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# A simple and efficient new algorithm to increase the regularity and vanishing moments of biorthogonal wavelets

Mehmet Ersoy <sup>a</sup>

<sup>a</sup>*Université de Savoie, Laboratoire de Mathématiques, 73376 Le Bourget-du-Lac Cedex, France*

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## Abstract

We present an elegant way to construct a highorder scheme to elevate the degree of regularity and the vanishing moments of wavelets at any order  $s$ . To this end, we consider a wavelet family orthonormal or biorthonormal and with an iterative process using a compact operator we build others wavelets (existing or none) which provides useful properties as the *elevation* of the regularity and vanishing moments. The created wavelets family are biorthogonal. Finally, we build a simple algorithm which computes new filters from the initial data.

*Key words:* Compact operator, Biorthogonal multi-resolution, Orthonormal multi-resolution, Wavelets' regularity, Wavelets' vanishing moments, fast algorithm

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## 1 Introduction

The aim of this paper is to present a new manner to construct biorthogonal wavelets with some properties of vanishing moments and regularity which could be a powerful tool in wavelet partial differential equations theory ; in particular in the case of the wavelet-Galerkin method (WG)(see e.g [1,3,5,10]) or the so-called hybrid method ([8]). The property of regularity and vanishing moment is more important in wavelets theory. To this end, we use an iterative process in order to reach the wished properties, for instance  $s$  vanishing moments. Then we get new wavelets or existing ones (if we consider for instance boxed spline wavelets as in section 1). The new wavelets have  $s$  vanishing moments while the biorthogonal ones are more regular. Moreover, this

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*Email address:* Mehmet.Ersoy@univ-savoie.fr (Mehmet Ersoy).

construction could be a way to recover all boxed spline wavelets. As in most works, we privilege compactly supported wavelets which are very attractive for such a construction.

The paper is organized as follows: in the first section we briefly recall some theoretical elements. The second section is devoted to the elevation of wavelets' regularity and vanishing moments. We use wavelets with  $p$  vanishing moments and following the idea in [13], we iterate the construction to obtain a biorthogonal system of wavelets (elevation means increasing of regularity and vanishing moments for the biorthogonal wavelets constructed). In Theorem 1 we present the main result to justify such a construction. The result lead naturally to the discussion on how this construction can be efficient in partial differential equation–wavelet theory: we explain how to possibly use the presented construction in the hybrid technique used by J. Liandrat and Pj Ponenti in [11] or in the case of wavelet-Galerkin method.

## 2 Some theoretical elements

We describe here some properties of Multi-Resolution Analysis (MRA) with wavelets. An MRA on  $\mathbb{R}$  is a sequence  $(V_j)_{j \in \mathbb{Z}}$  of increasing closed subset  $V_j \subset V_{j+1}$  of  $L^2(\mathbb{R})$ . Sets  $V_j$  are generated by a function called *scaling function* noted  $\phi$ . Generally  $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and we choose a normalisation as

$$\int_{\mathbb{R}} \phi(x) dx = 1. \tag{1}$$

Thanks to MRA properties, the family  $\{\phi(\cdot - k), k \in \mathbb{Z}\}$  forms a Riesz basis of  $L^2(\mathbb{R})$  and under particular hypothesis on  $\phi$ , we obtain an orthonormal or biorthogonal basis.

Let  $\phi(x)$  be a compactly supported *scaling function* which satisfies the *two-scale relation* (TSR)

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi(2x - k)$$

where  $\text{supp } \phi = [0, 2p - 1]$  for  $p \in \mathbb{N}$  (correspond to the degree of approximation, see definition 2.2) and  $h_k \in \mathbb{R}$ , for  $k = 0 \dots 2p - 1$  and  $h_k = 0$  otherwise.  $h_k$  are called *scaling discrete filters or coefficients*. By the equality (1), we have obviously  $\sum_{k \in \mathbb{Z}} h_k = 1$ .

This (TSR) is a simple consequence of  $V_0 \subset V_1$  (as we will see above).

Now, by a Fourier transform on (TSR), we obtain the following relation, noted  $\widehat{(\text{TSR})}$ :

$$\hat{\phi}(2w) = m_0(w)\hat{\phi}(w)$$

where  $m_0$  is the transfer (or characteristic) function defined by:

$$m_0(w) := \frac{1}{2} \sum_{k \in \mathbb{Z}} h_k e^{-ikw}.$$

If we assume that the set of functions  $(\phi(x - k))_{k \in \mathbb{Z}}$  forms a Riesz basis of  $L^2(\mathbb{R})$ , then we can define the space  $V_0$  as

$$V_0 = \overline{\text{Span}\{\phi(x - k), k \in \mathbb{Z}\}}$$

and

$$V_j := \overline{\text{Span}\{2^{j/2}\phi(2^j x - k), k \in \mathbb{Z}\}}$$

obtained by dilatation of  $2^j$  on  $\phi(\cdot - k)$ .

The spaces  $V_j$  and the scaling function  $\phi$  satisfy the following properties:

$$\forall j \in \mathbb{Z}, V_j \subset V_{j+1} \quad (2)$$

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\} \text{ and } \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \quad (3)$$

$$\forall j \in \mathbb{Z}, f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad (4)$$

$$f(x) \in V_0 \Leftrightarrow f(x - k) \in V_0, \forall k \in \mathbb{Z} \quad (5)$$

$$(\phi(x - k))_{k \in \mathbb{Z}} \text{ forms a Riesz basis } V_0. \quad (6)$$

These properties define a *Multi Resolution Analysis* (MRA).

Now to get a regular and localisation properties on the scaling function derivatives, we consider the following definition.

**Definition 2.1 (R-regular MRA)** *Under (MRA) hypothesis above and if we suppose that  $\phi$  is  $C^{R-1}$ ,  $\phi$  is almost everywhere differentiable, and for almost every  $x \in \mathbb{R}$ , for every integer  $\alpha \leq R$  and for all integer  $p$ , it exists  $C_p$  such that*

$$|\partial^\alpha \phi(x)| \leq C_p (1 + |x|)^{-p}.$$

*We call this set of properties an R-regular MRA.*

In addition, if the family  $(\phi(x - k))_{k \in \mathbb{Z}}$  is orthonormal, we call it an Orthonormal MRA (noted OMRA).

We recall here another useful equivalent definition of the vanishing moment,

**Definition 2.2** Let  $f \in L^2(\mathbb{R})$ . Let  $n \in \mathbb{N}$ . We say that  $f$  admits an approximation of degree  $n$  if and only if  $f$  satisfies a (TSR), i.e.

$$\exists (\alpha_k)_{k \in \mathbb{Z}} \subset \mathbb{R} \text{ such as } f(x) = \sum_{k \in \mathbb{Z}} \alpha_k f(2x - k)$$

and

$$\forall l = 0 \dots n - 1, \sum_{k \in \mathbb{Z}} (-1)^k k^l \alpha_k = 0.$$

Now, introducing the space  $W_j$  as a supplementary space to  $V_j$  in  $V_{j+1}$ , we get obviously  $W_0 \subset V_1$ . Thereby, we can write a similar relation to (TSR) called two details relation (TDR) on the function  $\psi$  called *wavelet*

$$\psi(x) = \sum_{k \in \mathbb{Z}} g_k \phi(2x - k).$$

Spaces  $W_j$  denotes the wavelet spaces, the coefficient  $g_k$  are the *wavelet discrete filters*. We see immediately that the support of  $\psi$  is  $\text{supp } \psi = [1 - p, p]$  since we have  $\text{supp } \phi = [0, 2p - 1]$ .

Applying Fourier transform on (TDR), we have the  $\widehat{\text{(TDR)}}$ :

$$\hat{\psi}(w) = m_1(w/2) \hat{\phi}(w/2)$$

where  $m_1$  is the transfer (or characteristic) function of the wavelet defined by:

$$m_1(w) = \frac{1}{2} \sum_{k \in \mathbb{Z}} g_k e^{-iw}.$$

Now, let us recall the definition of the vanishing moment,

**Definition 2.3** Let  $f \in L^2(\mathbb{R})$ ,  $n \in \mathbb{N}$ . We say that  $f$  admits  $n$  vanishing moments if

$$\forall k = 0 \dots n - 1, \int_{\mathbb{R}} x^k f(x) dx = 0.$$

Finally, we recall the notion of biorthonormal MRA (BMRA). Let us introduce two different non orthonormal MRA spanned by sequences of closed spaces  $V_j$  and  $\tilde{V}_j$ . Scaling functions are denoted by  $\phi$  and  $\tilde{\phi}$  respectively. A BMRA is an orthonormal relation given by  $(\phi, \tilde{\psi})$  and  $(\tilde{\phi}, \psi)$  where  $\psi$  and  $\tilde{\psi}$  are the corresponding wavelets of these MRA.

In what follows, the integer  $p$  represents the degree of approximation of the scaling function.

### 3 Main results

Our motivation is to construct a high order method to elevate the degree of vanishing moment and regularity of any given wavelet in order to get powerful wavelets for wavelet partial differential equations theory. In a first step, we show how to build these wavelets and then how the constructed ones could be efficient to solve partial differential equations.

For this purpose, let us introduce the compact operator  $T$  of kernel  $G(x, y) = \mathbb{1}_{[x-1, x]}(y)$  where its adjoint is defined by  $G^*(x, y) = \mathbb{1}_{[x, x+1]}(y)$ .

The operator  $T$  acting  $s$  times on the initial wavelet is an elegant way to build a new pair (or existing one) of biorthogonal wavelets. The process is called *an elevation at order  $s$* . To start the construction, we can choose initially biorthogonal wavelets or orthonormal ones (see Theorem 1 and Corollary 3.1 respectively).

**Theorem 1** *Let  $\phi, \tilde{\phi}$  be two compactly supported scaling functions of a BMRA. Let  $T$  be the operator described above and  $T^*$  its adjoint. Let  $s \in \mathbb{N}$  be the order of elevation and let us define  $\Phi^{(s)}$  and  $\tilde{\Phi}^{(s)}$  as*

$$\Phi^{(s)} := (T)^s \phi, \quad \tilde{\Phi}^{(s)} := (T^*)^{-s} \tilde{\phi}$$

where  $(X)^s$  designs the action of  $X$   $s$  times, and by definition  $X^0 := Id$ .

Then, the scaling functions  $\Phi^{(s)}$  and  $\tilde{\Phi}^{(s)}$  generates a BMRA. Moreover,

$$P_0^{(s)}(w) = \frac{1}{2^s} \frac{S_s(2w)}{S_s(w)} m_0(w), \quad \tilde{P}_0^{(s)}(w) = 2^s \frac{\overline{S_s(w)}}{S_s(2w)} \tilde{m}_0(w)$$

$$P_1^{(s)}(w) = 2^s \frac{m_1(w)}{S_s(w)}, \quad \tilde{P}_1^{(s)}(w) = \frac{1}{2^s} \overline{S_s(w)} \tilde{m}_1(w)$$

where  $m_0, \tilde{m}_0, P_0^{(s)}, \tilde{P}_0^{(s)}, m_1, \tilde{m}_1, P_1^{(s)}, \tilde{P}_1^{(s)}$  denote the transfer functions of  $\phi, \tilde{\phi}, \Phi^{(s)}, \tilde{\Phi}^{(s)}, \psi, \tilde{\psi}, \Psi^{(s)}, \tilde{\Psi}^{(s)}$  and  $S_s(w) = (1 - e^{-iw})^s$ .

To prove Theorem 1, let us recall the following results whose proof can be found in [13]

**Lemma 2** *Let  $\phi$  be a scaling function then*

(i)  $\Gamma > 0$  where  $\Gamma(w) = \sum_{k \in \mathbb{Z}} |\hat{\Phi}(w + 2k\pi)|^2$ .

(ii) The set  $\{\Phi(\cdot - k), k \in \mathbb{Z}\}$  forms a Riesz basis.

**Lemma 3** Let  $(\phi, \psi)$  be the scaling function and wavelet of an OMRA. We suppose  $\phi, \psi$  differentiable. Then the following equality holds

$$\langle \psi'(x), \psi'(x-k) \rangle + \langle \phi'(x), \phi'(x-k) \rangle = 4\langle \phi'(x), \psi'(x-2k) \rangle$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product of  $L^2(\mathbb{R})$ .

In order to complete the proof let us do the following remark.

**Remark 3.1** We have obviously,

$$\Psi(x) = 4 \int_{-\infty}^x \psi(t) dt, \quad 4 \int_{-\infty}^x \tilde{\Psi}(t) dt = \psi(x), \quad \forall x \in \mathbb{R}. \quad (7)$$

**Proof of theorem 1:** We proceed by induction on  $s$ . We start with  $s = 1$ . To simplify, we denote  $\Phi = \Phi^{(1)}$ .

We show firstly that the set  $\{\Phi(\cdot - k), k \in \mathbb{Z}\}$  cannot be orthonormal and thus there exists a biorthogonal function denoted by  $\tilde{\Phi}$  such that  $\langle \Phi(\cdot - k), \tilde{\Phi}(\cdot - l) \rangle = \delta_{k,l}$ . To this end, we proceed as follows:

**Step 1.**  $\Phi$  satisfies a the (TSR). Indeed, since

$$\Phi(x) = T\phi(x) = \int_{-\infty}^x (\phi(t) - \phi(t-1)) dt$$

and using the (TSR) on  $\phi$ . Furthermore, when  $\phi$  is compactly supported, we obtain the discrete filters of  $\Phi$  defined as:

$$H_k = \frac{h_{k-1} + h_k}{2}, \quad k = 0 \dots 2p \text{ with } h_{-1} = h_{2p} = 0.$$

**Step 2.** Now, it remains to show that  $(\Phi(\cdot - k))_{k \in \mathbb{Z}}$  forms a Riesz basis. For this purpose, we use Lemma 2 to find two constants  $0 < A \leq B$  such that

$$\forall w \in \mathbb{R}, \quad A \leq \sum_{k \in \mathbb{Z}} |\hat{\Phi}(w + 2k\pi)|^2 \leq B.$$

The fact that  $\Gamma$  is  $2\pi$ -periodic continuous with compacity property give us the constant  $B$  whereas  $A$  is obtained since  $\Gamma > 0$ .

**Step 3.** To conclude, we prove that  $(\Phi(\cdot - k))_{k \in \mathbb{Z}}$  never forms an orthonormal family but is nearly orthonormal, (i.e  $\langle \Phi(\cdot - k), \Phi(\cdot) \rangle_{L^2(\mathbb{R})} = 0$  for almost all integer  $k$ ) which ensures that the family  $\Phi$  cannot be orthonormal. Setting  $\tilde{\Phi}$  by  $\tilde{\phi} = T^*\tilde{\Phi}$ , using Property (7) and Lemma 3, we get the result.

Finally, by a straightforward computation, we get the formulas related to transfer functions.

Now it remains to iterate the process for  $s = 2$ . To this end, we use  $\Phi$  and  $\tilde{\Phi}$  instead of  $\phi$  and  $\tilde{\phi}$  and we repeat the process, i.e. for  $s = 3, 4, \dots$

□

Now, we present the effective simple construction of  $\Phi^{(s)}$  and  $\tilde{\Phi}^{(s)}$  by their filters.

**Theorem 4** *Under the hypothesis of theorem 1, if we suppose that  $\phi$  and  $\tilde{\phi}$  are compactly supported where  $\text{supp } \phi = [0, 2p - 1]$  and  $\text{supp } \tilde{\phi} = [0, 2\tilde{p} - 1]$  then writing  $H_k^{(s)}$ ,  $\tilde{H}_k^{(s)}$  the discrete filters of  $\Phi^{(s)}$ ,  $\tilde{\Phi}^{(s)}$  (resp.) and  $h_k^{(s)}$ ,  $\tilde{h}_k^{(s)}$  the discrete filters of  $\phi^{(s)}$ ,  $\tilde{\phi}^{(s)}$  we have,*

$$\begin{aligned}\text{supp } \Phi^{(s)} &= [0, 2p - 1 + s] \\ \text{supp } \tilde{\Phi}^{(s)} &= [0, 2\tilde{p} - 1 - s]\end{aligned}$$

with the following formulas to compute the new filters from the older

$$H_k^{(s)} = \frac{1}{2^s} \sum_{l=0}^s \binom{s}{l} h_{k-l}, \quad k = 0 \dots 2p - 1 + s \quad (8)$$

$$\sum_{l=0}^s \binom{s}{l} \tilde{H}_{k-l}^{(s)} = 2^s \tilde{h}_k, \quad k = 0 \dots 2\tilde{p} - 1 - s, \quad \tilde{p} \geq \frac{s+1}{2} \quad (9)$$

**Proof of Theorem 4:** We have already proved the result for  $s = 1$  (Step 1. of the proof of Theorem 1). Writing  $H_k^0 := h_k$ , we have

**for  $s = 1$ :**  $H_k^{(1)} = \frac{H_{k-1}^{(0)} + H_k^{(0)}}{2}$  and  $\Phi^{(1)}$  admits an approximation of degree  $p + 1$  and  $\text{supp } \Phi^{(1)} = [0, 2p]$  (to prove that  $\Phi^{(1)}$  admits an approximation of degree  $p + 1$  we use the definition and an induction on  $l$ ).

**for  $s = 2$ :**  $H_k^{(2)} = \frac{H_{k-1}^{(1)} + H_k^{(1)}}{2}$  and  $\Phi^{(2)}$  admits an approximation of degree  $p + 2$  and  $\text{supp } \Phi^{(1)} = [0, 2p + 1]$ . In addition, according to the first step  $s = 1$ , we can rewrite:

$$H_k^{(2)} = \frac{H_k^{(0)} + H_{k-1}^{(0)} + H_{k-1}^{(0)} + H_{k-2}^{(0)}}{2^2}.$$

By induction we obtain the result at order  $s$ .

To obtain  $\tilde{H}_k$ , we can compute it directly or it is sufficient to remark that : substituting  $h_k$  by  $\tilde{H}_k$  and  $\tilde{h}_k$  by  $H_k$  we obtain the result thanks to Property (7).



□

As said before, we can also perform this technique when the initial wavelet is orthonormal and we get the following result:

**Corollary 3.1** *Let  $\phi$  be the scaling functions of an OMRA. Let  $T$  be the operator described at the beginning of the section and  $T^*$  its adjoint. Let  $s \in \mathbb{N}$  be the order of elevation and let us define  $\Phi^{(s)}$  and  $\tilde{\Phi}^{(s)}$  as:*

$$\Phi^{(s)} = (T)^s \phi, \quad \tilde{\Phi}^{(s)} = (T^*)^{-s} \phi.$$

*Then, the scaling functions  $\Phi^{(s)}$  and  $\tilde{\Phi}^{(s)}$  generate a BMRA. Moreover, the transfer functions satisfy the relations:*

$$P_0^{(s)}(w) = \frac{1}{2^s} \frac{S_s(2w)}{S_s(w)} m_0(w), \quad \tilde{P}_0^{(s)}(w) = 2^s \frac{\overline{S_s(w)}}{S_s(2w)} m_0(w)$$

$$P_1^{(s)}(w) = 2^s \frac{m_1(w)}{S_s(w)}, \quad \tilde{P}_1^{(s)}(w) = \frac{1}{2^s} \overline{S_s(w)} m_1(w)$$

*where  $m_0, P_0^{(s)}, \tilde{P}_0^{(s)}, m_1, P_1^{(s)}, \tilde{P}_1^{(s)}$  are the transfer functions of  $\phi, \Phi^{(s)}, \tilde{\Phi}^{(s)}, \psi, \Psi^{(s)}, \tilde{\Psi}^{(s)}$  and  $S_s(w) = (1 - e^{-iw})^s$ .*

**Corollary 3.2** *Under the hypothesis of corollary 3.1, if we suppose that  $\phi$  is compactly supported where  $\text{supp } \phi = [0, 2p-1]$  then, writing  $H_k^{(s)}, \tilde{H}_k^{(s)}$  discrete filters of  $\Phi^{(s)}, \tilde{\Phi}^{(s)}$  ( resp. ) and  $h_k^{(s)}$  discrete filters of  $\phi$ , we have,*

$$\text{supp } \Phi^{(s)} = [0, 2p - 1 + s]$$

$$\text{supp } \tilde{\Phi}^{(s)} = [0, 2p - 1 - s]$$

*with the following formulas to compute the new filters from the older*

$$H_k^{(s)} = \frac{1}{2^s} \sum_{l=0}^s \binom{s}{l} h_{k-l}, \quad k = 0 \dots 2p - 1 + s \quad (10)$$

$$\sum_{l=0}^s \binom{s}{l} \tilde{H}_{k-l}^{(s)} = 2^s h_k, \quad k = 0 \dots 2p - 1 - s, \quad p \geq \frac{s+1}{2} \quad (11)$$

**Remark 3.2**

- (1) The wavelet  $\tilde{\Phi}^{(s)}$  is more regular (see Property 7) than  $\Phi^{(s)}$  but has less vanishing moments. Therefore, when we decompose a signal, we use  $\Phi^{(s)}$  for a better approximation and we reconstruct with  $\tilde{\Phi}^{(s)}$  (because when we commit errors, the errors are still regular).

- (2) We have gained vanishing moments and regularity but in counterpart we have increased the support of the scaling function. Nevertheless, the support is still **compact**. Furthermore, this operation preserves the parity of the initial wavelet. Moreover, if initial wavelets are curl free then so are the constructed wavelets. So, the presented wavelets could be used for construction of curl free wavelets for the problems as incompressible Navier-Stokes equations (see e.g. [6,2,7]).
- (3) From Theorem 1, we see that it is enough to apply iteratively the operator  $T$  in order to recover all boxed spline family (for instance as displayed on FIG. 1).

Now, if we know explicitly the initial wavelet  $(\psi, \tilde{\psi})$  then we can define  $(\Psi^{(s)}, \tilde{\Psi}^{(s)})$ . Indeed, knowing the analytical expression of the transfer functions  $m_0, m_1$  and/or  $\tilde{m}_0, \tilde{m}_1$ , we can write the analytical expression of  $\Phi$  and  $\tilde{\Phi}$  using  $(\widehat{\text{TSR}})$  to obtain:

$$\Phi^{(s)}(w) = \frac{S_s(w)}{\sigma(w)}\phi(w) \text{ and } \tilde{\Phi}^{(s)}(w) = \frac{\overline{\sigma(w)}}{S_s(w)}\tilde{\phi}(w)$$

where  $\sigma(w) = w^s$ . So, we can understand the function  $\sigma$  as the symbol of an homogeneous elliptic operator. Hence, identically to the work in [11], we can also use this construction to solve homogenous partial differential equations. One way to illustrate is the following: let us consider the following partial differential equation  $Lu = f$  where  $L$  is an elliptic operator. Let us consider  $(\psi_\alpha, \tilde{\psi}_{\tilde{\alpha}})$  a biorthogonal wavelet family. The expansion of  $f$  is given by  $\sum_{\alpha, \tilde{\alpha}} \langle f, \psi_\alpha \rangle \tilde{\psi}_{\tilde{\alpha}}$ . Applying  $L^{-1}$  to the previous expression of  $f$ , formally we get  $u = \sum_{\alpha, \tilde{\alpha}} \langle u, L^* \psi_\alpha \rangle L^{-1} \tilde{\psi}_{\tilde{\alpha}}$  where  $L^* \psi_\alpha$  and  $L^{-1} \tilde{\psi}_{\tilde{\alpha}}$  denotes the so-called pseudo-wavelets (see e.g. [4,8,12]). These wavelets are exactly the wavelets constructed in Theorem 1. Finally, the algorithm could be the following:

- (1) Interpolation step
- (2) Decomposition of  $f$  in the initial basis with the discrete filter  $h_k$  and  $\tilde{h}_k$ , we get  $d_f$  wavelet coefficient of  $f$
- (3) Decomposition of  $u$  obtained with  $d_u = 2^{(-j)s} d_f$  where  $d_u$  denotes the wavelet coefficient of  $u$  in the new basis with the discrete filter  $H_k$  and  $\tilde{H}_k$ .
- (4) Reconstruction of  $u$  with the discrete filter  $H_k$  and  $\tilde{H}_k$

Next, another approach to solve partial differential equation with wavelets is the wavelet-Galerkin method. As know, in such a method, the difficulty is to solve the algebraic system. One way to overpass this difficulty is to use the constructed wavelet in order to get simpler stiffness matrix ; here we use the *near orthonormality* property as we will see below (this idea is a generalisation of the work in [13]).

Let us recall the method in the case of  $Lu = f$  where  $L$  is a differential operator at order  $m$ . First, we consider the expansion of  $f$  and  $u$  in the constructed wavelets  $\Phi_k^{(s)}$  (at arbitrary order  $s$  where the family  $(\Phi_k^{(s)})_k$  is not orthonormal) basis such that

$$u = \sum_k \alpha_k \Phi_k^{(s)} \text{ and } f = \sum_k \beta_k \Phi_k^{(s)} \text{ where } \Phi_k^{(s)}(x) := \Phi^{(s)}(x - k),$$

thus we have

$$\sum_k \alpha_k L\Phi_k^{(s)} = \sum_k \beta_k \Phi_k^{(s)}.$$

Taking the scalar product with  $\Phi^{(s)}$  **instead of**  $\tilde{\Phi}^{(s)}$  (the biorthogonal one), we get the following algebraic system

$$\sum_k \alpha_k \langle L\Phi_k^{(s)}, \Phi_l^{(s)} \rangle = \sum_k \beta_k \langle \Phi_k^{(s)}, \Phi_l^{(s)} \rangle, l \in \mathbb{Z}. \quad (12)$$

Here, we prefer to use the nearly orthonormal property instead of the biorthogonality because a lot of terms vanish in the left hand side of the system (12). Therefore, we have some simplifications on the algebraic system, i.e.

$$\langle L\Phi_k^{(s)}, \Phi_l^{(s)} \rangle = 0 \text{ for almost any index } k.$$

As an illustration, we may consider the operator  $L = \frac{d^2}{dx^2}$ . Indeed, we have, for instance for  $s = 1$ ,  $\langle \frac{d^2}{dx^2}\Phi_0^{(1)}, \Phi_k^{(1)} \rangle = -\langle \frac{d}{dx}\Phi_0^{(1)}, \frac{d}{dx}\Phi_k^{(1)} \rangle$  and a simple computation give:

$$\langle \frac{d}{dx}\Phi_0^{(1)}, \frac{d}{dx}\Phi_k^{(1)} \rangle = 2\delta_{0,k} - \delta_{0,1+k} - \delta_{0,1-k}, k \in \mathbb{Z}.$$

To conclude this section, let us present the algorithms in biorthonormal and orthonormal case respectively (see Algorithm 1-2 and 3-4 related to the formulas (8)-(9) and (10)-(11)) and some correspondings constructed wavelets.

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Algorithm 1. Computation of the filters  $H_k^{(s)}$

**Require:**  $h_k, k = 0 \dots 2p - 1$

**for**  $k = 0$  to  $2p - 1 + s$  **do**

$$H_k^{(s)} = \frac{1}{2^s} \sum_{l=0}^s \binom{s}{l} h_{k-l}$$

**end for**

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Algorithm 2. Computation of the dual filters  $\tilde{H}_k^{(s)}$

**Require:**  $\tilde{h}_k, k = 0 \dots 2\tilde{p} - 1$

**for**  $k = 0$  to  $2\tilde{p} - 1 + s$  **do**

$$\tilde{H}_k^{(s)} = 2^s \tilde{h}_k - \sum_{l=1}^s \binom{s}{l} \tilde{H}_{k-l}$$

**end for**

---

Algorithm 3. Computation of the filters  $H_k^{(s)}$

**Require:**  $h_k, k = 0 \dots 2p - 1$

**for**  $k = 0$  to  $2p - 1 + s$  **do**

$$H_k^{(s)} = \frac{1}{2^s} \sum_{l=0}^s \binom{s}{l} h_{k-l}$$

**end for**

---

Algorithm 4. Computation of the dual filters  $\tilde{H}_k^{(s)}$

**Require:**  $h_k, k = 0 \dots 2\tilde{p} - 1$

**for**  $k = 0$  to  $2\tilde{p} - 1 + s$  **do**

$$\tilde{H}_k^{(s)} = 2^s h_k - \sum_{l=1}^s \binom{s}{l} \tilde{H}_{k-l}$$

**end for**

---

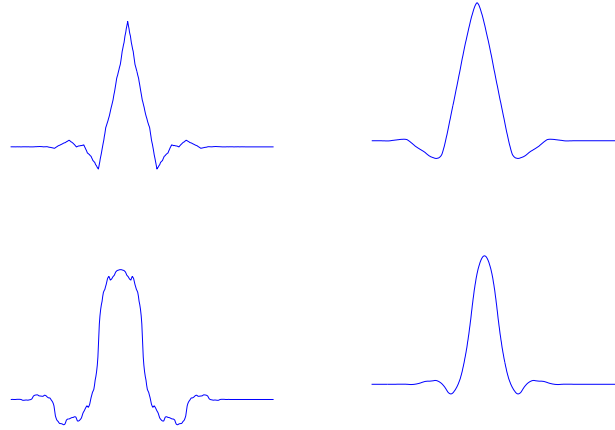


Fig. 1. Top left  $\phi$ , top right  $\tilde{\phi}$ , bottom left  $\Phi^{(1)}$ , bottom right  $\tilde{\Phi}^{(1)}$  where  $\phi, \tilde{\phi}$  are compactly supported B-spline 4 4

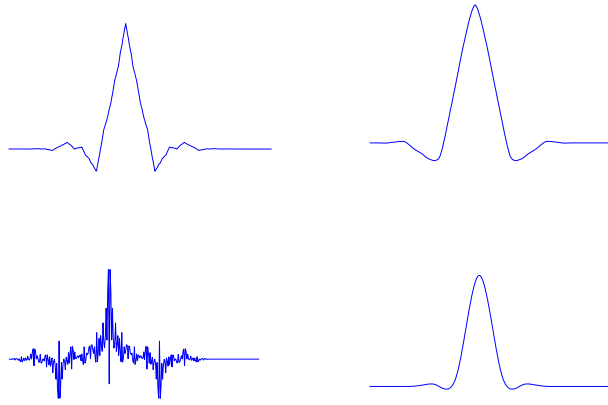


Fig. 2. Top left  $\phi$ , top right  $\tilde{\phi}$ , bottom left  $\Phi^{(2)}$ , bottom right  $\tilde{\Phi}^{(2)}$  where  $\phi, \tilde{\phi}$  are compactly supported B-spline 4 4

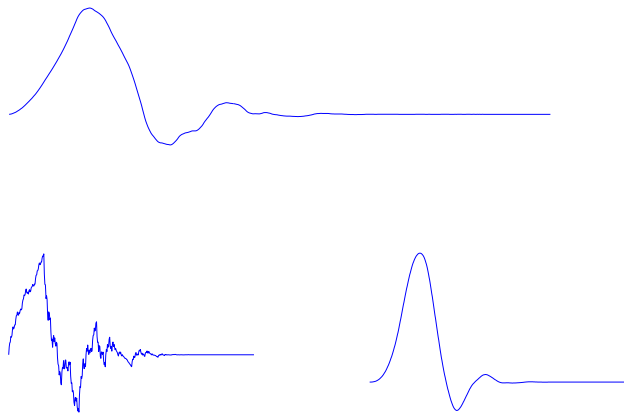


Fig. 3. Top left  $\phi$ , bottom left  $\Phi^{(1)}$ , bottom right  $\tilde{\Phi}^{(1)}$  where  $\phi$  is the Daubechies Wavelet with 4 vanishing moments

#### 4 Conclusion and perspectives

This construction permit to elevate the degree of vanishing moments and regularity which is important for decomposition-reconstruction problem for any given signal. Moreover, the constructed wavelets preserve some properties as parity, curl free of the initial wavelet ; indeed when the initial is curl free so is the constructed. So it may be possible to use this technique for the construction of curl free wavelet with more regularity and vanishing moments to solve

incompressible Navier-Stokes equations (see [6,7,9]). Furthermore, the computation of the new filters from the older are implemented easily (see (8)-(9) and (10)-(11)) and their applications can be the wavelet-Galerkin method using the property of near orthonormality of the created biorthogonal wavelets. Meanwhile, we could adapt the algorithm in a closed manner describe in [11] for more general partial differential problems. To this end, one possible approach is to follow the presented construction with an appropriate compact operator  $T$ .

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