above (1). Let ϕ be a C^∞ diffeomorphism of M such that $f = g \circ \phi$, set $X = \text{Sing } f$ and $Y = \text{Sing } g$, and let $\{X_i\}$ and $\{Y_i\}$ be the irreducible analytic components of X and Y , respectively. Let A and B be unions of some intersections of some X_i and Y_i , respectively. Assume that $\phi(A) = B$. Then we can choose a C^ω diffeomorphism π so that $f = g \circ \pi$ and $\pi(A) = B$. Consequently, theorem 3.1,(1) holds for the germs of f on A and g on B . Similar statements for (2) and (3) hold.

(ii) In the Nash case, C^∞ right equivalence does not imply Nash right equivalence. Indeed, let N be a compact contractible Nash manifold with non-simply connected boundary of dimension $n > 3$ (e.g., see [Mz]). Set $M = (\text{Int } N) \times (0, 1)$ and let $f : M \rightarrow (0, 1)$ denote the projection. Then M and f are of class Nash, and M is Nash diffeomorphic to \mathbf{R}^{n+1} for the following reason. Smooth the corners of $N \times [0, 1]$. Then $N \times [0, 1]$ is a compact contractible Nash manifold with simply connected boundary of dimension > 4 . Hence by the positive answers to Poincaré conjecture and Schönflies problem (Brown-Mazur Theorem) $N \times [0, 1]$ is C^∞ diffeomorphic to an $(n + 1)$ -ball. Hence by Theorem VI.2.2, [S₂] M is Nash diffeomorphic to an open $(n + 1)$ -ball. Let $g : M \rightarrow \mathbf{R}$ be a Nash function which is Nash right equivalent to the projection $: \mathbf{R}^n \times (0, 1) \rightarrow (0, 1)$. Then f and g are C^ω right equivalent since $\text{Int } N$ is C^ω diffeomorphic to \mathbf{R}^n , but they are not Nash equivalent because $\text{Int } N$ and \mathbf{R}^n are not Nash diffeomorphic, by Theorem VI 2.2, [S₂].

(iii) The theorem holds even if M has corners, which is clear by the following proof. (See the definition of normal crossing singularity in this case after corollary 3.3.)

Note that for the proof of parts (2) and (3) of the theorem, we use some results (from lemma 3.5 to proposition 3.12) that will be detailed in the next section dedicated to cardinality issues.

Proof of theorem 3.1. (1). The idea of proof is taken from [S₁]. The proof is divided into three steps. Denote by X and Y the extended critical sets of f and g , that is, $X = f^{-1}(f(\text{Sing } f))$ and $Y = g^{-1}(g(\text{Sing } g))$. Note that X and Y are not necessarily analytic sets. Let M be analytic and closed in the ambient Euclidean

space \mathbf{R}^N , and consider the Riemannian metric on M induced from that of \mathbf{R}^N . Set $n = \dim M$.

Step 1. Assume that X is an analytic set. Let ϕ denote a C^∞ diffeomorphism of M such that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M arbitrarily close to ϕ such that $\pi(X) = Y$.

Proof of step 1. Let $\{X_i : i = 0, \dots, n-1\}$ and $\{Y_i : i = 1, \dots, n-1\}$ be the canonical stratifications of X and Y respectively, and put $X_n = M - X$ and $Y_n = M - Y$.

Before beginning the proof, we give some definitions and two remarks. Fix X_i . Let $\{M_l\}$ be a family of compact C^∞ manifolds of dimension i possibly with boundary such that $\{\text{Int } M_l\}$ is a locally finite covering of $\cup_{j=0}^i X_j$. A function on $\cup_{j=0}^i X_j$ is called of class C^∞ if its restriction to each M_l is of class C^∞ . Thus $C^\infty(\cup_{j=0}^i X_j)$ is a subset of $\prod_l C^\infty(M_l)$. We give to $C^\infty(\cup_{j=0}^i X_j)$ the product topology of the C^∞ topology on each $C^\infty(M_l)$, i.e. the compact-open C^∞ topology. Then $C^\infty(\cup_{j=0}^i X_j)$ is a Fréchet space, and a remark is the following statement.

Let X' be a normal-crossing C^ω subset of M contained in X . Assume that the sheaf of \mathcal{O} -ideals on M defined by X' is generated by a single C^ω function ξ on M . Let V denote the subspace of $C^\infty(\cup_{j=0}^i X_j)$ consisting of functions which vanish on X' . Then the map $C^\infty(\cup_{j=0}^i X_j) \ni h \rightarrow \xi h \in V$ is open in the compact-open C^∞ topology (surjectiveness follows from Theorem VI,3.10, [Ml], and openness does from the open mapping theorem on Fréchet spaces).

We give the strong Whitney C^∞ topology on $C^\infty(\cup_{j=0}^i X_j)$ in the same way. In this topology also the above remark holds, which we easily prove by the above remark using a partition of unity of class C^∞ , though $C^\infty(\cup_{j=0}^i X_j)$ is not a Fréchet space if $\cup_{j=0}^i X_j$ is not compact (the second remark).

Now we begin the proof. By induction, for some $i \in \mathbf{N}$, assume that we have constructed a C^∞ diffeomorphism π_{i-1} of M close to ϕ such that $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j}$ is of class C^ω (in the sense that $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j} \in \prod_l C^\omega(M_l)$) and $\pi_{i-1}(X_j) = Y_j$ for any j . Let \mathcal{M} denote the sheaf of \mathcal{O} -ideals on M defined by $\cup_{j=0}^{i-1} X_j$, which is coherent because X is normal-crossing. Then $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j} \in H^0(M, \mathcal{O}/\mathcal{M})^N$ for the following reason. As the problem is local, we can assume that $M = \mathbf{R}^n$ and $X = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}$ for some $n' \leq n \in \mathbf{N}$. Moreover, we suppose that $i = n$ because if for each irreducible analytic component E of $\cup_{j=0}^i X_j$ we can extend $\pi_{i-1}|_{E \cap \cup_{j=0}^{i-1} X_j}$ to an analytic map on E then the extensions for all E define an analytic map from $\cup_{j=0}^i X_j$ to \mathbf{R}^N . Then this is easy (see the arguments before step 2). Consider any C^ω extension $\alpha : M \rightarrow \mathbf{R}^N$ of $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j}$ (corollary 2.4). Here we can choose α so as to be sufficiently close to π_{i-1} and so that $\text{Im } \alpha \subset M$ for the following reason. Let $\gamma_1, \dots, \gamma_k \in C^\omega(M)$ be generators of \mathcal{M} (corollary 2.2). Then there exist $\delta_1, \dots, \delta_k \in C^\infty(M, \mathbf{R}^N)$ such that $\pi_{i-1} - \alpha = \sum_{j=1}^k \gamma_j \delta_j$, which is proved by the above Malgrange theorem. Replace α with the composite of $\alpha + \sum \gamma_j (C^\omega \text{ approximation of } \delta_j)$ and the orthogonal projection of a neighborhood of M in \mathbf{R}^N to M . Then α becomes to satisfy the requirements. Let $p_j : U_j \rightarrow Y_j$ be the orthogonal projection of a tubular neighborhood of Y_j in \mathbf{R}^N . Here U_j is described as $\cup_{y \in Y_j} \{x \in \mathbf{R}^N : |x - y| < \epsilon(y), (x - y) \perp T_y Y_j\}$ for some positive C^0 function ϵ on Y_j where $T_y Y_j$ denotes the tangent space of Y_j at y , and we can

choose ϵ so large that $\epsilon(y) \geq \epsilon_0 \text{dis}(y, \cup_{k=0}^{j-1} Y_k)$ locally at each point of $\cup_{k=0}^{j-1} Y_k$ for some positive number ϵ_0 because Y is normal crossing. Then α can be so close to π_{i-1} that $\alpha(X_i) \subset U_i$ since $\alpha = \pi_{i-1}$ on $\cup_{j=0}^{i-1} X_j$ and hence $d\alpha_x v = d\pi_{i-1} v$ for any $x \in \cup_{j=0}^{i-1} X_j$ and for any tangent vector v at x tangent to $\cup_{j=0}^{i-1} X_j$. Define π_i on $\cup_{j=0}^i X_j$ to be α on $\cup_{j=0}^{i-1} X_j$ and $p_i \circ \alpha$ on X_i . Note that π_i is a C^ω map from $\cup_{j=0}^i X_j$ to $\cup_{j=0}^i Y_j \subset \mathbf{R}^N$ and close to $\pi_{i-1}|_{\cup_{j=0}^i X_j}$ because $\pi_{i-1}|_{\cup_{j=0}^i X_j} = \bar{p}_i \circ \pi_{i-1}|_{\cup_{j=0}^i X_j}$ and because the fact α is close to π_{i-1} implies that $\bar{p}_i \circ \alpha$ on $\cup_{j=0}^i X_j$ ($= \pi_i$) is close to $\bar{p}_i \circ \pi_{i-1}$ on $\cup_{j=0}^i X_j$, where $\bar{p}_i : \bar{U}_i \rightarrow \cup_{j=0}^i Y_j$ is the natural extension of p_i . We need to extend π_i to a C^∞ diffeomorphism of M which is close to π_{i-1} and carries each X_j to Y_j . Compare $\pi_i \circ \pi_{i-1}^{-1}|_{\cup_{j=0}^i Y_j}$ and the identity map of $\cup_{j=0}^i Y_j$. Then they are close each other and what we prove is the following statement.

Let τ be a C^∞ map between $\cup_{j=0}^i Y_j$ close to id. Then we can extend τ to a C^∞ diffeomorphism of M which is close to id and carries each Y_j to Y_j , $j = i+1, \dots, n$.

By the second induction it suffices to extend τ to a C^∞ map between $\cup_{j=0}^{i+1} Y_j$ close to id. We reduce the problem to a trivial case. First it is enough to extend τ to a C^∞ map from $\cup_{j=0}^{i+1} Y_j$ to \mathbf{R}^N close to id by virtue of $p_{i+1} : U_{i+1} \rightarrow Y_{i+1}$ as above. Secondly, if we replace τ with $\tau - \text{id}|_{\cup_{j=0}^i Y_j}$ then the problem is that for a C^∞ map $\tau : \cup_{j=0}^i Y_j \rightarrow \mathbf{R}^N$ close to the zero map we can extend τ to a C^∞ map from $\cup_{j=0}^{i+1} Y_j$ to \mathbf{R}^N close to 0. Thirdly, we can assume that τ is a function. Hence we can use a partition of unity of class C^∞ and the problem becomes local. So we assume that $M = \mathbf{R}^n$, Y is a union of some irreducible analytic components of $\{y_1 \cdots y_n = 0\}$ and $\tau \in C^\infty(\cup_{j=0}^i Y_j)$ is close to 0 and vanishes on $\{y \in \cup_{j=0}^i Y_j \subset \mathbf{R}^n : |y| \geq 1\}$, where $(y_1, \dots, y_n) \in \mathbf{R}^n$. Let ξ be a C^∞ function on M such that $\xi = 1$ on $\{y \in \mathbf{R}^n : |y| \leq 1\}$ and $\xi = 0$ on $\{|y| \geq 2\}$. If $n = 0$ or 1 we have nothing to do. Hence by the third induction on n , assume that we have a C^∞ extension $\tau_1(y_2, \dots, y_n)$ of $\tau|_{\{y_1=0\} \cap \cup_{j=0}^i Y_j}$ to $\{y_1 = 0\} \cap \cup_{j=0}^{i+1} Y_j$ close to 0. Regard $\tau_1(y_2, \dots, y_n)$ as a C^∞ function on $\cup_{j=0}^{i+1} Y_j$, which is possible because $\cup_{j=0}^{i+1} Y_j$ is contained in the product of \mathbf{R} and the image of $\{y_1 = 0\} \cap \cup_{j=0}^{i+1} Y_j$ under the projection $\mathbf{R} \times \mathbf{R}^{n-1} \rightarrow \mathbf{R}^{n-1}$. Replace τ with $\tau - \tau_1 \xi$, which vanishes on $\{y_1 = 0 \text{ or } |y| \geq 2\} \cap \cup_{j=0}^i Y_j$. Next consider $(\text{new } \tau)/y_1|_{\{y_2=0\} \cap \cup_{j=0}^i Y_j}$ and apply the second remark and the same arguments. Then we reduce the problem to the case where $\tau = 0$ on $\{y_1 y_2 = 0 \text{ or } |y| \geq 3\} \cap \cup_{j=0}^i Y_j$ and by the fourth induction to the case where $\tau = 0$ on $\cup_{j=0}^i Y_j$. Thus step 1 is proved.

Step 2. Assume that $X = Y$, X is an analytic set and there exists a C^∞ diffeomorphism ϕ of M close to id so that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M close to ϕ such that $f = g \circ \pi$. (To be precise, we fix g and modify f and ϕ so that ϕ is so close to id and that the following proof works, which is possible by step 1.)

Proof of step 2. We construct π by integrating along a well-chosen vector field on M . There exist analytic vector fields w_1, \dots, w_N on M which span the tangent space at each point of M , e.g., $w_{ix} = dp_x \frac{\partial}{\partial x_i}$, $x \in M$, where $(x_1, \dots, x_N) \in \mathbf{R}^N$ and p is the orthogonal projection of a tubular neighborhood of M in \mathbf{R}^N . Consider a vector field $v = \frac{\partial}{\partial t} + \sum_{i=1}^N a_i w_i$ on $M \times [0, 1]$ where $a_i \in C^\omega(M \times [0, 1])$ for $i \in \{1, \dots, N\}$. Put $F(x, t) = (1-t)f(x) + tg(x)$ for $(x, t) \in M \times [0, 1]$.

Assume that we have found such a_i , $i = 1, \dots, N$, that $v(F) = 0$ and $|\sum_{i=1}^N a_i w_i|$

is small. Then F is constant along integral curves of v , therefore, the flow of v furnishes an analytic diffeomorphism π so that $f = g \circ \pi$.

Therefore, what we have to do is to construct the relevant a_i , $i \in \{1, \dots, N\}$. First look at the local case. We will show that there exist a closed neighborhood U of each point of M and $a_i \in C^\omega(U \times [0, 1])$, $i = 1, \dots, N$, such that $v(F) = 0$ on $U \times [0, 1]$ and $|\sum_{i=1}^N a_i w_i|$ is small. If the point is in X , we can write $U = \{x \in \mathbf{R}^n : |x| \leq 1\}$, $g(x) - c = \prod_{i=1}^n x_i^{n_i}$ and $f(x) - c = \lambda(x)(g(x) - c)$ for $x \in U$, where $c \in \mathbf{R}$, λ is a C^ω function on U and close to 1, and at least one of n_i 's, say n_1 , is non-zero. Assume that $c = 0$ without loss of generality. Then there exists v of the form $\frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial x_1}$, $b_1 \in C^\omega(U \times [0, 1])$, which satisfies $v(F) = 0$. Actually

$$\begin{aligned} \left(\frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial x_1}\right)F(x, t) &= \\ (1 - \lambda)g(x) + b_1(x, t)[n_1(t + (1 - t)\lambda(x))\frac{g(x)}{x_1} + (1 - t)g(x)\frac{\partial \lambda}{\partial x_1}(x)] &= 0, \\ b_1(x, t) &= \frac{-(1 - \lambda(x))x_1}{n_1(t + (1 - t)\lambda(x)) + (1 - t)x_1\frac{\partial \lambda}{\partial x_1}(x)}, \end{aligned}$$

which is an analytic function in $U \times [0, 1]$ and close to 0. Shrink U if necessary. Then for some $0 < i_1 < \dots < i_n \leq N$, the vector fields w_{i_1}, \dots, w_{i_n} span the tangent space there, and $b_1 \frac{\partial}{\partial x_1}$ is described uniquely by $\sum_{j=1}^n a_{i_j} w_{i_j}$ for some C^ω functions a_{i_j} . Hence a_{i_j} , $j = 1, \dots, n$, and $a_i = 0$, $i \notin \{i_1, \dots, i_n\}$, fulfill the requirements.

?Next consider the situation at a point outside of X (note that the values of f and g at the point may be different, and hence the above arguments do not work). We can choose its local coordinate system so that $U = \{x \in \mathbf{R}^n : |x| \leq 1\}$ and $\frac{\partial g}{\partial x_1} = 1$ on U . Then

$$\begin{aligned} \left(\frac{\partial}{\partial t} + b \frac{\partial}{\partial x_1}\right)F(x, t) &= -f + g + b_1\left((1 - t)\frac{\partial f}{\partial x_1} + t\frac{\partial g}{\partial x_1}\right) = 0, \\ b_1(x, t) &= \frac{f - g}{(1 - t)\frac{\partial f}{\partial x_1} + t\frac{\partial g}{\partial x_1}}, \end{aligned}$$

and $(1 - t)\frac{\partial f}{\partial x_1} + t\frac{\partial g}{\partial x_1}$ and $f - g$ are close to 1 and 0, respectively. Hence there exist U and a_i as before.

Consequently, using a partition of unity of class C^∞ we obtain a C^∞ vector field $v' = \frac{\partial}{\partial t} + \sum_{i=1}^N a'_i w_i$ on $M \times [0, 1]$ such that $v'(F) = 0$ and $|\sum_{i=1}^N a'_i w_i|$ is small (to be precise, we start from a locally finite system of coordinate neighborhoods of M and a partition of unity subordinate to it, and modify f and ϕ so that $|\sum_{i=1}^N a'_i w_i|$ is small).

Now, to construct the global analytic vector field v on $M \times [0, 1]$ we use Cartan Theorems A and B. Consider the sheaf of relations \mathcal{J} on $M \times [0, 1]$ defined by

$$\begin{aligned} \mathcal{J} = \cup_{(x,t) \in M \times [0,1]} \{(\beta, \alpha_1, \dots, \alpha_N) \in \mathcal{O}_{(x,t)}^{N+1} : \\ \beta(f_x - g_x) + \sum_{i=1}^N \alpha_i(w_i((1 - t)f + tg))_{(x,t)} = 0\}. \end{aligned}$$

The sheaf \mathcal{J} is a coherent sheaf of \mathcal{O} -modules by Oka theorem 2.5. Later we will find $l \in \mathbf{N}$ and global cross-sections $(b_k, a_1^k, \dots, a_N^k) \in H^0(M \times [0, 1], \mathcal{J})$, $k \in \{1, \dots, l\}$, such that for any $(x, t) \in M \times [0, 1]$, any C^ω vector field germ ω at (x, t) in $M \times [0, 1]$ with $\omega(F_{(x,t)}) = 0$ is of the form $\sum_{k=1}^l \xi_k v_k(x,t)$ for some C^ω function germs ξ_k at (x, t) in $M \times [0, 1]$, where $v_k = b_k \frac{\partial}{\partial t} + \sum_{i=1}^N a_i^k w_i$. Assume that the existence of l and $(b_k, a_1^k, \dots, a_N^k)$. Then by the above method of construction of v' and by a partition of unity of class C^∞ there exist C^∞ functions θ_k on M such that $v' = \sum_{k=1}^l \theta_k v_k$. Approximate θ_k by C^ω functions $\tilde{\theta}_k$, and set $\tilde{v} = \sum_{k=1}^l \tilde{\theta}_k v_k$. Then \tilde{v} is a C^ω vector field such that $F(\tilde{v}) = 0$ and is described as $\tilde{a}_0 \frac{\partial}{\partial t} + \sum_{i=1}^N \tilde{a}_i w_i$, $\tilde{a}_i \in C^\omega(M \times [0, 1])$, for the following reason. Let \mathcal{I} denote the coherent sheaf of \mathcal{O} -submodules of the sheaf of \mathcal{O} -modules of germs of C^ω vector fields on $M \times [0, 1]$ defined by

$$\mathcal{I}_{(x,t)} = \{\omega : \omega(F_{(x,t)}) = 0\} \quad \text{for } (x, t) \in M \times [0, 1],$$

and define an \mathcal{O} -homomorphism $\delta : \mathcal{O}^l \rightarrow \mathcal{I}$ by

$$\delta(\gamma_1, \dots, \gamma_l) = \sum_{k=1}^l \gamma_k v_k(x,t) \quad \text{for } (\gamma_1, \dots, \gamma_l) \in \mathcal{O}_{(x,t)}^l, (x, t) \in M \times [0, 1].$$

Then δ is surjective, $H^0(M \times [0, 1], \mathcal{I})$ is the set of all C^ω vector fields w on $M \times [0, 1]$ with $w(F) = 0$, and hence by Cartan Theorem B the homomorphism $C^\omega(M \times [0, 1])^l \ni (d_1, \dots, d_l) \rightarrow \sum_{k=1}^l d_k v_k \in H^0(M \times [0, 1], \mathcal{I})$ is surjective, i.e., \tilde{v} is of the form $\sum_{k=1}^l d_k v_k$ for some C^ω functions d_k on $M \times [0, 1]$. Therefore, we have $\tilde{v} = \tilde{a}_0 \frac{\partial}{\partial t} + \sum_{i=1}^N \tilde{a}_i w_i$ for $\tilde{a}_0 = \sum_{k=1}^l d_k$ and $\tilde{a}_i = \sum_{k=1}^l d_k a_i^k$, $i = 1, \dots, N$. Here \tilde{a}_0 is unique and hence close to 1, and $|\sum_{i=1}^N \tilde{a}_i w_i|$ is small. Thus $v = \frac{\partial}{\partial t} + \sum_{i=1}^N (\tilde{a}_i / \tilde{a}_0) w_i$ is what we wanted.

It remains to find $(b_k, a_1^k, \dots, a_N^k)$, $k = 1, \dots, l$. That is equivalent to prove that $H^0(M \times [0, 1], \mathcal{I})$ is finitely generated by Cartan Theorem B because the homomorphism $\mathcal{J}_{(x,t)} \ni (\beta, \alpha_1, \dots, \alpha_N) \rightarrow \beta \frac{\partial}{\partial t} + \sum_{i=1}^N \alpha_i w_i(x,t) \in \mathcal{I}_{(x,t)}$ is surjective. Moreover, it suffices to see that each stalk $\mathcal{I}_{(x,t)}$ is generated by a uniform number of elements by corollary 2.2. Note that F is an analytic function with only normal crossing singularities. Hence we replace F with f to simplify notation. Let \mathcal{K} denote the sheaf of \mathcal{O} -modules of C^ω vector field germs on M such that

$$\mathcal{K}_x = \{\omega : \omega(f_x) = 0\} \quad \text{for } x \in M.$$

Then it suffices to choose $l \in \mathbf{N}$ so that for any $x_0 \in M$, \mathcal{K}_{x_0} is generated by l elements. Since the problem is local we can assume that $M = \mathbf{R}^n$, $x_0 = 0$ and $f(x) = \prod_{i=1}^k x_i^{n_i}$ with $n_1, \dots, n_k > 0$, $0 < k \leq n$. Write $\omega \in \mathcal{K}_0$ as $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, $\alpha_i \in \mathcal{O}_0$, and set $h(x) = \prod_{i=1}^k x_i$. Then $\omega(f_{x_0}) = 0$ means

$$\sum_{i=1}^k n_i \alpha_i f(x) / x_i = 0, \quad \text{hence} \quad \sum_{i=1}^k n_i \alpha_i h(x) / x_i = 0.$$

Therefore, each α_i is divisible by x_i . Hence, setting $\alpha'_i = \alpha_i / x_i$ we obtain $\sum_{i=1}^k n_i \alpha'_i = 0$. It is clear that $\{(\alpha'_1, \dots, \alpha'_n) \in \mathcal{O}_0^n : \sum_{i=1}^k \alpha'_i = 0\}$ is generated by $n-1$ elements, which proves step 2.

The proof of step 1 shows that any C^∞ diffeomorphism of M carrying $\text{Sing } f$ to $\text{Sing } g$ is approximated by an analytic diffeomorphism of M with the same property. Hence it suffices to prove the next statement.

Step 3. Assume that $\text{Sing } f = \text{Sing } g$, and there exists a C^∞ diffeomorphism ϕ of M arbitrarily close to id in the sense of step 2 such that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M such that $f = g \circ \pi$.

Proof of step 3. Set $Z = \text{Sing } f$ and let Z_i , $i = 1, 2, \dots$, be connected components of Z . Let U_i be disjoint small open neighborhoods of Z_i in M such that if $\phi(U_i) \cap U_{i'} \neq \emptyset$ then $i = i'$. Then by steps 1 and 2 there exist C^ω diffeomorphisms $\pi_i : U_i \rightarrow \phi(U_i)$ close to $\phi|_{U_i} : U_i \rightarrow \phi(U_i)$ such that $f = g \circ \pi_i$ on U_i . Note that if we define a map between M to be π_i on each U_i and ϕ elsewhere, then the map is a C^∞ diffeomorphism by the definition of the strong Whitney C^∞ topology. For $x_0 \in M$, let $m(x_0)$ denote the multiplicity of $g - g(x_0)$ at x_0 , i.e., $m(x_0) = |\alpha| = \alpha_1 + \dots + \alpha_n$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ such that $g(x) - g(x_0)$ is written as $\pm x^\alpha$ for some local coordinate system (x_1, \dots, x_n) at x_0 . There exists $h \in C^\omega(M)$ such that $h^{-1}(0) = Z$ and h is $m(x)$ -flat at each $x \in Z$ for the following reason. For each i , let $\{Z_{i,j}\}_j$ be the stratification of Z_i by multiplicity number, and for each $Z_{i,j}$, consider the smallest analytic set in U_i and hence in M containing each connected component of $Z_{i,j}$. Then we have a locally finite decomposition of Z_i into irreducible analytic sets $\{W_{i,j}\}_j$ in M such that $m(x)$ is constant, say $m_{i,j}$, on each $W_{i,j} - \cup_{j'} \{W_{i,j'} : \dim W_{i,j'} < \dim W_{i,j}\}$. By corollary 2.2 there exists $h_{i,j} \in C^\omega(M)$ —e.g., the $m_{i,j}$ th power of the square sum of a finite number of global generators of the sheaf of \mathcal{O} -ideals defined by $W_{i,j}$ —such that $h_{i,j}^{-1}(0) = W_{i,j}$ and $h_{i,j}$ is $m_{i,j}$ -flat at $W_{i,j}$, and then considering the sheaf of \mathcal{O} -ideals $\prod_{i,j} h_{i,j} \mathcal{O}$ we obtain h in the same way.

We will reduce the problem to the case where $\pi_i - \text{id}$ on U_i and $f - g$ are divisible by h . Since $\text{supp } \mathcal{O}/h\mathcal{O} = Z$, $\{\pi_i\}_i$ defines an element of $H^0(M, (\mathcal{O}/h\mathcal{O})^N)$. Hence applying Cartan Theorem B to the exact sequence $0 \rightarrow (h\mathcal{O})^N \rightarrow \mathcal{O}^N \rightarrow (\mathcal{O}/h\mathcal{O})^N \rightarrow 0$, we obtain $\pi' \in C^\omega(M)^N$ such that $\pi_i - \pi' \in hC^\omega(U_i)^N$ for each i . We need to modify π' to be a diffeomorphism of M . Let ξ be a C^∞ function on M such that $\xi = 0$ outside of a small neighborhood of Z and $\xi = 1$ on a smaller one. Approximate C^∞ maps $\sum_i \xi(\pi_i - \pi')/h$ and $(1 - \xi)(\phi - \pi')/h$ from M to \mathbf{R}^N by C^ω maps H_1 and H_2 , respectively. Then $hH_1 + hH_2 + \pi'$ is an analytic approximation of $\phi : M \rightarrow \mathbf{R}^N$ whose difference with π_i on U_i is divisible by h . Hence its composite π'' with the orthogonal projection of M in \mathbf{R}^N is an analytic approximation of $\phi : M \rightarrow M$ such that $\pi_i - \pi''$ is divisible by h because for $\theta_1, \theta_2 \in \mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle^m$, $\eta \in \mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle$ and $\rho \in \mathbf{R}\langle\langle y_1, \dots, y_m \rangle\rangle$ with $\theta_1(0) = \theta_2(0) = \eta(0) = \rho(0) = 0$, then $\rho \circ \theta_1 - \rho \circ (\theta_1 + \eta\theta_2) \in \mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle$ is divisible by η . Here, replace ϕ , π_i and f with $\phi \circ \pi''^{-1}$, $\pi_i \circ \pi''^{-1}$ and $f \circ \pi''^{-1}$, respectively. Then the equalities $f = g \circ \phi$ and $f = g \circ \pi_i$ continue to hold, and $\pi_i - \text{id}$ and hence $f - g$ are divisible by h and, moreover, by h^{3+s} by the same way, where $s \in \mathbf{N}$ is such that for $\psi_l(x) = \prod_{j=1}^l x_j \in \mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle$, $l \leq n$, and for $\alpha \in \mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle$ which vanishes on $\text{Sing } \psi_l$, then α^s is contained in the ideal of $\mathbf{R}\langle\langle x_1, \dots, x_n \rangle\rangle$ generated by $\frac{\partial \psi_l}{\partial x_1}, \dots, \frac{\partial \psi_l}{\partial x_n}$ (Hilbert Zero Point Theorem). Set $h_1 = (f - g)/h^{3+s} \in C^\omega$, which is close to 0 by the second remark.

As in the proof of step 2, we define C^ω vector fields w_i , $i = 1, \dots, N$, and a C^ω function F on $M \times [0, 1]$, and it suffices to find a C^ω vector field v of the form $\frac{\partial}{\partial t} + \sum_{i=1}^N a_i w_i$ on $M \times [0, 1]$ such that $v(F) = 0$ and $|\sum_{i=1}^N a_i w_i|$ is bounded. Since $f = g + h^{3+s} h_1$, then $F = g + (1 - t)h^{3+s} h_1$, and the equality $v(F) = 0$

becomes :

$$h^{3+s}h_1 = \sum_{i=1}^N a_i(w_i g + (1-t)h^{2+s}h_{2,i})$$

for some C^ω functions $h_{2,i}$ on M close to 0. This is solvable locally. Indeed, for each $x_0 \in M - Z$, at least one of $w_i g$, say $w_1 g$, does not vanish at x_0 . Hence $a_1 = h^{3+s}h_1/(w_1 g + (1-t)h^{2+s}h_{2,1})$, $a_2 = \dots a_N = 0$ is a solution on a neighborhood of x_0 . Assume that $x_0 \in Z$. Then choose an analytic local coordinate system (x_1, \dots, x_n) at x_0 in M so that $g(x) = \prod_{i=1}^n x_i^{n_i} + \text{const}$, where $\sum_{i=1}^n n_i = m(x_0) > 1$. Here we can assume that $x_0 = 0$, $\text{const} = 0$, $n_1, \dots, n_l > 0$ and $n_{l+1} = \dots = n_n = 0$. Note that $m(0, \dots, 0, x_{l+1}, \dots, x_n) = m(0)$ for $(x_{l+1}, \dots, x_n) \in \mathbf{R}^{n-l}$ near 0. What we prove is that for each $t_0 \in [0, 1]$, the ideal I of $\mathcal{O}_{(0,t_0)}$ generated by $\frac{\partial g(0,t_0)}{\partial x_i} + (1-t_{(0,t_0)})h_{(0,t_0)}^{2+s}h_{2,i(0,t_0)}$, $i = 1, \dots, l$, contains $h_{(0,t_0)}^{3+s}h_{1(0,t_0)}$. Let J denote the ideal of $\mathcal{O}_{(0,t_0)}$ generated by $\frac{\partial g(0,t_0)}{\partial x_i}$, $i = 1, \dots, l$. Then it suffices to see that $h_{(0,t_0)}^{1+s} \in J$ because if so, $J \supset I$, $J \ni h_{(0,t_0)}^{3+s}h_{1(0,t_0)}$, $J = I + \mathfrak{m}J$ and hence by Nakayama lemma $J = I$, where \mathfrak{m} is the maximal ideal of $\mathcal{O}_{(0,t_0)}$. Moreover, assuming $g(x) = x_1 \cdots x_l$ we prove that $h_{(0,t_0)}^s \in J$, which is sufficient because $\frac{\partial g(0,t_0)}{\partial x_i} = n_i \prod_{j=1}^l x_j^{n_j-1} \frac{\partial x_1 \cdots x_l}{\partial x_i}$ and $h_{(0,t_0)}$ is divisible by $\prod_{j=1}^l x_j^{n_j-1}$ by the definition of h . However, $h_{(0,t_0)}^s \in J$ is clear by the definition of s . Note that we can choose local $v = \frac{\partial}{\partial t} + \sum_{i=1}^N a_i w_i$ in any case so that $|\sum_{i=1}^N a_i w_i|$ is arbitrarily small.

We continue to proceed in the same way as in the proof of step 2. We obtain C^ω vector fields $v_k = b_k \frac{\partial}{\partial t} + \sum_{i=1}^N a_i^k w_i$, $k = 1, \dots, l$, by local existence and a C^∞ vector field $v' = \frac{\partial}{\partial t} + \sum_{i=1}^N a_i' w_i$ such that $v_k(F) = v'(F) = 0$, $|\sum_{i=1}^N a_i' w_i|$ is small and v' is of the form $\sum_{k=1}^l \theta_k w_k$. After then we approximate θ_k by C^ω functions $\tilde{\theta}_k$, and $v = \sum_{k=1}^l \tilde{\theta}_k v_k$ fulfills the requirements. Thus we complete the proof of (1).

(2). Set $X = f^{-1}(f(\text{Sing } f))$ and $Y = g^{-1}(g(\text{Sing } g))$, and let π be a C^2 diffeomorphism of M such that $f \circ \pi = g$. Then X and Y are normal crossing, $\pi(Y) = X$, and we assume that π is close to id in the Whitney C^2 topology by replacing f and π with $f \circ \pi'$ and $\pi'^{-1} \circ \pi$ for a C^∞ approximation π' of π in the Whitney C^2 topology. Hence by lemma 3.5 (see the next section), its proof and properness of f and g there exists a C^∞ diffeomorphism π'' of M close to id in the Whitney C^2 topology such that $\pi''(Y) = X$. Replace f and π , once more, by $f \circ \pi''$ and $\pi''^{-1} \circ \pi$. Then we can assume that, moreover, $X = Y$. We want to modify π to be of class C^∞ on a neighborhood of X . Set $B(\epsilon) = \{x \in \mathbf{R}^n : |x| \leq \epsilon\}$ for $\epsilon > 0 \in \mathbf{R}$. Let $\{U_i\}$ and $\{U_i'\}$ be locally finite open coverings of X in M such that $\overline{U_i'} \subset U_i$, $\pi(\overline{U_i'}) \subset U_i$, each $f|_{U_i}$ is C^∞ right equivalent to the function $\prod_{j=1}^n x_j^{\alpha_j} + \text{constant}$, $x = (x_1, \dots, x_n) \in \text{Int } B(\epsilon_i)$, for some $\epsilon_i > 0 \in \mathbf{R}$ and some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ depending on i with $\alpha_1 > 0, \dots, \alpha_{n'} > 0, \alpha_{n'+1} = \dots = \alpha_n = 0$ and that $U_i \cap X$ and U_i' are carried to $\text{Int } B(\epsilon_i) \cap \{x_1 \cdots x_{n'} = 0\}$ and $B(\epsilon_i/2)$ by the diffeomorphism of equivalence. Then by induction on i it suffices to prove the following statement (for simplicity of notation we assume that $\epsilon_i = 3$ and $\overline{U_i'}$ is carried to $B(1)$).

Let C be a closed subset of $B(3)$. Let f and g be C^∞ functions on \mathbf{R}^n such that f is of the form $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$ for the above α and g is of the form $x^\alpha g'$ for some positive C^∞ function g' on \mathbf{R}^n . Let π be a C^2 imbedding of $B(3)$ into \mathbf{R}^n such that $f \circ \pi = g$ on $B(3)$ and $\pi(X \cap B(3)) \subset X$ where $X = \{x^\alpha = 0\}$.

Let $\tau : B(3) \rightarrow \mathbf{R}^n$ be a C^2 approximation of π in the C^1 topology such that $\tau(X \cap B(3)) \subset X$, $f \circ \tau = g$ on a neighborhood of C in $B(3)$ and τ is of class C^∞ there. Then, fixing on $(B(3) - B(2)) \cup C$, we can approximate τ by a C^2 imbedding $\tilde{\tau} : B(3) \rightarrow \mathbf{R}^n$ in the C^1 topology so that $\tilde{\tau}(X \cap B(3)) \subset X$, $f \circ \tilde{\tau} = g$ on $B(1)$ and $\tilde{\tau}$ is of class C^∞ on $B(1)$.

We prove the statement. Set $\tau(x) = (\tau_1(x), \dots, \tau_n(x))$. Then $\tau_j(x)$ for each $1 \leq j \leq n'$ is divisible by x_j , to be precise, there exists a positive C^1 function F_j on $B(3)$ such that $\tau_j(x) = x_j F_j(x)$ since $\pi(X \cap B(3)) \subset X$, $X = \{0\} \times \mathbf{R}^{n-1} \cup \dots \cup \mathbf{R}^{n'-1} \times \{0\} \times \mathbf{R}^{n-n'}$ and π is close to id. The required approximation $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$ also has to have the form $(x_1 \tilde{F}_1, \dots, x_{n'} \tilde{F}_{n'}, \tilde{\tau}_{n'+1}, \dots, \tilde{\tau}_n)$ for some positive C^1 functions \tilde{F}_j and C^2 functions $\tilde{\tau}_{n'+1}, \dots, \tilde{\tau}_n$. Set $F = (F_1, \dots, F_{n'})$ and $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_{n'})$. Then F is of class C^∞ on a neighborhood of C , the condition $f \circ \tilde{\tau} = g$ on $B(1)$ coincides with the one $\tilde{F}^\alpha = g'$ on $B(1)$, and the other conditions which $\tilde{F}, \tilde{\tau}_{n'+1}, \dots, \tilde{\tau}_n$ satisfy are that $\tilde{F} = F$ on $(B(3) - B(2)) \cup C$, $(\tilde{F}, \tilde{\tau}_{n'+1}, \dots, \tilde{\tau}_n)$ is an approximation of $(F, \tau_{n'+1}, \dots, \tau_n)$ in the C^1 topology and that $\tilde{\tau}$ is of class C^2 on $B(3)$ and of class C^∞ on $B(1)$.

Set $Z = \{(x, y) \in B(3) \times \mathbf{R}^{n'} : y^\alpha = g'(x)\}$, which is a C^∞ submanifold with boundary of $B(3) \times \mathbf{R}^{n'}$ by the implicit function theorem since g' is positive. Note that $\tilde{F}^\alpha = g'$ on $B(1)$ if and only if $\text{graph } \tilde{F}|_{B(1)} \subset Z$ and that $\text{graph } F|_C \subset Z$. Let $p : W \rightarrow Z$ be a C^∞ projection of a tubular neighborhood of Z in $B(3) \times \mathbf{R}^{n'}$ such that $p(x, y)$ for $(x, y) \in W$ is of the form $(x, p_2(x, y))$, which is constructed as follows. Since g' is positive, $Z \cap \{x\} \times \mathbf{R}^{n'}$ for each $x \in B(3)$ is smooth and, moreover, the restriction to Z of the projection $B(3) \times \mathbf{R}^{n'} \rightarrow B(3)$ is submersive. Hence if we define $p(x, y)$ for each $(x, y) \in B(3) \times \mathbf{R}^{n'}$ near Z to be the orthogonal projection image of (x, y) to $Z \cap \{x\} \times \mathbf{R}^{n'}$, then p satisfies the requirements. Let $(\hat{F}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n)$ be a C^∞ approximation of $(F, \tau_{n'+1}, \dots, \tau_n)$ in the C^1 topology, fixed on a neighborhood of C , and let ϕ be a C^∞ function on $B(3)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B(1)$ and $\phi = 0$ on $B(3) - B(2)$. Define a C^2 map $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_{n'}) : B(3) \rightarrow \mathbf{R}^{n'}$ by

$$\tilde{F}(x) = \phi(x)p_2(x, \hat{F}(x)) + (1 - \phi(x))F(x) \quad \text{for } x \in B(3),$$

and set $\tilde{\tau} = (x_1 \tilde{F}_1, \dots, x_{n'} \tilde{F}_{n'}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n)$ on $B(3)$. Then $\text{graph } \tilde{F}|_{B(1)} \subset Z$ because $\tilde{F}|_{B(1)}$ coincides with the map $B(1) \ni x \rightarrow p_2(x, \tilde{F}(x)) \in \mathbf{R}^{n'}$ whose graph is contained in Z ; $\tilde{F} = F$ on $B(3) - B(2)$ since $\phi = 0$ there; $\tilde{F} = F$ on C since $\hat{F} = F$ there and since $p(x, F(x)) = (x, F(x))$ there; $(\tilde{F}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n)$ is an approximation of $(F, \tau_{n'+1}, \dots, \tau_n)$ in the C^1 topology since so is $(\hat{F}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n)$; $\tilde{\tau}$ is of class C^2 because if we set $p_2(x, y) = (p_{2,1}(x, y), \dots, p_{2,n'}(x, y))$ then

$$\tilde{\tau}_j(x) = \phi(x)x_j p_{2,j}(x, \hat{F}(x)) + (1 - \phi(x))\tau_j(x), \quad 1 \leq j \leq n';$$

$\tilde{\tau}$ is of class C^∞ on $B(1)$ since $\tilde{F}(x) = p_2(x, \hat{F}(x))$ on $B(1)$. Thus the statement is proved.

In conclusion, for some closed neighborhood V of $f(\text{Sing } f)$ in \mathbf{R} each of whose connected components contains one point of $f(\text{Sing } f)$, there exists a C^2 diffeomorphism τ of M sufficiently close to π in the Whitney C^1 topology such that τ is of class C^∞ on $f^{-1}(V)$ and $f \circ \tau = g$ on $f^{-1}(V)$. Then the restrictions of f

and g to $f^{-1}(\overline{\mathbf{R} - V})$ are proper and locally trivial maps onto $\overline{\mathbf{R} - V}$, $f \circ \pi = g$ on $f^{-1}(\overline{\mathbf{R} - V})$ and $\tau|_{f^{-1}(\overline{\mathbf{R} - V})}$ is an approximation of $\pi|_{f^{-1}(\overline{\mathbf{R} - V})}$ in the C^1 topology. Hence it is easy to modify τ so that $f \circ \tau = g$ and τ is of class C^∞ everywhere fixing on $f^{-1}(V)$. Therefore, f and g are C^∞ right equivalent, which proves (2).

(3). Note that here, we use more intensively some results and arguments from next section. Let $0 \ll l \in \mathbf{N}$. We prove first that f and g are semialgebraically C^l right equivalent and later that semialgebraic C^l right equivalence implies Nash right equivalence. We proceed with the former step as in the above proof of (2). Let π be a semialgebraic C^2 diffeomorphism of M such that $f \circ \pi = g$, and set $X = f^{-1}(f(\text{Sing } f))$ and $g^{-1}(g(\text{Sing } g))$. Let π' be a Nash approximation of π in the semialgebraic C^2 topology (Approximation Theorem VI.4.1, [S₂]). Then π' is a diffeomorphism of M and $\pi'^{-1} \circ \pi$ is a semialgebraic C^2 approximation of id in the semialgebraic C^2 topology. Hence by replacing f and π with $f \circ \pi'$ and $\pi'^{-1} \circ \pi$, we assume that π is close to id in the semialgebraic C^2 topology. Moreover, we suppose that $X = Y$ as in the proof of (2) by using lemma 3.6 and its remark in place of lemma 3.5. Furthermore, by using lemma 3.9 we can reduce the problem to the case where M is the interior of a compact Nash manifold possibly with boundary M_1 and for each $x \in \partial M_1$, (M_{1x}, X_x) is Nash diffeomorphic to the germ at 0 of $(\mathbf{R}^{n-1} \times [0, \infty), \{(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1} : x_1 \cdots x_{n'} = 0\} \times (0, \infty))$ for some $n' (< n) \in \mathbf{N}$.

We modify π on a semialgebraic neighborhood of X . By lemma 3.10 and proposition 3.11,(iii) there exist “finite” open semialgebraic coverings $\{U_i\}$ and $\{U'_i\}$ of X in M such that the closure $\overline{U'_i}$ in M is contained in U_i , $\pi(\overline{U'_i})$ is contained in U_i , $f|_{U_i}$ is Nash right equivalent to $x^\alpha + \text{constant}$ on $\text{Int } B_{\xi_i}(\epsilon_i)$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ depending on i with $\alpha_1 > 0, \dots, \alpha_{n'} > 0, \alpha_{n'+1} = \dots = \alpha_n = 0$, $n' (< n) \in \mathbf{N} - \{0\}$ and $B_{\xi_i}(\epsilon_i) = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_n > 0, |x^\alpha| \leq \xi_i(x_n), |x| \leq \epsilon_i\}$ for some $\epsilon_i > 0 \in \mathbf{R}$ and some positive Nash function ξ_i on $(0, \infty)$ and that $U_i \cap X$ and U'_i are carried to $\text{Int } B_{\xi_i}(\epsilon_i) \cap \{x_1 \cdots x_{n'} = 0\}$ and $\text{Int } B_{\xi_i/2}(\epsilon_i/2)$ by the diffeomorphism of equivalence. For modification of π on a semialgebraic neighborhood of X we need the following statement. Let $l' \in \mathbf{N}$ such that $l \leq l' \leq l + \#\{i\}$.

Let ξ be a small positive Nash function on $(0, \infty)$, and let C be a closed semialgebraic subset of $B_{3\xi}(3)$. Let f and g be Nash functions on $B_{4\xi}(4)$ such that f is of the form x^α for the above α and g is of the form $x^\alpha g'$ for some positive Nash function g' on $B_{4\xi}(4)$. Let π be a semialgebraic C^2 imbedding of $B_{3\xi}(3)$ into $B_{4\xi}(4)$ close to id in the semialgebraic C^2 topology such that $f \circ \pi = g$ on $B_{3\xi}(3)$ and $\pi(X \cap B_{3\xi}(3)) \subset X$ where $X = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}$. Let $\tau : B_{3\xi}(3) \rightarrow B_{4\xi}(4)$ be a semialgebraic C^2 approximation of π in the semialgebraic C^1 topology such that $\tau(X \cap B_{3\xi}(3)) \subset X$, $f \circ \tau = g$ on a closed semialgebraic neighborhood V of C in $B_{3\xi}(3)$ and τ is of class $C^{l'}$ there. Then, fixing on $(B_{3\xi}(3) - B_{2\xi}(2)) \cup C$ we can approximate τ by a semialgebraic C^2 imbedding $\tilde{\tau} : B_{3\xi}(3) \rightarrow B_{4\xi}(4)$ in the semialgebraic C^1 topology so that $\tilde{\tau}(X \cap B_{3\xi}(3)) \subset X$, $f \circ \tilde{\tau} = g$ on $B_\xi(1)$ and $\tilde{\tau}$ is of class $C^{l'-1}$ on $B_\xi(1)$.

We prove the statement. As before, set $\tau = (\tau_1, \dots, \tau_n)$, $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$, let F_j and \tilde{F}_j , $1 \leq j \leq n'$, be positive semialgebraic C^1 functions on $B_{3\xi}(3)$ such that $\tau_j = x_j F_j(x)$ and $\tilde{\tau}_j = x_j \tilde{F}_j(x)$ on $B_{3\xi}(3)$, and set $F = (F_1, \dots, F_{n'})$ and $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_{n'})$. Note that F_j are of class $C^{l'-1}$ on a semialgebraic neighborhood of C in $B_{3\xi}(3)$, which is different to F_j in the proof of (2) where they are of class C^∞ . Then the required conditions are $\tilde{F}^\alpha = g'$ on $B_\xi(1)$, $\tilde{F} = F$ on $(B_{3\xi}(3) - B_{2\xi}(2)) \cup$

C , $(\tilde{F}, \tilde{\tau}_{n'+1}, \dots, \tilde{\tau}_n)$ is a semialgebraic C^1 approximation of $(F, \tau_{n'+1}, \dots, \tau_n)$ in the semialgebraic C^1 topology, and $\tilde{\tau}$ is of class C^2 on $B_{3\xi}(3)$ and of class $C^{l'-1}$ on $B_\xi(1)$.

Set $Z = \{(x, y) \in B_{3\xi}(3) \times \mathbf{R}^{n'} : y^\alpha = g'(x)\}$, which is a Nash submanifold with boundary of $B_{3\xi}(3) \times \mathbf{R}^{n'}$, and let $p : W \rightarrow Z$ be a Nash projection of a semialgebraic tubular neighborhood of Z in $B_{3\xi}(3) \times \mathbf{R}^{n'}$ such that $p(x, y)$ for $(x, y) \in W$ is of the form $(x, p_2(x, y))$, which is constructed as before. Let $(\hat{F}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n)$ be a Nash approximation of $(F, \tau_{n'+1}, \dots, \tau_n)$ in the semialgebraic C^1 topology, and let ϕ and ψ be semialgebraic $C^{l'}$ functions on $B_{3\xi}(3)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_\xi(1)$, $\phi = 0$ on $B_{3\xi}(3) - B_{2\xi}(2)$, $0 \leq \psi \leq 1$, $\psi = 1$ on $B_{3\xi}(3) - V$ and $\psi = 0$ on a semialgebraic neighborhood of C in $B_{3\xi}(3)$ smaller than $\text{Int } V$. Set

$$\begin{aligned} \tilde{F}(x) &= \phi(x)p_2[x, \psi(x)\hat{F}(x) + (1 - \psi(x))F(x)] + (1 - \phi(x))F(x) \quad \text{for } x \in B_{3\xi}(3), \\ \text{and} \quad \tilde{\tau} &= (x_1\tilde{F}_1, \dots, x_{n'}\tilde{F}_{n'}, \hat{\tau}_{n'+1}, \dots, \hat{\tau}_n) \quad \text{on } B_{3\xi}(3). \end{aligned}$$

Then we see as before that the required conditions are satisfied. Hence the statement is proved.

By the statement and by a partition of unity of class semialgebraic C^l we obtain an open semialgebraic neighborhood U of X and a semialgebraic C^2 diffeomorphism τ of M close to π in the semialgebraic C^1 topology such that τ is of class C^l on U and $f \circ \tau = g$ on U (the point is that after fixing U we can choose τ so as to be arbitrarily close to id). Then it is easy to modify τ so that τ is of class semialgebraic C^l and $f \circ \tau = g$, i.e., f and g are semialgebraically C^l right equivalent as follows.

Let η be a semialgebraic C^l function on M such that $0 \leq \eta \leq 1$, $\eta = 0$ outside of U and $\eta = 1$ on a smaller semialgebraic neighborhood of X , and set $A = \{(x, y) \in (M - X)^2 : f(y) = g(x)\}$. Then A is a Nash manifold and there exists a Nash projection $q : Q \rightarrow A$ of a small semialgebraic tubular neighborhood of A in the square of the ambient Euclidean space of M of the form $q(x, y) = (x, q_2(x, y))$ for $x \in M - X$. Let $\check{\tau}$ be a Nash approximation of τ in the semialgebraic C^1 topology, and set

$$\check{\tau} = q_2[x, \eta(x)\tau(x) + (1 - \eta(x))\check{\tau}(x)] \quad \text{for } x \in M.$$

Then $\check{\tau}$ is well-defined because the graph of the map from M to the ambient Euclidean space of $M : x \rightarrow \eta(x)\tau(x) + (1 - \eta(x))\check{\tau}(x)$ is contained in Q , hence $\check{\tau}$ is a semialgebraic C^l diffeomorphism of M and $f \circ \check{\tau} = g$. Thus the former step of the proof is achieved

Let $0 \ll l^{(3)} \ll \dots \ll l \in \mathbf{N}$. For the latter step also we can assume that $X = Y$ and that there exists a semialgebraic C^l diffeomorphism π of M close to id in the semialgebraic C^l topology such that $f \circ \pi = g$. Let μ be a Nash function on \mathbf{R} such that $\mu^{-1}(0) = \text{Sing } f$ and μ is regular at $\mu^{-1}(0)$. Consider $\mu \circ f$ and $\mu \circ g$. Their zero sets are X , they have only normal crossing singularities at X , the same sign at each point of M and the same multiplicity at each point of X , and we see easily that the Nash function on M , defined to be $\mu \circ g / \mu \circ f$ on $M - X$, is close to 1 in the semialgebraic C^l topology. Hence the conditions in lemma 3.10 are satisfied and there exists a Nash diffeomorphism π' of M close to id in the semialgebraic $C^{l'}$ topology such that $\pi'(X) = X$ and $f \circ \pi' - g$ is l' -flat at X . Thus, replacing f and π with $f \circ \pi'$ and $\pi'^{-1} \circ \pi$, we assume that $f - g$ is l' -flat at X and π is close to id in the $C^{l'}$ topology.

By proposition 3.12 we can assume that M is the interior of a compact Nash manifold possibly with corners M_1 and f is the restriction to M of a Nash function f_1 on M_1 with only normal crossing singularities (see the definition of a function with only normal crossing singularities on a manifold with corners after corollary 3.3). Then by the definition of semialgebraic C^l topology, π is extensible to a semialgebraic $C^{l'}$ diffeomorphism π_1 of M_1 such that $\pi_1 - \text{id}$ is l' -flat at ∂M_1 . Hence g also is extensible to a semialgebraic $C^{l'}$ function g_1 on M_1 , and $f_1 - g_1$ is close to 0 in the $C^{l'}$ topology and l' -flat at ∂M_1 . Let v_i , $i = 1, \dots, N$, be Nash vector fields on M_1 spanning the tangent space of M_1 at each point, let ν_1 be a non-negative Nash function on M_1 with zero set ∂M_1 and regular there, and set $\nu_2 = \sum_{i=1}^N (v_i f_1)^2$ and $\nu = \nu_1^{l''} \nu_2$. Then the radical of $\nu_2 \mathcal{N}$ is the sheaf of \mathcal{N} -ideals defined by $X \cup \partial M_1$, and $f_1 - g_1$ is divisible by ν ; to be precise, there exists a semialgebraic $C^{l''}$ function β on M_1 such that $f_1 - g_1 = \nu \beta$. Moreover, β is close to 0 in the $C^{l^{(3)}}$ topology for the following reason.

As shown in step 1 of the proof of (1), the map $C^\infty(M_1) \ni h \rightarrow \nu h \in \nu C^\infty(M_1)$ is open. Hence for $h \in C^\infty(M_1)$, if νh is close to 0 in the $C^{l''}$ topology then h is close to 0 in the $C^{l^{(3)}}$ topology. This holds for $h \in C^{l''}(M_1)$ also because h of class $C^{l''}$ is approximated by a C^∞ function h' in the $C^{l''}$ topology and $\nu \cdot (h - h')$ and hence $\nu h'$ are close to 0 in the $C^{l''}$ topology. Therefore, β is close to 0 in the $C^{l^{(3)}}$ topology.

It follows from the definition of semialgebraic C^l topology that $\nu_1^{l''} \beta|_M$ is close to 0 in the semialgebraic $C^{l^{(3)}}$ topology. Then the conditions in proposition 3.11,(ii) for f and $g (= f - \nu_1^{l''} \beta \sum_{i=1}^N (v_i f_1)^2|_M)$ are satisfied. Hence f and g are Nash right equivalent. \square

3.2. Cardinality of the set of equivalence classes.

The main results of this section are theorem 3.2 and corollary 3.3.

Theorem 3.2. *Let M be a compact analytic (respectively, Nash) manifold of strictly positive dimension. Then the cardinality of analytic (resp., Nash) R-L equivalence classes of analytic (resp. Nash) functions on M with only normal crossing singularities is 0 or countable. In the Nash case we do not need to assume that M is compact, and if M is non-compact then the cardinality is always countable.*

We postpone proving theorem 3.2 until the end of this section. We reduce the C^ω case to the Nash case by proposition 3.4 and then the non-compact Nash case to the compact Nash case by proposition 3.12. Lemmas 3.6 and 3.7 together with Nash Approximation Theorem of C^ω solutions in [C-R-S₁] prove the compact Nash case. A large portion of this section is dedicated to the reduction of the non-compact case to the compact case. There the main tools are Nash sheaf theory, a Nash version of Hironaka Desingularization Theorem (theorems 2.9, 2.10) and the theorems in [S₂].

Remark. (i) The case where the cardinality is zero may appear, e.g. $M = S^2, \mathbf{P}(2)$ (for the proof, see the arguments in (v) below in case $M = \mathbf{R}^2$).

(ii) In the theorem we do not need to fix M , namely, the cardinality of equivalence classes of analytic or Nash functions on all compact analytic manifolds or Nash manifolds, respectively, with only normal crossing singularities is also countable. Indeed, the cardinality is clearly infinite, and there are only a countable number of compact analytic manifolds and (not necessarily compact) Nash manifolds up to

analytic diffeomorphism and Nash diffeomorphism, respectively, which will be clear in the proof of lemma 3.7.

(iii) Theorem 3.2 does not hold for analytic functions on a non-compact analytic manifold. To be precise, for a non-compact analytic manifold M , the cardinality of analytic R-L equivalence classes of (proper) analytic functions on M with only normal crossing singularities is of the continuum (0 or of the continuum, respectively,) as shown below.

(iv) There always exists a non-singular analytic (Nash) function on any non-compact, connected analytic (Nash, respectively,) manifold M , which we construct later.

(v) An example of non-compact M where there is no proper analytic (Nash) function with only normal crossing singularities is \mathbf{R}^2 . We see this by reduction to absurdity. Assume that there exists one f . Note that each level of f is a finite union of Jordan curves. Let $a_1 \in \mathbf{R}$ and $X_1 \subset \mathbf{R}^2$ be, respectively, a point of $\text{Im } f$ and a Jordan curve in $f^{-1}(a_1)$ whose does not intersect with $f^{-1}(a_1)$ inside U_1 . Next choose $a_2 \in f(U_1)$, a Jordan curve X_2 in $f^{-1}(a_2) \cap U_1$ and U_2 in the same way. If we continue these arguments, we arrive at a contradiction to the above note.

Proof of (iii) for proper functions. Assume that there exists a proper analytic function f on a non-compact analytic manifold M with only normal crossing singularities. Replace f with $\pi \circ f$ for some proper analytic function π on \mathbf{R} if necessary. Then we can assume that $f(\text{Sing } f) = \mathbf{N}$ because $f(\text{Sing } f)$ has no accumulating points in \mathbf{R} . Define a map $\alpha_f : \mathbf{N} \rightarrow \mathbf{N}$ so that for each $n \in \mathbf{N}$, $f - n$ is $\alpha_f(n)$ -flat at any point of $f^{-1}(n) \cap \text{Sing } f$ and not $(\alpha_f(n) + 1)$ -flat at some point of $f^{-1}(n) \cap \text{Sing } f$. If a proper analytic function g with $g(\text{Sing } g) = \mathbf{N}$ is C^ω R-L equivalent to f then $\alpha_f = \alpha_g$. Consider all proper C^ω functions π on \mathbf{R} such that $\text{Sing } \pi = \mathbf{N}$ and $\pi = \text{id}$ on \mathbf{N} . Then it is easy to see that the cardinality of $\{\alpha_{\pi \circ f}\}$ is of the continuum. Hence the cardinality of C^ω R-L equivalence classes of proper analytic functions on M with only normal crossing singularities is of the continuum. \square

Proof of (iv). Assume that $\dim M > 1$. We use the idea of handle body decomposition by Morse functions (see [Mi]). Let f be a non-negative proper C^∞ function on a non-compact connected C^ω manifold M with only singularities of Morse type. Approximate f and changing \mathbf{R} by some C^ω diffeomorphism of \mathbf{R} , we assume that f is of class C^ω , $f|_{\text{Sing } f}$ is injective and $f(\text{Sing } f) = 2\mathbf{N}$. For each $k \in \mathbf{N}$, let A_k be the union of $f^{-1}(k) \cap \text{Sing } f$ and one point in each connected component of $f^{-1}(k)$ not containing points of $\text{Sing } f$. Consider the 1-dimensional simplicial complex K whose 0-skeleton K^0 is $\cup_{k \in \mathbf{N}} A_k$ and whose 1-skeleton K^1 consists of 1-simplexes \overline{ab} , $a, b \in K^0$, such that $f(\{a, b\}) = \{k, k + 1\}$ for some $k \in \mathbf{N}$ and there exists a connected component C of $f^{-1}((k, k + 1))$ with $\overline{C} \ni a, b$. Note that such C is unique because $f_{f^{-1}((2k', 2k' + 2))} : f^{-1}((2k', 2k' + 2)) \rightarrow (2k', 2k' + 2)$ is a proper submersion for $k' \in \mathbf{N}$ and that conversely for each connected component C of $f^{-1}((k, k + 1))$ there exist $a, b \in K^0$ such that $f(\{a, b\}) = \{k, k + 1\}$ and $\overline{C} \ni a, b$. In other words we can identify K^1 as $\{\text{connected components of } f^{-1}((k, k + 1)) : k \in \mathbf{N}\}$. Moreover, for $\overline{ab} \in K^1$ there exist an injective C^ω map $l_{a,b} : [0, 1] \rightarrow M$ with $l_{a,b}(0) = a$, $l_{a,b}(1) = b$, $f \circ l_{a,b}(t) = f(a) \pm t$ and $\text{Im } l_{a,b} = \text{Im } l_{b,a}$. Here for $\overline{ab} \neq \overline{a'b'}$, $\text{Im } l_{a,b} \cap \text{Im } l_{a',b'} = \{a\}$ if $a = a'$ or $a = b'$, $\text{Im } l_{a,b} \cap \text{Im } l_{a',b'} = \{b\}$ if $b = a'$ or $b = b'$, and $\text{Im } l_{a,b} \cap \text{Im } l_{a',b'} = \emptyset$ otherwise. Hence we identify the underlying polyhedron $|K|$ with the subset $\cup_{\overline{ab} \in K^1} \text{Im } l_{a,b}$ of

M , i.e., K is realized in M . Note that also there exists an unique C^0 retraction $r : M \rightarrow \bigcup_{ab \in K^1} \text{Im } l_{a,b}$ such that $f \circ r = f$.

We will see that each $a \in K^0$ is the end of some half-polygon in $|K|$, i.e., there exist distinct $a_0 = a, a_1, a_2, \dots \in K^0$ such that $\overline{a_i a_{i+1}} \in K^1, i \in \mathbf{N}$. Note that $a_i \rightarrow \infty$ (i.e., $f(a_i) \rightarrow \infty$) as $i \rightarrow \infty$. Since M is non-compact and connected, there exists a proper C^1 map $l : [0, \infty) \rightarrow M$ such that $l(0) = a$. We can move $\text{Im } l$ into $|K|$ by r so that $\text{Im } l$ is the underlying polyhedron of some subcomplex of K . If there is a 1-simplex s in $K_l \stackrel{\text{def}}{=} K|_{\text{Im } l}$ with an end v such that v is not $l(0)$ nor an end of another 1-simplex in K_l , then remove s and v from K_l , and repeat this operation as many times as possible. Then K_l becomes a simplicial subcomplex of K and $|K_l|$ is the union of a half-polygon and Jordan curves. Remove, moreover, some vertices except $l(0)$ and 1-simplexes so that $|K_l|$ is a half-polygon. Then we obtain an injective simplicial map $l : \tilde{\mathbf{N}} \rightarrow K$ with $l(0) = a$, where $\tilde{\mathbf{N}} = \mathbf{N} \cup \{[i, i+1] : i \in \mathbf{N}\}$. Let L_a denote all of such l , and let l_a be such that $\min f \circ l_a = \max\{\min f \circ l : l \in L_a\}$ and $\#(f \circ l_a)^{-1}(\min f \circ l_a) \leq \#(f \circ l)^{-1}(\min f \circ l)$ for $l \in L_a$ with $\min f \circ l = \min f \circ l_a$.

Next we show that $\min f \circ l_a \rightarrow \infty$ as $a \rightarrow \infty$. Otherwise, there would exist distinct a_1, a_2, \dots in K^0 such that $\min f \circ l_{a_i} = \text{const}$, say m . Note that $a_i \rightarrow \infty$ as $i \rightarrow \infty$. Since $f^{-1}(m)$ is compact we have a subsequence of a_1, a_2, \dots where $\text{Im } l_{a_i}$ contain one point $b_0 \in K^0$ with $f(b_0) = m$. Next, choose a subsequence so that $\text{Im } l_{a_i}$ contain $\overline{b_0 b_1} \in K^0$ for some $b_1 \in K^0$ and $l_{a_i}(k_i + 1) = b_0$ and $l_{a_i}(k_i) = b_1$ for some $k_i \in \mathbf{N}$. Repeating these arguments we obtain sequences a_1, a_2, \dots and b_0, b_1, \dots in K^0 such that $l_{a_i}(k_i + i) = b_0, \dots, l_{a_i}(k_i) = b_i$ for some $k_i \in \mathbf{N}, i = 1, 2, \dots$. Then $\bigcup_{i \in \mathbf{N}} \overline{b_i b_{i+1}}$ is a half-polygon by the property of l_a . Fix i so large that $f(b_j) > m, j = i, i+1, \dots$, consider a polyhedron $l_{a_i}([0, k_i]) \cup \overline{b_i b_{i+1}} \cup \overline{b_{i+1} b_{i+2}} \cup \dots$ and remove vertices and open 1-simplexes from it as above construction of l so that the polyhedron becomes a half-polygon starting from a_i . This half-polygon defines a new $l \in L_{a_i}$. Clearly $\min f \circ l \geq m = \min f \circ l_{a_i}$ for this l by the definition of l . However, $\min f \circ l = \min f \circ l_{a_i}$ by the definition of l_{a_i} . Then the difference between this l and l_{a_i} is $\#(f \circ l)^{-1}(m) < \#(f \circ l_{a_i})^{-1}(m)$ since $f \circ l_{a_i}(k_i + 1) = m, f \circ l_{a_i}([0, k_i]) \supset f \circ l([0, k_i])$ and since $f(\overline{b_i b_{i+1}} \cup \overline{b_{i+1} b_{i+2}} \cup \dots) > m$, which contradicts the definition of l_{a_i} . Thus $\min f \circ l_a \rightarrow \infty$ as $a \rightarrow \infty$.

Note that M is C^∞ diffeomorphic to $M - \text{Im } l_a$ for any $a \in K^0$ and the diffeomorphism can be chosen to be id outside of a small neighborhood of $\text{Im } l_a$. Hence if $\text{Im } l_a \cap \text{Im } l_{a'} = \emptyset$ for any $a \neq a' \in \text{Sing } f$, there exists a C^ω diffeomorphism $\pi : M \rightarrow M - \bigcup_{a \in \text{Sing } f} \text{Im } l_a$ and $f \circ \pi$ is a non-singular analytic function on M . Consider the case of $\text{Im } l_a \cap \text{Im } l_{a'} \neq \emptyset$ for some $a \neq a' \in \text{Sing } f$. Set $\{a_0, a_1, \dots\} = \text{Sing } f, X_0 = Y_0 = \text{Im } l_{a_0}$ and $Z_0 = \emptyset$. Let $i \in \mathbf{N}$. Assume by induction that we have defined subpolyhedra $X_i \supset Y_i \supset Z_i$ of $|K|$. If $X_i \cap \text{Im } l_{a_{i+1}} = \emptyset$, set $X_{i+1} = X_i \cup \text{Im } l_{a_{i+1}}, Y_{i+1} = \text{Im } l_{a_{i+1}}$ and $Z_{i+1} = \emptyset$. Otherwise, set $X_{i+1} = X_i \cup \text{Im } l_{a_{i+1}}([0, k_{i+1}]), Z_{i+1} =$ the closure of the unbounded connected component of the set of difference of the connected component of X_i containing $l_{a_{i+1}}(k_{i+1})$ and of $l_{a_{i+1}}(k_{i+1})$, and set $Y_{i+1} = Z_{i+1} \cup \text{Im } l_{a_{i+1}}([0, k_{i+1}])$, where $k_{i+1} = \min\{k \in \mathbf{N} : X_i \cap l_{a_{i+1}}([0, k]) \neq \emptyset\}$. Then $X = \bigcup_{i \in \mathbf{N}} X_i$ is the underlying polyhedron of a subcomplex of K , and for each i there exists a C^∞ diffeomorphism $\pi_i : M - Z_i \rightarrow M - Y_i$ such that $\pi_i = \text{id}$ on $X_i - Z_i$ and outside of a small neighborhood of $Y_i - Z_i$ in $M - Z_i$. By the property $\min f \circ l_a \rightarrow \infty$ as $a \rightarrow \infty$, we see that $\dots \circ \pi_1 \circ \pi_0 : M \rightarrow M$ is a well-defined C^∞ diffeomorphism to $M - X$. Approximate it by a C^ω diffeomorphism $\pi : M \rightarrow M - X$. Then $f \circ \pi$ is

the required non-singular analytic function on M .

Consider the case where M is a non-compact connected Nash manifold. Then there exists a proper Nash function on M with only singularities of Morse type for the following reason. By Theorem VI.2.1, [S₂] M is Nash diffeomorphic to the interior of a compact Nash manifold with boundary M' , which is called a *compactification* of M . Then by using a partition of unity of class semialgebraic C^2 we obtain a nonnegative semialgebraic C^2 function ϕ on M' with zero set $\partial M'$ and with only singularities of Morse type. Approximate the semialgebraic C^2 function $1/\phi$ on M by a Nash function ψ in the semialgebraic C^2 topology (Approximation Theorem), which is the required function. Note that $\#\text{Sing } \psi < \infty$ because $\text{Sing } \psi$ is semialgebraic. Hence in the same way as in the analytic case, we find a Nash function on M without singularities by the following fact.

Let X be a 1-dimensional closed semialgebraic connected subset of M which is a union of smooth curves X_0, \dots, X_k such that any X_i is closed in M , any X_i and X_j intersect transversally and for each $a \in X$ there exists one and only one path from a to ∞ in X . Then M and $M - X$ are Nash diffeomorphic.

We prove this fact as follows. Assume that $M = \text{Int } M'$ for M' as above. Then the closure \overline{X} of X in M' intersects with $\partial M'$ at one point. By moving X by a semialgebraic C^1 diffeomorphism of M and then by a Nash diffeomorphism (Approximation Theorem) we assume that \overline{X} is smooth at $\overline{X} \cap \partial M'$ and \overline{X} and $\partial M'$ intersect transversally. Let ξ denote the function on M' which measures distance from \overline{X} , which is semialgebraic, and approximate $\xi|_{M' - \overline{X}}$ by a positive Nash function $\tilde{\xi}$ on $M' - \overline{X}$ so that $\tilde{\xi}(x) \rightarrow 0$ as $x \rightarrow$ a point of \overline{X} . Let $\epsilon > 0$ be small enough. Then $\tilde{\xi}|_{\tilde{\xi}^{-1}((0, \epsilon])} : \tilde{\xi}^{-1}((0, \epsilon]) \rightarrow (0, \epsilon]$ is a proper Nash submersion by [C-S₁]. Hence $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon])$ and $M' - \overline{X}$ are semialgebraically C^1 diffeomorphic and, moreover, Nash diffeomorphic by Approximation Theorem. On the other hand, $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon])$ is a compact Nash manifold with corners, and if we smooth the corners then $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon])$ is C^∞ and hence Nash (Theorem VI.2.2, [S₂]) diffeomorphic to M' by the assumptions on X , which implies that $M - X - \tilde{\xi}^{-1}((0, \epsilon])$ and M are Nash diffeomorphic. Therefore, $M - X = \text{Int}(M' - \overline{X})$ is Nash diffeomorphic to M . \square

Proof of (iii) for non-proper functions. As in the situation above, we have a non-bounded non-singular non-negative analytic function f on a non-compact connected analytic manifold M . Let π be a proper analytic function on \mathbf{R} such that $\text{Sing } \pi = \mathbf{N}$ and $\pi = \text{id}$ on \mathbf{N} . Then $\pi \circ f(\text{Sing } \pi \circ f) = \mathbf{N}$. Hence as in the case of proper functions, we see that the cardinality of C^ω R-L equivalence classes of analytic functions on M with only normal crossing singularities is of the continuum. \square

Let P denote the set of homogeneous polynomial functions on \mathbf{R}^2 of degree 4. It is easy to see that the analytic R-L equivalence classes of P has the cardinality of the continuum, almost blow-analytic R-L equivalence on analytic functions on \mathbf{R}^2 coincides with blow-analytic R-L equivalence and is an equivalence relation, and the blow-analytic R-L equivalence classes of P is finite. For general dimension, as noted in the remark of definition 1.1 we do not know whether almost blow-analytic (blow-Nash) R-L equivalence is an equivalence relation. So we say that analytic (Nash) functions f and g lie in the *same class* if there exists a sequence of analytic functions f_0, \dots, f_k such that $f_0 = f$, $f_k = g$ and f_i and f_{i+1} are almost blow-analytically (blow-Nash) R-L equivalent, $i = 0, \dots, k - 1$. Then

Corollary 3.3. *Let M be a compact analytic (resp. Nash) manifold of strictly*

positive dimension. Then the cardinality of the set of classes of analytic (resp. Nash) functions on M , classified by almost blow-analytic (resp. blow-Nash) R - L equivalence, is countable. In the Nash case we do not need to assume that M is compact.

Proof of corollary 3.3. The analytic case follows from theorem 3.2 and Hironaka Desingularization Theorem. The compact Nash case does from theorems 2.10 and 3.2. \square

For the proof of theorem 3.2 in the non-compact case we need to extend the notions of singularity and of normal crossing singularity. Let f be an analytic function on analytic manifold with corners M . We say f is *singular* at a point x_0 of ∂M if the restriction of f to the stratum of the canonical stratification of ∂M containing x_0 is singular at x_0 . (Hence f is singular at points of the stratum of dimension 0 of the canonical stratification of ∂M). We say f has *only normal crossing singularities* if $f|_{\text{Int } M}$ does so and the following condition is satisfied. We can construct an analytic manifold M' which contains M and is of the same dimension by extending a locally finite system of analytic local coordinate neighborhoods of M . (We call M' an *analytic manifold extension* of M . We naturally define also a *Nash manifold extension* of a Nash manifold with corners.) In the same way, shrinking M' if necessary we obtain a normal crossing analytic subset X of M' such that $\text{Int } M$ is a union of some connected components of $M' - X$, and f is extended to an analytic function f' on M' . Let ϕ be an analytic function on M' defined by X . Then the condition is that the germ of $(f' - f'(x_0))\phi$ at each point x_0 of X has only normal crossing singularities. We naturally define also a *normal crossing analytic* (Nash) subset of M and a *normal crossing sheaf* of \mathcal{O} - (\mathcal{N} -)ideals on M .

By the following proposition we reduce the C^ω case of theorem 3.2 to the Nash case.

Proposition 3.4. *Let M be a compact Nash manifold possibly with corners, and f a C^ω function on M with only normal crossing singularities. Then f is C^ω right equivalent to some Nash function.*

Remark. If M is a non-compact Nash manifold, proposition 3.4 does not hold. For example, $M = \mathbf{R}$ and $f(x) = \sin x$.

Proof of proposition 3.4. Set $X = f^{-1}(f(\text{Sing } f))$. Let $g : \tilde{X} \rightarrow M$ be a C^ω immersion of a compact C^ω manifold possibly with corners such that $\text{Im } g = X$, $g|_{g^{-1}(\text{Reg } X)}$ is injective and for each $x \in \tilde{X}$, $g_x(\tilde{X}_x)$ is an analytic subset germ of $M_{g(x)}$ (we construct \tilde{X} and g locally and then paste them). For a connected component C of \tilde{X} there are 2 possible cases to consider: $g(C) \subset \partial M$ or $g(C) \not\subset \partial M$. Assume that $g(C) \not\subset \partial M$ for any C . Then $g(\text{Int } \tilde{X}) \subset \text{Int } M$, $g(\partial \tilde{X}) \subset \partial M$ and X is a normal crossing analytic subset of M . Consider the family of all C^∞ maps $g' : \tilde{X} \rightarrow M$ with $g'(\text{Int } \tilde{X}) \subset \text{Int } M$ and $g'(\partial \tilde{X}) \subset \partial M$. Then

Lemma 3.5. *Let $r (> 0) \in \mathbf{N} \cup \{\infty\}$. Then g is C^∞ stable in the family in the sense that any such C^∞ map $g' : \tilde{X} \rightarrow M$ close to g in the C^r topology is C^∞ R - L equivalent to g .*

Remark. Even if M is a non-compact Nash (C^∞) manifold possibly with corners, lemma 3.5 holds in the Whitney C^r (strong Whitney C^∞ , respectively,) topology, which is clear by the following proof.

Proof of lemma 3.5. It suffices to find a C^∞ diffeomorphism of M which carries $\text{Im } g$ to $\text{Im } g'$. As usual, using a tubular neighborhood of M in its ambient Euclidean space, the orthogonal projection to M and a partition of unity of class C^∞ , we reduce the problem to the following local one.

Assume that $M = \mathbf{R}^n \times [0, \infty)^m$ and $X = \{x_1 \cdots x_l = 0\}$, $l \leq n$, and let $y_1 = y_1(x)$ be a C^∞ function on M which is close to the function x_1 in the Whitney C^r topology and coincides with x_1 outside of a neighborhood of 0. Then there exists a C^∞ diffeomorphism π of M which is id outside of a neighborhood of 0 and close to id in the Whitney C^r topology and carries $\{x_2 \cdots x_l = 0\} \cup \{y_1(x) = 0\}$ to X .

This is true since $\pi(x_1, \dots, x_{n+m}) = (y_1(x), x_2, \dots, x_{n+m})$ satisfies the requirements. \square

Continued proof of proposition 3.4. Let $0 \ll r' \ll r \in \mathbf{N}$. Case without corners. Give a Nash manifold structure to \tilde{X} (Theorem of Nash, see Theorem I.3.6, [S₂]). Let $g' : \tilde{X} \rightarrow M$ be a Nash approximation of g in the C^r topology, e.g., the composite of a polynomial approximation of the map g from \tilde{X} to the ambient Euclidean space of M and the orthogonal projection of a tubular neighborhood of the space, and set $X' = \text{Im } g'$. Then by lemma 3.5 there exists a C^∞ diffeomorphism π of M which carries X to X' , and by the above proof π can be arbitrarily close to the identity map in the C^r topology. Let t_1, \dots, t_l be the critical values of f . We assume that $t_i > 0$. We want to construct a Nash function f' on M such that $(f')^{-1}(f'(\text{Sing } f')) = X'$, $f' \circ \pi = f$ on X for some modified π and, moreover, $f' \circ \pi - t_i$ and $f - t_i$ have the same multiplicity at each point of $f^{-1}(t_i)$ for each i . For each t_i , let \mathcal{I}_i denote the sheaf of \mathcal{N} -ideals with zero set $\pi(f^{-1}(t_i))$ and having the same multiplicity (in the natural sense) as $f \circ \pi^{-1} - t_i$ at each point of $\pi(f^{-1}(t_i))$, which exists because if a non-singular analytic set germ is semialgebraic then it is a non-singular Nash set germ. Then \mathcal{I}_i is generated by a finite number of global Nash functions (theorem 2.7). Let ϕ_i denote the square sum of the generators and define a Nash function ψ_i on M so that $\psi_i^2 = \phi_i$ and ψ_i and $f \circ \pi^{-1} - t_i$ have the same sign everywhere. Note that $\psi_i^{-1}(0) = \pi(f^{-1}(t_i))$ and ψ_i and $f \circ \pi^{-1} - t_i$ have the same multiplicity at each point of $\psi_i^{-1}(0)$. Set $\phi = \prod \phi_i$. We have a global cross-section of the sheaf of \mathcal{N} -modules $\mathcal{N} / \prod_i \mathcal{I}_i^2$ whose value at each point x of $\psi_i^{-1}(0)$ equals $\psi_{ix} + t_i \text{ mod } \mathcal{I}_{ix}^2$. Apply theorem 2.8 to the homomorphism $\mathcal{N} \rightarrow \mathcal{N} / \prod_i \mathcal{I}_i^2$ and the global cross-section. Then there exists a Nash function ψ on M such that for each i , $\psi - t_i$ and $f \circ \pi^{-1} - t_i$ have the same sign at each point of a neighborhood of $\psi_i^{-1}(0)$ and the same multiplicity at each point of $\psi_i^{-1}(0)$. We need to modify ψ so that $X' = \psi^{-1}(\psi(\text{Sing } \psi))$. Let f'' be a C^∞ function on M such that $f'' = \psi$ on a small neighborhood of X' and $X' = (f'')^{-1}(f''(\text{Sing } f''))$, which is constructed by ψ , $f \circ \pi^{-1}$ and by a partition of unity of class C^∞ . Then $f'' - \psi$ is of the form $\phi \xi$ for some C^∞ function ξ on M . Set $f' = \psi + \phi \cdot$ (a strong Nash approximation of ξ in the C^∞ topology). Then f' is a Nash function, $X' = (f')^{-1}(f'(\text{Sing } f'))$ and $f' - t_i$ and $f \circ \pi^{-1} - t_i$ have the same multiplicity at each point of $\pi(f^{-1}(t_i))$ for each i .

By Theorem 3.1,(1) it suffices to see that f and $h \stackrel{\text{def}}{=} f' \circ \pi$ are C^∞ right equivalent. Note that $h^{-1}(h(\text{Sing } h)) = X'$, and $f - t_i$ and $h - t_i$ have the same multiplicity at each point of $f^{-1}(t_i)$ for each i . Remember that π is close to id in the C^r topology. We can choose f' also so that f and h are close each other in the C^r topology. Indeed, $f \circ \pi^{-1} - f'$ is of the form $\eta \prod_i \psi_i$ for some C^∞ function η on M . Replace f' with $f' + (\text{a strong Nash approximation of } \eta) \prod_i \psi_i$. Then f and

h are close. Hence we can reduce the problem, as usual, to the local one:

Let $M = \mathbf{R}^n$, $f(x) = x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ and $h(x) = a(x)x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ for some C^∞ function $a(x)$ on M close to 1 in the Whitney $C^{r'}$ topology (remember the second remark in the proof of theorem 3.1,(1)). Assume that $\alpha_1 > 0$. Then f and h are C^∞ right equivalent through a C^∞ diffeomorphism close to id in the Whitney $C^{r'}$ topology.

This is true since the C^∞ diffeomorphism $\mathbf{R}^n \ni (x_1, \dots, x_n) \rightarrow (a^{1/\alpha_1}(x)x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ satisfies the requirements. Thus the case without corners is proved.

Case with corners. Let M_1 be a Nash manifold extension of M . We can assume that M is the closure of a union of some connected components of $M_1 - Y$ for a normal crossing Nash subset Y of M_1 . Let U be an open semialgebraic neighborhood of M in M_1 so small that f is extensible to an analytic function f_1 on U with only normal crossing singularities. Shrinking U if necessary, we replace X in the above proof with $X_1 = f_1^{-1}(f_1(\text{Sing } f_1))$, define a C^ω manifold \tilde{X}_1 and a C^ω immersion $g_1 : \tilde{X}_1 \rightarrow U$ in the same way. For each connected component C of \tilde{X}_1 there are two possible cases to consider: $g_1(C) \subset Y$ or $g_1(C) \not\subset Y$. If $g_1(C) \subset Y$, C has locally a Nash manifold structure such that $g_1|_C$ is a local Nash diffeomorphism to $g_1(C)$. Moreover, C with this structure is a Nash manifold by fact 1 in the proof of lemma 3.8. Set $g'_1 = g_1$ on C . If $g_1(C) \not\subset Y$, give a Nash manifold structure to C , approximate $g_1|_C$ by a Nash immersion $g'_1|_C : C \rightarrow M_1$. In this way we define a Nash immersion $g'_1 : \tilde{X}_1 \rightarrow M_1$ and set $X' = \text{Im } g'_1 \cap M$. The rest proceeds in the same way as the case without corners. \square

The following lemma is the C^ω or Nash version of lemma 3.5 and is used to prove theorem 3.1 and lemma 3.7.

Lemma 3.6. *Let $r (> 0) \in \mathbf{N} \cup \{\infty\}$. Let M and N be compact C^ω manifolds possibly with corners such that $\dim M = 1 + \dim N$, and $\phi : N \rightarrow M$ a C^ω immersion such that $\phi(\text{Int } N) \subset \text{Int } M$, $\phi(\partial N) \subset \partial M$, $\text{Im } \phi$ is a normal crossing analytic subset of M and the restriction of ϕ to $\phi^{-1}(\text{Reg Im } \phi)$ is injective. Then ϕ is C^ω stable in the family of C^ω maps from N to M carrying ∂N to ∂M in the same sense as in lemma 3.5. If M , N and ϕ are of class Nash, ϕ is Nash stable in the family of Nash maps with the same property as above.*

Remark. In the case of a non-compact M and proper ϕ , we see easily that the former half part of lemma 3.6 holds in the Whitney C^r topology, $r (> 0) \in \mathbf{N} \cup \{\infty\}$. We can prove the latter half in the non-compact case in the semialgebraic C^r topology by reducing to the compact case by lemmas 3.8 and 3.9.

Proof of lemma 3.6. Let ψ be an analytic approximation of ϕ in family in the analytic case. Then by lemma 3.5 ψ is C^∞ R-L equivalent to ϕ , namely, there exists a C^∞ diffeomorphism π of M which carries $\text{Im } \phi$ to $\text{Im } \psi$. Note that we can choose π so as to be close to id in the C^r topology by the proof of lemma 3.5. Then by step 1 in the proof of theorem 3.1,(1) and its proof we can choose an analytic π even in the case with corners. the existence of an analytic diffeomorphism τ of N with $\psi \circ \tau = \pi \circ \phi$ is clear because $\tau = \psi^{-1} \circ \pi \circ \phi$ on $\phi^{-1}(\text{Reg Im } \phi)$. Thus ϕ and ψ are C^ω R-L equivalent.

Assume that M , N , ϕ and ψ are of class Nash. It suffices to find a Nash diffeomorphism of M which carries $\text{Im } \phi$ to $\text{Im } \psi$. Let π be such a diffeomorphism of M of class C^ω .

Case without corners. Let \mathcal{I}_ϕ and \mathcal{I}_ψ denote the sheaves of \mathcal{N} -ideals on M defined by $\text{Im } \phi$ and $\text{Im } \psi$, respectively, and let $\{f_i\}$ and $\{g_j\}$ be a finite number of their respective global generators. Then $\{g_j \circ \pi\}$ is a set of global generators of the sheaf of \mathcal{O} -ideals $\mathcal{I}_\phi \mathcal{O}$ on M . Hence in the same way as step 2 in the proof of theorem 3.1,(1) we obtain C^ω functions $\alpha_{i,j}$ and $\beta_{i,j}$ on M such that

$$f_i = \sum_j \alpha_{i,j} \cdot (g_j \circ \pi) \quad \text{and} \quad g_j \circ \pi = \sum_i \beta_{i,j} f_i.$$

Let $M \subset \mathbf{R}^n$ and h a Nash function on \mathbf{R}^n with zero set M . Extend g_j to Nash functions on \mathbf{R}^n and use the same notation g_j theorem 2.8. Consider equations of Nash functions in variables $(x, y, a_{i,j}, b_{i,j}) \in M \times \mathbf{R}^n \times \mathbf{R}^{n'} \times \mathbf{R}^{n'}$ for some n' :

$$h(y) = 0, \quad f_i(x) - \sum_j a_{i,j} g_j(y) = 0, \quad g_j(y) - \sum_i b_{i,j} f_i(x) = 0.$$

We have a C^ω solution $y = \pi(x)$, $a_{i,j} = \alpha_{i,j}(x)$ and $b_{i,j} = \beta_{i,j}(x)$. Hence by Theorem 1.1, [C-R-S₁] there exists a Nash solution $y = \pi'(x)$, $a_{i,j} = \alpha'_{i,j}(x)$ and $b_{i,j} = \beta'_{i,j}(x)$, which are close to π , $\alpha_{i,j}$ and $\beta_{i,j}$, respectively. Then

$$\text{Im } \pi' = M, \quad f_i = \sum_j \alpha'_{i,j} \cdot (g_j \circ \pi'), \quad g_j \circ \pi' = \sum_i \beta_{i,j} f_i.$$

Hence π' is a Nash diffeomorphism of M and carries $\text{Im } \phi$ to $\text{Im } \psi$.

Case with corners. We can assume that for Nash manifold extensions M_1 and N_1 of M and N , respectively, ϕ, ψ and π are extensible to Nash immersion ϕ_1 and ψ_1 of N_1 into M_1 and to a C^ω imbedding π_1 of a semialgebraic open neighborhood U of M in M_1 into M_1 , respectively, and there exist normal crossing Nash subsets Y of M_1 and Z of N_1 such that M and N are closures of unions of some connected components of $M_1 - Y$ and of $N_1 - Z$, respectively, $\phi_1(Z) \subset Y$ and $\psi_1(Z) \subset Y$. Let M_1 be contained and closed in an open semialgebraic set O in \mathbf{R}^n , and h_1 a Nash function on O with zero set M_1 . Take a small open semialgebraic neighborhood V of M in M_1 and shrink M_1, N_1 and U so that $\pi(U) \subset V$ and $U \cap \text{Im } \phi_1$ and $V \cap \text{Im } \psi_1$ are normal crossing Nash subsets of U and V , respectively. Then in the same way as above we obtain a finite number of global generators $\{f_{1,i}\}$ and $\{g_{1,j}\}$ of the sheaves of \mathcal{N} -ideals on U and V defined by $U \cap \text{Im } \phi_1$ and $V \cap \text{Im } \psi_1$, respectively, and analytic functions $\alpha_{1,i,j}, \beta_{1,i,j}$ on U such that

$$f_{1,i} = \sum_j \alpha_{1,i,j} \cdot (g_{1,j} \circ \pi_1) \quad \text{and} \quad g_{1,j} \circ \pi_1 = \sum_i \beta_{1,i,j} f_{1,i} \quad \text{on } U.$$

We need to describe the condition $\pi(\partial M) = \partial M$, i.e., $\pi_1(U \cap Y) \subset Y$. Let ξ' be the square sum of a finite number of global generators of the sheaf of \mathcal{N} -ideals \mathcal{I} on M_1 defined by Y . Then ξ' is a generator of \mathcal{I}^2 , and since M is a manifold with corners there exists an unique Nash function ξ on a semialgebraic neighborhood of M in M_1 such that $\xi^2 = \xi'$ and $\xi > 0$ on $\text{Int } M$. Shrink U once more. Then $\xi|_U$ and $\xi \circ \pi_1|_U$ are well-defined generators of $\mathcal{I}|_U$ and $\mathcal{I}\mathcal{O}|_U$, respectively, and we have a positive analytic function γ on U such that $\xi \circ \pi = \gamma\xi$ on U . We shrink O , and using the same notation we extend $g_{1,j}$ and ξ to Nash functions on O .

Consider the germs on $M \times O \times \mathbf{R}^{n'} \times \mathbf{R}^{n'} \times \mathbf{R}$ of equations of Nash functions in variables $(x, y, a_{i,j}, b_{i,j}, c) \in U \times O \times \mathbf{R}^{n'} \times \mathbf{R}^{n'} \times \mathbf{R}$ for some n' :

$$h_1(y) = 0, f_{1,i}(x) - \sum_j a_{i,j} g_{1,j}(y) = 0, g_{1,j}(y) - \sum_i b_{i,j} f_{1,i}(x) = 0, \xi(y) - c\xi(x) = 0.$$

Then, since Theorem 1.1, [C-R-S₁] holds in the case of germs, we have Nash germ solutions on M of the equations $y = \pi'_1(x)$, $a_{i,j} = \alpha'_{i,j}(x)$, $b_{i,j} = \beta'_{i,j}(x)$ and $c = \gamma'(x)$. The equation $\xi \circ \pi'_1 = \gamma'\xi$ means $\pi'_1(M) = M$. Thus $\pi'_1|_M$ is the required Nash diffeomorphism of M . \square

The following lemma shows countable cardinality of the normal crossing Nash (C^ω) subsets of a compact Nash (C^ω , respectively,) manifold possibly with corners.

Lemma 3.7. *Let M be a compact Nash manifold possibly with corners of strictly positive dimension. Consider Nash immersions ϕ from compact Nash manifolds possibly with corners of dimension equal to $\dim M - 1$ into M such that $\text{Im } \phi$ are normal crossing Nash subsets of M , $\phi|_{\phi^{-1}(\text{Reg } \text{Im } \phi)}$ are injective and ϕ carry the interior and the corners into the interior and the corners, respectively. Then the cardinality of Nash R-L equivalence classes of all the ϕ 's is countable. The analytic case also holds.*

Proof of lemma 3.7. Note that the cardinality is infinite because for any $k \in \mathbf{N}$ we can imbed k copies of a sphere of dimension $\dim M - 1$ in M . It suffices to treat only the Nash case for the following reason.

Let $\phi : M' \rightarrow M$ be an analytic immersion as in lemma 3.7 for analytic M' and M . Assume that M has no corners. Since a compact analytic manifold is C^ω diffeomorphic to a Nash manifold, we suppose that M' and M are Nash manifolds. Approximate ϕ by a Nash map. Then ϕ is C^ω R-L equivalent to the approximation by lemma 3.6. Hence we can replace ϕ by a Nash map.

Assume that M has corners. Let $M_1 \subset \mathbf{R}^n$ be an analytic manifold extension of M such that M is the closure of a union of some connected components of $M_1 - Y$ for a normal crossing analytic subset Y of M_1 . We can assume that M_1 is compact as follows. Let α denote the function on M_1 which measures distance from M . Approximate $\alpha|_{M_1 - M}$ by a C^ω function α' in the Whitney C^∞ topology, and let $\epsilon < \epsilon'$ be positive numbers so small that $M \cup \alpha'^{-1}((0, \epsilon'])$ is compact and such that the restrictions of α' to $\alpha'^{-1}((\epsilon, \epsilon'))$ and to its intersections with strata of the canonical stratification of Y are regular. Then $(M_1 \cap \alpha'^{-1}((\epsilon, \epsilon')), Y \cap \alpha'^{-1}((\epsilon, \epsilon')))$ is C^ω diffeomorphic to $(M_1 \cap \alpha'^{-1}((\epsilon + \epsilon')/2)), Y \cap \alpha'^{-1}((\epsilon + \epsilon')/2)) \times (\epsilon, \epsilon')$. Hence, replacing M_1 with the double of $M \cup \alpha'^{-1}((0, (\epsilon + \epsilon')/2])$, we assume that M_1 is compact. Next we reduce the problem to the case where M_1 and Y are of class Nash. Define, as in the proof of proposition 3.4, a C^ω immersion $g : \tilde{Y} \rightarrow M_1$ of a compact C^ω manifold so that $\text{Im } g = Y$, $g|_{g^{-1}(\text{Reg } \text{Im } g)}$ is injective and for each $y \in \tilde{Y}$, $g_y(\tilde{Y}_y)$ is an analytic subset germ of $M_{1g(y)}$, give Nash structures on M_1 and \tilde{Y} , and approximate g by a Nash map g' . Then by lemma 3.6 there exists a C^ω diffeomorphism π of M_1 which carries $\text{Im } g$ to $\text{Im } g'$. Hence we can replace Y with $\text{Im } g'$ and we assume from the beginning that M_1 , Y and M are of class Nash. By the same reason we suppose that M' is a Nash manifold possibly with corners and the closure of a union of some connected components of $M'_1 - Y'$ for a compact Nash manifold extension M'_1 of M' and a normal crossing Nash subset Y' of M'_1 .

Extend ϕ to a C^ω immersion ϕ_1 of a compact semialgebraic neighborhood U of M' in M'_1 into M_1 , choose U so small that $\phi_1(U \cap Y') \subset Y$, and approximate, as in the proof of step 1 in theorem 3.1,(1), ϕ_1 by a Nash map $\tilde{\phi}_1$ so that $\tilde{\phi}_1(U \cap Y') \subset Y$ (here we use theorems 2.7 and 2.8 in place of corollaries 2.2 and 2.4 in the proof in theorem 3.1,(1)). Then $\tilde{\phi}_1|_{M'}$ is a Nash immersion into M , $\text{Im } \tilde{\phi}_1|_{M'}$ is a normal crossing Nash subset of M , $\tilde{\phi}_1|_{(\tilde{\phi}_1|_{M'})^{-1}(\text{Reg Im } \tilde{\phi}_1|_{M'})}$ is injective, $\tilde{\phi}_1(\partial M') \subset \partial M$, and $\tilde{\phi}_1|_{M'}$ is C^ω R-L equivalent to ϕ by lemma 3.6. Thus we reduce the analytic case to the Nash one.

Consider the Nash case. Let $M \subset \mathbf{R}^n$ and $\phi : M' \rightarrow M$ be a Nash immersion as in lemma 3.7. Let M_1, M'_1, Y and $\phi_1 : M'_1 \rightarrow M_1$ be Nash manifold extensions of M and M' , a normal crossing Nash subset of M_1 and a Nash immersion, respectively, such that $M_1 \subset \mathbf{R}^n$, $\phi_1 = \phi$ on M' , M and M' are the closures of unions of some connected components of $M_1 - Y$ and $M'_1 - \phi_1^{-1}(Y)$, respectively, $U \cap \text{Im } \phi_1$ is a normal crossing Nash subset of an open semialgebraic neighborhood U of M in M_1 and $\phi_1|_{\phi_1^{-1}(\text{Reg}(U \cap \text{Im } \phi_1))}$ is injective. By Artin-Mazur Theorem (see the proof of theorem 2.9) we can regard M'_1 as an open semialgebraic subset of the regular point set of an algebraic variety in $\mathbf{R}^n \times \mathbf{R}^{n'}$ for some n' and ϕ_1 as the restriction to M'_1 of the projection $\mathbf{R}^n \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^n$. We will describe all of such $\phi_1 : M'_1 \rightarrow M_1$ with fixed complexity as follows. Any algebraic set in $\mathbf{R}^n \times \mathbf{R}^{n'}$, and its subset of the regular points where the projection to \mathbf{R}^n is regular, are, respectively, described by the common zero set of polynomial functions f_1, \dots, f_l on $\mathbf{R}^n \times \mathbf{R}^{n'}$ for some $l \in \mathbf{N}$ and

$$\bigcup_{\substack{I=\{i_1, \dots, i_k\} \subset \{1, \dots, l\} \\ I'=\{i'_1, \dots, i'_{k+n'}\} \subset \{1, \dots, l\}}} \{x = (x_1, \dots, x_{n+n'}) \in \mathbf{R}^{n+n'} : \\ f_1(x) = \dots = f_l(x) = 0, \quad \left| \frac{\partial(f_{i'_1}, \dots, f_{i'_{k+n'}})}{\partial(x_{i_1}, \dots, x_{i_k}, x_{n+1}, \dots, x_{n+n'})}(x) \right| \neq 0, \\ g_{I', i''} f_{i''} = \sum_{j=1}^{k+n'} g_{I', i'', j} f_{i'_j}, \quad g_{I', i''}(x) \neq 0, \quad i'' \in \{1, \dots, l\} - I'\}$$

for some polynomial functions $g_{I', i''}$ and $g_{I', i'', j}$ on $\mathbf{R}^n \times \mathbf{R}^{n'}$, where $k = n + 1 - \dim M$, and $\frac{\partial(\cdot)}{\partial(\cdot)}$ denotes the Jacobian matrix. Moreover, its open semialgebraic subset is its intersection with

$$\bigcup_{j'=1}^l \bigcap_{i=1}^l \{x \in \mathbf{R}^{n+n'} : h_{i, j'}(x) > 0\}$$

for some polynomial functions $h_{i, j'}$ on $\mathbf{R}^n \times \mathbf{R}^{n'}$. (Here we enlarge l if necessary.) Thus $\phi_1 : M'_1 \rightarrow \mathbf{R}^n$ is described by the family $f_i, g_{I', i''}, g_{I', i'', j}$ and $h_{i, j'}$ and conversely, any polynomial functions $f_i, g_{I', i''}, g_{I', i'', j}$ and $h_{i, j'}$ define in the above way a Nash manifold M'_1 in $\mathbf{R}^{n+n'}$ of dimension $= \dim M - 1$ such that the projection $\phi_1 : M'_1 \rightarrow \mathbf{R}^n$ is an immersion. If the degree of the polynomials are less than or equal to $d \in \mathbf{N}$ we say $\phi_1 : M'_1 \rightarrow \mathbf{R}^n$ is of complexity l, d, n' .

Furthermore, since a polynomial function on $\mathbf{R}^{n+n'}$ of degree less than or equal to d is of the form $\sum_{\alpha \in \mathbf{N}_d^{n+n'}} a_\alpha x^\alpha$, $a_\alpha \in \mathbf{R}$, where we regard $\mathbf{N}_d^{n+n'} = \{\alpha \in \mathbf{N}^{n+n'} : |\alpha| \leq d\}$, the family of $f_i, \dots, h_{i, j'}$ of degree less than or equal to d as an element

$a = (a_\alpha)$ of \mathbf{R}^N for some $N \in \mathbf{N}$. We write $\phi_1 : M'_1 \rightarrow \mathbf{R}^n$ as $\phi_{1a} : M'_{1a} \rightarrow \mathbf{R}^n$. Then the set $X = \cup_{a \in \mathbf{R}^N} \{a\} \times M'_{1a} \subset \mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'}$ is semialgebraic, and we can identify $\phi_{1a} : M'_{1a} \rightarrow \mathbf{R}^n$ with $p|_{(q \circ p)^{-1}(a)} : (q \circ p)^{-1}(a) \rightarrow \{a\} \times \mathbf{R}^n$, where $p : X \rightarrow \mathbf{R}^N \times \mathbf{R}^n$ and $q : \mathbf{R}^N \times \mathbf{R}^n \rightarrow \mathbf{R}^N$ are the projections.

Consider the condition $\text{Im } \phi_{1a} \subset M_1$. The subset of \mathbf{R}^N consisting of a such that $p|_{(q \circ p)^{-1}(a)}$ fails to satisfy this condition is $q \circ p(X \cap \mathbf{R}^N \times (\mathbf{R}^n - M_1) \times \mathbf{R}^{n'})$ and hence is semialgebraic. Let A denote its complement in \mathbf{R}^N . Thus $\text{Im } \phi_{1a} \subset M_1$ if and only if $a \in A$.

Next consider when $U \cap \text{Im } \phi_{1a}$ is normal crossing and $\phi_{1a}|_{\phi_{1a}^{-1}(\text{Reg}(U \cap \text{Im } \phi_{1a}))}$ is injective. For that, remember that the tangent space $T_x M'_{1a}$ of M'_{1a} at $x \in M'_{1a}$ for M'_{1a} described by $f_i, g_{I', i''}, \dots$ as above. We have

$$T_x M'_{1a} = \{y \in \mathbf{R}^{n+n'} : df_{1x}y = \dots = dg_{I', i''}y = 0\},$$

and hence the set $TX \stackrel{\text{def}}{=} \{(a, x, y) \in X \times \mathbf{R}^{n+n'} : y \in T_x M'_{1a}\}$ is semialgebraic. Assume that $a \in A$. Set

$$M''_{1a} = \{(x, x') \in M'_{1a} \times M'_{1a} : x \neq x', \phi_{1a}(x) = \phi_{1a}(x') \in U, \\ \dim(d\phi_{1ax}(T_x M'_{1a}) + d\phi_{1ax'}(T_{x'} M'_{1a})) = \dim M - 1\}.$$

Then M''_{1a} and $\cup_{a \in A} \{a\} \times M''_{1a}$ are semialgebraic, and $a \in A - q'(\cup_{a \in A} \{a\} \times M''_{1a})$ if and only if for any $x \neq x' \in M'_{1a}$ with $\phi_{1a}(x) \in U$, the germs of ϕ_{1a} at x and x' intersect transversally, where $q' : \mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'} \times \mathbf{R}^n \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^N$ denotes the projection. Repeating the same arguments on m -tuple of M'_{1a} for any $m \leq \dim M$ we obtain a semialgebraic subset B of A such that for $a \in A$, $a \in B$ if and only if $U \cap \text{Im } \phi_{1a}$ is normal crossing and $\phi_{1a}|_{\phi_{1a}^{-1}(\text{Reg}(U \cap \text{Im } \phi_{1a}))}$ is injective.

Let $\{B_i\}$ be a finite stratification of B into connected Nash manifolds such that $q \circ p$ is Nash trivial over each B_i [C-S₁], i.e., for each i there exists a Nash diffeomorphism $\pi_i : (q \circ p)^{-1}(B_i) \rightarrow (q \circ p)^{-1}(b_i) \times B_i$ of the form $\pi_i = (\pi'_i, q \circ p)$ for some $b_i \in B_i$. For $a \in B$, set $M'_a = \phi_{1a}^{-1}(M)$ and $\phi_a = \phi_{1a}|_{M'_a}$. Then $\{\phi_a : M'_a \rightarrow \mathbf{R}^n\}_{a \in B}$ is the family of all $\phi : M' \rightarrow M$ as in lemma 3.7 which are extensible to $\phi_1 : M'_1 \rightarrow M_1$ with fixed U and complexity l, d, n' , and if a and a' are in the same B_i , $i \in I$, then $\phi_a : M'_a \rightarrow M$ and $\phi_{a'} : M'_{a'} \rightarrow M$ are Nash R-L equivalent by lemma 3.6 for the following reason. As there exists a C^0 curve in B_i joining a and a' , considering a finite sequence of points on the curve we can assume that a' is close to a as elements of \mathbf{R}^N . We can replace ϕ_a and $\phi_{a'}$ with $\phi_a \circ (\pi'_i|_{\{a\} \times M'_a})^{-1} = p_n \circ (\pi'_i|_{\{a\} \times M'_a})^{-1} : \{b_i\} \times M'_{b_i} \rightarrow \mathbf{R}^n$ and $p_n \circ (\pi'_i|_{\{a'\} \times M'_{a'}})^{-1} : \{b_i\} \times M'_{b_i} \rightarrow \mathbf{R}^n$, where p_n denotes the projection $\mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'} \rightarrow \mathbf{R}^n$. Hence for application of lemma 3.6 it suffices to see that $(\pi'_i|_{\{a'\} \times M'_{a'}})^{-1}$ is close $(\pi'_i|_{\{a\} \times M'_a})^{-1}$ in the C^1 topology. That is clear because we can regard $(\pi'_i|_{\{a'\} \times M'_{a'}})^{-1}$ and $(\pi'_i|_{\{a\} \times M'_a})^{-1}$ as $\pi_i^{-1}|_{(q \circ p)^{-1}(b_i) \times \{a'\}}$ and $\pi_i^{-1}|_{(q \circ p)^{-1}(b_i) \times \{a\}}$, respectively, $(q \circ p)^{-1}(b_i)$ is compact and because of the following fact. For compact C^1 manifolds M_2 and M_3 and for a C^1 function $g : M_2 \times M_3 \rightarrow \mathbf{R}$ if 2 points u and v in M_3 are close each other then the functions $M_2 \ni x \rightarrow g(x, u) \in \mathbf{R}$ and $M_2 \ni x \rightarrow g(x, v) \in \mathbf{R}$ are close in the C^1 topology. Hence the cardinality of equivalence classes of $\phi_a : M'_a \rightarrow M$, $a \in B$, is finite. Until now we have fixed U . We need argue for all semialgebraic neighborhood

U of M in M_1 . However, it is sufficient to treat a countable number of U 's since M is compact. Thus the cardinality of all equivalence classes is countable. \square

The following lemmas 3.8, 3.9, 3.10 and proposition 3.11 are preparations for the proof of proposition 3.12. Lemmas 3.8 and 3.9 show that a normal crossing Nash subset of a non-compact Nash manifold is trivial at infinity.

Lemma 3.8. *Let X be a normal crossing Nash subset of a Nash manifold M and let $f : M \rightarrow \mathbf{R}^m$ be a proper Nash map whose restrictions to $M - X$ and to strata of the canonical stratification of X are submersions onto \mathbf{R}^m . Then f is Nash trivial, i.e., there exists a Nash diffeomorphism $\pi : M \rightarrow f^{-1}(0) \times \mathbf{R}^m$ of the form $\pi = (\pi', f)$, and π' is chosen so that $\pi'(X) = X \cap f^{-1}(0)$ and $\pi' = \text{id}$ on $f^{-1}(0)$.*

?The analytic case also holds.

This is shown in [C-S_{1,2}] in the case of empty X . We prove here the nonempty case. If M is a Nash manifold with corners and if the restrictions of f to strata of the canonical stratification $\{M_k\}$ of ∂M compatible with X are also submersions onto \mathbf{R}^m , the lemma holds, which is clear by the following proof. Note that, here, $\{M_k\}$ is defined as follows. For a semialgebraic set S , let $\text{Reg } S$ denote the subset of X consisting of points x such that S_x is a Nash manifold germ of dimension $= \dim S$. Then $M_{n-1} = \text{Reg}(\partial M - X)$, $M_{n-2} = \text{Reg}(\partial M - M_{n-1})$, $M_{n-3} = \text{Reg}(\partial M - M_{n-1} - M_{n-2})$, ... Note that $\{M_k\}$ is a stratification of ∂M to Nash manifolds of dimension k , $X \cap \partial M$ is a union of some connected components of M_0, \dots, M_{n-1} , and the method of construction of $\{M_k\}$ is ‘‘canonical’’.

Proof of lemma 3.8. Let $n = \dim M$, k an integer smaller than n , and let X_k denote the union of strata of the canonical stratification of X of dimension less than or equal to k . We define π' on X_k by induction on k , and then on M . To this aim, we can assume that π' is already given on X , say $\pi_X = (\pi'_X, f_X)$, by the following fact, where $f_A = f|_A$ for a subset A of M .

Fact 1. There exist a Nash manifold \tilde{X}_k of dimension k and a Nash immersion $p_{\tilde{X}_k} : \tilde{X}_k \rightarrow M$ such that $\text{Im } p_{\tilde{X}_k} = X_k$ and $p_{\tilde{X}_k}|_{p_{\tilde{X}_k}^{-1}(X_k - X_{k-1})}$ is injective.

Proof of fact 1. (Artin-Mazur Theorem. See the proof of theorem 2.9.) Let M be contained and closed in \mathbf{R}^N , and let X_k^Z denote the Zariski closure of X_k in \mathbf{R}^N . Then there exist an algebraic variety \widetilde{X}_k^Z (the normalization of X_k^Z) in $\mathbf{R}^N \times \mathbf{R}^{N'}$ for some $N' \in \mathbf{N}$ and a union of some connected components \tilde{X}_k of \widetilde{X}_k^Z such that \widetilde{X}_k^Z is non-singular at \tilde{X}_k , hence \tilde{X}_k is a Nash manifold and the restriction $p_{\tilde{X}_k}$ to \tilde{X}_k of the projection $p : \mathbf{R}^N \times \mathbf{R}^{N'} \rightarrow \mathbf{R}^N$ satisfies the requirements.

Let ϕ_i be a finite number of global generators of the sheaf of \mathcal{N} -ideals \mathcal{I} on M defined by X , and set $\phi = \sum \phi_i^2$. Then $\phi > 0$ on $M - X$ and ϕ is a global generator of \mathcal{I}^2 . For a subset A of M and $x \in \mathbf{R}^m$, set $A(x) = A \cap f^{-1}(x)$. We will extend π'_X to π' . For that it suffices to find π' of class semialgebraic C^l for a large integer l for the following reason.

Fact 2. Let g be a semialgebraic C^l function on M whose restriction to X is of class Nash. Then fixing g on X we can approximate g by a Nash function in the semialgebraic C^{l-n} topology.

Proof of fact 2. As in the proof of theorem 3.1,(1), step 1, $g|_X$ is extensible to a Nash function G on M by theorem 2.8. Replace g with $g - G$. Then we can assume that $g = 0$ on X and g is of the form $\sum g_i \phi_i$ for some semialgebraic C^{l-n} functions

g_i on M by using a partition of unity of class semialgebraic C^l and by reducing to the case where $(M, X) = (\mathbf{R}^n, \{x_1 \cdots x_{n'} = 0\})$ for some $n' \in \mathbf{N}$ as in the proof of lemma 3.9. As usual, approximate g_i by Nash functions \tilde{g}_i in the semialgebraic C^{l-n} topology we obtain the required approximation $\sum \tilde{g}_i \phi_i$ of g .

Moreover, it suffices to see that there exists a finite semialgebraic C^l stratification $\{B_i\}$ of \mathbf{R}^m such that for each i , $\pi_X|_{X \cap f^{-1}(B_i)}$ is extensible to a semialgebraic C^l diffeomorphism $\pi_i = (\pi'_i, f_{f^{-1}(B_i)}) : f^{-1}(B_i) \rightarrow M(b_i) \times B_i$ for some point $b_i \in B_i$. The reason is explained in [C-S₂] and in the proof of Theorem II.6.3, [S₃]. We do not repeat it. In the following arguments we need to stratify \mathbf{R}^m to $\{B_i\}$, each B_i to $\{B_{i,j} : j = 1, 2, \dots\}$ and once more. However, we always use notation \mathbf{R}^m for strata, which does not cause problems because we can choose stratifications so that strata are semialgebraically C^l diffeomorphic to Euclidean spaces.

We proceed to the construction of π as in the proof of Theorem II.6.7, [S₃], which is very long. Without loss of generality we assume that $\pi'_X|_{X(0)} = \text{id}$. First we can modify in order ϕ to a semialgebraic C^l function so that for each $x \in \mathbf{R}^m$, $\phi|_{M(x)-X}$ has only singularities of Morse type (Claim 2, *ibid.*) (here we need to stratify \mathbf{R}^m and the main method of proof is a semialgebraic version of Thom's transversality theorem), ϕ is constant on each connected component of $Z \stackrel{\text{def}}{=} \cup_{x \in \mathbf{R}^m} \text{Sing}(\phi|_{M(x)-X})$ and the values are distinct from each other (Claim 4, *ibid.*) (the second stratification). Next, let Y be a connected component of Z and set $\tilde{Y} = \phi^{-1}(\phi(Y))$. Then there exist an open semialgebraic neighborhood U of \tilde{Y} in M and a semialgebraic C^l imbedding $u = (u', f_U) : U \rightarrow U(0) \times \mathbf{R}^m$ such that $u'|_{U(0)} = \text{id}$ and $\phi \circ u' = \phi|_U$ (Claim 5, *ibid.*) (the third stratification). Thirdly, applying lemma 3.8 without X to the semialgebraic C^l map $(f, \phi)|_{\phi^{-1}(I)} : \phi^{-1}(I) \rightarrow \mathbf{R}^m \times I$ for each connected component I of $(0, \infty) - \phi(Z)$, we obtain a semialgebraic C^l diffeomorphism $\lambda = (\lambda', f_{\phi^{-1}(I)}, \phi|_{\phi^{-1}(I)}) : \phi^{-1}(I) \rightarrow \phi^{-1}(I)(0, 0) \times \mathbf{R}^m \times I$ such that $\lambda'|_{\phi^{-1}(I)(0,0)} = \text{id}$, where $\phi^{-1}(I)(0, 0) = \phi^{-1}(t_0) \cap M(0)$ for some $t_0 \in I$ (Claim 7, *ibid.*). Fourthly, we paste u and λ for all I and construct a semialgebraic C^l diffeomorphism $v = (v', f_{M-X}) : M - X \rightarrow (M(0) - X) \times \mathbf{R}^m$ such that $v'|_{M(0)-X} = \text{id}$ and $\phi \circ v' = \phi|_{M-X}$, *ibid.* Hence it suffices to prove the following fact by the same idea of pasting.

Fact 3. There exist an open semialgebraic neighborhood W of X in M and a semialgebraic C^l imbedding $w = (w', f_W) : W \rightarrow M(0) \times \mathbf{R}^m$ such that $w' = \pi'_X$ on X , $w'|_{W(0)} = \text{id}$ and $\phi \circ w' = \phi|_W$.

Proof of fact 3. Here the condition $\phi \circ w' = \phi|_W$ is not necessary. If there exists a semialgebraic C^l imbedding w without this condition, we change ϕ on W with $\phi \circ w'$, extend it to a semialgebraic C^l function on M positive on $M - X$, and repeat the above arguments from the beginning. Then fact 3 is satisfied by this w' .

If X is smooth, the problem becomes easy. Hence we reduce to the smooth case. Let $\tilde{X} \subset \mathbf{R}^N \times \mathbf{R}^{N'}$ and $p_{\tilde{X}} : \tilde{X} \rightarrow M$ be a Nash manifold and the restriction to \tilde{X} of the projection $p : \mathbf{R}^N \times \mathbf{R}^{N'} \rightarrow \mathbf{R}^N$ defined in the proof of fact 1 for $k = n - 1$. For a small positive semialgebraic C^0 function ϵ on \tilde{X} , let \tilde{Q} denote the ϵ -neighborhood of \tilde{X} in $M \times \mathbf{R}^{N'}$, i.e.,

$$\tilde{Q} = \bigcup_{z \in \tilde{X}} \{z' \in M \times \mathbf{R}^{N'} : \text{dis}(z, z') < \epsilon(z)\},$$

and let $\tilde{q} : \tilde{Q} \rightarrow \tilde{X}$ denote the orthogonal projection, which is a Nash submersion.

Set

$$\tilde{M} = \{(x, y) \in \tilde{Q} \subset M \times \mathbf{R}^{N'} : \tilde{q}(x, y) = (x', y) \text{ for some } x' \in X\}.$$

Then \tilde{M} is a Nash manifold of dimension n containing \tilde{X} , and $p_{\tilde{M}} : \tilde{M} \rightarrow M$ is a (not necessarily proper) Nash immersion, where $p_A = p|_A$ for a subset A of $M \times \mathbf{R}^{N'}$. Set $A(0) = A \cap M(0) \times \mathbf{R}^{N'}$, $f_A = f \circ p_A$ for the same A , and $\tilde{X} = p_{\tilde{M}}^{-1}(X)$. Then \tilde{X} is a normal crossing Nash subset of \tilde{M} , $p_{\tilde{X}} : \tilde{X} \rightarrow X$ is a (not necessarily proper) local Nash diffeomorphism at each point of \tilde{X} , $\pi_X = (\pi'_X, f_X)$ is lifted to $\pi_{\tilde{X}} = (\pi'_{\tilde{X}}, f_{\tilde{X}}) : \tilde{X} \rightarrow \tilde{X}(0) \times \mathbf{R}^m$, and there exists a Nash function $\tilde{\phi}$ on \tilde{M} with zero set \tilde{X} which is, locally at each point of \tilde{X} , the square of a regular function and such that

$$(1) \quad \tilde{\phi} = \tilde{\phi} \circ \pi'_{\tilde{X}} \quad \text{on } \tilde{X}$$

(we construct $\tilde{\phi}$ first on $\tilde{M}(0)$, and extend it to \tilde{X} so that (1) is satisfied and then to \tilde{M} as usual). Moreover $\pi'_{\tilde{X}} = \text{id}$ on $\tilde{X}(0)$.

Note that $X_{m-1} = \emptyset$ since $f_{X_k - X_{k-1}}$ is a submersion onto \mathbf{R}^m if $X_k \neq \emptyset$. Let $m \leq k < n$. Then by the definition of \tilde{X} , the map $p_{\tilde{X} \cap p^{-1}(X_k - X_{k-1})} : \tilde{X} \cap p^{-1}(X_k - X_{k-1}) \rightarrow X_k - X_{k-1}$ is a Nash $(n-k)$ -fold covering. Hence considering a semialgebraic triangulation of $X_k(0)$ compatible with X_{k-1} —a semialgebraic homeomorphism from the underlying polyhedron of some simplicial complex to $X_k(0)$ such that $X_{k-1}(0)$ is the image of the union of some simplexes—(Łojasiewicz) and small open semialgebraic neighborhoods of π_X^{-1} (open simplexes) in $M - X_{k-1}$, we obtain finite open semialgebraic coverings $\{Q_{k,i} : i\}$ of $X_k - X_{k-1}$ in $M - X_{k-1}$ and $\{\tilde{Q}_{k,i,j} : i, 1 \leq j \leq n-k\}$ of $\tilde{X} \cap p^{-1}(X_k - X_{k-1})$ in $\tilde{M} - \tilde{X} \cap p^{-1}(X_{k-1})$ such that $\pi_X^{-1}(X(0) \cap Q_{k,i}) = X \cap Q_{k,i}$, $(Q_{k,i}, X_k \cap Q_{k,i})$ are Nash diffeomorphic to $(\mathbf{R}^n, \{0\} \times \mathbf{R}^k)$, $p_{\tilde{Q}_{k,i,j}} : (\tilde{Q}_{k,i,j}, \tilde{X} \cap p^{-1}(X_k) \cap \tilde{Q}_{k,i,j}) \rightarrow (Q_{k,i}, X_k \cap Q_{k,i})$ are Nash diffeomorphisms, and $\tilde{Q}_{k,i,j} \cap \tilde{Q}_{k,i,j'} = \emptyset$ if $j \neq j'$. Define Nash functions $\phi_{k,i,j}$ on $Q_{k,i}$ to be $\tilde{\phi} \circ p_{\tilde{Q}_{k,i,j}}^{-1}$. Then $\phi_{k,i,j}$ are the squares of Nash functions, say $\phi_{k,i,j}^{1/2}$, and we can choose $Q_{k,i}$ so small that the maps $(f, \phi_{k,i,1}^{1/2}, \dots, \phi_{k,i,n-k}^{1/2}) : Q_{k,i} \rightarrow \mathbf{R}^{m+n-k}$ are submersions, that if $Q_{k,i} \cap Q_{k,i'} \neq \emptyset$ then

$$(2) \quad \{\phi_{k,i,j}|_{Q_{k,i} \cap Q_{k,i'}} : j = 1, \dots, n-k\} = \{\phi_{k,i',j}|_{Q_{k,i} \cap Q_{k,i'}} : j = 1, \dots, n-k\},$$

and that if $Q_{k,i} \cap Q_{k',i'} \neq \emptyset$ for $k < k'$ then

$$(3) \quad \{\phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} : j = 1, \dots, n-k\} \supset \{\phi_{k',i',j}|_{Q_{k,i} \cap Q_{k',i'}} : j = 1, \dots, n-k'\}.$$

Let $\Phi_{k,k',i,i'}$ denote the $k' - k$ Nash functions on $Q_{k,i} \cap Q_{k',i'}$ in the complement in (3). Note that (1) implies

$$(1)' \quad \phi_{k,i,j}^{1/2} \circ \pi'_X = \phi_{k,i,j}^{1/2} \quad \text{on } X \cap Q_{k,i}.$$

We work from now in the semialgebraic C^l category. ? Shrink again $Q_{k,i}$ (fixing always $X_k \cap Q_{k,i}$), and set $Q_k = \cup_i Q_{k,i}$. Then there are semialgebraic C^l submersive retractions $q_k : Q_k \rightarrow X_k - X_{k-1}$ such that

$$(4) \quad f \circ q_k = f \quad \text{on } Q_k,$$

$$(5) \quad q_k \circ \pi'_X = \pi'_X \circ q_k \quad \text{on } X \cap Q_k,$$

and the maps $(q_k|_{Q_{k,i}}, \phi_{k,i,1}^{1/2}, \dots, \phi_{k,i,n-k}^{1/2}) : Q_{k,i} \rightarrow (X_k - X_{k-1}) \times \mathbf{R}^{n-k}$ are semialgebraic C^l imbeddings (we construct q_k as follows. For a while, assume that q_k on $Q_k(0)$ are already given so that the conditions (14) on $Q_k(0) \cap Q_{k'}(0)$ and (15) on $Q_{k,i}(0) \cap Q_{k',i'}(0)$ below are satisfied. Extend each q_k to $q_k : X \cap Q_k \rightarrow X_k - X_{k-1}$ so that (4) and (5) are satisfied as follows, which is uniquely determined, though we need to choose Q_k so that $\pi'_X(X \cap Q_k) \subset Q_k(0)$. For $(x, y) \in Q_k^2$ with small $\text{dis}(x, y)$, let $r(x, y)$ denote the orthogonal projection image of x to $X_k(f(y)) - X_{k-1}$. Let $q'_k : Q_k \rightarrow X_k - X_{k-1}$ be any semialgebraic C^l extension of q_k , shrink Q_k and define $q_k(x)$ for $x \in Q_k$ to be $r(q'_k(x), x)$. Then q_k is a semialgebraic C^l submersive retraction of Q_k to $X_k - X_{k-1}$ and satisfies (4) and (5)). Hence for each $x \in X_k - X_{k-1}$, $(\phi_{k,i,1}^{1/2}, \dots, \phi_{k,i,n-k}^{1/2})$ is a local coordinate system of $q_k^{-1}(x) \cap Q_{k,i}$ at x . Therefore, by (3), for each $Q_{k,i}$ and $Q_{k',i'}$ with $k < k'$ there exists a unique semialgebraic C^l submersion $q_{k,k',i,i'} : Q_{k,i} \cap Q_{k',i'} \rightarrow X_{k'} \cap Q_{k,i}$ such that

$$(6) \quad q_k \circ q_{k,k',i,i'} = q_k \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ and}$$

$$(7) \quad \phi_{k,i,j}^{1/2} \circ q_{k,k',i,i'} = \phi_{k,i,j}^{1/2} \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'}$$

(to be precise, the domain of definition of $q_{k,k',i,i'}$ is $q_{k'}^{-1}(Q_{k,i}) \cap Q_{k,i} \cap Q_{k',i'}$. However, we omit $q_{k'}^{-1}(Q_{k,i})$ for simplicity of notation. In the following arguments also we simplify the domains of definition of many maps). Then we have the following equalities (4)', (5)' and (8).

$$(4)' \quad f \circ q_{k,k',i,i'} \stackrel{(4)}{=} f \circ q_k \circ q_{k,k',i,i'} \stackrel{(6)}{=} f \circ q_k \stackrel{(4)}{=} f \quad \text{on } Q_{k,i} \cap Q_{k',i'}.$$

$$q_k \circ q_{k,k',i,i'} \circ \pi'_X \stackrel{(6)}{=} q_k \circ \pi'_X \stackrel{(5)}{=} \pi'_X \circ q_k \stackrel{(6)}{=} \pi'_X \circ q_k \circ q_{k,k',i,i'} \stackrel{(5)}{=} q_k \circ \pi'_X \circ q_{k,k',i,i'} \\ \text{on } X \cap Q_{k,i} \cap Q_{k',i'},$$

$$\phi_{k,i,j}^{1/2} \circ q_{k,k',i,i'} \circ \pi'_X \stackrel{(7)}{=} \phi_{k,i,j}^{1/2} \circ \pi'_X \stackrel{(1)'}{=} \phi_{k,i,j}^{1/2} \stackrel{(7)}{=} \phi_{k,i,j}^{1/2} \circ q_{k,k',i,i'} \stackrel{(1)'}{=} \phi_{k,i,j}^{1/2} \circ \pi'_X \circ q_{k,k',i,i'} \\ \text{on } X \cap Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'},$$

hence by imbeddingness of $(q_k|_{Q_{k,i}}, \phi_{k,i,1}^{1/2}, \dots, \phi_{k,i,n-k}^{1/2})$

$$(5)' \quad q_{k,k',i,i'} \circ \pi'_X = \pi'_X \circ q_{k,k',i,i'} \quad \text{on } X \cap Q_{k,i} \cap Q_{k',i'}.$$

By assumption, (14) on $Q_k(0) \cap Q_{k'}(0)$ and (15) on $Q_{k,i}(0) \cap Q_{k',i'}(0)$ hold, then by (4) and (5), (14) on $X \cap Q_k \cap Q_{k'}$ and (15) on $X \cap Q_{k,i} \cap Q_{k',i'}$ do, and hence by the same imbeddingness

$$(8) \quad q_{k,k',i,i'} = q_{k'} \quad \text{on } X \cap Q_{k,i} \cap Q_{k',i'}.$$

Compare q_{k,k',i_1,i'_1} and q_{k,k',i_2,i'_2} . By (2) and (3)

$$q_{k,k',i_1,i'_1} = q_{k,k',i_1,i'_2} \quad \text{on } Q_{k,i_1} \cap Q_{k',i'_1} \cap Q_{k',i'_2},$$

$$q_{k,k',i_1,i'_2} = q_{k,k',i_2,i'_2} \quad \text{on } Q_{k,i_1} \cap Q_{k,i_2} \cap Q_{k',i'_2},$$

hence $q_{k,k',i_1,i'_1} = q_{k,k',i_2,i'_2} \quad \text{on } Q_{k,i_1} \cap Q_{k,i_2} \cap Q_{k',i'_1} \cap Q_{k',i'_2}.$

Therefore, we have semialgebraic C^l submersions $q_{k,k'} : Q_k \cap Q_{k'} \rightarrow X_{k'} - X_{k'-1}$, $k < k'$, such that

$$\begin{aligned}
(4)' \quad & f \circ q_{k,k'} = f \quad \text{on } Q_k \cap Q_{k'}, \\
(6) \quad & q_k \circ q_{k,k'} = q_k \quad \text{on } Q_k \cap Q_{k'}, \\
(7) \quad & \phi_{k,i,j}^{1/2} \circ q_{k,k'} = \phi_{k,i,j}^{1/2} \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'}, \\
(8) \quad & q_{k,k'} = q_{k'} \quad \text{on } X \cap Q_k \cap Q_{k'}.
\end{aligned}$$

We want to shrink the Q_k 's and modify the q_k 's keeping (4) and (5) so that

$$(9) \quad q_{k,k'} = q_{k'} \quad \text{on } Q_k \cap Q_{k'} \text{ for } k < k'.$$

We proceed by double induction. Let $m \leq k_1 < k_2 < n \in \mathbf{N}$, and assume that (9) holds for $k < k' < k_2$ and for $k_1 < k < k' = k_2$. Fix q_k , $k < k_2$. Then we need to modify q_{k_2} so that (9) holds for $k = k_1$ and $k' = k_2$. Let ξ be a semialgebraic C^l function on $M - X_{k_1-1}$ such that $0 \leq \xi \leq 1$, $\xi = 1$ outside of a small open semialgebraic neighborhood Q'_{k_1} ($\subset Q_{k_1}$) of $X_{k_1} - X_{k_1-1}$ in $M - X_{k_1-1}$ and $\xi = 0$ on a smaller one Q''_{k_1} . Shrink Q_{k_2} and define a semialgebraic C^l submersive retraction $q'_{k_2} : Q_{k_2} \rightarrow X_{k_2} - X_{k_2-1}$ by $q'_{k_2} = q_{k_2}$ on $Q_{k_2} - Q'_{k_1}$ and for $x \in Q_{k_2} \cap Q'_{k_1}$, let $q'_{k_2}(x)$ be the orthogonal projection image of $\xi(x)q_{k_2}(x) + (1 - \xi(x))q_{k_1,k_2}(x) \in \mathbf{R}^N$ to the Nash manifold $X_{k_2}(f(x)) - X_{k_2-1}$. Then q'_{k_2} satisfies (4) and (9) for $k = k_1$, $k' = k_2$ and for Q_k replaced by Q''_{k_1} , the map $(q'_{k_2}|_{Q_{k_2,i}}, \phi_{k_2,i,1}^{1/2}, \dots, \phi_{k_2,i,n-k_2}^{1/2}) : Q_{k_2,i} \rightarrow (X_{k_2} - X_{k_2-1}) \times \mathbf{R}^{n-k_2}$ continues to be a semialgebraic C^l imbedding if we shrink $Q_{k_2,i}$ (of course, fixing $Q_{k_2,i} \cap X_{k_2}$), $q'_{k_2} = q_{k_2}$ on $X \cap Q_{k_2}$ by (8), hence (5) for q'_{k_2} holds, and $q'_{k_2} = q_{k_2}$ on $Q_{k_2} \cap \cup_{k_1 < k < k_2} Q_k$ for the following reason. Let $k_1 < k_3 < k_2$. It suffices to see that $q_{k_1,k_2} = q_{k_2}$ on $Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}$, which is equivalent, by uniqueness of q_{k_1,k_2} , to

$$\begin{aligned}
(10) \quad & q_{k_1} \circ q_{k_2} = q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}, \\
(11) \quad & \phi_{k_1,i_1,j}^{1/2} \circ q_{k_2} = \phi_{k_1,i_1,j}^{1/2} \quad \text{on } Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3} \\
& \quad \text{for } \phi_{k_1,i_1,j}|_{Q_{k_1,i_1} \cap Q_{k_2,i_2}} \in \Phi_{k_1,k_2,i_1,i_2}.
\end{aligned}$$

By (6) for $k = k_1$ and $k' = k_3$ and for $k = k_3$ and $k' = k_2$

$$\begin{aligned}
q_{k_1} \circ q_{k_1,k_3} &= q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_3}, \\
q_{k_3} \circ q_{k_3,k_2} &= q_{k_3} \quad \text{on } Q_{k_2} \cap Q_{k_3}.
\end{aligned}$$

By (9) for $k = k_1$ and $k' = k_3$ and for $k = k_3$ and $k' = k_2$

$$\begin{aligned}
q_{k_1,k_3} &= q_{k_3} \quad \text{on } Q_{k_1} \cap Q_{k_3}, \\
q_{k_3,k_2} &= q_{k_2} \quad \text{on } Q_{k_2} \cap Q_{k_3}.
\end{aligned}$$

Hence

$$(12) \quad q_{k_1} \circ q_{k_3} = q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_3},$$

$$(13) \quad q_{k_3} \circ q_{k_2} = q_{k_3} \quad \text{on } Q_{k_2} \cap Q_{k_3}.$$

Therefore,

$$(10) \quad q_{k_1} \circ q_{k_2} \stackrel{(12)}{=} q_{k_1} \circ q_{k_3} \circ q_{k_2} \stackrel{(13)}{=} q_{k_1} \circ q_{k_3} \stackrel{(12)}{=} q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}.$$

We can prove (11) in the same way because if $Q_{k_1, i_1} \cap Q_{k_2, i_2} \cap Q_{k_3, i_3} \neq \emptyset$ then $\Phi_{k_1, k_2, i_1, i_2} |_{Q_{k_1, i_1} \cap Q_{k_2, i_2} \cap Q_{k_3, i_3}}$ is the disjoint union of $\Phi_{k_1, k_3, i_1, i_3} |_{Q_{k_1, i_1} \cap Q_{k_2, i_2} \cap Q_{k_3, i_3}}$ and $\Phi_{k_3, k_2, i_3, i_2} |_{Q_{k_1, i_1} \cap Q_{k_2, i_2} \cap Q_{k_3, i_3}}$. Thus the induction process works, and we assume that (9) is satisfied. Consequently, the following ‘‘controlledness’’ conditions are satisfied by (6), (7) and (9):

$$(14) \quad q_k \circ q_{k'} = q_k \quad \text{on } Q_k \cap Q_{k'} \text{ for } k < k',$$

$$(15) \quad \phi_{k, i, j}^{1/2} \circ q_{k'} = \phi_{k, i, j}^{1/2} \quad \text{on } Q_{k, i} \cap Q_{k', i'} \text{ for } k < k' \text{ and } \phi_{k, i, j} |_{Q_{k, i} \cap Q_{k', i'}} \in \Phi_{k, k', i, i'}.$$

It remains to construct q_k on $Q_k(0)$. First define r as above, i.e., for $(x, y) \in Q_k^2(0)$ with small $\text{dis}(x, y)$, let $r(x, y)$ denote the orthogonal projection image of x to $X_k(0) - X_{k-1}$. Set $q_k(x) = r(x, x)$ for $x \in Q_k(0)$. Then $q_k : Q_k(0) \rightarrow X_k(0) - X_{k-1}$ are Nash submersive retractions. We need to modify them so that (14) on $Q_k(0) \cap Q_{k'}(0)$ and (15) on $Q_{k, i}(0) \cap Q_{k', i'}(0)$ are satisfied. This is clearly possible by the above arguments.

Now we define W and w as in fact 3. Set $W = \cup_{k=m}^{n-1} Q_k$ and consider each $Q_{k, i}$. Shrink $Q_{k, i}$ so that

$$(\pi'_X \circ q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2})(Q_{k, i}) \subset (q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2})(Q_{k, i}(0)).$$

Then for each $x \in Q_{k, i}$ there exists a unique $y \in Q_{k, i}(0)$ such that

$$(q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2})(y) = (\pi'_X \circ q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2})(x).$$

The correspondence $w'_{k, i}$ from x to y is a semialgebraic C^l map such that $w_{k, i} = (w'_{k, i}, f_{Q_{k, i}}) : Q_{k, i} \rightarrow Q_{k, i}(0) \times \mathbf{R}^m$ is a semialgebraic C^l imbedding by (4), $w'_{k, i} = \pi'_X$ on $X \cap Q_{k, i}$ because

$$(q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2}) \circ \pi'_X(x) \stackrel{(1)', (5)}{=} (\pi'_X \circ q_k, \phi_{k, i, 1}^{1/2}, \dots, \phi_{k, i, n-k}^{1/2})(x) \text{ for } x \in X \cap Q_{k, i},$$

and $w'_{k, i} |_{Q_{k, i}(0)} = \text{id}$ by (4) and by the equality $\pi'_X = \text{id}$ on $X(0)$. Hence it suffices to see that $w'_{k, i} = w'_{k', i'}$ on $Q_{k, i} \cap Q_{k', i'}$. This is clear by (2) if $k = k'$ and $Q_{k, i} \cap Q_{k', i'} \neq \emptyset$. Assume that $k < k'$ and $Q_{k, i} \cap Q_{k', i'} \neq \emptyset$. By (3) we suppose that

$$\phi_{k', i', j}^{1/2} = \phi_{k, i, j+k'-k}^{1/2} \quad \text{on } Q_{k, i} \cap Q_{k', i'}, \quad j = 1, \dots, n - k'.$$

Then by the definition of $w'_{k, i}$ and $w'_{k', i'}$ we only need to show that

$$q_{k'} \circ w'_{k, i} = \pi'_X \circ q_{k'} \quad \text{on } Q_{k, i} \cap Q_{k', i'},$$

which is equivalent to:

$$\begin{aligned} q_k \circ q_{k'} \circ w'_{k, i} &= q_k \circ \pi'_X \circ q_{k'} \quad \text{on } Q_{k, i} \cap Q_{k', i'} \text{ and} \\ \phi_{k, i, j}^{1/2} \circ q_{k'} \circ w'_{k, i} &= \phi_{k, i, j}^{1/2} \circ \pi'_X \circ q_{k'} \quad \text{on } Q_{k, i} \cap Q_{k', i'}, \quad j = 1, \dots, k' - k. \end{aligned}$$

We have

$$\begin{aligned} q_k \circ q_{k'} \circ w'_{k,i} &\stackrel{(14)}{=} q_k \circ w'_{k,i} = \pi'_X \circ q_k \stackrel{(14)}{=} \pi'_X \circ q_k \circ q_{k'} \stackrel{(5)}{=} q_k \circ \pi'_X \circ q_{k'}, \\ \phi_{k,i,j}^{1/2} \circ q_{k'} \circ w'_{k,i} &\stackrel{(15)}{=} \phi_{k,i,j}^{1/2} \circ w'_{k,i} = \phi_{k,i,j}^{1/2} \stackrel{(15)}{=} \phi_{k,i,j}^{1/2} \circ q_{k'} \stackrel{(1)'}{=} \phi_{k,i,j}^{1/2} \circ \pi'_X \circ q_{k'}, \\ & j = 1, \dots, k' - k. \end{aligned}$$

Thus we complete the proof of fact 3 and hence of the lemma. \square

Lemma 3.9. *Let M be a non-compact Nash manifold contained and closed in \mathbf{R}^N and X a normal crossing Nash subset of M . Let $B(r)$ denote the closed ball in \mathbf{R}^N with center 0 and radius $r \in \mathbf{R}$. Then there exists a Nash diffeomorphism $\tau : M \rightarrow M \cap \text{Int } B(r)$, for some large r , such that $\tau(X) = X \cap \text{Int } B(r)$.*

This does not necessarily hold in the analytic case.

Proof of lemma 3.9. Assume that M is of dimension n . Set $X_n = M - X$. Choose r so large that the $p|_{X_i - B(r/2)}$ are submersions onto $(r/2, \infty)$, where $\{X_i : i = 0, \dots, n-1\}$ denotes the canonical stratification of X and $p(x) = |x|$ for $x \in M$. Then by lemma 3.8 there exists a Nash diffeomorphism $\rho : M - B(r/2) \rightarrow (B \cap p^{-1}(r)) \times (r/2, \infty)$ of the form $\rho = (\rho', p)$ such that $\rho'(X - B(r/2)) = X \cap p^{-1}(r)$. Let $\alpha : (-\infty, r) \rightarrow \mathbf{R}$ be a semialgebraic C^l diffeomorphism such that $\alpha = \text{id}$ on $(-\infty, r/2)$, where l is a sufficiently large integer. Set

$$\tau_0(x) = \begin{cases} x & \text{for } x \in M \cap B(r/2) \\ \rho^{-1}(\rho'(x), \alpha^{-1} \circ p(x)) & \text{for } x \in M - B(r/2). \end{cases}$$

Then τ_0 is a semialgebraic C^l diffeomorphism from M to $M \cap \text{Int } B(r)$ such that $\tau_0(X) = X \cap \text{Int } B(r)$. We only need to approximate τ_0 by a Nash diffeomorphism keeping the last property. Let $\pi : M \rightarrow M \cap \text{Int } B(r)$ be a Nash approximation of τ_0 in the semialgebraic C^l topology. Replace τ_0 with $\pi \circ \tau_0^{-1}$. Then what we prove is:

Let \tilde{M} be a compact Nash manifold with boundary in \mathbf{R}^N , \tilde{X} a normal crossing Nash subset of \tilde{M} with $\partial\tilde{M} \not\subset \tilde{X}$, $\tilde{\tau}_0$ a semialgebraic C^l diffeomorphism of $\text{Int } \tilde{M}$ arbitrarily close to id in the semialgebraic C^l topology such that $\tilde{\tau}_0(\tilde{X} \cap \text{Int } \tilde{M})$ is a normal crossing Nash subset of $\text{Int } \tilde{M}$. Then we can approximate $\tilde{\tau}_0$ by a Nash diffeomorphism $\tilde{\tau}$ of $\text{Int } \tilde{M}$ in the semialgebraic C^1 topology so that $\tilde{\tau}(\tilde{X} \cap \text{Int } \tilde{M}) = \tilde{\tau}_0(\tilde{X} \cap \text{Int } \tilde{M})$.

We proceed as in the proof of step 1, theorem 3.1,(1). Let $\{\tilde{X}_j : j = 0, \dots, n-1\}$ denote the canonical stratification of \tilde{X} and set $\tilde{X}_n = \tilde{M} - X$. By induction, for some $i \in \mathbf{N}$, assume that $\tilde{\tau}_0|_{\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}}$ is of class Nash. Let $l' \in \mathbf{N}$. Then it suffices to choose l large enough and to approximate $\tilde{\tau}_0$ by a semialgebraic C^l diffeomorphism $\tilde{\tau}$ of $\text{Int } \tilde{M}$ in the semialgebraic $C^{l'}$ topology so that $\tilde{\tau}(\tilde{X} \cap \text{Int } \tilde{M}) = \tilde{\tau}_0(\tilde{X} \cap \text{Int } \tilde{M})$ and $\tilde{\tau}|_{\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}}$ is of class Nash. Let \mathcal{I} denote the sheaf of \mathcal{N} -ideals on $\text{Int } \tilde{M}$ defined by $\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}$. By theorem 2.7, \mathcal{I} is generated by a finite number of global cross-sections ξ_1, \dots, ξ_k of \mathcal{I} . Then $\tilde{\tau}_0|_{\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}}$ is an element of $H^0(\text{Int } \tilde{M}, \mathcal{N}/\mathcal{I})^N$ by the same reason as in the proof in step 1. Hence by theorem 2.8 we have a Nash map $h : \text{Int } \tilde{M} \rightarrow \mathbf{R}^N$ such that $h = \tilde{\tau}_0$ on $\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}$.

Here we can choose h so as to be sufficiently close to $\tilde{\tau}_0$ in the semialgebraic $C^{l'}$ topology for the following reason. To this aim, it suffices to see that $\tilde{\tau}_0 - h$ is of the form $\sum_{j=1}^k \xi_j \beta_j$ for some semialgebraic $C^{l'}$ maps $\beta_j : \text{Int } \tilde{M} \rightarrow \mathbf{R}^N$ because $h + \sum_{j=1}^k \xi_j \cdot$ (Nash approximation of β_j in the semialgebraic $C^{l'}$ topology) fulfills the requirements. Hence we will prove the following statement.

Let β be a semialgebraic C^l function on $\text{Int } \tilde{M}$ vanishing on $\cup_{j=0}^{i-1} \tilde{X}_j \cap \text{Int } \tilde{M}$. Then β is of the form $\sum_{j=1}^k \beta_j \xi_j$ for some semialgebraic $C^{l'}$ functions β_j on $\text{Int } \tilde{M}$.

By the second induction, assume that the statement holds for manifolds of dimension strictly less than n . The problem is reduced to the Euclidean case as follows. There exists a finite open semialgebraic covering $\{O_s\}$ of $\text{Int } \tilde{M}$ such that each $(O_s, O_s \cap \tilde{X})$ is Nash diffeomorphic to $(\mathbf{R}^n, \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n_s} = 0\})$ for some $n_s \in \mathbf{N}$. Let $\{\eta_s\}$ and $\{\eta'_s\}$ be a partition of unity of class semialgebraic C^l subordinate to $\{O_s\}$, and semialgebraic C^l functions on $\text{Int } \tilde{M}$, respectively, such that $\eta'_s = 1$ on $\text{supp } \eta_s$ and $\text{supp } \eta'_s \subset O_s$. If each $(\beta \eta_s)|_{O_s}$ is described to be of the form $\sum_j \beta_{j,s} \xi_j|_{O_s}$ for some semialgebraic $C^{l'}$ functions $\beta_{j,s}$ on O_s then the naturally defined $\sum_s \beta_{j,s} \eta'_s$, $j = 1, \dots, k$, are semialgebraic $C^{l'}$ functions on $\text{Int } \tilde{M}$ and $\beta = \sum_j (\sum_s \beta_{j,s} \eta'_s) \xi_j$. Hence we can assume that $(\text{Int } \tilde{M}, \text{Int } \tilde{M} \cap X) = (\mathbf{R}^n, \{x_1 \cdots x_{n'} = 0\})$ for some $n' \in \mathbf{N}$, and then $n' > 0$. Apply the induction hypothesis to $\beta|_{\{x_1=0\}}$. Then there exist semialgebraic C^{l_1} functions β'_j on \mathbf{R}^{n-1} such that

$$\beta(0, x_2, \dots, x_n) = \sum_{j=1}^k \beta'_j(x_2, \dots, x_n) \xi_j(0, x_2, \dots, x_n)$$

because $\mathcal{I}|_{\{x_1=0\}}$ is the sheaf of \mathcal{N} -ideals on $\{x_1 = 0\}$ defined by $\cup_{j=0}^{i-1} \tilde{X}_j \cap \{x_1 = 0\}$ (here $l_1 > 0$ is arbitrarily given and l depends on l_1). Regard naturally β'_j as semialgebraic C^{l_1} functions on \mathbf{R}^n and replace β with $\beta - \sum \beta'_j \xi_j$. Then we can suppose that $\beta = 0$ on $\{x_1 = 0\}$ from the beginning. It is easy to see that β/x_1 is a well-defined semialgebraic C^{l_1-1} function. Consider β/x_1 and $\{x_2 \cdots x_{n'} = 0\}$ in place of β and $\{x_1 \cdots x_{n'} = 0\}$, and repeat the same arguments for $\{x_2 = 0\}$ and so on. Then we finally arrive at the case of $\tilde{X} = \emptyset$. Thus the statement is proved, and h is chosen to be close to $\tilde{\tau}_0$ in the semialgebraic $C^{l'}$ topology.

Set $Y = \tilde{\tau}_0(\tilde{X} \cap \text{Int } \tilde{M})$ and $Y_j = \tilde{\tau}_0(\tilde{X}_j \cap \text{Int } \tilde{M})$. Then Y is a normal crossing Nash subset of $\text{Int } \tilde{M}$, $\{Y_j : j = 0, \dots, n-1\}$ is its canonical stratification, and \bar{Y} is a normal crossing semialgebraic C^l subset of \tilde{M} in the sense that \tilde{M} has a semialgebraic C^l local coordinate system (x_1, \dots, x_n) at each point of $\partial \tilde{M}$ with $\bar{Y} = \{x_1 \geq 0, x_2 \cdots x_{n'} = 0\}$ for some $n' > 0 \in \mathbf{N}$ by the definition of semialgebraic C^l topology. Hence there exists a tubular neighborhood U_i of Y_i in \mathbf{R}^N such that for some $\epsilon > 0 \in \mathbf{R}$

$$U_i = \cup_{y \in Y_i} \{x \in \mathbf{R}^N : |x - y| < \epsilon \text{ dis}(y, \cup_{j=0}^{i-1} Y_j), (x - y) \perp T_y Y_i\}.$$

Let $q_i : U_i \rightarrow Y_i$ denote the orthogonal projection. Choose h so close to $\tilde{\tau}_0$ that $h(\tilde{X}_i \cap \text{Int } \tilde{M}) \subset U_i$. Then $q_i \circ h|_{\cup_{j=0}^i \tilde{X}_j \cap \text{Int } \tilde{M}}$ is a Nash map to $\cup_{j=0}^i Y_j$ close to $\tilde{\tau}_0|_{\cup_{j=0}^i \tilde{X}_j \cap \text{Int } \tilde{M}}$ in the semialgebraic $C^{l'}$ topology. Note that the map is a diffeomorphism by Lemma II.1.7, [S₂]. Hence it remains only to extend it to a semialgebraic

C^l approximation $\tilde{\tau} : \text{Int } \tilde{M} \rightarrow \text{Int } \tilde{M}$ of $\tilde{\tau}_0$ in the semialgebraic C^l topology so that $\tilde{\tau}(\tilde{X} \cap \text{Int } \tilde{M}) = Y$. However, we have already proved it without the last condition. Moreover, the proof shows also that the condition is furnished inductively. Thus we complete the construction of τ . \square

Lemma 3.10. *Let f and g be Nash functions on a Nash manifold M which have the same sign at each point of M , only normal crossing singularities at $X \stackrel{\text{def}}{=} f^{-1}(0)$ and the same multiplicity at each point of X . Let $l \in \mathbf{N}$. Then there exists a Nash diffeomorphism π of M such that $\pi(X) = X$ and $f - g \circ \pi$ is l -flat at X .*

If f is fixed and g is chosen such that the Nash function on M , defined to be g/f on $M - X$, is close to 1 in the Nash ($\stackrel{\text{def}}{=} \text{semialgebraic } C^\infty$) topology, then π is chosen to be close to id in the Nash topology.

?The analytic case also holds.

When M , f and g are of class C^ω , assume that the multiplicities of f and g are bounded. Then the same statement holds as above, whose proof is easier than the above.

Proof of lemma 3.10. Let $M \subset \mathbf{R}^N$, $n = \dim M$ and l be sufficiently large. For each $k (< n) \in \mathbf{N}$, let X_k denote the union of the strata of the canonical stratification of X of dimension less than or equal to k . By induction, assume that $f - g$ is l -flat at X_{k-1} for some k . Then we need only to find a Nash diffeomorphism π of M such that $\pi - \text{id}$ is l -flat at X_{k-1} , $\pi(X) = X$ and $f - g \circ \pi$ is l -flat at X_k (to be precise, we will construct π so that $\pi - \text{id}$ and $f - g \circ \pi$ are $l^{(4)}$ -flat at X_{k-1} and X_k , respectively, for some $0 \ll l^{(4)} \ll \dots \ll l' \ll l$).

We proceed as in the proof of lemma 3.8. Let (\tilde{M}, \tilde{X}) and $(\tilde{M}_k, \tilde{X}_k)$ be pairs of Nash manifolds and Nash submanifolds, $p : \tilde{M} \rightarrow M$ and $p_k : \tilde{M}_k \rightarrow M$ Nash immersions and $q_k : \tilde{M}_k \rightarrow \tilde{X}_k$ a Nash submersive retraction such that $\dim \tilde{M} = \dim \tilde{M}_k = n$, $p(\tilde{X}) = X$, $p_k(\tilde{X}_k) = X_k$, $p|_{\tilde{X} - p^{-1}(X_{n-2})}$ and $p_k|_{\tilde{X}_k - p_k^{-1}(X_{k-1})}$ are injective, and $p_k(q_k^{-1}(\tilde{X}_k \cap p_k^{-1}(X_{k-1}))) \subset X$. Shrink \tilde{M}_k if necessary. Then we have an open semialgebraic neighborhood U of $\tilde{X} \cap p^{-1}(X_k)$ in \tilde{M} and a Nash $(n - k)$ -fold covering map $r : U \rightarrow \tilde{M}_k$ such that $p_k \circ r = p$ on U . Let $\tilde{\phi}$ be a Nash function on \tilde{M} with zero set \tilde{X} which is, locally at each point of \tilde{X} , the square of a regular function. Then for each $x \in \tilde{M}_k$, $\tilde{\phi}(r^{-1}(x))$ is a family of $(n - k)$ -numbers possibly with multiplicity, and there exist Nash functions $\tilde{\phi}_{k,1}, \dots, \tilde{\phi}_{k,n-k}$ on an open semialgebraic neighborhood of each point of \tilde{M}_k such that $\tilde{\phi}(r^{-1}(x)) = \{\tilde{\phi}_{k,1}(x), \dots, \tilde{\phi}_{k,n-k}(x)\}$ for x in the neighborhood. For simplicity of notation we assume that $\tilde{\phi}_{k,1}(x), \dots, \tilde{\phi}_{k,n-k}(x)$ are defined globally, which causes no problem because the following arguments are done locally and do not depend on the order of $\tilde{\phi}_{k,1}(x), \dots, \tilde{\phi}_{k,n-k}(x)$. Moreover, we suppose that each $\tilde{\phi}_{k,i}$ is the square of a regular Nash function, say $\tilde{\phi}_{k,i}^{1/2}$, by the same first reason. Set $\tilde{f}_k = f \circ p_k$ and $\tilde{g}_k = g \circ p_k$.

We want to construct a Nash diffeomorphism $\tilde{\pi}_k$ between semialgebraic neighborhoods of \tilde{X}_k in \tilde{M}_k such that $\tilde{\pi}_k(p_k^{-1}(X)) \subset p_k^{-1}(X)$, $\tilde{\pi}_k - \text{id}$ is l'' -flat at $q_k^{-1}(\tilde{X}_k \cap p_k^{-1}(X_{k-1}))$ and $\tilde{f}_k - \tilde{g}_k \circ \tilde{\pi}_k$ is l'' -flat at \tilde{X}_k . Assume that \tilde{X}_k is connected without loss of generality. Since \tilde{f}_k and \tilde{g}_k have only normal crossing singularities at $p_k^{-1}(X)$, the same sign at each point of \tilde{M}_k and the same multiplicity at each point of $p_k^{-1}(X)$, and since $\tilde{f}_k^{-1}(0) = \tilde{g}_k^{-1}(0) = \cup_{i=1}^{n-k} (\tilde{\phi}_{k,i}^{1/2})^{-1}(0) \cup q_k^{-1}(\tilde{X}_k \cap p_k^{-1}(X_{k-1}))$,

we have Nash functions F and G on \tilde{M}_k and $\alpha = (\alpha_1, \dots, \alpha_{n-k}) \in (\mathbf{N} - \{0\})^{n-k}$ such that $\tilde{f}_k = F\tilde{\phi}_k^{1/2\alpha}$, $\tilde{g}_k = G\tilde{\phi}_k^{1/2\alpha}$, $FG \geq 0$ on \tilde{M}_k and $FG > 0$ on $\tilde{M}_k - q_k^{-1}(\tilde{X}_k \cap p_k^{-1}(X_{k-1}))$, where $\tilde{\phi}_k^{1/2\alpha} = \prod_{i=1}^{n-k} \tilde{\phi}_{k,i}^{1/2\alpha_i}$. Assume that $F \geq 0$ and hence $G \geq 0$ because the other cases are proved in the same way as follows. Note that F and G have zero set $q_k^{-1}(\tilde{X}_k \cap p_k^{-1}(X_{k-1}))$, only normal crossing singularities there and the same multiplicity at each point. Shrink \tilde{M}_k so that the map $(q_k, \tilde{\phi}_{k,1}^{1/2}, \dots, \tilde{\phi}_{k,n-k}^{1/2}) : \tilde{M}_k \rightarrow \tilde{X}_k \times \mathbf{R}^{n-k}$ is a Nash imbedding, let V denote its image, identify \tilde{M}_k and \tilde{X}_k with V and $\tilde{X}_k \times \{0\}$ through the map, set $\tilde{p}_k = p_k|_{\tilde{X}_k}$, regard p_k as an immersion of V into M and \tilde{f}_k and \tilde{g}_k as functions on V , and let $(z, y) = (z, y_1, \dots, y_{n-k}) \in V \subset \tilde{X}_k \times \mathbf{R}^{n-k}$. Then

$$\tilde{f}_k(z, y) = F(z, y)y^\alpha \quad \text{and} \quad \tilde{g}_k(z, y) = G(z, y)y^\alpha.$$

Set

$$F' = \sum_{\beta \in \mathbf{N}_l^{n-k}} \frac{\partial^{|\beta|} F}{\partial y^\beta}(z, 0)y^\beta / \beta!, \quad G' = \sum_{\beta \in \mathbf{N}_l^{n-k}} \frac{\partial^{|\beta|} G}{\partial y^\beta}(z, 0)y^\beta / \beta!,$$

$$\tilde{f}'_k = F'y^\alpha \quad \text{and} \quad \tilde{g}'_k = G'y^\alpha,$$

where $\mathbf{N}_l^{n-k} = \{\beta \in \mathbf{N}^{n-k} : |\beta| \leq l\}$ and $\beta! = \prod_{i=1}^{n-k} \beta_i!$. Then \tilde{f}'_k and \tilde{g}'_k are Nash functions on V , $\tilde{f}_k - \tilde{f}'_k$ and $\tilde{g}_k - \tilde{g}'_k$ are l -flat at $\tilde{X}_k \times \{0\}$, and F' and G' have the same properties as F and G . Hence for the construction of $\tilde{\pi}_k$, we can replace \tilde{f}_k and \tilde{g}_k with \tilde{f}'_k and \tilde{g}'_k . An advantage of \tilde{f}'_k and \tilde{g}'_k is the fact (*) $F' - G'$ is l' -flat at $V \cap \tilde{p}_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$, though $F - G$ is l' -flat only at $\tilde{p}_k^{-1}(X_{k-1}) \times \{0\}$. Write

$$\tilde{f}'_k = \prod_{i=1}^{n-k} (F'^{1/|\alpha|} y_i)^{\alpha_i} \quad \text{and} \quad \tilde{g}'_k = \prod_{i=1}^{n-k} (G'^{1/|\alpha|} y_i)^{\alpha_i}.$$

Then there exists a unique Nash diffeomorphism $\tilde{\pi}_k$ between semialgebraic neighborhoods of $\tilde{X}_k \times \{0\}$ in V of the form $\tilde{\pi}_k(z, y) = (z, \tilde{\pi}'_k(z, y)y)$, for some positive Nash function $\tilde{\pi}'_k$ on the neighborhood of source, such that $\tilde{f}'_k = \tilde{g}'_k \circ \tilde{\pi}_k$ on the neighborhood because we can reduce the problem to the case where $\tilde{g}'_k = z^\beta y^\alpha$ for some $\beta \in \mathbf{N}^k$ and some local Nash coordinate system $z = (z_1, \dots, z_k)$ of \tilde{X}_k such that $\tilde{p}_k^{-1}(X_{k-1}) = \{z^\beta = 0\}$ (by considering two pairs $(\tilde{f}'_k, z^\beta y^\alpha)$ and $(\tilde{g}'_k, z^\beta y^\alpha)$) and then $\tilde{\pi}'_k(z, y) = (F'/z^\beta)^{1/|\alpha|}$ is the unique solution. Such a $\tilde{\pi}_k$ fulfills the requirements. Indeed, $\tilde{\pi}_k(p_k^{-1}(X)) \subset p_k^{-1}(X)$ by the form of $\tilde{\pi}_k$ because $p_k^{-1}(X)$ in V is of the form $\tilde{X}_k \times \{y_1 \cdots y_{n-k} = 0\} \cup \tilde{p}_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$; $\tilde{\pi}_k - \text{id}$ is l'' -flat at $V \cap \tilde{p}_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$ because of (*); $\tilde{f}_k - \tilde{g}_k \circ \tilde{\pi}_k$ is l'' -flat at $\tilde{X}_k \times \{0\}$ because

$$\tilde{f}_k - \tilde{g}_k \circ \tilde{\pi}_k = (\tilde{f}_k - \tilde{f}'_k) + (\tilde{f}'_k - \tilde{g}'_k \circ \tilde{\pi}_k) + (\tilde{g}'_k \circ \tilde{\pi}_k - \tilde{g}_k \circ \tilde{\pi}_k).$$

Let W be an open semialgebraic neighborhood of $X_k - X_{k-1}$ in M so small that there exists an open semialgebraic neighborhood of $(\tilde{X}_k - \tilde{p}_k^{-1}(X_{k-1})) \times \{0\}$ in the intersection of the domain of definition of $\tilde{\pi}_k$ and the range of values to which the restriction of p_k is a diffeomorphism onto W . Then $\tilde{\pi}_k$ induces a Nash imbedding

$\pi_k : W \rightarrow M \subset \mathbf{R}^n$ such that $\pi_k(X \cap W) \subset X$, $\pi_k = \text{id}$ on $X_k - X_{k-1}$ and $f - g \circ \pi_k$ is l'' -flat at $X_k - X_{k-1}$. Though π_k is not necessarily extensible to a neighborhood of X_k , there exists a Nash map $\eta : M \rightarrow \mathbf{R}^N$ such that $\eta - \text{id}$ is l'' -flat at X_{k-1} , $\eta - \pi_k$ is l'' -flat at $X_k - X_{k-1}$, and hence $f - g \circ \eta$ is l'' -flat at X_k for the following reason. Let \mathcal{I} denote the sheaf of \mathcal{N} -ideals on M defined by X_k . Then by theorem 2.8, it suffices to find an element $\bar{\eta}$ in $H^0(M, \mathcal{N}/\mathcal{I}'')^N$ such that $\bar{\eta}_x$ is the image of π_{kx} under the natural map $\mathcal{N}_x^n \rightarrow (\mathcal{N}_x/\mathcal{I}_x'')^N$ for $x \in W$ and $\bar{\eta}_x = \text{id}$ for $x \in X_{k-1}$. This is clearly possible by the property that $\tilde{\pi}_k - \text{id}$ is l'' -flat at $V \cap \tilde{p}_k^{-1}(X_k) \times \mathbf{R}^{n-k}$.

We modify $\tilde{\pi}_k$ to show that η can be a diffeomorphism of M . Assume that $(**)$ $\tilde{\pi}'_k \leq 1$ holds for simplicity of notation, which is possible if we consider the third function h on M with the same properties as f and g , $h/f \geq 1$ on $M - X$ and $h/g \geq 1$ on $M - X$. Let ψ be a non-negative small Nash function on \tilde{X}_k with zero set $\tilde{p}_k^{-1}(X_{k-1})$ such that

$$(3*) \quad \begin{aligned} Z &\stackrel{\text{def}}{=} \{(z, y) \in \tilde{X}_k \times \mathbf{R}^{n-k} : |y| \leq \psi(z)\} \subset \text{domain of } \tilde{\pi}_k, \\ \tilde{\pi}'_k(z, sy) &> \left| \frac{\partial \tilde{\pi}'_k(z, sy)}{\partial s} s \right| \\ &\text{for } (z, y, s) \in \tilde{X}_k \times \mathbf{R}^{n-k} \times \mathbf{R} \text{ with } (z, sy) \in Z \text{ and } |y| = 1 \end{aligned}$$

and $p_k|_Z$ is injective, which exists by the Łojasiewicz inequality. Let $\rho(t)$ be a semialgebraic $C^{l''}$ function on \mathbf{R} such that $(4*)$ $0 \leq \rho \leq 1$, $(5*)$ $\frac{d\rho}{dt} \leq 0$, $\rho = 1$ on $(-\infty, 1/2]$ and $\rho = 0$ on $[1, \infty)$. Set

$$\tilde{\tau}'_k = \begin{cases} 1 & \text{for } (z, y) \in Z \cap p_k^{-1}(X_{k-1}) \\ \rho(|y|/\psi(z))\tilde{\pi}'_k(z, y) + 1 - \rho(|y|/\psi(z)) & \text{for } (z, y) \in Z - p_k^{-1}(X_{k-1}), \\ \tilde{\tau}_k(z, y) = (z, \tilde{\tau}'_k(z, y)y) & \text{for } (z, y) \in Z. \end{cases}$$

Then $\tilde{\tau}'_k$ and hence $\tilde{\tau}_k$ are of class semialgebraic $C^{l^{(3)}}$ and $\tilde{\tau}_k - \text{id}$ is $l^{(3)}$ -flat at $Z \cap p_k^{-1}(X_{k-1}) = \tilde{p}_k^{-1}(X_{k-1}) \times \{0\}$ since $\tilde{\pi}'_k(z, y) - 1$ is $(l'' - 1)$ -flat at $Z \cap p_k^{-1}(X_{k-1})$. Clearly $\tilde{\tau}_k = \text{id}$ on a semialgebraic neighborhood of $\partial Z - p_k^{-1}(X_{k-1})$ in Z . Moreover, $\tilde{\tau}_k$ is a diffeomorphism of Z as follows. To see it we can assume that $n - k = 1$ because $\tilde{\tau}_k = \tilde{\pi}_k$ on a neighborhood of $(\tilde{X}_k - \tilde{p}_k^{-1}(X_{k-1})) \times \{0\}$ in Z and because $\tilde{\pi}_k$ and hence $\tilde{\tau}_k$ carry each segment $\{z\} \times \{\mathbf{R}y\} \cap Z$ for $(z, y) \in (\tilde{X}_k - \tilde{p}_k^{-1}(X_{k-1})) \times \mathbf{R}^{n-k}$ with $|y| = 1$ to itself. Then

$$\begin{aligned} \frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y} &= \tilde{\tau}'_k(z, y) + \frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y}, \\ \tilde{\tau}'_k(z, y) &= \rho(|y|/\psi(z))\tilde{\pi}'_k(z, y) + 1 - \rho(|y|/\psi(z)) \stackrel{(**), (4*)}{\geq} \tilde{\pi}'_k(z, y), \\ \frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y} &= \frac{d\rho}{dt}(|y|/\psi(z))(\tilde{\pi}'_k(z, y) - 1)|y|/\psi(z) + \rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k(z, y)y}{\partial y} \stackrel{(**), (5*)}{\geq} \\ &\quad \rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k(z, y)y}{\partial y}, \quad \text{hence} \\ \frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y} &\geq \tilde{\pi}'_k(z, y) + \rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k(z, y)y}{\partial y} \stackrel{(3*), (4*)}{>} 0 \quad \text{for } (z, y) \in Z. \end{aligned}$$

Define a semialgebraic $C^{l(3)}$ diffeomorphism τ_k of M so that $\tau_k \circ p_k = p_k \circ \tilde{\tau}_k$ on Z and $\tau_k = \text{id}$ on $M - p_k(Z)$. Then $\tau_k = \pi_k$ on Z if we shrink Z , hence $\tau_k - \eta$ is $l(3)$ -flat at X_k , $\tau_k(X) = X$ and $f - g \circ \tau_k$ is $l(3)$ -flat at X_k . Let ω be a non-negative-valued global generator of the square of \mathcal{I} —the sheaf of \mathcal{N} -ideals defined by X_k . Then there exists a semialgebraic $C^{l(4)}$ map $\xi : M \rightarrow \mathbf{R}^N$ such that $\tau_k - \eta = \omega\xi$. Approximate ξ by a Nash map ξ' in the semialgebraic $C^{l(4)}$ topology, and set $\pi = (\eta + \omega\xi') \circ o$, where o denotes the orthogonal projection to M of its semialgebraic tubular neighborhood in \mathbf{R}^N . Then π is a Nash diffeomorphism of M such that $\pi - \text{id}$ is $l(4)$ -flat at X_{k-1} and $f - g \circ \pi$ is $l(4)$ -flat at X_k . We can modify π so that $\pi(X) = X$ in the same way as in the proof of step 1, theorem 3.1,(1) and lemma 3.9 because π is an approximation of τ_k and $\tau_k(X) = X$. Thus we complete the proof of the former half of lemma 3.10.

The latter half is clear by the above proof (though (**)) does not necessarily hold, π'_k is close to 1 in the Nash topology, which is sufficient to proceed). \square

The following lemma is also a globalization of Proposition 2, Chapter II, [T] and shows sufficient conditions for two functions to be right equivalent.

Proposition 3.11. (i) *Let f be a C^ω function on a C^ω manifold M . Let v_i , $i = 1, \dots, k$, be C^ω vector fields on M , and I denote the ideal of $C^\infty(M)$ or $C^\omega(M)$ generated by $v_i f$, $i = 1, \dots, k$. Let ϕ be a small C^∞ or C^ω function on M contained in I^2 in the strong Whitney C^∞ topology. Then f and $f + \phi$ are C^∞ or C^ω right equivalent, respectively, and the diffeomorphism of equivalence can be chosen to be close to id in the same topology.*

(ii) *If f , M and v_i are of class Nash or C^∞ or C^ω , and ϕ is of the form $\sum_{i,j=1}^k \phi_{i,j} v_i f \cdot v_j f$ for some small Nash or C^∞ or C^ω functions $\phi_{i,j}$ in the Nash or (strong) Whitney C^∞ topology, then f and $f + \phi$ are Nash or C^∞ or C^ω right equivalent, respectively, by a Nash or C^∞ or C^ω diffeomorphism close to id in the same topology.*

(iii) *Assume that M is a Nash manifold and f is a Nash function on M with only normal crossing singularities. Set $X = f^{-1}(f(\text{Sing } f))$. Let ϕ be a Nash function on M r -flat at X for some large $r \in \mathbf{N}$. Then there exists a Nash diffeomorphism $\pi : V_1 \rightarrow V_2$ between closed semialgebraic neighborhoods of X in M close to id in the semialgebraic $C^{r'}$ topology, for $0 < r' (\ll r) \in \mathbf{N}$, such that $f \circ \pi = f + \phi$ on V_1 , $\pi - \text{id}$ is r' -flat at X , and π is extensible to a semialgebraic C^r diffeomorphism of M .*

Proof of proposition 3.11. Consider the analytic case. We want to reduce (i) to (ii). For a while we proceed in the strong Whitney C^∞ topology. In the same way as the remarks in the proof of step 1, theorem 3.1,(1) we can prove that for “ C^ω ” functions ξ_1, \dots, ξ_l on M , the map $\Xi^\infty : C^\infty(M)^l \ni (h_1, \dots, h_l) \rightarrow \sum_{i=1}^l \xi_i h_i \in \sum_{i=1}^l \xi_i C^\infty(M)$ is open. Moreover, the map $\Xi^\omega : C^\omega(M)^l \ni (h_1, \dots, h_l) \rightarrow \sum_{i=1}^l \xi_i h_i \in \sum_{i=1}^l \xi_i C^\omega(M)$ is open as follows.

Let $(h_1, \dots, h_l) \in C^\omega(M)^l$ such that $\sum_{i=1}^l \xi_i h_i$ is small. Then by openness of Ξ^∞ there exists small $(h'_1, \dots, h'_l) \in C^\infty(M)^l$ such that $\sum_{i=1}^l \xi_i h'_i = \sum_{i=1}^l \xi_i h_i$ and hence $(h_1 - h'_1, \dots, h_l - h'_l) \in \text{Ker } \Xi^\infty$. Therefore, it suffices to see that $\text{Ker } \Xi^\omega$ is dense in $\text{Ker } \Xi^\infty$. Let $H = (h_1, \dots, h_l) \in \text{Ker } \Xi^\infty$. We want to approximate H by an element of $\text{Ker } \Xi^\omega$.

Let \mathcal{J} denote the kernel of the homomorphism $: \mathcal{O}^l \supset \mathcal{O}_a^l \ni (\phi_1, \dots, \phi_l) \rightarrow$

$\sum_{i=1}^l \xi_{ia} \phi_i \in \mathcal{O}_a \subset \mathcal{O}$, $a \in M$, which is a coherent sheaf of \mathcal{O} -submodules of \mathcal{O}^l by theorem 2.5. Let $M^{\mathbf{C}}$ and $\mathcal{J}^{\mathbf{C}}$ be Stein and coherent complexifications of M and \mathcal{J} which are complex conjugation preserving. Let $\{U_i\}$ be a locally finite open covering of $M^{\mathbf{C}}$ such that each $\overline{U_i}$ is compact, and let $H_{i,j} = (h_{1,i,j}, \dots, h_{l,i,j})$, $j = 1, \dots, n_i$, $i = 1, 2, \dots$, be global cross-sections of \mathcal{J} such that $H_{i,j}|_M$ are real valued and for each i , $H_{i,1}, \dots, H_{i,n_i}$ generate $\mathcal{J}^{\mathbf{C}}$ on U_i (theorem 2.1). Then $H_{i,1}|_{U_i \cap M}, \dots, H_{i,n_i}|_{U_i \cap M}$ generate $\text{Ker } \Xi^\infty|_{U_i \cap M}$ for the following reason. By Theorem VI,1.1', [Ml] it is equivalent to prove that $\text{Ker } \Xi_a = \mathcal{F}_a \text{Ker } \Xi_a^\omega$, $a \in U_i$, where \mathcal{F}_a is the completion of \mathcal{O}_a in the (the maximal ideal of \mathcal{O}_a)-adic topology and the homomorphisms $\Xi_a^\omega : \mathcal{O}_a^l \rightarrow \mathcal{O}_a$ and $\Xi_a : \mathcal{F}_a^l \rightarrow \mathcal{F}_a$ are naturally defined. However, this condition is the same as the definition of flatness of \mathcal{F}_a over \mathcal{O}_a , and flatness is well-known (see [Ml]). Thus $H_{i,1}|_{U_i \cap M}, \dots, H_{i,n_i}|_{U_i \cap M}$ generate $\text{Ker } \Xi^\infty|_{U_i \cap M}$. Let $\{\rho_i\}$ be a partition of unity of class C^∞ subordinate to $\{U_i \cap M\}$. Then $\rho_i H \in \text{Ker } \Xi^\infty|_{U_i \cap M}$ and we have C^∞ functions $\chi_{i,j}$ on M , $j = 1, \dots, n_i$, $i = 1, 2, \dots$, such that $\text{supp } \chi_{i,j} \subset U_i \cap M$ and $\rho_i H = \sum_{j=1}^{n_i} \chi_{i,j} H_{i,j}|_M$. As in [W] we can approximate $\chi_{i,j}$ by analytic functions $\chi'_{i,j}$ so that each $\chi'_{i,j}$ can be complexified to a complex analytic function $\chi'_{i,j}{}^{\mathbf{C}}$ on $M^{\mathbf{C}}$ and $\sum_{i,j} |\chi'_{i,j}{}^{\mathbf{C}} H_{i,j}|$ is locally uniformly bounded. Then $\sum_{i,j} \chi'_{i,j}{}^{\mathbf{C}} H_{i,j}$ is a complex analytic map from $M^{\mathbf{C}}$ to \mathbf{C}^l , and its restriction to M is an approximation of H and an element of $\text{Ker } \Xi^\omega$.

Thus Ξ^ω is open (we do not know whether this holds in the Nash case). Hence for ϕ in (i) there exist small $\phi_{i,j} \in C^\omega(M)$, $i, j = 1, \dots, k$, such that $\phi = \sum_{i,j=1}^k \phi_{i,j} v_i f \cdot v_j f$. Consequently, (i) is reduced to (ii). From now on, we work in the Whitney C^r topology for any $r > 0 \in \mathbf{N}$, though we can do in the strong Whitney C^∞ topology. We can assume that M is open in its ambient Euclidean space for the following reason.

Let $p : \tilde{M} \rightarrow M$ denote the orthogonal projection of a tubular neighborhood of M in its ambient Euclidean space. Assume that proposition 3.11,(ii) (in the analytic case) holds for \tilde{M} . The map $C^\omega(\tilde{M}) \ni \Psi \rightarrow \Psi|_M \in C^\omega(M)$ is obviously continuous, surjective by corollary 2.4 and open as follows. Let $\xi \in C^\infty(\tilde{M})$ with $\xi = 1$ on M and $\xi = 0$ outside of a small neighborhood of M in \tilde{M} . Then the map $\xi C^\omega(\tilde{M}) \ni \xi \Psi \rightarrow \xi \Psi|_M \in C^\omega(M)$ is open because for $\psi \in C^\omega(M)$ and for $\Psi_0 \in C^\infty(\tilde{M})$, we have $\psi \circ p \in C^\omega(\tilde{M})$ and $(\xi \cdot \psi \circ p)|_M = \psi$ and the map $C^\omega(M) \ni \psi \rightarrow \xi \cdot \psi \circ p + \xi \Psi_0 - \xi \cdot \Psi_0|_M \circ p \in \xi C^\omega(\tilde{M})$ is continuous and carries $\Psi_0|_M$ to $\xi \Psi_0$. Hence for small $\psi \in C^\omega(M)$ there exists small $\xi \Psi \in \xi C^\omega(\tilde{M})$ such that $\Psi|_M = \psi$. Approximate ξ by an analytic function ξ' on \tilde{M} so that $\xi' = 1$ on M . Then $\xi' \Psi$ is analytic on \tilde{M} , close to $\xi \Psi$ and hence small since the map $C^\infty(\tilde{M})^2 \ni (\alpha, \beta) \rightarrow \alpha \beta \in C^\infty(\tilde{M})$ is continuous, and $\xi' \Psi|_M = \psi$. Consequently, the above restriction map $\Psi \rightarrow \Psi|_M$ is open by linearity. Let \tilde{v}_i , $i = 1, \dots, k$, be C^ω vector field extensions of v_i to \tilde{M} , and let $\tilde{\phi}_{i,j}$ be C^ω extensions of $\phi_{i,j}$ to \tilde{M} so small that $f \circ p$ and $f \circ p + \sum_{i,j} \tilde{\phi}_{i,j} \tilde{v}_i(f \circ p) \cdot \tilde{v}_j(f \circ p)$ satisfy the condition in proposition 3.11,(ii) and hence are C^ω right equivalent by a C^ω diffeomorphism $\tilde{\pi}$ close to id, i.e.,

$$f \circ p \circ \tilde{\pi} = f \circ p + \sum_{i,j=1}^k \tilde{\phi}_{i,j} \tilde{v}_i(f \circ p) \cdot \tilde{v}_j(f \circ p) \quad \text{on } \tilde{M}.$$

Set $\pi = p \circ \tilde{\pi}|_M$. Then π is a C^ω diffeomorphism of M close to id, and

$$f \circ \pi = f + \sum_{i,j=1}^k \phi_{i,j} v_i f \cdot v_j f.$$

Thus proposition 3.11,(ii) is proved for M . Hence we assume that M is open in \mathbf{R}^n .

Next we can suppose that $k = n$ and $v_i = \frac{\partial}{\partial x_j}$, $i = 1, \dots, n$, because each v_i is written as $\sum_{j=1}^n \alpha_{i,j} \frac{\partial}{\partial x_j}$ for some C^ω functions $\alpha_{i,j}$ on M .

Let η denote the function on M which measures distance from $\partial M \stackrel{\text{def}}{=} \overline{M} - M$ (if $\partial M = \emptyset$, $\eta \equiv +\infty$). Set $V = \{(x, y) \in M \times \mathbf{R}^n : |y| < \eta(x)\}$ and consider the C^ω function

$$g(x, y) = f(x + y) - f(x) - \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i}(x) \quad \text{for } (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n) \in V.$$

Then g is a global cross-section of the sheaf of \mathcal{O} -ideals \mathcal{I} on V generated by $y_i y_j$, $i, j = 1, \dots, n$. Hence applying theorem 2.3 to the surjective homomorphism $\mathcal{O}^{n^2} \ni (\alpha_{i,j}) \rightarrow \sum_{i,j=1}^n \alpha_{i,j} y_i y_j \in \mathcal{I}$ we obtain C^ω functions $g_{i,j}$ on V , $i, j = 1, \dots, n$, such that $g(x, y) = \sum_{i,j=1}^n y_i y_j g_{i,j}(x, y)$. Then

$$(*) \quad f(x + y) = f(x) + \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i}(x) + \sum_{i,j=1}^n y_i y_j g_{i,j}(x, y).$$

Let $\alpha = (\alpha_{i,j})_{i,j=1,\dots,n}$ be new variables in \mathbf{R}^{n^2} , set

$$\langle \alpha, \partial f \rangle = \left(\sum_{i=1}^n \alpha_{i,1} \frac{\partial f}{\partial x_i}(x), \dots, \sum_{i=1}^n \alpha_{i,n} \frac{\partial f}{\partial x_i}(x) \right),$$

and let W be a small open neighborhood of $M \times \{0\}$ in $M \times \mathbf{R}^{n^2}$ such that

$$(x, \langle \alpha, \partial f \rangle) \in V \quad \text{for } (x, \alpha) \in W.$$

Take y to be $\langle \alpha, \partial f \rangle$ in (*). Then

$$f(x + \langle \alpha, \partial f \rangle) = f(x) + \sum_{i,j}^n \alpha_{i,j} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) + \sum_{i,i',j,j'=1}^n \alpha_{i,i'} \alpha_{j,j'} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) G_{i',j'}(x, \alpha)$$

for C^ω functions $G_{i',j'}(x, \alpha) = g_{i',j'}(x, \langle \alpha, \partial f \rangle)$ on W . Consider the map

$$B : W \ni (x, \alpha) \rightarrow \left(x, \alpha_{i,j} + \sum_{i',j'=1}^n \alpha_{i,i'} \alpha_{j,j'} G_{i',j'}(x, \alpha) \right) \in M \times \mathbf{R}^{n^2}.$$

Then B is id and regular at $M \times \{0\}$. Hence, shrinking W , we assume that B is a diffeomorphism onto an open neighborhood O of $M \times \{0\}$ in $M \times \mathbf{R}^{n^2}$. Set

$B(x, \alpha) = (x, B_{i,j}(x, \alpha))$, and $B^{-1}(x, \beta) = (x, A'(x, \beta))$ for $(x, \beta) \in O$. Then A' is a C^ω map from O to \mathbf{R}^{n^2} ,

$$f(x + \langle \alpha, \partial f \rangle) = f(x) + \sum_{i,j}^n B_{i,j}(x, \alpha) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \quad \text{for } (x, \alpha) \in W,$$

$$f(x + \langle A'(x, \beta), \partial f \rangle) = f(x) + \sum_{i,j}^n \beta_{i,j} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \quad \text{for } (x, \beta) \in O.$$

Choose $\Phi = (\phi_{i,j})$ so small that its graph is contained in O . Then $\pi(x) = x + \langle A'(x, \Phi(x)), \partial f \rangle$ fulfills the requirements in (ii). Here if $\phi_{i,j}$ are small in the Whitney C^r or the strong Whitney C^∞ topology, π is close to id in the respective topology.

If f , M and v_i are of class C^ω and if ϕ is of class C^∞ , the same arguments as above work and the diffeomorphism of equivalence is of class C^∞ . Thus we complete the proof of (ii) in the analytic case. (ii) in the C^∞ or Nash case also is clear by the same proof. The difference is only that existence of C^∞ or Nash $g_{i,j}$ follows from a partition of unity of class C^∞ or theorem 2.8, respectively.

Consider (iii). Assume that M is not compact because the compact case is clear by the following arguments. Let M be imbedded in a Euclidean space so that its closure is a compact Nash manifold with boundary. Now, we consider an open semialgebraic tubular neighborhood of \overline{M} and extend f to the neighborhood as before. Then we can assume that M is open in \mathbf{R}^n and \overline{M} is a compact Nash manifold with corners, and for the construction of π it suffices to see that ϕ is of the form $\sum_{i,j}^n \phi_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$ for some Nash functions $\phi_{i,j}$ on M r' -flat at X , where $0 \ll r' \ll r \in \mathbf{N}$, for the following reason. Assume that there exist such $\phi_{i,j}$. Then by the above proof, we only need to find small semialgebraic $C^{r''}$ functions $\phi'_{i,j}$ on M in the semialgebraic $C^{r''}$ topology such that $\phi'_{i,j} = \phi_{i,j}$ on some semialgebraic neighborhood of X for $0 < r'' \ll r' \in \mathbf{N}$.

Consider only the case $r'' = 1$ because the general case can be proven in the same way. Set $g(x) = \prod_{a \in f(X)} (f(x) - a)^2$, and let h be a Nash function on M extensible to a Nash function \overline{h} on \overline{M} such that $0 < h \leq 1/2$, (1) $|\frac{\partial h}{\partial x_k}| \leq 1$, $k = 1, \dots, n$, and $\overline{h}^{-1}(0) = \overline{M} - M$, which exists since \overline{M} is a compact Nash manifold with corners. Let $\psi(t)$ be a semialgebraic C^1 function on \mathbf{R} such that $0 \leq \psi \leq 1$, $\psi = 1$ on $(-\infty, 1]$ and $\psi = 0$ on $[2, \infty)$. Set $c = \max |\frac{d\psi}{dt}|$. Then for some $m \in \mathbf{N}$, $\phi'_{i,j} = \phi_{i,j} \psi(g/h^m)$ fulfill the requirements as follows. Clearly $\phi'_{i,j} = \phi_{i,j}$ on a semialgebraic neighborhood $\{x \in M : g(x) \leq h^m(x)\}$ of X in M , and $\phi'_{i,j} = 0$ on $\{g(x) \geq 2h^m(x)\}$. Hence we prove that each $\phi'_{i,j}$ is small on $V \stackrel{\text{def}}{=} \{g(x) \leq 2h^m(x)\}$ in the semialgebraic C^1 topology. Let $\epsilon > 0 \in \mathbf{R}$. Let ξ denote the Nash function on M defined to be $\phi_{i,j}/g^2$ on $M - X$ and 0 on X . Then ξ , $\frac{\partial g}{\partial x_k}$ and $\frac{\partial \phi_{i,j}}{\partial x_k}$, $k = 1, \dots, n$, vanish at X . Hence there exists a semialgebraic neighborhood W of X in M where

$$(2) |\phi_{i,j}| \leq \epsilon g^2, \quad (3) \left| \frac{\partial g}{\partial x_k} \right| \leq 1, \quad (4) \left| \frac{\partial \phi_{i,j}}{\partial x_k} \right| \leq \epsilon.$$

By the Lojasiewicz inequality, we have $V \subset W$ for large m . Note that (5) $g \leq$

$1/2^{m-1}$ on V since $h \leq 1/2$. Then on V

$$\begin{aligned}
|\phi'_{i,j}| &= |\phi_{i,j}\psi(\frac{g}{h^m})| \stackrel{(2)}{\leq} \epsilon g^2 \stackrel{(5)}{<} \epsilon, \\
|\frac{\partial \phi'_{i,j}}{\partial x_k}| &\leq |\frac{\partial \phi_{i,j}}{\partial x_k}\psi(\frac{g}{h^m})| + |\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})|(|\frac{\partial g}{\partial x_k}|/h^m + m|g\frac{\partial h}{\partial x_k}|/h^{m+1}), \\
|\frac{\partial \phi_{i,j}}{\partial x_k}\psi(\frac{g}{h^m})| &\stackrel{(4)}{\leq} \epsilon, \\
|\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})\frac{\partial g}{\partial x_k}|/h^m &\stackrel{(3)}{\leq} \frac{c|\phi_{i,j}|}{h^m} \stackrel{\text{by def. of } V}{\leq} \frac{2c|\phi_{i,j}|}{g} \stackrel{(2),(5)}{\leq} c\epsilon, \\
m|\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})g\frac{\partial h}{\partial x_k}|/h^{m+1} &\stackrel{(1)}{\leq} \frac{mc|\phi_{i,j}|g}{h^{m+1}} \stackrel{(2)}{\leq} 2^{\frac{m+1}{m}}mceg^{2-\frac{1}{m}} \stackrel{(5)}{\leq} 2^{4+\frac{1}{m}-2m}mce.
\end{aligned}$$

Hence $\phi'_{i,j}$ is small on V for large m .

It remains to find $\phi_{i,j}$. Let \mathcal{K} denote the sheaf of \mathcal{N} -ideals on M defined by X . Then ϕ is a cross-section of \mathcal{K}^r since ϕ is r -flat at X and since X is normal crossing. On the other hand, $\sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathcal{N} \supset \mathcal{K}^{r'}$ since f has only normal crossing singularities. Hence ϕ is a cross-section of $\sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \mathcal{K}^{r'}$ because of $r' \ll r$. Let g_l , $l = 1, \dots, k'$, be global generators of $\mathcal{K}^{r'}$ (theorem 2.7). Apply theorem 2.8 to the surjective \mathcal{N} -homomorphism $\mathcal{N}^{n^2 k'} \supset \mathcal{N}_a^{n^2 k'} \ni (\alpha_{i,j,l}) \rightarrow \sum \alpha_{i,j,l} g_l a (\frac{\partial f}{\partial x_i})_a (\frac{\partial f}{\partial x_j})_a \in \sum_{i,j=1}^n (\frac{\partial f}{\partial x_i})_a (\frac{\partial f}{\partial x_j})_a \mathcal{K}_a^{r'} \subset \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \mathcal{K}^{r'}$, $a \in M$. Then there exist Nash functions $\phi_{i,j}$, $i, j = 1, \dots, n$, in $H^0(M, \mathcal{K}^{r'})$ such that $\phi = \sum_{i,j=1}^n \phi_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$. It follows that $\phi_{i,j}$ are r' -flat at X . \square

Proposition 3.12. (*Compactification of a Nash function with only normal crossing singularities*) *Let f be a bounded Nash function on a non-compact Nash manifold M with only normal crossing singularities. Then there exist a compact Nash manifold with corners M' and a Nash diffeomorphism $\pi : M \rightarrow \text{Int } M'$ such that $f \circ \pi^{-1}$ is extensible to a Nash function on M' with only normal crossing singularities.*

The analytic case does not necessarily hold.

?We cannot necessarily choose M' with smooth boundary. For example any compact Nash manifold with boundary whose interior is Nash diffeomorphic to $M = \mathbf{R}^3$ is Nash diffeomorphic to a closed ball in \mathbf{R}^3 (Theorem VI.2.2, [S₂]). But there is no Nash function on a 2-sphere with only normal crossing singularities (see remark (v) after theorem 3.2).

Extensibility of a Nash function to a compact Nash manifold with corners is shown in Proposition VI.2.8, [S₂]. Hence the problem is to impose to the extension to have only normal crossing singularities.

Proof of proposition 3.12. Set $n = \dim M$, $X = f^{-1}(f(\text{Sing } f))$, $B^N = \{x \in \mathbf{R}^N : |x| \leq 1\}$ for a positive integer N , and $S^{N-1} = \partial B^N$. Since there exists a Nash imbedding of M into \mathbf{R}^N such that the image is closed in \mathbf{R}^N , we can assume by lemma 3.9 that $M \subset \text{Int } B^N$, $\overline{M} - M \subset S^{N-1}$, \overline{M} is a compact Nash manifold with boundary, \overline{M} intersects transversally with S^{N-1} (in the sense that some Nash manifold extension \tilde{M} of \overline{M} intersects transversally with S^{N-1}), \overline{X} is a normal crossing Nash subset of \overline{M} , and there exists a Nash function g on \tilde{M} with only normal crossing singularities such that $g(\text{Sing } g) = f(\text{Sing } f)$, $g^{-1}(g(\text{Sing } g)) \cap \overline{M} =$

\overline{X} , $g = f$ on X , for each $a \in X$, $g(x) - g(a)$ has the same multiplicity as $f(x) - f(a)$ at a and $g(b) > g(a)$ if and only if $f(b) > f(a)$ for $b \in M$. We do not know whether $g|_M$ is Nash right equivalent to f . We will modify \overline{M} and g so that this is the case and $g|_{\overline{M}}$ has only normal crossing singularities.

Let ϕ be a polynomial function on \mathbf{R} such that $\phi^{-1}(0) = f(\text{Sing } f)$ and ϕ is regular at $\phi^{-1}(0)$. Let $r \in \mathbf{N}$ be large enough. Apply lemma 3.10 to $\phi \circ f$ and $\phi \circ g|_M$. Then we have a Nash diffeomorphism τ_1 of M such that $\tau_1(X) = X$ and $f \circ \tau_1 - g|_M$ is r -flat at X . Hence replacing f with $f \circ \tau_1$, we assume that $f - g$ is r -flat at X . Next, by proposition 3.11, (iii) there exists a semialgebraic C^r diffeomorphism τ_2 of M such that $g = f \circ \tau_2$ on a semialgebraic neighborhood V of X in M and τ_2 is of class Nash on V . We can choose V of the form $\{x \in M : \phi^2 \circ g(x) \leq c(x)\xi^m(x)\}$ by the Lojasiewicz inequality, where $\xi(x) = (1 - |x|^2)/2$ for $x \in \tilde{M}$, c is a positive Nash function on \tilde{M} such that c depends on only $|x|$ and m is a large odd integer. Shrink \tilde{M} so that $\xi < 0$ on $\tilde{M} - \overline{M}$. We can choose, moreover, c and m so that $\phi^2 \circ g - c\xi^m$ is regular at (its zero set) $-S^{N-1}$ and hence V is a Nash manifold with boundary $\{x \in M : \phi^2 \circ g(x) = c(x)\xi^m(x)\}$ for the following reason. Let $0 < \epsilon_0 \in \mathbf{R}$ be small. Then for any $0 < \epsilon \in \mathbf{R}$ with $\epsilon < \epsilon_0$, $\xi^{-1}(\epsilon) \cup (\phi \circ g)^{-1}(0)$ is normal crossing in \tilde{M} , and hence for small c and large m , $\phi^2 \circ g$ on $\{x \in \xi^{-1}(\epsilon) : 0 < \phi^2 \circ g(x) < 2c(x)\xi^m\}$ is regular. We can choose c and m independently of ϵ . Therefore, $\phi^2 \circ g - c\xi^m$ is regular at (its zero set) $\cap \xi^{-1}((0, \epsilon_0))$. Moreover, if we choose c and m so that $c\xi^m$ is close to a small constant on $M - \xi^{-1}((0, \epsilon_0/2])$, then $\phi^2 \circ g - c\xi^m$ is regular at (its zero set) $-\xi^{-1}((0, \epsilon_0/2])$. Hence $\phi^2 \circ g - c\xi^m$ can be regular at (its zero set) $-S^{N-1}$. However, we omit c for simplicity of notation. We want first to modify M so that \overline{V} is a neighborhood of \overline{X} in \overline{M} .

Apply theorem 2.10 to the two sheaves of \mathcal{N} -ideals on \tilde{M} defined by $(\phi \circ g)^{-1}(0)$ and generated by $\xi \cdot (\phi^2 \circ g - \xi^m)$. Note that the former sheaf is normal crossing, the stalk of the latter is not generated by one regular function germ at a point of $\overline{X} - X$ only, and at least one of the 2 stalks of the both sheaves at each $x \notin \overline{X} - X$ is \mathcal{N}_x . Then we have a composition of a finite sequence of blowings-up $\tau_3 : \hat{M} \rightarrow \tilde{M}$ along smooth Nash centers such that $\tau_3|_{\tau_3^{-1}(\tilde{M} - (\overline{X} - X))} : \tau_3^{-1}(\tilde{M} - (\overline{X} - X)) \rightarrow \tilde{M} - (\overline{X} - X)$ is a Nash diffeomorphism and $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3$ has only normal crossing singularities at its zero set, say Y . It follows that $(\hat{M}, Y, \tau_3^{-1}(M))$, locally at each point of $\overline{\tau_3^{-1}(M)}$, is Nash diffeomorphic to $(\mathbf{R}^n, \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\})$, the closure of a union of some connected components of $\{(x_1 \cdots x_{n'} \neq 0)\}$ for some $n' (\leq n) \in \mathbf{N}$ and $\overline{\tau_3^{-1}(M)} - \tau_3^{-1}(\overline{X} - X)$ is a Nash manifold with boundary. However, $\tau_3^{-1}(M)$ is not necessarily a manifold with corners. The case where $\tau_3^{-1}(M)$ is locally diffeomorphic to a union of more than one connected components of $\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} \neq 0\}$ at some point of $\tau_3^{-1}(\overline{X} - X)$, $0 < n' (\leq n) \in \mathbf{N}$ may appear. Then we need to separate these connected components. That is possible as shown in the proof of Theorem VI.2.1, [S₂]. Namely, there exist a compact Nash manifold L with corners and a Nash immersion $\tau_4 : L \rightarrow \tau_3^{-1}(M)$ such that $\tau_4|_{L - \text{Sing } \partial L}$ is a Nash diffeomorphism to its image and the image contains $\overline{\tau_3^{-1}(M)} - \tau_3^{-1}(\overline{X} - X) (\supset \tau_3^{-1}(M))$.

Clearly $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$ has only normal crossing singularities at its zero set $\tau_4^{-1}(Y)$ since τ_4 is an immersion. Set $\tau = \tau_2 \circ \tau_3 \circ \tau_4|_{\text{Int } L}$, $h = g \circ \tau_3 \circ \tau_4$, $W = (\tau_3 \circ \tau_4)^{-1}(V)$, which is a non-compact Nash manifold with boundary, $W' = \overline{W} - \partial W$ and $Z = (\tau_3 \circ \tau_4)^{-1}(X)$. Then τ is a semialgebraic C^r

diffeomorphism from $\text{Int } L$ to M and of class Nash on W , h is a Nash function on L , $h = f \circ \tau$ on W , h is regular on $\text{Int } L - Z$, $h|_{\text{Int } L \cup W'}$ has only normal crossing singularities at \overline{Z} though h is not necessarily so globally (note that W' and \overline{W} are Nash manifolds with corners by the next fact and the normal crossing property of $(\xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$), and \overline{W} is a neighborhood of \overline{Z} in L because if it were not, $\overline{Z} \cap (\tau_3 \circ \tau_4)^{-1}(\{x \in M : \phi^2 \circ g(x) = \xi^m(x)\})$ could be not empty and of dimension $n - 2$ but contained in $(\tau_3 \circ \tau_4)^{-1}(\xi^{-1}(0))$, which contradicts the normal crossing property of $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$. Thus \overline{V} is changed to \overline{W} —a neighborhood of \overline{Z} in L . We consider h on L in place of g on \overline{M} .

We replace τ by a Nash diffeomorphism. Let $0 \ll r \in \mathbf{N}$, set $\overline{\psi} = (\phi^r \circ h \cdot \xi^r) \circ \tau_3 \circ \tau_4$ on L and $\psi = \overline{\psi}|_{\text{Int } L}$, and let \mathcal{I} denote the sheaf of \mathcal{N} -ideals on $\text{Int } L$ generated by ψ . Then we regard τ as an element of $H^0(\text{Int } L, \mathcal{N}/\mathcal{I})^N$ because $\text{supp } \mathcal{N}/\mathcal{I} = Z$ and τ is of class Nash near there. Hence by theorem 2.8 there exists a Nash map $\tau' : \text{Int } L \rightarrow \mathbf{R}^N$ such that $\tau - \tau' = \psi\theta$ for some semialgebraic C^r map $\theta : \text{Int } L \rightarrow \mathbf{R}^N$ of class Nash on W . Approximate θ by a Nash map $\theta' : \text{Int } L \rightarrow \mathbf{R}^N$ in the semialgebraic C^r topology, and set $\tau'' = p \circ (\tau' + \psi\theta')$, where p denotes the orthogonal projection of a semialgebraic tubular neighborhood of M in \mathbf{R}^N . Then τ'' is a well-defined Nash diffeomorphism from $\text{Int } L$ to M and close to τ in the semialgebraic C^r topology, $f \circ \tau'' - h|_{\text{Int } L} = \psi\delta$ for some semialgebraic C^r function δ on $\text{Int } L$ though $f \circ \tau'' - h|_{\text{Int } L}$ does not necessarily vanish on W , and, moreover, δ is extensible to a semialgebraic $C^{r'}$ function $\overline{\delta}$ on $\text{Int } L \cup W'$ for $0 \ll r' (\ll r) \in \mathbf{N}$ by the definition of the semialgebraic $C^{r'}$ topology, by the fact that a small semialgebraic C^r function on $\text{Int } L$ is extensible to a semialgebraic C^r function on L and by

$$f \circ \tau'' - h|_{\text{Int } L} = f \circ p \circ (\tau + \psi \cdot (\theta' - \theta)) - f \circ p \circ \tau \quad \text{on } W.$$

The last equality implies also that δ is of class Nash on W , and hence on $\text{Int } L$ since $f \circ \tau''$ and h are Nash functions and $\psi^{-1}(0) \subset W$.

Next we modify h . Let $\overline{\delta}'$ be a Nash approximation on $\text{Int } L \cup W'$ of $\overline{\delta}$ in the semialgebraic $C^{r'}$ topology, and set $\delta' = \overline{\delta}'|_{\text{Int } L}$ and $h' = h + \psi\overline{\delta}'$ on $\text{Int } L \cup W'$. Then h' is a Nash function on $\text{Int } L \cup W'$ and has only normal crossing singularities at \overline{Z} by the same property of $h|_{\text{Int } L \cup W'}$ and by the definition of h' , and $f \circ \tau'' - h'|_{\text{Int } L}$ is of the form $\psi \cdot (\delta - \delta')$. Hence f and $h' \circ \tau''^{-1}$ satisfy the conditions in proposition 3.11,(ii) because $\phi^r \circ h \circ \tau''^{-1}$ is of the form $\sum_{i,j=1}^k \psi_{i,j} v_i \cdot v_j f$ for some Nash functions $\psi_{i,j}$ on M and Nash vector fields v_i , $i = 1, \dots, k$, on M which span the tangent space of M at each point of M and because $\xi^r|_M \cdot (\delta - \delta') \circ \tau''^{-1}$ is small as a semialgebraic $C^{r'}$ function on M . Consequently, f and $h' \circ \tau''^{-1}$ are Nash right equivalent, and we can replace f with $h'|_{\text{Int } L}$.

We can assume that $W' \cap \partial L$ is the union of some connected components σ of strata of the canonical stratification $\{L_i\}$ of ∂L such that $\overline{\sigma} \cap \overline{Z} \neq \emptyset$ for the following reason. Let ψ_L be a non-negative Nash function on L with zero set \overline{Z} , and let $\epsilon \in \mathbf{R}$ be such that the restriction of ψ_L to $\psi_L^{-1}((0, 2\epsilon))$ is regular. Then $\psi_L^{-1}(\epsilon)$ is a compact Nash manifold with corners $= \partial L \cap \psi_L^{-1}(\epsilon)$. Let $\{L_{\epsilon,i}\}$ denote the canonical stratification of $\partial L \cap \psi_L^{-1}(\epsilon)$. We blow up $L_{\epsilon,i}$ as follows. For a compact Nash submanifold possibly with corners L' of L and for some Nash manifold extension \tilde{L}' of L' , if $\tilde{L}' \cap L = L'$ and if (L, \tilde{L}') is locally diffeomorphic to $(\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \geq 0, \dots, x_{n'} \geq 0\}, \{x_{n_1} = \dots = x_{n_k} = 0\})$ for some

$n' (\leq n), 1 \leq n_1 < \dots < n_k \leq n \in \mathbf{N}$, we say L' has the property $(*)$. For L' with $(*)$, consider $\gamma : \Gamma \rightarrow L$ —the restriction of the blowing-up of a small Nash manifold extension \tilde{L} of L along center $\tilde{L} \cap \tilde{L}'$ to the closure of inverse image of $L - L'$, modify $\gamma : \Gamma \rightarrow L$ so that Γ is a compact Nash manifold with corners by the idea in the proof of Theorem VI.2.1, [S₂] as before, use the same notation $\gamma : \Gamma \rightarrow L$, and call it the $(*)$ -blowing-up of L along center L' . Note that $\gamma^{-1}(L')$ is the closure of a union of some connected components of $\text{Reg } \partial\Gamma$. Set $\Gamma_{-1} = L$ and let $0 \leq k \leq n-2$. Inductively we define $(*)$ -blowing-up $\gamma_k : \Gamma_k \rightarrow \Gamma_{k-1}$ of Γ_{k-1} along center $L_{\epsilon,0}$ if $k = 0$ and center $\overline{(\gamma_0 \circ \dots \circ \gamma_{k-1})^{-1}(L_{\epsilon,k})}$ if $k > 0$, which is possible because $L_{\epsilon,0}$ and $\overline{(\gamma_0 \circ \dots \circ \gamma_k)^{-1}(L_{\epsilon,k+1})}$ for $0 \leq k \leq n-3$ are compact Nash submanifolds with corners of Γ_{-1} and Γ_k with $(*)$, respectively. Thus we assume that the above condition on W holds considering $(\Gamma_{n-2}, (\psi_L \circ \gamma_0 \circ \dots \circ \gamma_{n-2})^{-1}([0, \epsilon]) - \partial\Gamma_{n-2})$ in place of (L, W) . Here we choose ϵ so small that $(**)$ h' is extensible to a Nash function on an open semialgebraic neighborhood of $\text{Int } L \cup \overline{W}$ in L with only normal crossing singularities.

Moreover, we can assume that the closure of each connected component of $\text{Reg } \partial L$ is a Nash manifold possibly with corners. Indeed, that follows if we repeat the same arguments as above to the canonical stratification of ∂L compatible with $\{x \in \partial L : \text{dis}(x, L_k) = \epsilon_k, \text{dis}(x, L_i) \geq \epsilon_i, i = 0, \dots, k-1\}, k = 0, \dots, n-2$ (here we naturally define the canonical stratification of ∂L compatible with the above family in the same way as the remark after lemma 3.8). After this modification of L , the property $(**)$ continues to hold.

Let $M_j, j \in J$, be the set of closures of the connected components of $\text{Reg } \partial L$, and let J_0 denote the subset of J consisting of j with $M_j \cap \overline{Z} = \emptyset$. Let \tilde{L} and \tilde{M}_j be Nash manifold extensions of L and M_j , respectively, which are contained and closed in a small open semialgebraic neighborhood U of L in the ambient Euclidean space such that $\cup_{j \in J} \tilde{M}_j$ is normal crossing in \tilde{L} and for each $j \in J$ there is one and only one connected component of $\tilde{L} - \tilde{M}_j$ which does not intersect with L . Let \tilde{Z} denote the smallest Nash subset of \tilde{L} containing Z . Then \tilde{Z} is normal crossing in \tilde{L} , and there exist Nash functions χ_j on \tilde{L} with zero set \tilde{M}_j , regular there and with $\chi_j > 0$ on $\text{Int } L$.

By $(**)$ we can choose a sufficiently small U so that h' can be extended to a Nash function h'_+ on $L_+ \stackrel{\text{def}}{=} \{x \in \tilde{L} : \chi_j(x) > 0, j \in J_0\}$, $h'_+(\text{Sing } h'_+) = h'(Z)$ and h'_+ having only normal crossing singularities. Now we smooth h'_+ at $\overline{L_+} - L_+$ as in the proof of Proposition VI.2.8, [S₂]. Let $\tilde{L} \subset \mathbf{R}^N$, set $G = \text{graph } h'_+ \subset L_+ \times \mathbf{R}$, and G^Z be the Zariski closure of G in $\mathbf{R}^N \times \mathbf{R}$ and Q be the normalization of G^Z in $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'}$ for some $N' \in \mathbf{N}$, and let $r : Q \rightarrow \mathbf{R}^N \times \mathbf{R}$ and $q : Q \rightarrow \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \rightarrow \mathbf{R}^N \times \mathbf{R}$ and $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \rightarrow \mathbf{R}^N$, respectively. Then it is known that r is a proper map to G^Z , and by Artin-Mazur Theorem there exists a union of connected components R of $Q - r^{-1}(\overline{G} - G)$ such that $R \subset \text{Reg } Q$ and $r|_R$ is a Nash diffeomorphism onto G . Here we can replace $r^{-1}(\overline{G} - G)$ with a Nash subset $q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0))$ of Q because $r^{-1}(\overline{G} - G) \subset q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0))$ and $R \cap q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0)) = \emptyset$, $q|_R$ is a Nash diffeomorphism onto L_+ , $h'_+ \circ q|_R$ is the restriction of the projection $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \rightarrow \mathbf{R}^N$ and hence extensible to a smooth rational function on Q , $\overline{R \cap q^{-1}(L)}$ is compact because r is proper and because $\overline{G \cap L \times \mathbf{R}}$ is compact by boundedness of f , and $h'_+ \circ q|_R$ has only normal crossing singularities because of

the same property of h'_+ (however, $\chi_j \circ q$ are now not necessarily regular at their zero sets). By theorem 2.8, $R \cap q^{-1}(\tilde{Z})$ is a Nash subset of $\text{Reg } Q$ and there exists a Nash function α on $\text{Reg } Q$ whose zero set is $R \cap q^{-1}(\tilde{Z})$ and which has only normal crossing singularities there since $R \cap q^{-1}(\tilde{Z})$ is a Nash subset of R and since its closure in $\text{Reg } Q$ does not intersect with $\overline{R} - R$.

Thus replacing \tilde{L} , L_+ , h'_+ , χ_j and \tilde{Z} with $\text{Reg } Q$, R , $h'_+ \circ q|_R$, $\chi_j \circ q|_{\text{Reg } Q}$ and $R \cap q^{-1}(\tilde{Z})$ we assume from the beginning that M and f satisfy moreover the following conditions.

- (i) \tilde{f} and χ_j , $j \in J$, are a finite number of Nash functions on a Nash manifold \tilde{M} , and \tilde{X} is a normal crossing Nash subset of \tilde{M} .
- (ii) M is a union of some connected components of $\tilde{M} - (\prod_{j \in J} \chi_j)^{-1}(0)$, \overline{M} is compact, $f = \tilde{f}|_M$, and $X = \tilde{X} \cap M$ (we do not assume that \overline{M} is a manifold with corners).

We make $\prod_{j \in J} \chi_j$ normal crossing at its zero set. Apply theorem 2.10 to the sheaf of \mathcal{N} -ideals on \tilde{M} defined by \tilde{X} and the sheaf of \mathcal{N} -ideals $\prod_{j \in J} \chi_j \mathcal{N}$. Then via blowings-up, $\prod_{j \in J} \chi_j$ becomes to have only normal crossing singularities at its zero set, and the conditions (i) and (ii) do not change because the subset of \tilde{M} where we modify by blowings-up is contained in $(\prod_{j \in J} \chi_j)^{-1}(0)$.

It remains to make \tilde{f} together with $(\prod_{j \in J} \chi_j)^{-1}(0)$ normal crossing. Let $\{\tilde{M}_i\}$ denote the canonical stratification of $(\prod_{j \in J} \chi_j)^{-1}(0)$, set $\tilde{M}_n = \tilde{M} - (\prod_{j \in J} \chi_j)^{-1}(0)$, and let $\tilde{\phi}$ be a polynomial function on \mathbf{R} such that $\tilde{\phi}^{-1}(0) = \cup_{i=0}^n \tilde{f}(\text{Sing } \tilde{f}|_{\tilde{M}_i})$ and $\tilde{\phi}$ is regular at $\tilde{\phi}^{-1}(0)$. Once more, apply theorem 2.10 to the sheaf of \mathcal{N} -ideals on \tilde{M} defined by $\tilde{X} \cup (\prod_{j \in J} \chi_j)^{-1}(0)$ and the sheaf of \mathcal{N} -ideals $[\tilde{\phi} \circ \tilde{f} \mathcal{N} : \cap_i \mathcal{I}_i^{\alpha_i}] \stackrel{\text{def}}{=} \cup_{x \in \tilde{M}} \{\rho \in \mathcal{N}_x : \rho \cap_i \mathcal{I}_i^{\alpha_i} \subset \tilde{\phi} \circ \tilde{f} \mathcal{N}_x\}$, where $\cap_i \mathcal{I}_i$ is the decomposition of the sheaf of \mathcal{N} -ideals on \tilde{M} defined by \tilde{X} to irreducible finite sheaves of \mathcal{N} -ideals and each α_i is the maximal integer such that $\tilde{\phi} \circ \tilde{f} \mathcal{N}$ is divisible by $\mathcal{I}_i^{\alpha_i}$. Then $(\tilde{f} - \tilde{f}(x_0)) \prod_{j \in J} \chi_j$ becomes to have only normal crossing singularities at its zero set for each $x_0 \in \tilde{M}$ and the subset of \tilde{M} where we modify now by blowings-up does not intersect with M because the stalk of the latter sheaf at each point of M is generated by a regular function germ and because $(\tilde{X} \cup (\prod_{j \in J} \chi_j)^{-1}(0) \cap \text{supp } \mathcal{N}) / [\tilde{\phi} \circ \tilde{f} \mathcal{N} : \cap_i \mathcal{I}_i^{\alpha_i}] \cap M = \emptyset$. Finally, we separate as before M at the points of \overline{M} where M is not locally connected so that \overline{M} is a compact Nash manifold with corners. Then $\tilde{f}|_{\overline{M}}$ has only normal crossing singularities, and we complete the proof. \square

Proof of theorem 3.2. By proposition 3.4 it suffices to prove the Nash case and, moreover, that the cardinality of Nash R-L equivalence classes of Nash functions with only normal crossing singularities on a compact Nash manifold possibly with corners is zero or countable because we can restrict functions to being bounded (remember the fact that \mathbf{R} is Nash diffeomorphic to $(0, 1)$) and because by proposition 3.12, for a bounded Nash function f with only normal crossing singularities on a non-compact Nash manifold M , we can regard M and f as the interior of a compact Nash manifold with corners M' and the restriction to M of a Nash function on M' with only normal crossing singularities. Assume that there is at least one Nash function f on M with only normal crossing singularities. Then the cardinality is infinite because we can increase arbitrarily the cardinality of the critical value set, which is finite, by replacing f with $\pi \circ f$ for some Nash function

π on \mathbf{R} . Let $\{X_\alpha\}_{\alpha \in A}$ denote all normal crossing Nash subsets of M . We define α and α' in A to be equivalent if there exists a Nash diffeomorphism of M which carries X_α to $X_{\alpha'}$. Then by lemma 3.7 the cardinality of equivalence classes of A is countable. Hence it suffices to see that for each X_α there exist at most a countable number of Nash R-L equivalence classes of Nash functions f on M with only normal crossing singularities such that $f^{-1}(f(\text{Sing } f)) = X_\alpha$. Let F_α denote all such Nash functions. Clearly there are a finite number of equivalence classes of $\{f|_{X_\alpha} : X_\alpha \rightarrow \mathbf{R} : f \in F_\alpha\}$ under the Nash left equivalence relation since the value sets are finite. Moreover, there are at most a countable number of choices of multiplicity of $f - f(a)$ at a for $f \in F_\alpha$ and $a \in X_\alpha$. Hence we reduce the problem to the following one. Fix $f \in F_\alpha$, and let F_f denote the family of $g \in F_\alpha$ such that $g = f$ on X_α and $g - g(a)$ has the same multiplicity as $f - f(a)$ at each point a of X_α . Then the cardinality of Nash right equivalence classes of functions in F_f is finite. Moreover, it suffices to prove that each element of F_f , say f , is stable in F_f in the sense that any $g \in F_f$ near f in the C^∞ topology is Nash right equivalent to f because there are only a finite number of connected components in F_f .

Set $n = \dim M$, imbed M in \mathbf{R}^N , and let $\{M_i\}$ denote the canonical stratification of M . There exist Nash vector fields v_1, \dots, v_k on M such that for each $x \in M_i$, v_{1x}, \dots, v_{kx} span the tangent space $T_x M_i$ of M_i at x and if we regard M as $\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \geq 0, \dots, x_{n'} \geq 0\}$ by its local coordinate system then for each $1 \leq i \leq n'$, $x_i \frac{\partial}{\partial x_i}$ is contained in the linear space over $N(M)$ spanned by v_1, \dots, v_k for the following reason.

Set $L_i = \cup_{j=0}^i M_j$, $i = 0, \dots, n-1$, and choose a Nash manifold extension \tilde{M} of M and Nash subset extensions \tilde{L}_i of L_i in \tilde{M} so that \tilde{L}_{n-1} is normal crossing in \tilde{M} and $\{\tilde{L}_i - \tilde{L}_{i-1}\}$ is the canonical stratification of \tilde{L}_{n-1} . Set $L_n = M$ and $\tilde{L}_n = \tilde{M}$ also. Then when we describe $(\tilde{M}, \tilde{L}_{n-1})$ by a local coordinate system as $(*) (\mathbf{R}^n, \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\})$,

$$\tilde{L}_i = \bigcup_{1 \leq j_1 < \dots < j_{n-i} \leq n'} \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_{j_1} = \dots = x_{j_{n-i}} = 0\}, \quad n - n' \leq i \leq n.$$

We consider on \tilde{M} in place of on M because existence of v_1, \dots, v_k follows from existence of Nash vector fields on \tilde{M} with the same properties.

First, let $w_{n,1}, \dots, w_{n,k_n}$ be Nash vector fields on \tilde{M} which span the tangent space of \tilde{M} at each point, and let α_n be a global generator of the sheaf of \mathcal{N} -ideals on \tilde{M} defined by \tilde{L}_{n-1} —we can choose \tilde{M} so that α_n exists because M is a manifold with corners. Then $v_{n,1} = \alpha_n w_{n,1}, \dots, v_{n,k_n} = \alpha_n w_{n,k_n}$ are Nash vector fields on \tilde{M} , span the tangent space of \tilde{M} at each point of $\tilde{M} - \tilde{L}_{n-1}$ and vanish at \tilde{L}_{n-1} , and in the case $(*)$, for each $1 \leq i \leq n'$, $x_i \frac{\partial}{\partial x_i}$ on $\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_j \neq 0 \text{ for } 1 \leq j \leq n' \text{ with } j \neq i\}$ is contained in the linear space over the Nash function ring on the set spanned by $v_{n,1}, \dots, v_{n,k_n}$.

Next fix $i < n$ and consider on \tilde{L}_i . Then it suffices to prove the following two statements. (i) There exist Nash vector fields $v_{i,1}, \dots, v_{i,k_i}$ on \tilde{L}_i —Nash cross-sections of the restrictions to \tilde{L}_i of the tangent bundle of \mathbf{R}^N , i.e. the restrictions to \tilde{L}_i of Nash vector fields on \mathbf{R}^N by theorem 2.8—which span the tangent space of $\tilde{L}_i - \tilde{L}_{i-1}$ at its each point and vanish at \tilde{L}_{i-1} and such that in the case of $(*)$ the condition on each irreducible component $\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_{j_1} = \dots = x_{j_{n-i}} = 0\}$, same as on \tilde{M} , is satisfied for $1 \leq j_1 < \dots < j_{n-i} \leq n'$; to be precise, for any

$1 \leq j \leq n'$ other than j_1, \dots, j_{n-i} , then $x_j \frac{\partial}{\partial x_j}$ on $\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_{j_1} = \dots = x_{j_{n-i}} = 0, x_l \neq 0 \text{ for } 1 \leq l \leq n' \text{ other than } j_1, \dots, j_{n-i}, j\}$ is contained in the linear space over the Nash function ring on the set spanned by $v_{i,1}, \dots, v_{i,k_i}$. (ii) Any Nash vector field on \tilde{L}_i tangent to $\tilde{L}_j - \tilde{L}_{j-1}$ at its each point for $j \leq i$ is extensible to a Nash vector field on \tilde{L}_{i+1} tangent to $\tilde{L}_{i+1} - \tilde{L}_i$ at each its point.

Proof of (i). By considering the Zariski closure of \tilde{L}_i and its normalization and by Artin-Mazur Theorem we have a Nash manifold P_i and a Nash immersion $\xi_i : P_i \rightarrow \tilde{L}_i$ such that $\xi_i|_{P_i - \xi_i^{-1}(\tilde{L}_{i-1})}$ is a Nash diffeomorphism onto $\tilde{L}_i - \tilde{L}_{i-1}$. Note that $\xi_i^{-1}(\tilde{L}_{i-1})$ is normal crossing in P_i . Apply the same arguments to $(P_i, \xi_i^{-1}(\tilde{L}_{i-1}))$ as on $(\tilde{M}, \tilde{L}_{n-1})$ (the difference is only that we need a finite number of global generators $\alpha_{i,1}, \alpha_{i,2}, \dots$ of the sheaf of \mathcal{N} -ideals on P_i defined by $\xi_i^{-1}(\tilde{L}_{i-1})$). Then there exist Nash vector fields $w_{i,1}, \dots, w_{i,k_i}$ on P_i with the corresponding properties, and they induce semialgebraic C^0 vector fields $v_{i,1}, \dots, v_{i,k_i}$ on \tilde{L}_i through ξ_i because $w_{i,1}, \dots, w_{i,k_i}$ vanish on $\xi_i^{-1}(\tilde{L}_{i-1})$. It is easy to see that $v_{i,1}, \dots, v_{i,k_i}$ are of class Nash by the normal crossing property of \tilde{L}_{n-1} in \tilde{M} and satisfy the conditions in (i).

Proof of (ii). Let v be a Nash vector field on \tilde{L}_i in (ii), let $\xi_{i+1} : P_{i+1} \rightarrow \tilde{L}_{i+1}$ be the same as above. Then since ξ_{i+1} is an immersion, v pulls back a Nash cross-section w of the restriction to $\xi_{i+1}^{-1}(\tilde{L}_i)$ of the tangent bundle of the Nash manifold P_{i+1} , and by theorem 2.8 we obtain a Nash vector field on P_{i+1} whose restriction to $\xi_{i+1}^{-1}(\tilde{L}_i)$ is w . This vector field induces a Nash vector field of \tilde{L}_{i+1} through ξ_{i+1} , which is an extension of v , by the same reason as in the proof of (i).

Let $g \in F_f$ near f . It suffices to see that f and g are C^ω right equivalent for the following reason. Assume that there exists a C^ω diffeomorphism π of M such that $f = g \circ \pi$. Let \tilde{M} and \tilde{L}_{n-1} be the same as above and so small that f and g are extensible to Nash functions \tilde{f} and \tilde{g} on \tilde{M} , respectively. Extend π to a C^ω diffeomorphism $\tilde{\pi} : U_1 \rightarrow U_2$ between open neighborhoods of M in \tilde{M} so that $\tilde{\pi}(U_1 \cap \tilde{L}_{n-1}) \subset \tilde{L}_{n-1}$ and $\tilde{f} = \tilde{g} \circ \tilde{\pi}$. As above, let α_n be a global generator of the sheaf of \mathcal{N} -ideals on \tilde{M} defined by \tilde{L}_{n-1} . Then $\alpha_n \circ \tilde{\pi} = \beta \alpha_n$ on U_1 for some positive C^ω function β on U_1 . Consider the following equations in variables $(x, y, z) \in \tilde{M}^2 \times \mathbf{R}$:

$$f(x) - g(y) = 0 \quad \text{and} \quad \alpha_n(y) - z\alpha_n(x) = 0$$

(here the second equation means that if $x \in \tilde{L}_{n-1}$ then $y \in \tilde{L}_{n-1}$). Then $y = \tilde{\pi}(x)$ and $z = \beta(x)$ are C^ω solutions. Hence by Theorem 1.1, [C-R-S₁] there exist Nash germ solutions on M $y = \pi'(x)$ and $z = \beta'(x)$ which are approximations of the germs of $\tilde{\pi}$ and β on M . Thus $\pi'|_M$ is a Nash diffeomorphism of M and $f = g \circ \pi'$ on M .

Now we show that C^ω right equivalence of f and g . Set $G(x, t) = (1-t)f(x) + tg(x)$ for $(x, t) \in M \times [0, 1]$. Then $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$. Hence by the same reason as in the proof of theorem 3.1, (1) it suffices to find a C^ω vector field v on $M \times [0, 1]$ of the form $\frac{\partial}{\partial t} + \sum_{i=1}^k a_i v_i$ for some C^ω functions a_i on $M \times [0, 1]$ such that $vG = 0$ on $M \times [0, 1]$, i.e.,

$$(**) \quad f - g = \sum_{i=1}^k a_i (v_i f + t v_i (g - f)).$$

Moreover, as shown there, we only need to solve this equations locally at each point (x_0, t_0) of $M \times [0, 1]$ since M is compact.

If $x_0 \notin X_\alpha$, $(v_i f)(x_0) \neq 0$ for some i and hence $a_j = 0$ for $j \neq i$ and $a_i = (f - g)/(v_i f + t v_i(g - f))$ around (x_0, t_0) are solutions of (**) because $g - f$ and hence $t v_i(g - f)$ are small in the C^∞ topology.

Let $x_0 \in X_\alpha$. Then we can assume that $M = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : |x| \leq 1, x_1 \geq 0, \dots, x_{n'} \geq 0\}$ for some $n' (\leq n) \in \mathbf{N}$, $x_0 = 0$, $f(x) = x^\beta$ for some $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$ with $|\beta| > 0$, $k = n$, $v_1 = x_1 \frac{\partial}{\partial x_1}, \dots, v_{n'} = x_{n'} \frac{\partial}{\partial x_{n'}}$, $v_{n'+1} = \frac{\partial}{\partial x_{n'+1}}, \dots, v_n = \frac{\partial}{\partial x_n}$ and $f - g = b x^\beta$ for some small C^ω function b on M (remember that the map $C^\infty(M) \ni \psi \rightarrow \psi x^\beta \in x^\beta C^\infty(M)$ is open in the C^∞ topology). Let i be such that $\beta_i \neq 0$. Then $v_i f = \beta_i x^\beta / x_i$ and $v_i(f - g) = b \beta_i x^\beta / x_i + \frac{\partial b}{\partial x_i} x^\beta$ if $i > n'$, and $v_i f = \beta_i x^\beta$ and $v_i(f - g) = b \beta_i x^\beta + x_i \frac{\partial b}{\partial x_i} x^\beta$ if $i \leq n'$. In any case (**) is solved as before. Thus theorem 3.2 is proved. \square

REFERENCES

- [B-C-R] J. Bochnak, M. Coste and M.F. Roy, *Real algebraic geometry*, Springer, 1998.
- [C] H. Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France, **85** (1957), 77–99.
- [C-R-S₁] M. Coste, J.M. Ruiz and M. Shiota, *Approximation in compact Nash manifolds*, Amer. J. Math., **117** (1995), 905–927.
- [C-R-S₂] ———, *Separation, factorization and finite sheaves on Nash manifolds*, Comp. Math., **103** (1996), 31–62.
- [C-S₁] M. Coste and M. Shiota, *Nash triviality in families of Nash manifolds*, Invent. Math., **108** (1992), 349–368.
- [C-S₂] ———, *Thom’s first isotopy lemma: a semialgebraic version, with uniform bound*, Real analytic and algebraic geometry, Walter de Gruyter, (1995), 83–101.
- [C-S₃] ———, *Nash functions on noncompact Nash manifolds*, Ann. Sci. Éc. Norm. Sup., **33** (2000), 139–149.
- [E] G. Efrogmson, *A Nullstellensatz for Nash rings*, Pacific J. Math., **54** (1974), 101–112.
- [Fi] G. Fichou, *Motivic invariants of arc-symmetric sets and blow-Nash equivalence*, Compositio Math., **141** (2005), 655–688.
- [Fu] T. Fukui, *Seeking invariants for blow-analytic equivalence*, Compositio Math., **105** (1997), 95–107.
- [F-K-K] T. Fukui, S. Koike and T.C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, Real Analytic and Algebraic Singularities, Pitman Research Notes in Mathematics Series, **381** (1998), 8–29.
- [F-P] T. Fukui and L. Paunescu, *On blow-analytic equivalence in “Arc spaces and additive invariants in real algebraic geometry”*, Proceedings of Winter School “Real algebraic and analytic geometry and motive integration”, Aussoie 2003, Panoramas et Synthèses, SMF **26** (2008), 87–125.
- [G-R] C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, 1965.
- [Hd] R. Hardt, *Semialgebraic local-triviality in semialgebraic mappings*, Amer. J. Math., **102** (1980), 291–302.
- [Ht] R. Hartshorne, *Algebraic geometry*, Springer, 1997.
- [Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math., **79** (1964), 109–326.
- [Hu] J. Hubbard, *On the cohomology of Nash sheaves*, Topology, **11** (1972), 265–270.
- [Ko] S. Koike, *Modified Nash triviality theorem for a family of zero-sets of weighted homogeneous polynomial mappings*, J. Math. Soc. Japan, **49** (1997), 617–631.
- [Ku] T.-C. Kuo, *On classification of real singularities*, Invent. Math., **82** (1985), 257–262.
- [K-W] T.C. Kuo and J. Ward, *A theorem on almost analytic equisingularities*, J. Math. Soc. Japan, **33** (1981), 471–484.
- [Ml] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, 1966.

- [Mz] B Mazur, *A note on some contractible 4-manifolds*, Ann. of Math., **173** (1956), 221-228.
- [Mi] J.W. Milnor, *Lectures on the h-cobordism theorem*, Princeton U. P., 1965.
- [Ri] J.J. Risler, *Sur l'anneau des fonctions de Nash globales*, Ann. Sci. Ecole Norm. Sup., **8** (1975), 365-378.
- [S₁] M. Shiota, *Equivalence of differentiable mappings and analytic mappings*, Publ. Math. IHES, **54** (1981), 237-322.
- [S₂] ———, *Nash manifolds*, Lecture Notes in Math., **1269** (1987), Springer.
- [S₃] ———, *Geometry of subanalytic and semialgebraic sets*, Birkhauser, 1997.
- [Sp] M. Spivakovsi, *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms*, J. Amer. Math. Soc., **12** (1999), 381-444.
- [T] J.C. Tougeron, *Idéaux de fonctions différentiables I*, Ann. Inst. Fourier (1968), 177-240.
- [W] W. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., **36** (1934), 63-89.

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