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Utility maximization in incomplete markets with default

Thomas LIM * Marie-Claire QUENEZ †

Abstract

We address the maximization problem of expected utility from terminal wealth. The special feature of this paper is that we consider a financial market where the price process of risky assets can have a default time. Using dynamic programming, we characterize the value function with a backward stochastic differential equation and the optimal portfolio policies. We separately treat the cases of exponential, power and logarithmic utility.

Key words : Utility maximization, dynamic programming, backward stochastic differential equation, default time, incomplete market, exponential utility, power utility, logarithmic utility.

1 Introduction

We consider an incomplete financial model with one bond and one risky asset. The price process S of the risky asset is assumed to be a local martingale driven by a brownian motion and a default indicating process. In such a context, we solve the portfolio optimization problem when investors want to maximize the expected utility from terminal wealth.

The utility maximization problem has been largely studied in the literature. Originally introduced by Merton (1971) in the context of constant coefficients and treated by markovian methods via Bellman equation of dynamic programming, it was developed for general process by martingal duality approach. For the case of complete markets, we refer to Karatzas *et al.* (1987), Cox and Huang (1989). For the case of incomplete and/or constrained markets, we refer to Karatzas *et al.* (1991), He and Pearson (1991) and Cvitanič and Karatzas (1992). Lukas (2001) considers the case of incomplete markets with a default in the markovian case. In contrast to these papers, in Hu *et al.* (2004), the authors do not use the duality approach, and they directly characterize the solution of the primal problem as the solution of a backward stochastic differential equation (BSDE), by using a verification theorem of the same spirit as El Karoui *et al.* (1997). Since they work in a Brownian filtration, they can use directly some results on quadratic BSDEs (Kobylanski (2000)). For the case of a

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discontinuous framework, we refer to Morlais (2008). She supposes that the price process of stocks is modeled by a local martingale driven by an independent one dimensional brownian motion and a Poisson point process. In using the same approach as in Hu *et al.* (2004), she obtains formally a BSDE for which there is none existence and uniqueness results. She proves the existence of a solution of this BSDE using an approximation method but she does not obtain uniqueness result, so it is not possible to characterize the value function as the solution of a BSDE. To be able to solve completely the problem, she restrains the admissible portfolio set to a compact set so that in this case the value function can be proved to be the unique solution of a BSDE.

The method we propose in order to obtain value function and optimal strategy is simple. We propose to study directly the value function with few dynamical programming technics. Then it is possible to prove that the value function is a particular solution of a quadratic BSDE.

The outline of the paper is organized as follows. In Section 2, we present the market model and the problem. In Section 3, we carry out the calculation of the value function and an optimal strategy for exponential utility and in Section 4 we define the indifference price for a contingent claim with the results of Section 3. In Section 5, we consider logarithmic utility, and in the final section we treat the power utility to complete the spectrum of important utility functions.

2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space equipped with a Brownian motion $(W_t)_{0\leq t\leq T}$. The filtration $\mathbb{F}$ is the completion of the filtration generated by W. We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and one risky asset with price process S. We suppose that the risky asset admits a default time τ. We introduce the jump process $N_t = \mathbb{1}_{\tau\leq t}$, we denote by $\mathbb{H}$ the filtration generated by this process and by $\mathbb{G}$ the enlarged filtration $\mathbb{F}\vee\mathbb{H}$ (we suppose that $\mathcal{G} = \mathcal{G}_T$). For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{P}\{\tau \leq t|\mathcal{F}_t\}$, and we denote by G the $\mathbb{F}$-survival process of τ with respect to the filtration $\mathbb{F}$, given as

$$G_t \ := 1 - F_t = \mathbb{P}\{\tau > t|\mathcal{F}_t\}\,,\ \forall t \in \mathbb{R}_+.$$

Definition 2.1. *Assume that $F_t < 1$ for all $t \in \mathbb{R}_+$. The $\mathbb{F}$-hazard process of τ under $\mathbb{P}$, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$. Equivalently, $\Gamma_t = -\ln G_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.*

Let us assume that this process is absolutely continuous with respect to the Lebesgue measure, so that there exists a process γ such that $\Gamma_t = \int_0^t \gamma_s ds$ for all $t \in \mathbb{R}_+$. It can be shown (see, for instance, Bielecki and Rutkowski (2004), chap. 6) that the process given by the formula :

$$M_t = N_t - \int_0^{t\wedge\tau} \gamma_s ds \tag{2.1}$$

is a $\mathbb{G}$-martingale. For the sake of brevity we shall denote $\lambda_t = (1 - N_t)\gamma_t$ thereafter. Recall that the filtrations $\mathbb{F}$ and $\mathbb{G}$ satisfy the following property called H-hypothesis (see Bremaud and Yor (1978) for a detailed study)

Proposition 2.1. Martingale invariance property *Every $\mathbb{F}$-martingale under $\mathbb{P}$ is also a $\mathbb{G}$-martingale under $\mathbb{P}$.*

Proof. By construction of the process N with $\mathbb{G}$-predictable intensity, $\mathcal{G}_t$ and $\mathcal{F}_T$ are independent, given $\mathcal{F}_t$. The result now follows from the observation that this property is equivalent to $E[X|\mathcal{F}_t] = E[X|\mathcal{G}_t]$ for every $\mathcal{F}_T$-measurable random variable X. □

The martingale invariance property is a common assumption in the literature on default risk modeling (see Elliott *et al.* (2000)) and as well as hedging and portfolio choice with jumps (see Blanchet-Scalliet and Jeanblanc (2004)). We also recall the useful martingale representation theorem (see Jeanblanc (2001))

Proposition 2.2. Martingale representation theorem *Let m be any $(\mathbb{P}, \mathbb{G})$-local martingale with $m_0 = 0$. Then, there exist two valued processes ϕ and ψ which are $\mathbb{G}$-predictable processes, $\mathbb{P}$-a.s. square integrable and such that*

$$m_t = \int_0^t \phi_s dW_s + \int_0^t \psi_s dM_s, \ 0 \leq t \leq T.$$

The price process S evolves according to the equation :

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t). \tag{2.2}$$

Assumption 2.1. *(i) μ, σ and β are $\mathbb{G}$-predictable and uniformly bounded stochastic processes.*

(ii) The process β satisfied $\beta_t > -1$ for all $t \in \mathbb{R}_+$.

(iii) $\left(\frac{\mu_t}{\sigma_t}\right)_{0 \leq t \leq T}$ is uniformly bounded.

A $\mathbb{G}$-predictable process $\pi = (\pi_t)_{0 \leq t \leq T}$ is called trading strategy if $\int_0^T \frac{\pi_t}{S_{t^-}} dS_t$ is well defined, e.g. $\int_0^T |\pi_t \sigma_t|^2 dt < \infty$ $\mathbb{P}$-a.s. and $\int_0^T |\pi_t \beta_t|^2 \lambda_t dt < \infty$ $\mathbb{P}$-a.s. The process $(\pi_t)_{0 \leq t \leq T}$ describes the amount of money invested in the risky asset S at time t. The wealth process $X^{x,\pi}$ of a trading strategy π with initial capital x satisfies the equation :

$$X_t^{x,\pi} = x + \int_0^t \frac{\pi_s}{S_{s^-}} dS_s$$

and by self-financing, we get :

$$dX_t^{x,\pi} = \pi_t(\mu_t dt + \sigma_t dW_t + \beta_t dN_t). \tag{2.3}$$

A function $U : (0,\infty) \to \mathbb{R}$ will be called utility function if it is strictly increasing, strictly concave, of class C^1, and satisfies the Inada's conditions :

$$U'(0^+) = \lim_{x\downarrow 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x\to\infty} U'(x) = 0.$$

The optimization problem is to maximize the expected utility from terminal wealth over the class $\mathcal{A}(x)$ of admissible portfolios, provided that the expectation is well defined. More precisely, the value function of this problem is defined by :

$$V(x) = \sup_{\pi\in\mathcal{A}(x)} E\left[U(X_T^{x,\pi})\right].$$

In the following, we will characterize the value function $V(x)$ and the optimal strategy.

3 Exponential utility

In this section, we specify the sense of optimality for trading strategies by stipulating that the investor wants to maximize his expected utility with respect to the exponential utility from his terminal wealth. Let us recall that for $\gamma > 0$ the exponential utility function is defined as :

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R}.$$

Definition 3.1. (Admissible strategies) *The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0\le t\le T}$ which satisfy $\int_0^T |\pi_t\sigma_t|^2 dt < \infty$ a.s. and $\int_0^T |\pi_t\beta_t|^2\lambda_t dt < \infty$ a.s. and there exists a constant K_π such that $X_t^{x,\pi} \ge K_\pi$ for all $t \in [0,T]$. We denote $\mathcal{A}_t$ if the initial time is t.*

For the sake of brevity we shall denote $\mathcal{A}$ instead of $\mathcal{A}(x)$ if there is no confusion.
Let $\xi \in \mathcal{G}_T$ be a given non-negative contingent claim, and let x be the initial endowment of the investor. Our first goal is to solve an optimization problem for an agent who buys a contingent claim ξ. To this end it suffices to find a strategy that maximizes

$$V(x,\xi) = \sup_{\pi\in\mathcal{A}} E\left[-\exp(-\gamma\left(X_T^{x,\pi} + \xi\right))\right], \quad \gamma > 0, \tag{3.1}$$

V is called value function. The maximization problem is evidently equivalent to :

$$V(x,\xi) = -e^{-\gamma x} \inf_{\pi\in\mathcal{A}} E\left[\exp\left(-\gamma\left(X_T^{0,\pi} + \xi\right)\right)\right].$$

We denote $V(x)$ the value function if the investor does not buy the contingent claim ξ and invests only in the risk-free asset and in the risky asset.
To solve this problem we define the value function $J(t)$ at t by the following formula :

$$J(t) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp(-\gamma(X_T^{x,\pi} - X_t^{x,\pi} + \xi))|\mathcal{G}_t\right] \ a.s. \tag{3.2}$$

Remark 3.1. *The function J is independent of the initial wealth x, then we can take $x = 0$ for the next.*

For the sake of brevity, we shall denote now X_t^π instead of $X_t^{0,\pi}$. In the following, we want to characterize J by a BSDE. Let us define the function $\Gamma(t,\pi)$ by the formula :

$$\Gamma(t,\pi) = E\left[\exp(-\gamma(X_T^\pi - X_t^\pi + \xi))|\mathcal{G}_t\right].$$

Proposition 3.1. *The set $\{\Gamma(t,\pi),\ \pi \in \mathcal{A}_t\}$ is stable by infimum, i.e. for every $\pi^0,\ \pi^1 \in \mathcal{A}_t$, there exists $\pi \in \mathcal{A}_t$ such that $\Gamma(t,\pi) = \Gamma(t,\pi^0) \wedge \Gamma(t,\pi^1)$.*

Proof. Let us define the set E :

$$E = \left\{\Gamma(t,\pi^0) \leq \Gamma(t,\pi^1)\right\}.$$

Thus $E \in \mathcal{G}_t$. Let us define π by the formula :

$$\forall\ s \in [t,T],\ \pi_s = \pi_s^0 \mathbb{1}_E + \pi_s^1 \mathbb{1}_{E^c}.$$

Then $X_s^\pi = X_s^{\pi^0}\mathbb{1}_E + X_s^{\pi^1}\mathbb{1}_{E^c}$, therefore $X_s^\pi \geq K_{\pi^0} \wedge K_{\pi^1}$, thus $\pi \in \mathcal{A}_t$. By construction of π, we have that $\Gamma(t,\pi) = \Gamma(t,\pi^0) \wedge \Gamma(t,\pi^1)$. □

With Proposition 3.1 the value function can be characterized with the function Γ

Corollary 3.1. *For all t, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{A}_t$, such that :*

$$J(t) = \lim_{n\to\infty} \downarrow \Gamma(t,\pi^n) \quad a.s.$$

Let us define the $\mathbb{G}$-adapted process $(J'_t)_{0\leq t\leq T}$ by the formula :

$$J'_t = \begin{cases} J(t) & \text{if } J \text{ is defined at } t \\ 0 & \text{else} \end{cases}$$

With the dynamic programming we obtain few properties about the process J'

Proposition 3.2. *For all strategies $\pi \in \mathcal{A}$, $(e^{-\gamma X_t^\pi} J'_t)_{0\leq t\leq T}$ is a submartingale.*

Proof. It is sufficient to show that :

$$E\left[e^{-\gamma X_t^\pi} J'_t|\mathcal{G}_s\right] \geq e^{-\gamma X_s^\pi} J'_s, \quad \forall\ t \geq s$$

$$i.e.\ \ E\left[e^{-\gamma(X_t^\pi - X_s^\pi)} J'_t|\mathcal{G}_s\right] \geq J'_s, \quad \forall\ t \geq s.$$

With equality (2.3) and Corollary 3.1, we have by monotone convergence theorem :

$$E\left[\exp(-\gamma(X_t^\pi - X_s^\pi))J'_t|\,\mathcal{G}_s\right] = \lim_{n\to\infty} \downarrow E\left[\exp\left(-\gamma\left(\int_s^t \pi_u \frac{dS_u}{S_{u^-}} + \int_t^T \pi_u^n \frac{dS_u}{S_{u^-}} + \xi\right)\right)\middle|\,\mathcal{G}_s\right]$$

Let us define the strategy $\tilde{\pi}^n$ by

$$\tilde{\pi}^n_u = \begin{cases} \pi_u & \text{if} \quad s \leq u < t \\ \pi^n_u & \text{if} \quad t \leq u \leq T \end{cases}$$

We can easily show that $\tilde{\pi}^n \in \mathcal{A}_s$. By definition of J'_s, we have $\lim_{n\to\infty} \Gamma(s, \tilde{\pi}^n) \geq J'_s$ a.s. Therefore :

$$E\left[\exp(-\gamma(X^\pi_t - X^\pi_s))J'_t|\mathcal{G}_s\right] = \lim_{n\to\infty} \downarrow \Gamma(s, \tilde{\pi}^n) \geq J'_s \ a.s.$$

Then, $\left(e^{-\gamma X^\pi_t} J'_t\right)_{0\leq t\leq T}$ is a submartingale for all $\pi \in \mathcal{A}$. □

We can so characterize the process J' by the following proposition

Proposition 3.3. (**Dynamic Programming principle**) $(J'_t)_{0\leq t\leq T}$ *is the largest* $\mathbb{G}$*-adapted process for the almost surely equality such that for all strategies* $\pi \in \mathcal{A}$, $\left(e^{-\gamma X^\pi_t} J'_t\right)_{0\leq t\leq T}$ *is a submartingale and* $J'_T = \exp(-\gamma\xi)$.

Proof. Let $(Y_t)_{0\leq t\leq T}$ be a $\mathbb{G}$-adapted process such that $\forall\ \pi \in \mathcal{A}$, $\left(e^{-\gamma X^\pi_t} Y_t\right)_{0\leq t\leq T}$ is a submartingale and $Y_T = \exp(-\gamma\xi)$. For all $t \in [0, T]$ and for all $\pi \in \mathcal{A}$, we have :

$$E\left[e^{-\gamma X^\pi_T} Y_T|\mathcal{G}_t\right] \geq e^{-\gamma X^\pi_t} Y_t$$

Then

$$E\left[\exp\left(-\gamma\left(\int_t^T \mu_s\pi_s ds + \sigma_s\pi_s dW_s + \beta_s\pi_s dN_s + \xi\right)\right)\middle|\mathcal{G}_t\right] \geq Y_t.$$

Therefore we have :

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp\left(-\gamma\left(\int_t^T \mu_s\pi_s ds + \sigma_s\pi_s dW_s + \beta_s\pi_s dN_s + \xi\right)\right)\middle|\mathcal{G}_t\right] \geq Y_t$$

Then we get :

$$J'_t \geq Y_t \quad a.s.$$

□

We now show that there exists a càd-làg version of the value function. More precisely,

Proposition 3.4. *There exists a càd-làg* $\mathbb{G}$*-adapted process* $(J_t)_{0\leq t\leq T}$ *such that for all* $t \in [0, T]$ *:*

$$J_t = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp(-\gamma(X^{x,\pi}_T - X^{x,\pi}_t + \xi))|\mathcal{G}_t\right] \quad a.s.$$

Proof. Let $\mathbb{D} = [0, T] \cap \mathbb{Q}$. Because (J'_t) is a submartingale, we have for almost every $\omega \in \Omega$, that the mapping $t \to J'_t(\omega)$ defined on $\mathbb{D}$ has at each point t of $[0, T[$ a finite right limit :

$$J'_{t^+}(\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J'_s(\omega)$$

and at each point of $]0,T]$ a finite left limit

$$J'_{t^-}(\omega) = \lim_{s\in\mathbb{D}, s\uparrow t} J'_s(\omega).$$

Thanks to Dellacherie and Meyer (1980), chap. 6, the process J'_{t^+} is a $\mathcal{G}_{t^+}$-submartingale. We can show that the process $\left(\exp\left(-\gamma X_t^\pi\right) J'_{t^+}\right)_{0\le t<T}$ is a $\mathcal{G}_{t^+}$-submartingale and because the filtration is right continuous, it is a $\mathcal{G}_t$-submartingale. Consequently, from Proposition 3.3, for all $t\in[0,T]$ $J'_{t^+}\le J'_t$ a.s. We have also $J'_t \le E\left[J'_{t^+}|\mathcal{G}_t\right]$. Thus : $J'_{t^+} = J'_t$ a.s. and

$$\forall\ t\in[0,T],\ J'_{t^+} = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp(-\gamma(X_T^{x,\pi} - X_t^{x,\pi} + \xi))|\mathcal{G}_t\right]\ a.s.$$

The result follows in taking J_t equals to the above process J'_{t^+}. □

Remark 3.2. *The property of dynamic programming can be written for the process J under the form : $(J_t)_{0\le t\le T}$ is the largest càd-làg $\mathbb{G}$-adapted process such that for all $\pi\in\mathcal{A}$, $\left(e^{-\gamma X_t^\pi} J_t\right)_{0\le t\le T}$ is a submartingale and $J_T = \exp(-\gamma\xi)$.*

We now show that the value function is bounded, that is interesting for the following to use the Doob-Meyer decomposition. More precisely,

Lemma 3.1. *For all $0\le t\le T$, the process J verifies :*

$$0 < J_t \le 1 \quad a.s.$$

Proof. By definition of process J, we know that for all t, $J_t > 0$. Moreover the strategy $\pi_s = 0$ for all $s\in[t,T]$ is admissible following Definition 3.1, hence $J_t \le \exp(-\gamma\xi)$ for all t. As we suppose that the contingent claim ξ is non negative, we have that $J_t \le 1$. □

With the dynamic programming principle, we can give a classical characterization of the optimal strategies:

Proposition 3.5. (**Characterization of optimal strategies**)
Let $\hat\pi\in\mathcal{A}$, the two following assertions are equivalent :

(i) $\hat\pi$ is an optimal strategy, i.e. $J_0 = \inf_{\pi\in\mathcal{A}} E\left[\exp(-\gamma(X_T^\pi+\xi))\right] = E\left[\exp(-\gamma(X_T^{\hat\pi}+\xi))\right]$.

(ii) $\left(e^{-\gamma X_t^{\hat\pi}} J_t\right)_{0\le t\le T}$ is a martingale.

Proof. Suppose (i) that is $\hat\pi$ is an optimal strategy, hence we have

$$J_0 = E\left[\exp(-\gamma(X_T^{\hat\pi}+\xi))\right]. \tag{3.3}$$

As $\left(e^{-\gamma X_t^{\hat{\pi}}} J_t\right)_{0\leq t\leq T}$ is a submartingale from Remark 3.2 and by (i), we have $J_0 = E\left[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))\right]$ which implies (ii).
To show the converse, suppose that $\left(e^{-\gamma X_t^{\hat{\pi}}} J_t\right)_{0\leq t\leq T}$ is a martingale, then we have :

$$E\left[e^{-\gamma X_T^{\hat{\pi}}} e^{-\gamma\xi}\right] = J_0.$$

which gives (i). □

Remark 3.3. *Note that we can obtain a quite general verification theorem for the value function : let $(Y_t)_{0\leq t\leq T}$ be a $\mathbb{G}$-adapted process which is equal to $\exp(-\gamma\xi)$ at T and such that for all strategies $\pi \in \mathcal{A}$, $\left(\exp\left(-\gamma\int_0^t \frac{\pi_s}{S_{s^-}} dS_s\right) Y_t\right)_{0\leq t\leq T}$ is a submartingale and that there exists a strategy $\hat{\pi} \in \mathcal{A}$ satisfying $\left(\exp\left(-\gamma\int_0^t \frac{\hat{\pi}_s}{S_{s^-}} dS_s\right) Y_t\right)_{0\leq t\leq T}$ is a martingale, then we have $Y_t = J_t$ a.s. for each $t \in [0, T]$.*

Remark 3.4. *In the Brownian filtration case, see Hu* et al. *(2004), the authors use a similar verification theorem but they look for Y as the solution of a BSDE of the form*

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds$$

for which some existence and uniqueness results hold. They characterize easily the function f with the two properties of verification theorem. In the case of jumps, see Morlais (2008), the author uses the same approach as in Hu et al. *(2004), she obtains formally a BSDE for which there is none existence and uniqueness results. She proves the existence of a solution of this BSDE with an approximation method but she does not obtain uniqueness result, so it is not possible to characterize the value function as the solution of a BSDE. In order to be able to solve completely the problem, she restrains the admissible portfolio set to a compact set so that in this case, the value function can be proved to be the unique solution of her BSDE.*

Now, we will define the process J as a solution of a BSDE. For that we need to define the following sets S^∞, $L^2(W)$ and $L^2(M)$:

$$S^\infty = \left\{Y \text{ càd-làg } \mathbb{G}\text{-adapted such that } \operatorname*{ess\,sup}_{\omega,t}(|Y_t(\omega)|) < \infty\right\}$$

$$L^2(W) = \left\{Z\ \mathbb{G}\text{-predictable},\ E\left[\int_0^T |Z_t|^2 dt\right] < \infty\right\}$$

$$L^2(M) = \left\{U\ \mathbb{G}\text{-predictable},\ E\left[\int_0^T |U_t|^2 \lambda_t dt\right] < \infty\right\}$$

With Remark 3.2 and Lemma 3.1, $(J_t)_{0\leq t\leq T}$ is a submartingale of class D and admits a unique Doob-Meyer decomposition thanks to Dellacherie and Meyer (1980), chap. 7 :

$$dJ_t = dm_t + dA_t$$

with m is a square integrable martingale and A is an increasing càd-làg $\mathbb{G}$-predictable process with $A_0 = 0$. From Theorem 3.3, the Doob-Meyer decomposition can be written under the form

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t \tag{3.4}$$

with $Z \in L^2(W)$, $U \in L^2(M)$.
In using Remark 3.2 and Proposition 3.5, it is possible to determine the process A of 3.4 and we prove that the value function is solution of a BSDE with a quadratic driver.

Proposition 3.6. *The process $(J_t, Z_t, U_t)_{0\leq t\leq T} \in S^\infty \times L^2(W) \times L^2(M)$ is solution of the following BSDE :*

$$\begin{cases} dJ_t &= Z_t dW_t + U_t dM_t \\ & \quad + \sup_{\pi\in\mathcal{A}} \left\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(J_t + U_t) \right\} dt \\ J_T &= \exp(-\gamma\xi) \end{cases} \tag{3.5}$$

Proof. To prove this proposition, we use the property that for each π, the process $(e^{-\gamma X_t^\pi} J_t)$ is a submartingale and that for at least one $\hat{\pi}$ this process is a martingale. It follows the finite variation part which appears in the decomposition of the semi-martingale $(e^{-\gamma X_t^\pi} J_t)$ (resp. $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$) is an increasing process (resp. null process). More precisely, let us calculate the derivative of process $e^{-\gamma X_t^\pi} J_t$, we first use Itô's formula

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi}\right) &= -\gamma e^{-\gamma X_{t^-}^\pi} dX_t^{\pi,c} + \frac{\gamma^2}{2} e^{-\gamma X_{t^-}^\pi} d<X^{\pi,c}>_t + \left(e^{-\gamma X_t^\pi} - e^{-\gamma X_{t^-}^\pi}\right) dN_t \\ &= e^{-\gamma X_{t^-}^\pi} \left[-\gamma(\pi_t\mu_t dt + \pi_t\sigma_t dW_t) + \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 dt + \left(e^{-\gamma\pi_t\beta_t} - 1\right) dN_t \right] \\ &= e^{-\gamma X_{t^-}^\pi} \left[\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t \right) dt - \gamma\pi_t\sigma_t dW_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right) dN_t \right] \end{aligned}$$

with $X^{\pi,c}$ the continuous part of X^π.
The product rule yields

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi} J_t\right) &= e^{-\gamma X_{t^-}^\pi} dJ_t + J_{t^-} d\left(e^{-\gamma X_t^\pi}\right) + d\left[e^{-\gamma X_t^\pi}, J_t\right] \\ &= e^{-\gamma X_{t^-}^\pi}(Z_t dW_t + U_t dM_t + dA_t) + e^{-\gamma X_{t^-}^\pi} J_{t^-} \left[\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t \right) dt \right. \\ & \quad \left. -\gamma\pi_t\sigma_t dW_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right) dN_t \right] - e^{-\gamma X_{t^-}^\pi} \gamma\pi_t\sigma_t Z_t dt \\ & \quad + e^{-\gamma X_{t^-}^\pi} \left(e^{-\gamma\pi_t\beta_t} - 1\right) U_t dN_t. \end{aligned}$$

We have with equality (2.1) :

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi} J_t\right) &= e^{-\gamma X_{t^-}^\pi}\Bigg[(Z_t - \gamma\pi_t\sigma_t J_t)dW_t + (U_t + (e^{-\gamma\pi_t\beta_t} - 1)(U_t + J_{t^-}))dM_t + dA_t \\ &\quad + \left\{\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) - \gamma\pi_t\sigma_t Z_t\right\} dt\Bigg] \\ &= \text{local martingale } + dA_t^\pi. \end{aligned}$$

With $dA_t^\pi = dA_t + \left\{\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) - \gamma\pi_t\sigma_t Z_t\right\} dt$.
By the dynamic programming principle (Remark 3.2), the process $\left(e^{-\gamma X_t^\pi} J_t\right)_{0\le t\le T}$ is a submartingale for all strategies $\pi \in \mathcal{A}$, then $dA_t^\pi \ge 0$ for all strategies $\pi \in \mathcal{A}$ and we get

$$dA_t \ge \left[\left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\pi_t\sigma_t Z_t\right] dt.$$

From Theorem 2.2 of Kramkov and Schachermayer (1999), there exists an optimal strategy $\hat\pi \in \mathcal{A}$ for problem 3.1. By Proposition 3.5, this optimal strategy $\hat\pi$ is such that $\left(e^{-\gamma X_t^{\hat\pi}} J_t\right)_{0\le t\le T}$ is a martingale, which implies that a.s.,

$$dA_t = \left[\left(-\frac{\gamma^2}{2}\hat\pi_t^2\sigma_t^2 + \gamma\hat\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\hat\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\hat\pi_t\sigma_t Z_t\right] dt.$$

Therefore we have

$$\begin{aligned} &\left(-\frac{\gamma^2}{2}\hat\pi_t^2\sigma_t^2 + \gamma\hat\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\hat\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\hat\pi_t\sigma_t Z_t \\ &\qquad = \sup_{\pi\in\mathcal{A}}\left\{\left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\pi_t\sigma_t Z_t\right\}. \end{aligned}$$

Thus decomposition (3.4) of $(J_t)_{0\le t\le T}$ is given by :

$$\begin{aligned} dJ_t &= Z_t dW_t + U_t dM_t \\ &\quad + \sup_{\pi\in\mathcal{A}}\left\{\left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\pi_t\sigma_t Z_t\right\} dt. \end{aligned}$$

□

The problem is that we can not prove that BSDE (3.5) admits a unique solution in $S^\infty \times L^2(W) \times L^2(M)$. We will see in the next that if the set of admissible portfolios is restricted to some bounded sets then BSDE (3.5) admits a unique solution in $S^\infty \times L^2(W) \times L^2(M)$. But the value function can be characterized as the largest solution of a BSDE

Theorem 3.1. *$(J_t)_{0\le t\le T}$ is the largest solution in the set $S^\infty \times L^2(W) \times L^2(M)$ of the BSDE :*

$$\begin{cases} dJ_t &= Z_t dW_t + U_t dM_t \\ & \quad + \sup_{\pi\in\mathcal{A}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)(J_t + U_t)\right\} dt \\ J_T &= \exp(-\gamma\xi) \end{cases} \tag{3.6}$$

The optimal strategy $\hat{\pi}$ to problem (3.1) is defined by :

$$\hat{\pi}_t = \arg\max_{\pi\in\mathcal{A}} \left\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t \left(1 - e^{-\gamma\pi_t\beta_t}\right)(J_t + U_t) \right\}.$$

Proof. Let (J'_t, Z'_t, U'_t) be a solution of equation (3.6) in $S^\infty \times L^2(W) \times L^2(M)$. Let us prove that for all strategies $\pi \in \mathcal{A}$ we have $\left(e^{-\gamma X_t^\pi} J'_t\right)_{0\le t\le T}$ is a submartingale

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi} J'_t\right) &= e^{-\gamma X_{t-}^\pi}\Bigg[(Z'_t - \gamma\pi_t\sigma_t J'_{t-})\, dW_t + \left(U'_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right)(U'_t + J'_{t-})\right) dM_t + dA_t \\ &\quad + \left\{ \left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J'_{t-} + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U'_t + J'_t) - \gamma\pi_t\sigma_t Z'_t \right\} dt \Bigg] \\ &= dM_t^\pi + dA_t^\pi \end{aligned}$$

With :

$$\begin{cases} dA_t^\pi = e^{-\gamma X_t^\pi}\left\{ dA_t + \left[\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J'_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U'_t + J'_t) - \gamma\pi_t\sigma_t Z'_t\right] dt \right\} \\ dM_t^\pi = e^{-\gamma X_{t-}^\pi}\left[(Z'_t - \gamma\pi_t\sigma_t J'_t) dW_t + \left(U'_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right)\left(U'_t + J'_{t-}\right)\right) dM_t\right] \end{cases}$$

$dA_t^\pi \ge 0$, as $dA_t = \sup_{\pi\in\mathcal{A}} \left\{ \left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J'_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U'_t + J'_t) + \gamma\pi_t\sigma_t Z'_t \right\} dt.$
Thus $M_t^\pi \le e^{-\gamma X_t^\pi} J'_t - J'_0$ and by definition of an admissible strategy there exists a constant K_π such that $X_t^\pi \ge K_\pi$ and J' is a positive bounded process, therefore M^π is upper bounded by a constant.
Thus $\left(e^{-\gamma X_t^\pi} J'_t\right)_{0\le t\le T}$ is a submartingale, because it is the sum of a submartingale and a nondecreasing process.
From Remark 3.2, $(J_t)_{0\le t\le T}$ is the largest process such that $\forall\ \pi \in \mathcal{A}$, $\left(e^{-\gamma X_t^\pi} J_t\right)_{0\le t\le T}$ is a submartingale and $J_T = \exp(-\gamma\xi)$. Therefore we get :

$$\forall\ t \in [0,T],\ \ J'_t \le J_t \ \ a.s.$$

□

Remark 3.5. *If we suppose that the set $\mathcal{A}$ is closed and there is no default (i.e. we consider the brownian motion case), then this result corresponds to that obtained in Hu et al. (2004).*

In the rest of this section, we show another characterization of the value function. For each k we consider the value function J^k defined by

$$J_t^k = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^k} E\left[\exp\left(-\gamma(X_T^{x,\pi} - X_t^{x,\pi} + \xi)\right)|\mathcal{G}_t\right] \quad a.s.$$

with $\mathcal{A}^k = \left\{\pi \in \mathcal{A},\ |\pi_t| \le k\ \ \forall\ t \in [0,T]\ a.s.\right\}$. Then the value function J can be characterized to be the limit of the value functions J^k.

These processes $(J_t^k)_{0\le t\le T}$ have the same properties as the process $(J_t)_{0\le t\le T}$: for all strategies $\pi \in \mathcal{A}^k$, $\left(e^{-\gamma X_t^\pi} J_t^k\right)_{0\le t\le T}$ is a submartingale and there exits a strategy $\hat{\pi}$ such that $\left(e^{-\gamma X_t^{\hat{\pi}}} J_t^k\right)_{0\le t\le T}$ is a martingale.

For each $k \in \mathbb{N}^*$, the process J^k can be showed to be characterized as the unique solution of a BSDE. Note that the uniqueness follows from the fact that the admissible set of strategies $\mathcal{A}^k$ is bounded. More precisely

Proposition 3.7. *The process $\left(J_t^k, Z_t^k, U_t^k\right)_{0\le t\le T}$ is the unique solution in $S^\infty \times L^2(W) \times L^2(M)$ of the following BSDE :*

$$\left\{\begin{array}{rcl} dJ_t^k & = & Z_t^k dW_t + U_t^k dM_t + \sup\limits_{\pi\in\mathcal{A}^k}\Big\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t^k + \gamma\pi_t(\mu_t J_t^k + \sigma_t Z_t^k) \\ & & \quad + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(J_t^k + U_t^k\right)\Big\} dt \\ J_T^k & = & \exp(-\gamma\xi) \end{array}\right. \tag{3.7}$$

Proof. To prove that the process $\left(J_t^k, Z_t^k, U_t^k\right)$ is solution of (3.7) we do as in the proof of Proposition 3.6. Let us show the uniqueness of the solution of BSDE (3.7). In BSDE (3.7) the driver is equal $d\mathbb{P}\otimes dt$ a.s. to

$$\inf_{\pi\in\mathcal{A}^k}\left\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 y - \gamma\pi_t(\mu_t y + \sigma_t z) - \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)(y+u)\right\}.$$

We can easily show that the driver is Lipschitz w.r.t. y, z, u because $\mathcal{A}^k$ is bounded (not necessarily compact). Then thanks to classical results (see Tang and Li (1994) or Barles *et al.* (1997)), BSDE (3.7) admits a unique solution. □

Remark 3.6. *Note that in the case $\mathcal{A}$ is compact, the result we have derived in a more simple way stated in Morlais (2008).*

We can characterize the value function J as the limit of the processes J^k

Proposition 3.8. $J_t = \lim\limits_{k\to\infty} \downarrow J_t^k$ *a.s.* $\forall\ t \in [0,T]$.

Proof. It is obvious with the definitions of sets $\mathcal{A}$ and $\mathcal{A}^k$ that

$$\forall\ k \in \mathbb{N} \quad J_t \le J_t^k \quad a.s.$$

and as for all t the sequence $\left(J_t^k\right)_{k\ge 0}$ is nonincreasing and lower bounded, we get the existence of a limit that we denote $\bar{J}_t$ for all t and $J_t \le \bar{J}_t$ a.s. (we can suppose that $\bar{J}$ is càd-làg as in the proof of Proposition 3.4).
In order to prove that for all t we have $J_t \ge \bar{J}_t$, we first prove that for each $\pi \in \mathcal{A}$, $\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)_{0\le t\le T}$ is a submartingale. As J^k is a submartingale, we get

$$\forall\ k \in \mathbb{N} \quad E[J_t^k|\mathcal{G}_s] \ge J_s^k \ge \bar{J}_s \quad a.s.$$

By monotone convergence theorem, we have :

$$E[\bar{J}_t|\mathcal{G}_s] \geq \bar{J}_s \quad a.s.$$

Then $\left(\bar{J}_t\right)_{0\leq t\leq T}$ is a submartingale of class D and admits the following Doob-Meyer decomposition : $d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t + d\bar{A}_t$ where $\bar{Z} \in L^2(W)$, $\bar{U} \in L^2(M)$ and $\bar{A}$ is a nondecreasing càd-làg $\mathbb{G}$-predictable process where $\bar{A}_0 = 0$.
We prove that for all bounded strategies π, the process $\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)_{0\leq t\leq T}$ is a submartingale from monotone convergence theorem. By similar arguments of the proof of Proposition 3.6 we get

$$d\bar{A}_t \geq \sup_{\pi\in\bar{\mathcal{A}}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \gamma\pi_t\left(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t\right) + \lambda_t\left(1-e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t+\bar{U}_t\right)\right\}dt$$

where $\bar{\mathcal{A}}$ is the set of admissible and bounded strategies.
We show that for all admissible strategies $\pi \in \mathcal{A}$ that $\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)_{0\leq t\leq T}$ is a submartingale : Let $\pi \in \mathcal{A}$, we have

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi}\bar{J}_t\right) &= e^{-\gamma X_{t^-}^\pi}\Big[\left(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t\right)dW_t + \left(\bar{U}_t + \left(e^{-\gamma\pi_t\beta_t}-1\right)\left(\bar{U}_t+\bar{J}_{t^-}\right)\right)dM_t + d\bar{A}_t \\ &\quad + \left\{\tfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t\left(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t\right) - \lambda_t\left(1-e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t+\bar{U}_t\right)\right\}dt\Big] \\ &= dM_t^\pi + dA_t^\pi \end{aligned}$$

with

$$\begin{cases} dA_t^\pi = e^{-\gamma X_t^\pi}\left\{d\bar{A}_t + \left[\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t}-1\right)\left(\bar{U}_t+\bar{J}_t\right) - \gamma\pi_t\left(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t\right)\right]dt\right\} \\ dM_t^\pi = e^{-\gamma X_{t^-}^\pi}\left[\left(\bar{Z}_t-\gamma\pi_t\sigma_t\bar{J}_t\right)dW_t + \left(\bar{U}_t + \left(e^{-\gamma\pi_t\beta_t}-1\right)\left(\bar{U}_t+\bar{J}_{t^-}\right)\right)dM_t\right] \end{cases}$$

by definition of $d\bar{A}$ we have $dA_t^\pi \geq 0$. Then :

$$M_t^\pi = e^{-\gamma X_t^\pi}\bar{J}_t - \bar{J}_0 - (A_t^\pi - A_0^\pi)$$

$$i.e \quad M_t^\pi \leq e^{-\gamma X_t^\pi}\bar{J}_t - \bar{J}_0.$$

As $\bar{J}_t \geq J_t \geq 0$ and $\bar{J}_t \leq 1$, we get :

$$M_t^\pi \leq e^{-\gamma X_t^\pi}.$$

By the definition of an admissible strategy we have that $M_t^\pi \leq e^{-\gamma K_\pi}$. Thus $(M_t^\pi)_{0\leq t\leq T}$ is an upper bounded local martingale, therefore it is a submartingale. As $(M_t^\pi)_{0\leq t\leq T}$ is a submartingale and $(A_t^\pi)_{0\leq t\leq T}$ is nondecreasing, $\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)_{0\leq t\leq T}$ is a submartingale. Because $\left(\bar{J}_t\right)_{0\leq t\leq T}$ is càd-làg $\mathbb{G}$-adapted and $\bar{J}_T = \exp(-\gamma\xi)$, we have from Remark 3.2 :

$$\bar{J}_t \leq J_t \quad a.s.$$

□

Remark 3.7. *Note that we have derived in a simple way the same approximation result as the one stated in Morlais (2008). By using BSDEs technics, she also proves that the processes Z^k and U^k converge in L^2 to the processes Z and U.*

4 Indifference pricing

We present a general framework of the Hodges and Neuberger (1989) approach with some utility functions as define in Section 2. We solve explicitly the problem in the case of exponential utility.

The Hodges approach for pricing of unhedgeable claims is a utility-based approach and can be summarized as follows : the issue at hand is to assess the value of some claim ξ as seen from the perspective of an investor who optimizes his behavior relative to some utility function, say u. The investor has two choices : else he invests only in the risk-free asset and in the risky asset or he invests also in the contingent claim. We can define the Hodges price:

Definition 4.1. *For a given initial endowment x, the Hodges buying price of a contingent claim ξ is the real number p such that $V(x) = V(x-p,\xi)$.*

The Hodges price p can be derived explicitly in the case of exponential utility by applying the results of Section 3. If the investor buys the contingent claim at the price p and invests the rest of his wealth the value function is equal to

$$V(x-p,\xi) = -\exp(-\gamma(x-p))J_0^{\xi}.$$

If he invests all his wealth in the risk-free asset and in the risky asset the value function is equal to

$$V(x) = -\exp(-\gamma x)J_0^0.$$

Where J_t^{ξ} is the largest solution of

$$\begin{cases} dJ_t &= Z_t dW_t + U_t dM_t \\ & \quad + \sup\limits_{\pi\in\mathcal{A}} \left\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t \left(1 - e^{-\gamma\pi_t\beta_t}\right)(J_t + U_t) \right\} dt \\ J_T &= \exp(-\gamma\xi) \end{cases}$$

The Hodges price for the contingent claim ξ is given by the formula :

$$p = \frac{1}{\gamma}\ln\left(\frac{J_0^0}{J_0^{\xi}}\right).$$

We can also define the Hodges price of the contingent claim ξ at time t by :

$$p_t = \frac{1}{\gamma}\ln\left(\frac{J_t^0}{J_t^{\xi}}\right).$$

Remark 4.1. *If we consider the admissible strategies set $\mathcal{A}^k$, the price of indifference p_k can be also defined by the same method. More precisely*

$$p^k = \frac{1}{\gamma}\ln\left(\frac{J_0^{k,0}}{J_0^{k,\xi}}\right) \quad (resp.\ p_t^k = \frac{1}{\gamma}\ln\left(\frac{J_t^{k,0}}{J_t^{k,\xi}}\right)).$$

We remark that

$$p = \lim_{k\to\infty} p^k \quad (resp.\ p_t = \lim_{k\to\infty} p_t^k).$$

5 Logarithmic utility

In this section we calculate the value function and characterize the optimal strategy for the utility maximization problem with respect to

$$U(x) = \log(x).$$

This time, we shall use a somewhat different notion of trading strategy : p_t denotes the part of the wealth $X_t^{x,p}$ invested in stock S, that is advantageous for the calculus. The number of shares of stock is given by the formula $\frac{p_t X_t^{x,p}}{S_t}$. A $\mathbb{G}$-predictable process $p = (p_t)_{0 \leq t \leq T}$ is said to be a trading strategy if the following wealth process is well defined :

$$X_t^{x,p} = x + \int_0^t \frac{p_s X_{s^-}^{x,p}}{S_{s^-}} dS_s.$$

By self-financing we have the following relation :

$$dX_t^{x,p} = X_{t^-}^{x,p} p_t(\mu_t dt + \sigma_t dW_t + \beta_t dN_t),$$

and from Dolean's formula, we get the expression of the wealth $X_t^{x,p}$:

$$X_t^{x,p} = x \exp\left(\int_0^t p_s \mu_s ds + \int_0^t p_s \sigma_s dW_s - \frac{1}{2} \int_0^t |p_s \sigma_s|^2 ds + \int_0^t \log(1 + p_s \beta_s) dN_s \right). \quad (5.1)$$

Definition 5.1. *The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes p satisfying $E\left[\int_0^T |p_t \sigma_t|^2 dt + \int_0^T |\log(1 + p_t \beta_t)|^2 \lambda_t dt\right] < \infty$ and $X_t^{x,p} > 0$ for all $0 \leq t \leq T$.*

Remark 5.1. *The condition $X_t^{x,p} > 0$ for all $0 \leq t \leq T$ is equivalent to $p_t \beta_t > -1$ for all $0 \leq t \leq \tau$.*

The optimization problem is given by

$$V(x) = \sup_{p \in \mathcal{A}} E\left[\log\left(X_T^{x,p}\right)\right]. \quad (5.2)$$

Let us define the value function $J_0 = \sup_{p \in \mathcal{A}} E\left[\log\left(\frac{X_T^{x,p}}{x}\right)\right]$ to solve the problem (5.2). To solve this problem, we need few assumptions

Assumption 5.1. *The intensity λ is uniformly bounded.*

Assumption 5.2. *The process β^{-1} is uniformly bounded.*

Contrary to the previous section, it is possible to characterize directly the value function without BSDE

Theorem 5.1. *The solution of problem (5.2) is given by $V(x) = \log(x) + J_0$ with :*

$$J_0 = E\left[\int_0^T \left(\hat{p}_t\mu_t - \frac{\hat{p}_t^2\sigma_t^2}{2} + \lambda_t \log(1+\hat{p}_t\beta_t)\right) dt\right]$$

where $\hat{p}$ is the optimal trading strategy and given by

$$\hat{p}_t = \begin{cases} \frac{\mu_t}{2\sigma_t^2} - \frac{1}{2\beta_t} + \frac{\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{if } t < \tau \\ \frac{\mu_t}{\sigma_t^2} \quad \text{if } t \geq \tau \end{cases}$$

Proof. With equality (5.1) we get the following expression for the process J :

$$J_0 = \sup_{p\in\mathcal{A}} E\left[\int_0^T \left(p_s\mu_s - \frac{|p_s\sigma_s|^2}{2} + \lambda_s \log(1+p_s\beta_s)\right) ds\right],$$

then

$$J_0 \leq E\left[\int_0^T \sup_{p_s\beta_s>-1} \left\{p_s\mu_s - \frac{|p_s\sigma_s|^2}{2} + \lambda_s \log(1+p_s\beta_s)\right\} ds\right].$$

In the following we want to look for the strategy $\hat{p}$ whose maximizes for each $s \in [0,T]$ $p_s\mu_s - \frac{p_s^2\sigma_s^2}{2} + \lambda_s\log(1+p_s\beta_s)$ with the unique condition that $\hat{p}_s\beta_s > -1$ before the default, for that we study the function f :

$$f(x) = \mu_s x - \frac{\sigma_s^2}{2}x^2 + \lambda_s \log(1+\beta_s x)$$

and her derivative

$$f'(x) = \mu_s - \sigma_s^2 x + \frac{\lambda_s\beta_s}{1+\beta_s x}. \tag{5.3}$$

After the default, it is easy to see that the optimal strategy is $\hat{p}_t = \frac{\mu_t}{\sigma_t^2}$. Now we are interested by the optimal strategy before the default. Let $y = 1+\beta_s x$:

$$f'(x) = 0 \Leftrightarrow \begin{cases} \mu_s y - \frac{\sigma_s^2}{\beta_s}y(y-1) + \lambda_s\beta_s = 0 \\ y = 1+\beta_s x \end{cases}$$

Let y_- and y_+ be the roots of $\mu_s y - \frac{\sigma_s^2}{\beta_s}y(y-1) + \lambda_s\beta_s$ with $y_- \leq y_+$, then :

$$y_- y_+ = -\frac{\lambda_s\beta_s^2}{\sigma_s^2}.$$

Thus $y_- < 0 < y_+$ and by taking $\hat{p}_s = \frac{y_+-1}{\beta_s}$ we have that $\hat{p}_s\beta_s > -1$ and

$$\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s\log(1+\hat{p}_s\beta_s) = \sup_{p_s>-\frac{1}{\beta_s}} \left\{p_s\mu_s - \frac{p_s^2\sigma_s^2}{2} + \lambda_s\log(1+p_s\beta_s)\right\}.$$

From (5.3) and the condition $p_t\beta_t > -1$ we obtain:

$$\hat{p}_s = \frac{\sigma_s^2 - \mu_s\beta_s - \sqrt{(\mu_s\beta_s+\sigma_s^2)^2 + 4\lambda_s\beta_s^2\sigma_s^2}}{-2\beta_s\sigma_s^2}.$$

Then we have the following inequality

$$J_0 \leq E\left[\int_0^T \left(\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s \log(1+\hat{p}_s\beta_s)\right) ds\right].$$

Now it is sufficient to show that the strategy $\hat{p}$ is admissible, but that is easy with Assumption 5.1 and Assumption 5.2. Thus the previous inequality is an equality

$$J_0 = E\left[\int_0^T \left(\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s \log(1+\hat{p}_s\beta_s)\right) ds\right]$$

and the strategy $\hat{p}$ is an optimal strategy. □

If we substitute $\hat{p}_t$ by its value in the expression of the value function J_0, we get

$$\begin{aligned} J_0 &= E\left[\int_0^T \left(\frac{\mu_t^2}{4\sigma_t^2} - \frac{\mu_t^2}{2\beta_t^2} - \frac{\sigma_t^2}{4\beta_t^2} - \frac{\lambda_t}{2} + \frac{(\mu_t\beta_t+\sigma_t^2)^2\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{4\beta_t^2\sigma_t^2}\right.\right. \\ &\quad \left.\left. +\lambda_t \log\left(\frac{1}{2} + \frac{\mu_t\beta_t}{2\sigma_t^2} + \sqrt{(\mu_t\beta_t+\sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}\right)\right)\right]. \end{aligned}$$

Remark 5.2. *Assumption 5.1 can be reduced to the fact that the strategy $\hat{p}$ is an admissible strategy.*

Remark 5.3. *Recall that in the case of no default, the optimal strategy is given by*

$$p_t^0 = \frac{\mu_t}{\sigma_t^2}.$$

Thus, in the case of default, the optimal strategy $\hat{p}$ can be written under the form

$$\hat{p}_t = p_t^0 - \epsilon_t$$

where ϵ_t is an additional term given by

$$\epsilon_t = \begin{cases} \frac{\mu_t}{2\sigma_t^2} + \frac{1}{2\beta_t} - \frac{\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2}, & \forall t \leq T \\ 0, \quad \forall t \geq T \end{cases}$$

Note that $\epsilon_t \geq 0$, which is expected because of the default. After the default the optimal strategy is equal to the optimal strategy in a model without default.

Remark 5.4. *Note that if the coefficient β converges to 0, then the optimal strategy converges to $\frac{\mu_t}{\sigma_t^2}$, which is expected because if β converges to 0, all happens as if there is no default.*

6 Power utility

To complete the spectrum of important utility functions, in this section we calculate the value function and characterize the optimal strategy for the utility maximization problem with respect to

$$U(x) = x^\gamma, \quad x \geq 0, \quad \gamma \in (0,1).$$

Trading strategies and wealth process have the same meaning as in Section 5. We have

$$X_t^{x,p} = x + \int_0^t \frac{p_s X_{s^-}^{x,p}}{S_{s^-}} dS_s.$$

and by self-financing property we get :

$$dX_t^{x,p} = X_{t^-}^{x,p} p_t(\mu_t dt + \sigma_t dW_t + \beta_t dN_t).$$

By using Dolean's formula, we get an expression of the wealth $X_t^{x,p}$:

$$X_t^{x,p} = x \exp\left(\int_0^t p_s \mu_s ds + \int_0^t p_s \sigma_s dW_s - \frac{1}{2}\int_0^t |p_s\sigma_s|^2 ds + \int_0^t \log(1+p_s\beta_s) dN_s \right).$$

Definition 6.1. *The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $p = (p_t)_{0\leq t\leq T}$ that satisfy $\int_0^T |p_t\sigma_t|^2 dt + \int_0^T |\log(1+p_t\beta_t)|^2 \lambda_t dt < \infty$ and $X_t^{x,p} > 0$ for all $0 \leq t \leq T$.*

The investor faces the maximization problem

$$V(x) = \sup_{p\in\mathcal{A}} E[(X_T^{x,p})^\gamma]. \tag{6.1}$$

In order to find the value function and an optimal strategy we apply the same method as for the exponential utility function. Most of the proofs are identical to Section 3 and are given in the annex.
As in Section 3 and Section 5, we give a dynamic extension of the initial problem and define the value function for each time t. More precisely, we have

$$J_t = \operatorname*{ess\,sup}_{p\in\mathcal{A}} E\left[\frac{(X_T^{x,p})^\gamma}{(X_t^{x,p})^\gamma} \middle| \mathcal{G}_t \right] \quad a.s.$$

Now, for the sake of brevity we shall denote X_t^p instead of $X_t^{x,p}$.
As in Section 3, we have a characterization for the process J by dynamic programming. More precisely

Proposition 6.1. *$(J_t)_{0\leq t\leq T}$ is the smallest càd-làg $\mathbb{G}$-adapted process such that for all $p \in \mathcal{A}$ $((X_t^p)^\gamma J_t)_{0\leq t\leq T}$ is a supermartingale and $J_T = 1$.*

And we have also a characterization for the optimal strategy

Proposition 6.2. *Let $\hat{p} \in \mathcal{A}$, the two following assertions are equivalent :*

(i) $\hat{p}$ is an optimal strategy, i.e. $J_0 = \sup_{p\in\mathcal{A}} E[(X_T^p)^\gamma] = E\left[\left(X_T^{\hat{p}}\right)^\gamma\right]$.

(ii) $\left(\left(X_t^{\hat{p}}\right)^\gamma J_t\right)_{0\le t\le T}$ *is a martingale.*

Now we will characterize the process J as the solution of a BSDE. For that we define the following sets :

$$S^{+,\infty} = \left\{Y \text{ càd-làg } \mathbb{G}\text{-adapted such that } Y_t > 0 \ \forall\, t\in[0,T] \text{ and } \operatorname*{ess\,sup}_{\omega,t}(|Y_t(\omega)|) < \infty\right\}$$

$$L^2_{loc}(W) = \left\{Z \ \mathbb{G}\text{-predictable}, \ \int_0^T |Z_t|^2 dt < \infty \ a.s.\right\}$$

$$L^2_{loc}(M) = \left\{U \ \mathbb{G}\text{-predictable}, \ \int_0^T |U_t|^2 \lambda_t dt < \infty \ a.s.\right\}$$

From Proposition 6.1, $(J_t)_{0\le t\le T}$ is a supermartingale and we can write it under the following form with Doob-Meyer decomposition thanks to Dellacherie and Meyer (1980) :

$$dJ_t = dm_t - dA_t$$

with m a local martingale and A a càd-làg $\mathbb{G}$-predictable nondecreasing process where $A_0 = 0$. With a local martingale representation theorem, there exist $Z \in L^2_{loc}(W)$ and $U \in L^2_{loc}(M)$, such that :

$$dJ_t = Z_t dW_t + U_t dM_t - dA_t. \tag{6.2}$$

From Proposition 6.1 and Proposition 6.2 we can give a characterization of the process J with a BSDE

Theorem 6.1. *The value function $(J_t)_{0\le t\le T}$ is the smallest solution in $S^{+,\infty}\times L^2_{loc}(W)\times L^2_{loc}(M)$ of the following BSDE :*

$$\begin{cases} dJ_t &= Z_t dW_t + U_t dM_t - \sup_{p\in\mathcal{A}}\left\{\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)J_t + \gamma p_t\sigma_t Z_t \right. \\ & \qquad \left. + \lambda_t((1+p_t\beta_t)^p - 1)(J_t+U_t)\right\} dt \\ J_T &= 1 \end{cases} \tag{6.3}$$

The optimal strategy $\hat{p}$ to problem (6.1) is characterized by the following formula :

$$\hat{p}_t = \arg\max_{p\in\mathcal{A}}\left\{\frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2 J_t + \gamma p_t\left(\mu_t J_t + \sigma_t Z_t\right) + \lambda_t((1+p_t\beta_t)^p - 1)(J_t+U_t)\right\} \ dt\otimes d\mathbb{P} \ a.s.$$

Unfortunately we can not say if BSDE (6.3) admits a unique solution. But we have another characterization for the process J, this one is obtained by approximation using the following processes J^k

Proposition 6.3. *As previously for the exponential utility, for all $k \in \mathbb{N}$ there exists a bounded positive càd-làg $\mathbb{G}$-adapted process $(J_t^k)_{0\leq t\leq T}$ such that :*

$$J_t^k = \operatorname*{ess\,sup}_{p\in\mathcal{A}^k} E\left[\left(\frac{X_T^p}{X_t^p}\right)^\gamma \middle| \mathcal{G}_t\right] \quad a.s.$$

The process (J^k, Z^k, U^k) is the unique solution in $S^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE :

$$\begin{cases} dJ_t^k &= Z_t^k dW_t + U_t^k dM_t - \sup\limits_{p\in\mathcal{A}^k} \left\{ \left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) J_t^k + \gamma p_t \sigma_t Z_t^k \right. \\ & \left. \quad + \lambda_t((1+p_t\beta_t)^p - 1)(J_t^k + U_t^k) \right\} dt \\ J_T^k &= 1 \end{cases} \tag{6.4}$$

We get another characterization of process J with the processes J^k. More precisely

Proposition 6.4. $J_t = \lim\limits_{k\to\infty} \uparrow J_t^k$ *a.s.*

Appendix A : Proof of Proposition 6.1 and Proposition 6.2

The technics are similar to Section 3. We first want to show that J is the smallest process such that for all $p \in \mathcal{A}$, $((X_t^p)^\gamma J_t)_{0\leq t\leq T}$ is a supermartingale and $J_T = 1$. Let $(Y_t)_{0\leq t\leq T}$ be a $\mathbb{G}$-adapted process such that $\forall\ p \in \mathcal{A}$, $((X_t^p)^\gamma Y_t)_{0\leq t\leq T}$ is a supermartingale and $Y_T = 1$. For all $t \in [0,T]$ and for all $p \in \mathcal{A}$, we have :

$$E\left[(X_T^p)^\gamma Y_T | \mathcal{G}_t\right] \leq (X_t^p)^\gamma Y_t$$

Then

$$E\left[\frac{(X_T^p)^\gamma}{(X_t^p)^\gamma} \middle| \mathcal{G}_t\right] \leq Y_t.$$

Therefore we have :

$$\operatorname*{ess\,sup}_{p\in\mathcal{A}_t} E\left[\frac{(X_T^p)^\gamma}{(X_t^p)^\gamma} \middle| \mathcal{G}_t\right] \leq Y_t.$$

Then we get :

$$J_t \leq Y_t \quad a.s.$$

We want now to prove the equivalence for the optimal strategy : suppose that $\hat{p}$ is an optimal strategy, hence we have

$$J_0 = \sup_{p\in\mathcal{A}} E\left[(X_T^p)^\gamma\right] = E\left[(X_T^{\hat{p}})^\gamma\right].$$

As $\left((X_t^{\hat{p}})^\gamma J_t\right)_{0\le t\le T}$ is a supermartingale, we have the following inequalities :

$$E\left[(X_t^{\hat{p}})^\gamma J_t\right] \le J_0$$

and

$$E\left[(X_T^{\hat{p}})^\gamma | \mathcal{G}_t\right] \le (X_t^{\hat{p}})^\gamma J_t.$$

If we take the expectation in this last inequality we get

$$E\left[(X_T^{\hat{p}})^\gamma\right] \le E\left[(X_t^{\hat{p}})^\gamma J_t\right] \le J_0.$$

Therefore we have

$$E\left[(X_t^{\hat{p}})^\gamma J_t\right] = J_0.$$

Hence $\left((X_t^{\hat{p}})^\gamma J_t\right)_{0\le t\le T}$ is a supermartingale with a constant expectation value, then it is a martingale.
To show the converse, suppose that $\left((X_t^{\hat{p}})^\gamma J_t\right)_{0\le t\le T}$ is a martingale, then we have :

$$E\left[(X_T^{\hat{p}})^\gamma\right] = J_0.$$

Because $((X_t^p)^\gamma J_t)_{0\le t\le T}$ is a supermartingale for all $p \in \mathcal{A}$, we have :

$$E\left[(X_T^p)^\gamma\right] \le J_0$$

Thus we have :

$$J_0 = \sup_{p\in\mathcal{A}} E\left[(X_T^p)^\gamma\right] = E\left[(X_T^{\hat{p}})^\gamma\right].$$

Appendix B : Proof of Theorem 6.1

We first characterize the process A of Doob-Meyer decomposition (6.2). For that we use the properties of process J. By Itô's formula we get :

$$\begin{aligned}
d((X_t^p)^\gamma J_t) &= (X_{t^-}^p)^\gamma J_{t^-}\left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)dt + \gamma p_t\sigma_t dW_t + ((1+p_t\beta_t)^\gamma - 1)dN_t\right] \\
&\quad +(X_{t^-}^p)^\gamma(Z_t dW_t + U_t dM_t - dA_t) + \gamma(X_{t^-}^p)^\gamma p_t\sigma_t Z_t dt \\
&\quad +(X_{t^-}^p)^\gamma((1+p_t\beta_t)^\gamma - 1)U_t dt \\
&= \text{ local martingale } - (X_t^p)^\gamma\left\{dA_t - \left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)J_t\right.\right. \\
&\quad \left.\left. +\gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t + U_t)\right]dt\right\}
\end{aligned}$$

From Proposition 6.1, $((X_t^p)^\gamma J_t)_{0\le t\le T}$ is a supermartingale for all strategies $(p_t)_{0\le t\le T} \in \mathcal{A}$, thus :

$$dA_t - \left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right) J_t + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t+U_t)\right] dt \ge 0.$$

Then we have :

$$dA_t \ge \left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right) J_t + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t+U_t)\right] dt.$$

From Theorem 2.2 of Kramkov and Schachermayer (1999), there exists an optimal strategy to problem (6.1). With Proposition 6.2, this optimal strategy $\hat{p} \in \mathcal{A}$ is such that $\left(\left(X_t^{\hat{p}}\right)^\gamma J_t\right)_{0\le t\le T}$ is a martingale, thus we get :

$$dA_t - \left[\left(\gamma \hat{p}_t\mu_t + \frac{\gamma(\gamma-1)}{2}\hat{p}_t^2\sigma_t^2\right) J_t + \gamma \hat{p}_t\sigma_t Z_t + \lambda_t((1+\hat{p}_t\beta_t)^\gamma - 1)(J_t+U_t)\right] dt = 0.$$

Therefore we have :

$$dA_t = \sup_{p\in\mathcal{A}}\left\{\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right) J_t + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^p - 1)(J_t+U_t)\right\} dt.$$

Thus the process J is solution of the BSDE

$$\begin{cases} dJ_t &= Z_t dW_t + U_t dM_t - \sup\limits_{p\in\mathcal{A}}\left\{\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right) J_t + \gamma p_t\sigma_t Z_t \right. \\ & \qquad \left. + \lambda_t((1+p_t\beta_t)^p - 1)(J_t+U_t)\right\} dt \\ J_T &= 1 \end{cases}$$

Now we want to show that the process J is the smallest solution of this BSDE : let (Y, Z, U) be a solution of BSDE (6.3), we show that for all $p \in \mathcal{A}$ we have that $\left((X_t^p)^\gamma J_t\right)_{0\le t\le T}$ is a supermartingale :

$$\begin{aligned} d\left((X_t^p)^\gamma Y_t\right) &= Y_{t^-} d\left((X_t^p)^\gamma\right) + \left(X_{t^-}^p\right)^\gamma dY_t + d\left[(X_t^p)^\gamma, Y_t\right] \\ &= Y_{t^-}\left(X_{t^-}^p\right)^\gamma \left[\gamma p_t\mu_t dt + \gamma p_t\sigma_t dW_t + \tfrac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2 dt + \left[(1+p_t\beta_t)^\gamma - 1\right] dN_t\right] \\ &\quad + \left(X_{t^-}^p\right)^\gamma (Z_t dW_t + U_t dM_t - dA_t) + \gamma Z_t \left(X_{t^-}^p\right)^\gamma p_t\sigma_t dt \\ &\quad + U_t\left(X_{t^-}^p\right)^\gamma \left[(1+p_t\beta_t)^\gamma - 1\right] dN_t \\ &= dm_t - (X_t^p)^\gamma \left[dA_t - dA_t^p\right] \end{aligned}$$

with

$$dA_t^p = \left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right) Y_t + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^p - 1)(Y_t+U_t)\right] dt$$

and $(m_t)_{0\le t\le T}$ is a local martingale :

$$dm_t = (X_{t^-})^p \left[(\gamma p_t\sigma_t Y_{t^-} + Z_t)\, dW_t + \left[\left[(1+p_t\beta_t)^\gamma - 1\right](U_t + Y_{t^-}) + U_t\right] dM_t\right].$$

Then :

$$(X_t^p)^\gamma Y_t - (X_0^p)^\gamma Y_0 = m_t - m_0 - \int_0^t (X_s^p)^\gamma (dA_s - dA_s^p)$$

$$m_t \geq m_0 + (X_t^p)^\gamma Y_t - (X_0^p)^\gamma Y_0$$

$$m_t \geq m_0 - (X_0^p)^\gamma Y_0$$

Therefore $(m_t)_{0\leq t\leq T}$ is a lower bounded local martingale, thus $(m_t)_{0\leq t\leq T}$ is a supermartingale and $((X_t^p)^\gamma J_t)_{0\leq t\leq T}$ is a supermartingale for all $p \in \mathcal{A}$. From Proposition 6.1, we can affirm that $J_t \leq Y_t$ $a.s.$ for all $t \in [0, T]$.

Appendix C : Proof of Proposition 6.3

The proof of existence of process J^k is similar to the proof of Proposition 3.4. Thus we show that the process J^k is bounded

$$J_t^k = \operatorname*{ess\,sup}_{p\in\mathcal{A}^k} E\left[\exp\left(\int_t^T \gamma\mu_s p_s ds + \int_t^T \gamma\sigma_s p_s dW_s \right.\right.$$
$$\left.\left. - \frac{1}{2}\int_t^T \gamma|\sigma_s p_s|^2 ds + \int_t^T \gamma\log(1+p_s\beta_s)dN_s\right)\middle| \mathcal{G}_t\right].$$

Let Q^p be the equivalent probability to $\mathbb{P}$ defined by the formula :

$$\frac{dQ^p}{d\mathbb{P}} = \exp\left(\int_t^T \gamma\sigma_s p_s dW_s - \frac{1}{2}\int_t^T |\gamma\sigma_s p_s|^2 ds\right).$$

Then we have :

$$J_t^k = \operatorname*{ess\,sup}_{p\in\mathcal{A}^k} E_{Q^p}\left[\exp\left(\int_t^T \gamma\mu_s p_s ds + \frac{\gamma^2-\gamma}{2}\int_t^T |\sigma_s p_s|^2 ds + \int_t^T \gamma\log(1+p_s\beta_s)dN_s\right)\middle| \mathcal{G}_t\right].$$

As the processes μ, σ et β are supposed bounded, the process $(J_t^k)_{0\leq t\leq T}$ is bounded.
These processes $(J_t^k)_{0\leq t\leq T}$ have the same properties as the process $(J_t)_{0\leq t\leq T}$: for all strategies $p \in \mathcal{A}^k$, $\left((X_t^p)^\gamma J_t^k\right)_{0\leq t\leq T}$ is a supermartingale and there exits a strategy $\hat{p}$ such that $\left(\left(X_t^{\hat{p}}\right)^\gamma J_t^k\right)_{0\leq t\leq T}$ is a martingale.
As the process $(J_t^k)_{0\leq t\leq T}$ is a bounded supermartingale, so it admits a Doob-Meyer decomposition :

$$dJ_t^k = Z_t^k dW_t + U_t^k dM_t - dA_t^k$$

with $Z^k \in L^2(W)$, $U^k \in L^2(M)$ and A^k is a càd-làg $\mathbb{G}$-predictable nondecreasing process where $A_0 = 0$.
In using the properties of process J^k we can determine the form of process A^k as in the previous appendix and we obtain a BSDE for which the process J^k is solution. Since the set $\mathcal{A}^k$ is bounded this BSDE admits a unique solution.

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